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Limit Sets as Examples in Noncommutative Geometry

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Abstract. The fundamental group of a hyperbolic manifold acts on the limit set, giving rise to a cross-product C^*-algebra. We construct nontrivial K-cycles for the cross-product algebra, thereby extending some results of Connes and Sullivan to higher dimensions. We also show how the Patterson–Sullivan measure on the limit set can be interpreted as a center-valued KMS state.

1. Introduction

If M is a complete oriented $(n+1)$-dimensional hyperbolic manifold then its fundamental group Γ acts on the sphere-at-infinity S^n of the hyperbolic space H^{n+1}. The limit set Λ is a closed Γ-invariant subset of S^n which is the locus for the complicated dynamics of Γ on S^n. It is self-similar and often has noninteger Hausdorff dimension.

One can associate a cross-product C^*-algebra $C^*(\Gamma, \Lambda)$ to the action of Γ on Λ. It is then of interest to see how the geometry of M relates to properties of $C^*(\Gamma, \Lambda)$. In this paper we study two aspects of this problem. One aspect is an interpretation of the Patterson–Sullivan measure [40] in the framework of noncommutative geometry. The second aspect is the construction and study of K-cycles for $C^*(\Gamma, \Lambda)$.

The Patterson–Sullivan measure is an important tool in the study of the Γ-action on Λ. If $x \in H^{n+1}$ then the Patterson–Sullivan measure $d\mu_x$ on Λ describes how Λ is seen by an observer at x. In the first part of this paper we give an algebraic interpretation of the Patterson–Sullivan measure. If a C^*-algebra is equipped with a one-parameter group of $*$-automorphisms then there is a notion of a β-KMS (Kubo–Martin–Schwinger) state on the algebra. This notion arose from quantum statistical mechanics, where β is the inverse temperature. For each $x \in H^{n+1}$, we construct a one-parameter group of $*$-automorphisms of $C^*(\Gamma, \Lambda)$ and show that $d\mu_x$ gives rise to a $\delta(\Gamma)$-KMS state (up to normalization), where $\delta(\Gamma)$ is the critical exponent of Γ.

Putting these together for various x, we obtain a picture of a field of C^*-algebras over M with fiber isomorphic to $C^*(\Gamma, \Lambda)$. The different copies of $C^*(\Gamma, \Lambda)$ have different one-parameter automorphism groups. The global KMS state is defined on the algebra A of continuous sections of the field and takes value in the center $Z(A) = C(M)$. One can translate some geometric statements to algebraic statements. For example, if M is convex-cocompact then $\delta(\Gamma)$ is the unique β so that there is a β-KMS state.

The bulk of the paper is concerned with constructing cycles that represent nontrivial classes in the K-homology $\mathrm{KK}_*(C^*(\Gamma, \Lambda); \mathbb{C})$ of $C^*(\Gamma, \Lambda)$, or equivalently, in the equivariant K-homology $\mathrm{KK}_*^\Gamma(C(\Lambda); \mathbb{C})$ of $C(\Lambda)$. This program was started by Connes [11, Chapter IV.3]. A motivation comes from the goal of doing analysis on the self-similar set Λ. One can give various meanings to this phrase. What is relevant to this paper is the idea of Atiyah that the K-homology of a compact Hausdorff space has cycles given by abstract elliptic operators on the space [5]. This has developed into the K-homology of C^*-algebras, for which we refer to the book of Higson and Roe [18]. Cycles for $\mathrm{KK}_*^\Gamma(C(\Lambda); \mathbb{C})$ can be considered to be something like elliptic operators on Λ.

Such cycles are pairs (H, F) satisfying certain properties, where H is a Hilbert space on which $C(\Lambda)$ and Γ act, and F is a self-adjoint operator on H. In the bounded formalism F is bounded and commutes with the elements of Γ up to compact operators, while in the unbounded formalism F is generally unbounded and commutes with the elements of Γ up to bounded operators.

The computation of $\mathrm{KK}_*^\Gamma(C(\Lambda); \mathbb{C})$ can be done by established techniques. Our goal is to find *explicit* and *canonical* cycles (H, F) which represent nontrivial elements in $\mathrm{KK}_*^\Gamma(C(\Lambda); \mathbb{C})$. To make an analogy, a compact oriented Riemannian manifold has a signature class in its K-homology, but it also has a signature *operator*. Clearly the study of the signature operator leads to issues that go beyond the study of the corresponding K-homology class.

In order to get canonical cycles in the limit set case, we will require them to commute with Γ on the nose. This is quite restrictive. In particular, to get natural examples of such cycles we must use the bounded formalism. In effect, we will construct signature-type operators on limit sets. There are two issues: first to show that there is a nontrivial signature-type equivariant K-homology class on Λ, and second to find an explicit equivariant K-cycle within the K-homology class. Connes and Sullivan described a natural cycle when the limit set is a quasicircle in S^2 and studied its properties. As their construction used some special features of the two-dimensional case, it is not immediately evident how to extend their methods to higher dimension.

In Section 6, we compute $\mathrm{KK}_i^{\Gamma}(C(\Lambda);\mathbb{C})$ in terms of equivariant K-cohomology, giving $\mathrm{K}_{\Gamma}^{n-i}(S^n, S^n - \Lambda)$. The appearance of the smooth manifold $S^n - \Lambda$ indicates its possible relevance for constructing K-cycles when $\Lambda \neq S^n$.

As Γ acts conformally on S^n, we construct our K-cycles in the framework of conformal geometry. We start with the case $n = 2k$. In Section 7, we consider an arbitrary oriented manifold X of dimension $2k$, equipped with a conformal structure. The Hilbert space H of square-integrable k-forms on X is conformally invariant. We consider a certain conformally invariant operator F on H that was introduced by Connes–Sullivan–Teleman in the compact case [12]. Under a technical assumption (which will be satisfied in the cases of interest), we show that (H, F) gives a K-cycle for $C_0(X)$ whose K-homology class is that of the signature operator $d + d^*$. We then prove the invariance of the K-homology class under quasiconformal homeomorphisms of X. This will be relevant for limit sets, as a hyperbolic manifold has a deformation space consisting of new hyperbolic manifolds whose dynamics on S^n are conjugated to the old one by quasiconformal homeomorphisms.

If Λ is the entire sphere-at-infinity S^{2k} then the pair (H, F) gives a nontorsion class in $\mathrm{KK}_{2k}^{\Gamma}(C(S^{2k});\mathbb{C})$. If $\Lambda \neq S^{2k}$ then the idea will be to sweep topological charge from $S^{2k} - \Lambda$ to Λ. More precisely, we have an isomorphism $\mathrm{KK}_{2k}^{\Gamma}(C_0(S^{2k} - \Lambda);\mathbb{C}) \cong \mathrm{KK}_{2k}^{\Gamma}(C(S^{2k}), C(\Lambda);\mathbb{C})$ and a boundary map $\mathrm{KK}_{2k}^{\Gamma}(C(S^{2k}), C(\Lambda);\mathbb{C}) \to \mathrm{KK}_{2k-1}^{\Gamma}(C(\Lambda);\mathbb{C})$. We can then form a cycle in $\mathrm{KK}_{2k-1}^{\Gamma}(C(\Lambda);\mathbb{C})$ starting from the above cycle (H, F) for $\mathrm{KK}_{2k}^{\Gamma}(C_0(S^{2k} - \Lambda);\mathbb{C})$. Twisting the construction by Γ-equivariant vector bundles on $S^n - \Lambda$ gives cycles for the rational part of $\mathrm{KK}_{2k-1}^{\Gamma}(C(\Lambda);\mathbb{C})$ represented by $\mathrm{Im}\left(\mathrm{K}_{\Gamma}^{0}(S^n - \Lambda) \to \mathrm{K}_{\Gamma}^{1}(S^n, S^n - \Lambda)\right)$.

To make this explicit, in Section 8, we consider a manifold X as above equipped with a partial compactification $\overline{X}$. Putting $\partial\overline{X} = \overline{X} - X$, for appropriate $\overline{X}$ the pair (H, F) also gives a cycle for $\mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X});\mathbb{C})$. The boundary map $\mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X});\mathbb{C}) \to \mathrm{KK}_{2k-1}(C(\partial\overline{X});\mathbb{C})$ was described by Baum and Douglas in terms of Ext classes [6]. In our case it will involve the L^2-harmonic k-forms on X. If $\overline{X}$ is a smooth manifold-with-boundary then we show that the ensuing class in $\mathrm{Ext}(C(\partial\overline{X}))$ is given by certain homomorphisms from $C(\partial\overline{X})$ to the Calkin algebra of a Hilbert space of exact k-forms on $\partial\overline{X}$. If $\overline{X}$ is the closed $2k$-ball then the Hilbert space is the $H^{-1/2}$ Sobolev space of such forms on S^{2k-1}, and is Möbius-invariant.

A Fuchsian group has limit set $S^{n-1} \subset S^n$. A quasiFuchsian group is conjugate to a Fuchsian group by a quasiconformal homeomorphism ϕ of S^n. In particular, $\phi(S^{n-1}) = \Lambda$. In the case of a quasiFuchsian group with $n = 2k$, we show in Section 9 that the element of $\mathrm{KK}_{2k-1}^{\Gamma}(C(\Lambda);\mathbb{C})$

constructed by the Baum–Douglas boundary map is represented by the pushforward under $\phi\big|_{S^{2k-1}}$ of the Fuchsian Ext class. If $k = 1$ then we recover the K-homology class on a quasicircle considered by Connes and Sullivan. We also describe the Ext class when M is an acylindrical convex-cocompact hyperbolic 3-manifold with incompressible boundary, in which case Λ is a Sierpinski curve.

Section 10 deals with the case when the sphere-at-infinity S^n has dimension $n = 2k - 1$. If $\Lambda \neq S^{2k-1}$ then we consider how to go from such an odd cycle on $S^{2k-1} - \Lambda$ to an even K-cycle on Λ. Our discussion here is somewhat formal and uses smooth forms. In the case $k = 1$ we recover the K-cycle on a Cantor set considered by Connes and Sullivan. We also describe a K-cycle in the quasiFuchsian case and some other convex-cocompact cases.

For a quasiFuchsian limit set $\Lambda \subset S^n$, with n odd or even, the K-cycle for $C(\Lambda)$ is essentially the same as the K-cycle for $C(S^{n-1})$ in the Fuchsian case $S^{n-1} \subset S^n$, after pushforward by $\phi\big|_{S^{n-1}}$. As an example of the analytic issues concerning the K-cycle, in Section 11 we consider the subalgebra $\mathcal{A} = \phi^* C^\infty(S^n)\big|_{S^{n-1}}$ of $C(S^{n-1})$. We show that the Fredholm module $(\mathcal{A}, H, F)$ is p-summable for sufficiently large p. In the case $n = 2$, Connes and Sullivan showed that the infimum of such p equals $\delta(\Gamma)$. An interesting analytic question is how this result extends to $n > 2$.

Some related papers about limit sets are [1,2,16,27,39].

2. Hyperbolic Manifolds and the Patterson–Sullivan Measure

For background information on hyperbolic manifolds and conformal dynamics, we refer to McMullen [30]. For background information on the Patterson–Sullivan measure, we refer to Nicholls [31] and Sullivan [40].

Let Γ be a torsion-free discrete subgroup of $\mathrm{Isom}^+(H^{n+1})$, the orientation-preserving isometries of the hyperbolic space H^{n+1}. We will generally assume that Γ is nonelementary, i.e. not virtually abelian, although some statements will be clearly valid for elementary groups. Put $M = H^{n+1}/\Gamma$, an oriented hyperbolic manifold.

We write S^n for the sphere-at-infinity of H^{n+1}, and put $\overline{H^{n+1}} = H^{n+1} \cup S^n$, with the topology of the closed unit disk. Let Λ denote the *limit set* of Γ. It is the minimal nonempty closed Γ-invariant subset of S^n. In particular, given $x_0 \in H^{n+1}$, Λ can be constructed as the set of accumulation points of $x_0\Gamma$ in $\overline{H^{n+1}}$. The *domain of discontinuity* is defined to be $\Omega = S^n - \Lambda$, an open subset of S^n. There are right Γ-actions on Λ and Ω, with the action on Ω being free and properly discontinuous. The quotient Ω/Γ is called the *conformal boundary* of M. We denote the action of $g \in \Gamma$ on Λ by $R_g \in \mathrm{Homeo}(\Lambda)$. This induces a left action of Γ on $C(\Lambda)$, by $g \cdot f = R_g^* f$. That is, for $g \in \Gamma$, $f \in C(\Lambda)$ and $\xi \in \Lambda$,

$$(g \cdot f)(\xi) = f(\xi g). \tag{2.1}$$

The *convex core* of M is the Γ-quotient of the convex hull (in H^{n+1}) of Λ. The group Γ is *convex-cocompact* if the convex core of M is compact. If Γ is convex-cocompact then it is Gromov-hyperbolic and Λ equals its Gromov boundary.

Let x_0 be a basepoint in H^{n+1}. The critical exponent $\delta = \delta(\Gamma)$ is defined by

$$\delta = \inf \left\{ s : \sum_{\gamma \in \Gamma} \mathrm{e}^{-s\, d(x_0, x_0\gamma)} < \infty \right\}. \tag{2.2}$$

For each $x \in H^{n+1}$, the Patterson–Sullivan measure $\mathrm{d}\mu_x$ is a certain measure on Λ. If Γ is such that $\sum_{\gamma \in \Gamma} \mathrm{e}^{-\delta\, \mathrm{d}(x_0, x_0\gamma)} = \infty$ then $\mathrm{d}\mu_x$ is a weak limit

$$\mathrm{d}\mu_x = \lim_{s \to \delta^+} \frac{\sum_{\gamma \in \Gamma} \mathrm{e}^{-s\, d(x, x_0\gamma)}\, \delta_{x_0\gamma}}{\sum_{\gamma \in \Gamma} \mathrm{e}^{-s\, d(x_0, x_0\gamma)}} \tag{2.3}$$

of measures on $\overline{H^{n+1}}$. If $\sum_{\gamma \in \Gamma} \mathrm{e}^{-\delta\, d(x_0, x_0\gamma)} < \infty$ then one proceeds slightly differently [40, Section 1].

Given $x, x' \in H^{n+1}$ and $\xi \in \Lambda$, put

$$D(x, x', \xi) = \lim_{x'' \to \xi} \left(d(x, x'') - d(x', x'') \right). \tag{2.4}$$

Formally one can think of $D(x, x', \xi)$ as $d(x, \xi) - d(x', \xi)$, although the two terms do not make individual sense. One has

$$\begin{aligned} &D(x, x', \xi) = -D(x', x, \xi), \\ &D(x, x', \xi) + D(x', x'', \xi) = D(x, x'', \xi), \\ &D(x\gamma, x'\gamma, \xi\gamma) = D(x, x', \xi). \end{aligned} \tag{2.5}$$

One can verify from (2.3) that

$$\mathrm{d}\mu_x = \mathrm{e}^{-\delta\, D(x, x', \cdot)}\, \mathrm{d}\mu_{x'} \tag{2.6}$$

and

$$(R_g)_* \, \mathrm{d}\mu_x = \mathrm{d}\mu_{xg}. \tag{2.7}$$

From (2.6) and (2.7),

$$(R_g)_* \, \mathrm{d}\mu_x = \mathrm{e}^{\delta\, D(x, xg, \cdot)}\, \mathrm{d}\mu_x. \tag{2.8}$$

We note that if we have (2.7) for a fixed x, and then define $\mathrm{d}\mu_{x'}$ by (2.6), it follows that (2.7) also holds for $\mathrm{d}\mu_{x'}$. We also note that the Patterson–Sullivan measure is not a single Γ-invariant measure. Rather, it is a Γ-invariant conformal density in the sense of [40, Section 1].

3. The Cross-Product C^*-Algebra

The *algebraic cross-product* $C(\Lambda) \rtimes \Gamma$ consists of finite formal sums $f = \sum_{g\in\Gamma} f_g g$, with $f_g \in C(\Lambda)$. The product of $f, f' \in C(\Lambda) \rtimes \Gamma$ is given by

$$\left(\sum_{g\in\Gamma} f_g g\right)\left(\sum_{g'\in\Gamma} f'_{g'} g'\right) = \sum_{\gamma\in\Gamma}\sum_{gg'=\gamma} f_g\,(g\cdot f'_{g'})\,\gamma, \tag{3.1}$$

or

$$(ff')_\gamma(\xi) = \sum_{gg'=\gamma} f_g(\xi)\, f'_{g'}(\xi g). \tag{3.2}$$

The $*$-operator is given by

$$(f^*)_g = g\cdot \overline{f_{g^{-1}}}, \tag{3.3}$$

or

$$(f^*)_g(\xi) = \overline{f_{g^{-1}}}(\xi g). \tag{3.4}$$

For each $\xi \in \Lambda$, there is a $*$-homomorphism $\pi^\xi : C(\Lambda) \rtimes \Gamma \to B(l^2(\Gamma))$ given by saying that for $f = \sum_{g\in\Gamma} f_g g$ and $c \in l^2(\Gamma)$,

$$(\pi^\xi(f)\,c)_\gamma = \sum_{\gamma'\in\Gamma} k_{\gamma,\gamma'}(\xi)\, c_{\gamma'}, \tag{3.5}$$

where

$$k_{\gamma,\gamma'}(\xi) = f_{\gamma(\gamma')^{-1}}(\xi\gamma^{-1}). \tag{3.6}$$

The *reduced cross-product C^*-algebra* $C_r^*(\Gamma, \Lambda)$ is the completion of $C(\Lambda) \rtimes \Gamma$ with respect to the norm

$$f \to \sup_{\xi\in\Lambda} \| \pi^\xi \|_{l^2(\Gamma)}\,. \tag{3.7}$$

The homomorphism π^ξ extends to $C_r^*(\Gamma, \Lambda)$. For $f \in C_r^*(\Gamma, \Lambda)$, $\pi^\xi(f)$ acts on $l^2(\Gamma)$ by a matrix $k_{\gamma,\gamma'}(\xi)$ which comes as in (3.6) from a formal sum $f = \sum_{g\in\Gamma} f_g\, g$ with each f_g in $C(\Lambda)$ (although if Γ is infinite then one loses the finite support condition when taking the completion). The product in $C_r^*(\Gamma, \Lambda)$ is given by the same formula (3.2).

The *maximal cross-product C^*-algebra* $C^*(\Gamma, \Lambda)$ is given by completing $C(\Lambda) \rtimes \Gamma$ with respect to the supremum of the norms of all $*$-representations on a separable Hilbert space. There is an obvious homomorphism $C^*(\Gamma, \Lambda) \to C_r^*(\Gamma, \Lambda)$.

LEMMA 3.8. *In our case* $C^*(\Gamma, \Lambda) = C_r^*(\Gamma, \Lambda)$. *Furthermore,* $C_r^*(\Gamma, \Lambda)$ *is nuclear, simple, and purely infinite.*

Proof. It follows from [35, Theorem 3.1] and [3, Theorem 3.37] that Γ acts topologically amenably on S^n, and hence also on Λ. Then [3, Proposition 6.1.8] implies that $C^*(\Gamma, \Lambda) = C_r^*(\Gamma, \Lambda)$ and [3, Corollary 6.2.14] implies that $C_r^*(\Gamma, \Lambda)$ is nuclear. From [1, Proposition 3.1], $C_r^*(\Gamma, \Lambda)$ is simple and purely infinite. (We are assuming here that Γ is nonelementary.) □

Thus $C_r^*(\Gamma, \Lambda)$ is a Kirchberg algebra [36]. In addition, it lies in the so-called bootstrap class $\mathcal{N}$, as follows for example from [45, Section 10]. Thus $C_r^*(\Gamma, \Lambda)$ falls into a class of C^*-algebras that can be classified by their K-theory.

4. An Automorphism Group and a Positive Functional on $C_r^*(\Gamma, \Lambda)$

In this section, for each $x \in H^{n+1}$, we construct a corresponding one-parameter group of $*$-automorphisms of $C^*(\Gamma, \Lambda)$. We show that the Patterson–Sullivan measure $\mathrm{d}\mu_x$ gives rise to a $\delta(\Gamma)$-KMS state (up to normalization).

Propositions 4.2 and 4.8 of the present section are special cases of general results about quasi-invariant measures and KMS states [34, Chapter II.5]. We include the proofs, which are quite direct in our case, for completeness.

Fix $x \in H^{n+1}$. Given $t \in \mathbb{R}$ and $f \in C_r^*(\Gamma, \Lambda)$, put

$$(\alpha_t f)_g(\xi) = \mathrm{e}^{itD(x,xg^{-1},\xi)} f_g(\xi). \tag{4.1}$$

PROPOSITION 4.2. α *is a strongly-continuous one-parameter group of* $*$*-automorphisms of* $C_r^*(\Gamma, \Lambda)$.

Proof. For $f \in C(\Lambda) \rtimes \Gamma$, the kernel k^t corresponding to $\alpha_t f$ is

$$\begin{aligned} k^t_{\gamma,\gamma'}(\xi) &= (\alpha_t f)_{\gamma(\gamma')^{-1}}(\xi\gamma^{-1}) = \mathrm{e}^{itD(x,x\gamma'\gamma^{-1},\xi\gamma^{-1})} f_{\gamma(\gamma')^{-1}}(\xi\gamma^{-1}) \\ &= \mathrm{e}^{itD(x,x\gamma'\gamma^{-1},\xi\gamma^{-1})} k_{\gamma,\gamma'}(\xi) \\ &= \mathrm{e}^{itD(x\gamma,x\gamma',\xi)} k_{\gamma,\gamma'}(\xi) = \mathrm{e}^{it(D(x\gamma,x,\xi)-D(x\gamma',x,\xi))} k_{\gamma,\gamma'}(\xi). \end{aligned} \tag{4.3}$$

Thus $\pi^\xi(\alpha_t f) = U(t,\xi)\pi^\xi(f)U(t,\xi)^{-1}$, where $U(t,\xi)$ is the unitary operator that acts on $c \in l^2(\Gamma)$ by

$$(U(t,\xi)c)_g = \mathrm{e}^{itD(xg,x,\xi)} c_g. \tag{4.4}$$

This shows that if $f \in C_r^*(\Gamma, \Lambda)$ then $\alpha_t f \in C_r^*(\Gamma, \Lambda)$, and that $\alpha_t f$ is strongly-continuous in t.

Given $f, f' \in C_r^*(\Gamma, \Lambda)$,

$$\begin{aligned}(\alpha_t(ff'))_g(\xi) &= \mathrm{e}^{itD(x,xg^{-1},\xi)}\,(ff')_g(\xi) = \mathrm{e}^{itD(x,xg^{-1},\xi)} \sum_{\gamma\gamma'=g} f_\gamma(\xi)\, f'_{\gamma'}(\xi\gamma)\\ &= \sum_{\gamma\gamma'=g} \mathrm{e}^{itD(x,x\gamma^{-1},\xi)}\, f_\gamma(\xi)\, \mathrm{e}^{itD(x\gamma^{-1},xg^{-1},\xi)}\, f'_{\gamma'}(\xi\gamma)\\ &= \sum_{\gamma\gamma'=g} \mathrm{e}^{itD(x,x\gamma^{-1},\xi)}\, f_\gamma(\xi)\, \mathrm{e}^{itD(x,x(\gamma')^{-1},\xi\gamma)}\, f'_{\gamma'}(\xi\gamma)\\ &= ((\alpha_t f)(\alpha_t f'))_g(\xi). \end{aligned} \tag{4.5}$$

Thus $\alpha_t(ff') = (\alpha_t(f))(\alpha_t(f'))$. Next, given $f \in C_r^*(\Gamma, \Lambda)$,

$$\begin{aligned}(\alpha_t f)^*_g(\xi) &= \overline{\mathrm{e}^{itD(x,xg,\xi g)}\, f_{g^{-1}}(\xi g)} = \mathrm{e}^{-itD(x,xg,\xi g)}\, \overline{f_{g^{-1}}(\xi g)}\\ &= \mathrm{e}^{-itD(xg^{-1},x,\xi)}\, \overline{f_{g^{-1}}(\xi g)}\\ &= \mathrm{e}^{itD(x,xg^{-1},\xi)}\, \overline{f_{g^{-1}}(\xi g)} = (\alpha_t f^*)_g(\xi). \end{aligned} \tag{4.6}$$

Thus $(\alpha_t(f))^* = \alpha_t(f^*)$. This shows that α_t is a $*$-automorphism of $C_r^*(\Gamma, \Lambda)$. Finally, it is clear that for $t, t' \in \mathbb{R}$, $\alpha_t \circ \alpha_{t'} = \alpha_{t+t'}$. □

Define a positive functional $\tau : C_r^*(\Gamma, \Lambda) \to \mathbb{C}$ by

$$\tau(f) = \int_\Lambda f_e \, \mathrm{d}\mu_x. \tag{4.7}$$

It may not be a state, as $\mathrm{d}\mu_x$ may not be a probability measure. (See Lemma 5.14. One could imagine normalizing $\mathrm{d}\mu_x$ by dividing it by its mass, but this would cause further complications.)

For background on KMS states, we refer to [33, Chapter 8.12]. We now show that τ satisfies the KMS condition.

PROPOSITION 4.8. *Given $f, f' \in C_r^*(\Gamma, \Lambda)$, and $t \in \mathbb{R}$, put*

$$F(t) = \tau(f\,\alpha_t(f')). \tag{4.9}$$

Then F has a continuous bounded continuation to $\{z \in \mathbb{C} : 0 \leqslant \mathrm{Imag}(z) \leqslant \delta\}$ that is analytic in $\{z \in \mathbb{C} : 0 < \mathrm{Imag}(z) < \delta\}$, with

$$F(t + i\delta) = \tau(\alpha_t(f')\, f). \tag{4.10}$$

Proof. From [33, Proposition 8.12.3], it is enough to show that (4.10) holds when $f' \in C(\Lambda) \rtimes \Gamma$. In this case,

$$\begin{aligned} F(t) &= \int_\Lambda (f(\alpha_t f'))_e(\xi)\, \mathrm{d}\mu_x(\xi) = \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,(\alpha_t f')_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,\mathrm{e}^{itD(x,xg,\xi g)}\, f'_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,\mathrm{e}^{itD(xg^{-1},x,\xi)}\, f'_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi). \end{aligned} \tag{4.11}$$

Then

$$\begin{aligned} F(t+i\delta) &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,\mathrm{e}^{itD(xg^{-1},x,\xi)}\, f'_{g^{-1}}(\xi g)\,\mathrm{e}^{-\delta D(xg^{-1},x,\xi)}\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,\mathrm{e}^{itD(xg^{-1},x,\xi)}\, f'_{g^{-1}}(\xi g)\,\mathrm{d}\mu_{xg^{-1}}(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi)\,\mathrm{e}^{itD(xg^{-1},x,\xi)}\, f'_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi g) \\ &= \int_\Lambda \sum_{g\in\Gamma} f_g(\xi g^{-1})\,\mathrm{e}^{itD(xg^{-1},x,\xi g^{-1})}\, f'_{g^{-1}}(\xi)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} \mathrm{e}^{itD(xg,x,\xi g)}\, f'_g(\xi)\, f_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} \mathrm{e}^{itD(x,xg^{-1},\xi)}\, f'_g(\xi)\, f_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda \sum_{g\in\Gamma} (\alpha_t f')_g(\xi)\, f_{g^{-1}}(\xi g)\,\mathrm{d}\mu_x(\xi) \\ &= \int_\Lambda ((\alpha_t f')f)_e(\xi)\,\mathrm{d}\mu_x(\xi) = \tau(\alpha_t(f')f). \end{aligned} \tag{4.12}$$

This proves the claim. □

5. Center-Valued KMS State

In this section we allow the point $x \in H^{n+1}$ to vary. We construct a field of C^*-algebras over M, each isomorphic to $C^*(\Gamma, \Lambda)$. The global KMS state is defined on the algebra A of continuous sections of the field and takes value in the center $Z(A) = C(M)$. We translate some statements about the conformal dynamics of Γ on S^n to algebraic statements about the KMS state on A.

Let $C(H^{n+1}, C_r^*(\Gamma, \Lambda))$ denote the continuous maps from H^{n+1} to $C_r^*(\Gamma, \Lambda)$. We write an element of $C(H^{n+1}, C_r^*(\Gamma, \Lambda))$ as $F \equiv \sum_{g\in\Gamma} F_{x,g} g$, with $F_{x,g} \in C(\Lambda)$. Then Γ acts by automorphisms on $C(H^{n+1}, C_r^*(\Gamma, \Lambda))$, by the formula

$$(\gamma \cdot F)_{x,g} = R_\gamma^* F_{x\gamma, \gamma^{-1} g\gamma}. \tag{5.1}$$

Define an one-parameter group of $*$-automorphisms $\mathcal{A}_t$ of $C(H^{n+1}, C_r^*(\Gamma, \Lambda))$ by

$$(\mathcal{A}_t F)_{x,g}(\xi) = \mathrm{e}^{itD(x, xg^{-1}, \xi)}\, F_{x,g}(\xi). \tag{5.2}$$

LEMMA 5.3. *$\mathcal{A}_t$ is Γ-equivariant.*

Proof. Given $\gamma \in \Gamma$ and $F \in C(H^{n+1}, C_r^*(\Gamma, \Lambda))$,

$$\begin{aligned}(\mathcal{A}_t(\gamma \cdot F))_{x,g}(\xi) &= \mathrm{e}^{itD(x, xg^{-1}, \xi)}\, F_{x\gamma, \gamma^{-1}g\gamma}(\xi\gamma) \\ &= \mathrm{e}^{itD(x\gamma, x\gamma\gamma^{-1}g^{-1}\gamma, \xi\gamma)}\, F_{x\gamma, \gamma^{-1}g\gamma}(\xi\gamma) \\ &= (\gamma \cdot (\mathcal{A}_t F))_{x,g}(\xi). \qquad \square\end{aligned} \tag{5.4}$$

We write the positive functional τ of (4.7) as τ_x. For $F \in C(H^{n+1}, C_r^*(\Gamma, \Lambda))$, define $\mathcal{T}(F) \in C(H^{n+1})$ by

$$(\mathcal{T}(F))(x) = \tau_x \left(\sum_{g\in\Gamma} F_{x,g}\, g \right) = \int_\Lambda F_{x,e}(\xi)\, \mathrm{d}\mu_x(\xi). \tag{5.5}$$

LEMMA 5.6. *$\mathcal{T}$ is Γ-equivariant.*

Proof. Given $\gamma \in \Gamma$ and $F \in C(H^{n+1}, C_r^*(\Gamma, \Lambda))$,

$$\begin{aligned}(R_\gamma^*(\mathcal{T}(F)))(x) &= (\mathcal{T}(F))(x\gamma) = \tau_{x\gamma} \left(\sum_{g\in\Gamma} F_{x\gamma,g}\, g \right) = \int_\Lambda F_{x\gamma,e}\, \mathrm{d}\mu_{x\gamma} \\ &= \int_\Lambda F_{x\gamma,e}\, (R_\gamma)_* \mathrm{d}\mu_x = \int_\Lambda (R_\gamma)^* F_{x\gamma,e}\, \mathrm{d}\mu_x \\ &= \int_\Lambda (\gamma \cdot F)_{x,e}\, \mathrm{d}\mu_x = (\mathcal{T}(\gamma \cdot F))(x). \qquad \square\end{aligned} \tag{5.7}$$

Let A be the Γ-invariant subspace $\left(C(H^{n+1}, C_r^*(\Gamma, \Lambda))\right)^\Gamma$. Then A consists of the continuous sections of a field of C^*-algebras over M in the sense of [13, Definition 10.3.1] with each fiber A_m isomorphic to $C_r^*(\Gamma, \Lambda)$. The center of A is $Z(A) = C(M)$. By Lemma 5.3, the automorphisms $\mathcal{A}_t$ restrict to a one-parameter group $\mathcal{B}_t$ of $*$-automorphisms of A.

By Lemma 5.6, the map $\mathcal{T}$ restricts to a map $\mathcal{S}: A \to Z(A)$. For $F, F' \in A$, put

$$\mathcal{F}(t) = \mathcal{S}(F\,\mathcal{B}_t(F')). \tag{5.8}$$

As in Proposition 4.8, $\mathcal{F}$ has a continuous extension to $\{z \in \mathbb{C} : 0 \leqslant \mathrm{Imag}(z) \leqslant \delta\}$ that is analytic (in the sense of [37, Definition 3.30]) in $\{z \in \mathbb{C} : 0 < \mathrm{Imag}(z) < \delta\}$, with

$$\mathcal{F}(t + i\delta) = \mathcal{S}(\mathcal{B}_t(F')\,F). \tag{5.9}$$

LEMMA 5.10. *For all $\sigma \in Z(A)$ and $F \in A$, $\mathcal{S}(\sigma F) = \sigma \mathcal{S}(F)$.*

We will call a linear map $\mathcal{S}: A \to Z(A)$ satisfying the preceding properties a center-valued δ-KMS state for the pair $(A, \mathcal{B}_t)$, or just a δ-KMS state. We do not require that $\mathcal{S}(1_A)$ be 1.

PROPOSITION 5.11. (a) *If Γ is convex-cocompact then the pair $(A, \mathcal{B}_t)$ has a $\delta(\Gamma)$-KMS state, and this is the only β for which $(A, \mathcal{B}_t)$ has a β-KMS state. Furthermore, the KMS-state is unique up to multiplication by positive elements of $Z(A)$.*

(b) *If Γ is not convex-cocompact then for each $\beta \in [\delta(\Gamma), \infty)$ the pair $(A, \mathcal{B}_t)$ has a β-KMS state.*

(c) *If Γ is not convex-cocompact and has no parabolic elements then the set of β for which $(A, \mathcal{B}_t)$ has a β-KMS state is $[\delta(\Gamma), \infty)$.*

Proof. Existence of β: for all Γ, the Patterson–Sullivan measure gives rise to a $\delta(\Gamma)$-KMS state on $(A, \mathcal{B}_t)$. Fix $x \in H^{n+1}$. From [42, Theorem 2.19(i)], if Γ is not convex-cocompact then for each $\beta \in [\delta(\Gamma), \infty)$, there is a positive measure $\mathrm{d}\nu_x$ on Λ satisfying

$$(R_g)_* \,\mathrm{d}\nu_x = \mathrm{e}^{\beta D(x, xg, \cdot)}\,\mathrm{d}\nu_x. \tag{5.12}$$

Given such a measure, for $x' \in H^{n+1}$, define $\mathrm{d}\nu_{x'}$ by

$$\mathrm{d}\nu_{x'} = \mathrm{e}^{\beta D(x, x', \cdot)}\,\mathrm{d}\nu_x. \tag{5.13}$$

Then we can form a β-KMS state for the pair $(A, \mathcal{B}_t)$ in the same way as with the Patterson–Sullivan measure.

Uniqueness of β : suppose that Γ has no parabolic elements. Fix $x \in H^{n+1}$. Consider the cross-product groupoid $G = \Lambda \rtimes \Gamma$. Define the cocycle $c(\xi, g) = D(x, xg, \xi)$. Suppose that $\xi g = g$ and $c(\xi, g) = 0$. Take an upper half-plane model for H^{n+1} in which ξ is the point at infinity. Then the hyperbolic element g translates by a signed length $d(g)$ in the $(n+1)$-th coordinate (along with a possible rotation in the other coordinates), and

$|D(x, xg, \xi)| = |d(g)|$. It follows that g is the identity element of Γ. Thus the subgroupoid $c^{-1}(0)$ is principal.

Suppose that we have a β-KMS state for the pair $(A, \mathcal{B}_t)$. From [26, Proposition 3.2], the KMS state arises from a positive measure $d\nu_x$ on Λ which satisfies (5.12). Then from [42, Theorem 2.19], if Γ is convex-cocompact then $\beta = \delta(\Gamma)$, while if Γ is not cocompact then $\beta \in [\delta(\Gamma), \infty)$. Furthermore, if Γ is convex-cocompact then $d\nu_x$ is proportionate to the Patterson–Sullivan measure $d\mu_x$. □

LEMMA 5.14. *$\mathcal{S}(1)$ is a positive eigenfunction of $\triangle_M$ with eigenvalue $\delta(\Gamma)\,(n - \delta(\Gamma))$.*

Proof. The function Φ on H^{n+1}, given by setting $\Phi(x)$ to be the mass of $d\mu_x$, is the pullback to H^{n+1} of a positive eigenfunction ϕ of $\triangle_M$ with eigenvalue $\delta(\Gamma)\,(n - \delta(\Gamma))$ [40, Theorem 28]. □

In general, $\mathcal{S}(1)$ is not bounded on M.

LEMMA 5.15. *If Γ is convex-cocompact then $\mathcal{S}(1) \in C_0(M)$.*

Proof. With reference to the proof of Lemma 5.14, if Γ is convex-cocompact then [42, Theorem 2.13(a)] implies that $\phi \in C_0(M)$, from which the result follows. □

In the rest of this section we assume that Γ is convex-cocompact. Let $\pi_m : A \to A_m$ be the homomorphism from A to the fiber over $m \in M$. Let A_0 be the subalgebra of A consisting of elements a so that the function $m \to \| \pi_m(a) \|$ lies in $C_0(M)$. Then A_0 is the C^*-algebra associated to the continuous field of C^*-algebras on M, in the sense of [13, Section 10.4.1]. From Lemma 5.15, the map $\mathcal{S}\colon A \to Z(A)$ restricts to a map $\mathcal{S}_0\colon A_0 \to Z(A_0)$, for which (5.8) and (5.9) again hold. Also, for all $\sigma \in Z(A_0)$ and $F \in A_0$, $\mathcal{S}_0(\sigma F) = \sigma \mathcal{S}_0(F)$.

6. K-Homology of the Cross-Product Algebra

In this section we compute $\mathrm{KK}_i^{\Gamma}(C(\Lambda); \mathbb{C})$ in terms of the equivariant K-cohomology, in the sense of the Borel construction, of the pair (S^n, Ω).

We let $\mathrm{K}^*(\cdot, \cdot)$ denote the representable (i.e. homotopy-invariant) K-cohomology of a topological pair [43, Chapter 7.68, Remark in Chapter 8.43, Chapter 11]. We let $\mathrm{K}_*(\cdot)$ denote the unreduced Steenrod K-homology of a compact metric space [17, p. 161], [22]. Put $\overline{M} = (H^{n+1} \cup \Omega)/\Gamma$, so $\overline{M} = M \cup \partial \overline{M}$, where $\partial \overline{M}$ is the conformal boundary.

For background on analytic K-homology and (equivariant) KK-theory, we refer to Higson and Roe [18] and Blackadar [8]. We recall that

$KK_i^\Gamma(C(\Lambda);\mathbb{C})$ is isomorphic to $KK_i(C^*(\Gamma,\Lambda);\mathbb{C})$ [8, Theorem 20.2.7]. We wish to compute $KK_i^\Gamma(C(\Lambda);\mathbb{C})$ in term of classical homotopy-invariant topology (as opposed to proper homotopy invariance).

If X and $A \subset X$ are manifolds then the relative K-group $K^0(X, A)$ has generators given by virtual vector bundles on X that are trivialized over A, and similarly for $K^1(X, A)$. We let $K_\Gamma^*(X, A)$ denote the relative K-theory of the Borel construction, e.g. $K_\Gamma^*(S^n, \Omega) = K^*((E\Gamma \times S^n)/\Gamma, (E\Gamma \times \Omega)/\Gamma)$. A model for $E\Gamma$ is H^{n+1}. There is a Γ-equivariant diffeomorphism $SH^{n+1} \to H^{n+1} \times S^n$ that sends a unit vector $\widehat{v}$ at a point $x \in H^{n+1}$ to the pair (x, ξ), where ξ is the point on the sphere-at-infinity hit by the geodesic starting at x with initial vector $\widehat{v}$. Passing to Γ-quotients gives a diffeomorphism $SM \to (E\Gamma \times S^n)/\Gamma$. The subspace $(E\Gamma \times \Omega)/\Gamma$ can be identified with the unit tangent vectors $v \in SM$ with the property that the geodesic generated by v goes out the conformal boundary $\partial\overline{M}$. We note that $(E\Gamma \times \Omega)/\Gamma$ is homotopy-equivalent to $\partial\overline{M}$.

PROPOSITION 6.1. $KK_i(C(\Lambda);\mathbb{C}) \cong K^{n-i}(S^n, \Omega)$ *and* $KK_i^\Gamma(C(\Lambda);\mathbb{C}) \cong K_\Gamma^{n-i}(S^n, \Omega)$.

Proof. We have $KK_i(C(\Lambda);\mathbb{C}) \cong K_i(\Lambda)$ [22, Theorem C]. By Alexander duality [22, Theorem B],

$$K_i(\Lambda) \cong K^{n-i}(S^n, \Omega). \tag{6.2}$$

(The statement of [22, Theorem B] is in terms of reduced homology and cohomology, but is equivalent to (6.2) if Ω is nonempty. The case when Ω is empty is more standard [43, Theorem 14.11].)

There is a spectral sequence to compute $KK_{-i}^\Gamma(C(\Lambda);\mathbb{C})$, with differential of degree $+1$ and E_2-term given by $E_2^{p,q} = H^p(\Gamma, KK_{-q}(C(\Lambda);\mathbb{C})) = H^p(\Gamma, K_{-q}(\Lambda))$ [23, Theorem 6, [24, p. 199]. As $B\Gamma$ has a model that is a finite-dimensional CW-complex, there is no problem with convergence of the spectral sequence. By (6.2), $K_{-q}(\Lambda) \cong K^{n+q}(S^n, \Omega)$. Then $E_2^{p,q} \cong H^p(\Gamma, K^{n+q}(S^n, \Omega))$. This will be the same as E_2-term of the Leray spectral sequence [43, Theorem 15.27, Remarks 2 and 3, p. 351–352] to compute $K_\Gamma^{n+i}(S^n, \Omega)$ from the fibration $((E\Gamma \times S^n)/\Gamma, (E\Gamma \times \Omega)/\Gamma) \to B\Gamma$, with the same differentials. Changing the sign of i gives the claim. □

The significance of Proposition 6.1 is that when $\Omega \neq \emptyset$, it indicates that it should be possible to construct elements of $KK_*^\Gamma(C(\Lambda);\mathbb{C})$ by means of the smooth manifold Ω. More precisely, we have an isomorphism $KK_n^\Gamma(C_0(\Omega);\mathbb{C}) \cong KK_n^\Gamma(C(S^n), C(\Lambda);\mathbb{C})$ and a boundary map $KK_n^\Gamma(C(S^n), C(\Lambda);\mathbb{C}) \to KK_{n-1}^\Gamma(C(\Lambda);\mathbb{C})$. We can then start with an explicit cycle (H, F) for $KK_n^\Gamma(C_0(\Omega);\mathbb{C})$ and follow these maps to construct the corresponding cycle in $KK_{n-1}^\Gamma(C(\Lambda);\mathbb{C})$.

If $\Lambda = S^n$ then the signature class $\sigma \in \mathrm{KK}_n(C(S^n); \mathbb{C})$ satisfies $\sigma = C_n [S^n]$, where $[S^n] \in \mathrm{KK}_n(C(S^n); \mathbb{C})$ is the fundamental K-homology class, represented by the Dirac operator, and C_n is a power of 2. Under the isomorphism (6.2), σ goes over to $*\sigma = C_n [1] \in \mathrm{K}^0(S^n)$. Applying the Chern character gives $\mathrm{ch}(*\sigma) = C_n \cdot 1 \in \mathrm{H}^0(S^n; \mathbb{Q})$.

There is a natural transformation $f: \mathrm{K}^*(X, A) \otimes \mathbb{Q} \to \mathrm{K}^*(X, A; \mathbb{Q})$, where the right-hand-side is K-theory with coefficients. For general topological spaces, f need not be injective or surjective. If X and A are finite-dimensional CW-complexes then the Atiyah-Hirzebruch spectral sequence implies that f is injective and has dense image in the sense that the annihilator of $\mathrm{Im}(f)$, in the dual space $\left(\mathrm{K}^*(X, A; \mathbb{Q})\right)^*$, vanishes. (Note that tensoring with $\mathbb{Q}$ does not commute with arbitrary direct products.) If in addition $\mathrm{K}^*(X, A; \mathbb{Q})$ is finite-dimensional then f is an isomorphism. From the proof of Proposition 6.1 there is an injective map $\mathrm{KK}_i^\Gamma(C(\Lambda); \mathbb{C}) \otimes \mathbb{Q} \to \mathrm{K}_\Gamma^{n-i}(S^n, \Omega; \mathbb{Q})$ with dense image, which is an isomorphism when the right-hand side is finite-dimensional.

The Chern character gives an isomorphism between $\mathrm{K}^*(X, A; \mathbb{Q})$ and $\mathrm{H}^*(X, A; \mathbb{Q})$, after 2-periodization of the latter, and similarly for $\mathrm{K}_\Gamma^*(X, A; \mathbb{Q})$. One can compute $\mathrm{H}_\Gamma^*(S^n, \Omega; \mathbb{Q})$ using the Leray spectral sequence, with E_2-term $E_2^{p,q} = \mathrm{H}^p(\Gamma; \mathrm{H}^q(S^n, \Omega; \mathbb{Q}))$. If $\Lambda = S^n$ then $E_2^{0,0} = \mathrm{H}^0(\Gamma; \mathrm{H}^0(S^n; \mathbb{Q})) = \mathrm{H}^0(\Gamma; \mathbb{Q}) = \mathbb{Q}$. This term is unaffected by the differentials of the spectral sequence, and so it passes to the limit. In particular, the element $C_n \cdot 1 \in \mathrm{H}^0(S^n; \mathbb{Q})$ is Γ-invariant and gives a nonzero element of $\mathrm{H}_\Gamma^0(S^n; \mathbb{Q}) = \mathbb{Q}$. Hence there is a corresponding element of $\mathrm{K}_\Gamma^0(S^n; \mathbb{Q})$.

If $\Lambda \neq S^n$ and $n > 1$ then the exact sequence

$$0 \to \mathrm{H}^0(S^n, \Omega; \mathbb{Q}) \to \mathrm{H}^0(S^n; \mathbb{Q}) \to \mathrm{H}^0(\Omega; \mathbb{Q}) \to \mathrm{H}^1(S^n, \Omega; \mathbb{Q}) \to \mathrm{H}^1(S^n; \mathbb{Q}) \to \cdots \tag{6.3}$$

implies that $\mathrm{H}^0(S^n, \Omega; \mathbb{Q}) = 0$ and $\mathrm{H}^1(S^n, \Omega; \mathbb{Q}) = \mathbb{Q}^{|\pi_0(\Omega)|}/\mathbb{Q}$. Then the $E_2^{p,0}$-term of the spectral sequence for $\mathrm{H}_\Gamma^*(S^n, \Omega; \mathbb{Q})$ vanishes, and the $E_2^{0,1}$-term is $\mathrm{H}^0(\Gamma; \mathrm{H}^1(S^n; \mathbb{Q})) = \mathrm{H}^0(\Gamma; \mathbb{Q}^{|\pi_0(\Omega)|}/\mathbb{Q}) \cong \mathbb{Q}^{|\pi_0(\partial \overline{M})|}/\mathbb{Q}$. This term is unaffected by the differentials of the spectral sequence, and so it passes to the limit to give a contribution to $\mathrm{H}_\Gamma^1(S^n, \Omega; \mathbb{Q})$. There is a corresponding component of $\mathrm{K}_\Gamma^1(S^n, \Omega; \mathbb{Q})$.

7. An Even K-Cycle on a Manifold

In this section we consider an arbitrary oriented manifold X of dimension $2k$, equipped with a conformal structure. The Hilbert space H of square-integrable k-forms on X is conformally invariant. We consider a certain conformally invariant operator F that was introduced by Connes–Sullivan–

Teleman in the compact case [12]. Under a technical assumption, we show that (H, γ, F) gives a K-cycle for $C_0(X)$ whose K-homology class is that of the signature operator $d + d^*$. We then show the invariance of the K-homology class under quasiconformal homeomorphisms.

As a short digression, let us discuss why we use the operator F. It is well-known that the bounded K-cycle $\left(L^2(X; \Lambda^*), \frac{d+d^*}{\sqrt{1+\triangle}}\right)$ represents a nontrivial class in $K_{2k}(C_0(X))$. In the case $X = S^{2k}$, equipped with the action of a discrete group Γ by Möbius transformations, this operator gives rise to an element of $\mathrm{K}_{2k}^{\Gamma}(C(S^{2k}); \mathbb{C})$ [23], but at the price of making some modifications. Namely, there is a natural action of Γ on the L^2-forms on S^{2k} which is unitary on $L^2(S^{2k}; \Lambda^k)$ but is nonunitary on $L^2(S^{2k}; \Lambda^*)$ (as we are using a Riemannian structure). One has to modify the Γ-action in order to make it unitary. After doing so, the Γ-action commutes with $\frac{d+d^*}{\sqrt{1+\triangle}}$ *up to compact operators*. In later sections we will take $X = \Omega = S^{2k} - \Lambda$, on which the relevant group Γ acts conformally. We want a K-cycle that *commutes* with Γ. The Connes–Sullivan–Teleman operator is well-suited for this purpose. In addition, the conformal *invariance* of the Connes–Sullivan–Teleman operator will lead to the quasiconformal invariance of its K-homology class. This will be important when we consider quasiconformal deformations of Γ-actions.

For notation, if X is a Riemannian manifold then we let $L^2(X; \Lambda^q)$ denote the square-integrable q-forms on X, and similarly for $L^p(X; \Lambda^q)$, $L^p_c(X; \Lambda^q)$, $C^\infty(X; \Lambda^q)$, $C^\infty_c(X; \Lambda^q)$, and $H^s(X; \Lambda^q)$, where the c-subscript denotes compact support.

7.1. SOME CONFORMALLY-INVARIANT CONSTRUCTIONS

In this section, we define the operator F and introduce the technical Assumption 7.11.

As for the role of Assumption 7.11, if X is compact then one can use a pseudodifferential calculus to see that (H, γ, F) gives a K-cycle for $C(X)$. If X is noncompact then there is a local pseudodifferential calculus on X, but it will be insufficient to verify the K-cycle conditions. Instead we use finite-propagation-speed arguments for Dirac-type operators. Assumption 7.11 effectively arises in interpolating between our operator F and the Dirac-type operator $D = d + d^*$.

Let X be an oriented $2k$-dimensional manifold with a given conformal class $[g]$ of Riemannian metrics.

LEMMA 7.1. *There is a complete Riemannian metric in the conformal class.*

Proof. Without loss of generality, we may assume that X is connected. Choose a Riemannian metric g_0 in the conformal class. There is an exhaustion $K_0 \subset K_1 \subset \cdots$ of X by smooth compact manifolds-with-boundary, with $K_i \subset \mathrm{int}(K_{i+1})$. For $i > 1$, choose a nonnegative smooth function ϕ_i with $\mathrm{supp}(\phi_i) \subset \mathrm{int}(K_{i+1}) - K_{i-2}$ so that for any path $\{\gamma_i(t)\}_{t\in[0,1]}$ from ∂K_{i-1} to ∂K_i, $\int_0^1 e^{\phi_i(\gamma_i(t))}\, g_0(\gamma_i', \gamma_i')^{1/2}\, dt \geqslant 1$. Put $\phi = \sum_i \phi_i$. Then $g = e^{2\phi}\, g_0$ is complete. □

We now make some constructions that are independent of the choice of the complete Riemannian metric g in the conformal class $[g]$. Consider the complex Hilbert space $H = L^2(X; \Lambda^k)$ of square-integrable k-forms on X, with its conformally invariant inner product. There is an obvious action of $C_0(X)$ on H. Let γ be the conformally invariant $\mathbb{Z}_2$-grading operator on H given by

$$\gamma = i^k *. \tag{7.2}$$

Let $H = H_+ \oplus H_-$ be the corresponding orthogonal decomposition. There are operators

$$d : C_c^\infty(X; \Lambda^{k-1}) \to C_c^\infty(X; \Lambda^k) \tag{7.3}$$

and

$$d^* : C_c^\infty(X; \Lambda^{k+1}) \to C_c^\infty(X; \Lambda^k). \tag{7.4}$$

Then

$$\mathrm{Im}(d^*) = \gamma\ \mathrm{Im}(d). \tag{7.5}$$

There is a conformally invariant orthogonal decomposition

$$H = \overline{\mathrm{Im}(d)} \oplus \overline{\mathrm{Im}(d^*)} \oplus \mathcal{H}, \tag{7.6}$$

where

$$\mathcal{H} = \{\omega \in H \cap C^\infty(X; \Lambda^k) : d\omega = d^*\omega = 0\}. \tag{7.7}$$

Furthermore, $\mathcal{H}$ is an orthogonal direct sum $\mathcal{H}_+ \oplus \mathcal{H}_-$ of its self-dual and anti-self-dual subspaces.

We note that the normed vector space $L_c^{2k/k-1}(X; \Lambda^{k-1})$ is conformally invariant.

LEMMA 7.8. *$\overline{\mathrm{Im}(d)}$ equals the closure of the image of d on $\{\eta \in L_c^{2k/k-1}(X; \Lambda^{k-1}) : d\eta \in L^2(X; \Lambda^k)\}$.*

Proof. Clearly $\overline{\mathrm{Im}(d)}$ is contained in the closure of the image of d on $\{\eta \in L_c^{2k/k-1}(X; \Lambda^{k-1}) : d\eta \in L^2(X; \Lambda^k)\}$. Conversely, suppose that $\eta \in L_c^{2k/k-1}(X; \Lambda^{k-1})$ has $d\eta \in L^2(X; \Lambda^k)$. Let $\rho \in C_c^\infty(\mathbb{R})$ be an even function with support in $[-1, 1]$ and $\int_{\mathbb{R}} \rho(s)\,\mathrm{d}s = 1$. Put $\triangle = dd^* + d^*d$. For $\epsilon > 0$, put

$$\widehat{\rho}(\epsilon^2\triangle) = \int_{\mathbb{R}} \mathrm{e}^{is\epsilon(d+d^*)}\,\rho(s)\,\mathrm{d}s = \int_{\mathbb{R}} \cos(s\epsilon\sqrt{\triangle})\,\rho(s)\,\mathrm{d}s. \tag{7.9}$$

By elliptic theory, $\widehat{\rho}(\epsilon^2\triangle)\eta \in C^\infty(X; \Lambda^{k-1})$. By finite propagation speed arguments [18, Proposition 10.3.1], the support of $\widehat{\rho}(\epsilon^2\triangle)\eta$ lies within distance ϵ of the essential support of η, so $\widehat{\rho}(\epsilon^2\triangle)\eta \in C_c^\infty(X; \Lambda^{k-1})$. Finally, by the functional calculus, $\lim_{\epsilon\to 0} d\left(\widehat{\rho}(\epsilon^2\triangle)\eta\right) = \lim_{\epsilon\to 0} \widehat{\rho}(\epsilon^2\triangle)d\eta = d\eta$ in $L^2(X; \Lambda^k)$. □

Define $F \in B(H)$ by

$$F(\omega) = \begin{cases} \omega & \text{if } \omega \in \overline{\mathrm{Im}(d)}, \\ -\omega & \text{if } \omega \in \overline{\mathrm{Im}(d^*)}, \\ 0 & \text{if } \omega \in \mathcal{H}. \end{cases} \tag{7.10}$$

Then $F^* = F$ and F anticommutes with γ.

ASSUMPTION 7.11. *There is a complete Riemannian metric in the conformal class such that for each $\omega \in \overline{\mathrm{Im}(d)}$, there is an $\eta \in L^2(X; \Lambda^{k-1})$ with $d\eta = \omega$.*

We do not know if Assumption 7.11 is really necessary for what follows, but it is required for our proofs. It is equivalent to saying that there is a gap away from zero in the spectrum of the Laplacian on $L^2(X; \Omega^k)$ [28, Proposition 1.2].

EXAMPLE 1. Assumption 7.11 is satisfied for the conformal class of the unit ball in $\mathbb{R}^{2k}$, by taking the hyperbolic metric. More generally, it is satisfied when X is the interior of a compact manifold-with-boundary $\overline{X}$, and the conformal class comes from a smooth Riemannian metric g_0 on $\overline{X}$. One can see this by using the complete asymptotically hyperbolic metric on X given by $g = \rho^{-2} g_0$, where near the boundary $\partial\overline{X}$, $\rho \in C^\infty(\overline{X})$ equals the distance function to the boundary with respect to g_0. Then the essential spectrum of the k-form Laplacian on X will be the same as that of the essential spectrum of the k-form Laplacian on H^{2k}, which has a gap away from zero.

EXAMPLE 2. Assumption 7.11 is satisfied for the conformal class of the standard Euclidean metric on $\mathbb{R}^{2k}$. Consider a radially symmetric metric on $\mathbb{R}^{2k}$ of the form $g = \sigma^2(r)\,(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2)$, where $\sigma\colon (0, \infty) \to (0, \infty)$ is a smooth function satisfying

$$\sigma(r) = \begin{cases} 1 & \text{if } r < 1, \\ \frac{1}{r \ln r} & \text{if } r > 2. \end{cases} \tag{7.12}$$

From [15, Theorem 2.2], the essential spectrum of the k-form Laplacian on $(\mathbb{R}^{2k}, g)$ is bounded below by a positive constant. (In the case $k=1$, $(\mathbb{R}^2, g)$ has a hyperbolic cusp at infinity.)

EXAMPLE 3. Suppose that a discrete group Γ acts properly and cocompactly on X. Considering metrics on X that pullback from the orbifold X/Γ, whether or not Assumption 7.11 is satisfied for these metrics is topological, i.e. independent of the metric on X/Γ.

7.2. A CONFORMALLY-INVARIANT K-CYCLE

In this section, under Assumption 7.11, we show that (H, γ, F) gives a K-cycle for $C_0(X)$ whose K-homology class is that of the signature operator $d + d^*$.

For notation, if H is a Hilbert space then we denote the bounded operators on H by $B(H)$, the compact operators on H by $K(H)$ and the Calkin algebra by $Q(H) = B(H)/K(H)$. We recall that a cycle for $\mathrm{KK}_0(C_0(X); \mathbb{C})$ is given by a triple (H, γ, F) where

(1) H is a separable Hilbert space with $\mathbb{Z}_2$-grading operator $\gamma \in B(H)$,
(2) There is a $*$-homomorphism $C_0(X) \to B(H)$, and
(3) $F \in B(H)$ is such that $F\gamma + \gamma F = 0$ and for all $a \in C_0(X)$, we have $a(F^2 - I) \in K(H)$, $a(F - F^*) \in K(H)$, and $[F, a] \in K(H)$.

We now consider the triple (H, γ, F) of Section 7.1. We let $P_{\overline{\mathrm{Im}(d)}}$, $P_{\overline{\mathrm{Im}(d^*)}}$, and $P_{\mathcal{H}}$ denote orthogonal projections onto $\overline{\mathrm{Im}(d)}$, $\overline{\mathrm{Im}(d^*)}$, and $\mathcal{H}$, respectively. We let G denote the Green's operator for $\triangle$ on $L^2(X; \Lambda^k)$, so $\triangle G = G\triangle = I - P_{\mathcal{H}}$.

PROPOSITION 7.13. *For all $a \in C_0(X)$, $a(F^2 - I)$ is compact.*

Proof. We may assume that $a \in C_c^\infty(X)$. This is because for any $a \in C_0(X)$, there is a sequence $\{a_i\}_{i=1}^\infty$ in $C_c^\infty(X)$ with $\lim_{i\to\infty} a_i = a$ in the sup norm. Then $a(F^2 - I)$ will be the norm limit of the compact operators $a_i(F^2 - I)$, and hence compact.

We have $I - F^2 = P_{\mathcal{H}}$. Let K be the support of a. Choose a complete Riemannian metric g in the given conformal class. Applying Gårding's inequality [18, 10.4.4] with $D = d + d^*$, there is a $c > 0$ so that for all $\omega \in H$,

$$c \,\|\, P_{\mathcal{H}}\omega \,\|_{H^1(K;\Lambda^k)} \leqslant \|\, P_{\mathcal{H}}\omega \,\|_{L^2(M;\Lambda^k)} \leqslant \|\, \omega \,\|_{L^2(M;\Lambda^k)} \,. \tag{7.14}$$

It follows that the map $\omega \to a(P_{\mathcal{H}}\omega)\big|_K$ is bounded from $L^2(M;\Lambda^k)$ to $H^1(K;\Lambda^k)$. By Rellich's Lemma [18, 10.4.3], the inclusion map from $H^1(K;\Lambda^k)$ to $L^2(M;\Lambda^k)$ is compact. The proposition follows. □

PROPOSITION 7.15. *If Assumption 7.11 is satisfied then for all* $a \in C_0(X)$, $[F,a]$ *is compact.*

Proof. It is enough to prove the proposition for $a \in C_c^\infty(X)$. We may assume that a is real. Write the action of a on H as a (3×3)-matrix with respect to the decomposition (7.6). Then we must show that its off-diagonal entries are compact. By the self-adjointness of a, it is enough to show that $\left(I - P_{\overline{\mathrm{Im}(d)}}\right) a P_{\overline{\mathrm{Im}(d)}} : \overline{\mathrm{Im}(d)} \to \overline{\mathrm{Im}(d^*)} \oplus \mathcal{H}$ and $\left(I - P_{\overline{\mathrm{Im}(d^*)}}\right) a P_{\overline{\mathrm{Im}(d^*)}} : \overline{\mathrm{Im}(d^*)} \to \overline{\mathrm{Im}(d)} \oplus \mathcal{H}$ are compact.

Given $\eta \in C_c^\infty(X;\Lambda^{k-1})$,

$$\begin{aligned} a\,d\eta &= d(a\eta) - da \wedge \eta \\ &= d(a\eta) - da \wedge (P_{\mathcal{H}}\eta + dGd^*\eta + d^*Gd\eta) \\ &= d(a(\eta - P_{\mathcal{H}}\eta - dGd^*\eta)) - da \wedge G^{1/2}d^*G^{1/2}d\eta. \end{aligned} \tag{7.16}$$

Thus

$$(I - P_{\overline{\mathrm{Im}(d)}})\,aP_{\overline{\mathrm{Im}(d)}} = -(I - P_{\overline{\mathrm{Im}(d)}})\,\mathrm{d}a \wedge G^{1/2}d^*G^{1/2}P_{\overline{\mathrm{Im}(d)}}. \tag{7.17}$$

As $d^*G^{1/2}$ is bounded, to show that $(I - P_{\overline{\mathrm{Im}(d)}})\,aP_{\overline{\mathrm{Im}(d)}}$ is compact, it suffices to show that $da \wedge G^{1/2} : (\overline{\mathrm{Im}(d^*)} \subset L^2(X;\Lambda^{k-1})) \to L^2(X;\Lambda^k)$ is compact. Put $D = d + d^*$, so $D^2 = \triangle$. By Assumption 7.11, there is an even function $\rho \in C_0(\mathbb{R})$ so that when acting on $\overline{\mathrm{Im}(d^*)} \subset L^2(X;\Lambda^{k-1})$, we have $G^{1/2} = \rho(D)$. We can assume that $\rho(x) = \frac{1}{|x|}$ for $|x|$ large. The compactness now follows from the fact that $da \wedge \rho(D)$: $L^2(X;\Lambda^{k-1}) \to L^2(X;\Lambda^k)$ is compact [18, Proposition 10.5.2].

Let $(da)_\sharp$ denote the vector field that is dual to da, with respect to g. Given $\eta \in C_c^\infty(X;\Lambda^{k+1})$,

$$\begin{aligned} a\,d^*\eta &= d^*(a\eta) + i_{(da)_\sharp}\eta \\ &= d^*(a\eta) + i_{(da)_\sharp}(P_{\mathcal{H}}\eta + dGd^*\eta + d^*Gd\eta) \\ &= d^*(a(\eta - P_{\mathcal{H}}\eta - d^*Gd\eta)) + i_{(da)_\sharp}G^{1/2}dG^{1/2}d^*\eta. \end{aligned} \tag{7.18}$$

Thus

$$(I - P_{\overline{\mathrm{Im}(d^*)}})\,aP_{\overline{\mathrm{Im}(d^*)}} = (I - P_{\overline{\mathrm{Im}(d^*)}})\,i_{(da)_\sharp}G^{1/2}dG^{1/2}P_{\overline{\mathrm{Im}(d^*)}}. \tag{7.19}$$

Following the previous line of proof, we conclude that $(I - P_{\overline{\mathrm{Im}(d^*)}})\,aP_{\overline{\mathrm{Im}(d^*)}}$: $\overline{\mathrm{Im}(d^*)} \to \overline{\mathrm{Im}(d)} \oplus \mathcal{H}$ is compact. □

Thus the triple (H,γ,F) is a cycle for $\mathrm{KK}_0(C_0(X);\mathbb{C}) \cong \mathrm{KK}_{2k}(C_0(X);\mathbb{C})$. We extend γ to the usual $\mathbb{Z}_2$-grading on $L^2(X;\Lambda^*)$.

PROPOSITION 7.20. *If Assumption* 7.11 *is satisfied then the cycles* (H,γ,F) *and* $(L^2(X;\Lambda^*),\gamma,\frac{d+d^*}{\sqrt{1+\triangle}})$ *represent the same class in* $\mathrm{KK}_{2k}(C_0(X);\mathbb{C})$.

Proof. Define $\overline{F}\in B(L^2(X;\Lambda^*))$ by

$$\overline{F}\omega=\begin{cases}\omega & \text{if } \omega\in L^2(X;\Omega^j), \quad j<k,\\ F\omega & \text{if } \omega\in L^2(X;\Omega^k),\\ -\omega & \text{if } \omega\in L^2(X;\Omega^j), \quad j>k.\end{cases} \tag{7.21}$$

Then $\overline{F}$ anticommutes with γ, and the cycle $(L^2(X;\Lambda^*),\gamma,\overline{F})$ differs from (H,γ,F) by the addition of a degenerate cycle. Hence they define the same class in $\mathrm{KK}_{2k}(C_0(X);\mathbb{C})$. Now $\overline{F}$ commutes with $\frac{d+d^*}{\sqrt{1+\triangle}}$, so it anticommutes with $i\gamma\frac{d+d^*}{\sqrt{1+\triangle}}$. Then the cycles with $F_t=\cos(t)\,\overline{F}+i\,\sin(t)\,\gamma\frac{d+d^*}{\sqrt{1+\triangle}}$, $t\in[0,\frac{\pi}{2}]$, homotop from $(L^2(X;\Lambda^*),\gamma,\overline{F})$ to $(L^2(X;\Lambda^*),\gamma,i\,\gamma\,\frac{d+d^*}{\sqrt{1+\triangle}})$. Finally, the cycles with $F_t=(i\,\gamma\,\cos(t)+\sin(t))\,\frac{d+d^*}{\sqrt{1+\triangle}}$, $t\in[0,\frac{\pi}{2}]$, homotop from $(L^2(X;\Lambda^*),\gamma,i\,\gamma\,\frac{d+d^*}{\sqrt{1+\triangle}})$ to $(L^2(X;\Lambda^*),\gamma,\frac{d+d^*}{\sqrt{1+\triangle}})$. The proposition follows. □

Remark. If X is compact then Proposition 7.20 was previously proved in [12, p. 677] by a different argument.

7.3. QUASICONFORMAL INVARIANCE

In this section we show that the K-homology class of (H,γ,F) is invariant under quasiconformal homeomorphisms of X.

PROPOSITION 7.22. *If* $\phi: X_1\to X_2$ *is an orientation-preserving* K*-quasiconformal homeomorphism, for some* $K<\infty$*, and* X_1 *and* X_2 *satisfy Assumption* 7.11*, then* $\phi_*[(H_1,\gamma_1,F_1)]=[(H_2,\gamma_2,F_2)]$ *in* $\mathrm{KK}_{2k}(C_0(X_2);\mathbb{C})$.

Proof. The pushforward $\phi_*[(H_1,\gamma_1,F_1)]\in\mathrm{KK}_{2k}(C_0(X_2);\mathbb{C})$ is represented by a K-cycle using H_1, γ_1, and F_1, where $C_0(X_2)$ acts on H_1 via the pullback ϕ^*: $C_0(X_2)\to C_0(X_1)$. As ϕ is K-quasiconformal, $(\phi^{-1})^*H_1$ and H_2 are the same as topological vector spaces. By naturality, we can represent $\phi_*[(H_1,\gamma_1,F_1)]$ by letting $C_0(X_2)$ act on $(\phi^{-1})^*H_1$, equipped with the transported operator $(\phi^{-1})^*F_1$. From Lemma 7.8, $(\phi^{-1})^*\overline{\mathrm{Im}(d)}=\overline{\mathrm{Im}(d)}$. Then $(\phi^{-1})^*F_1$ is the operator constructed using d and the transported grading operator $(\phi^{-1})^*\gamma_1$. Hence it suffices to work on a fixed manifold X and consider two conformal structures that are K-quasiconformal. Equivalently, we can consider the corresponding grading operators γ_1 and γ_2 [14, Lemma 2.3].

There is a measurable bundle homomorphism μ_+: $\Lambda^k_-\to\Lambda^k_+$ with $\sup_{x\in X}|\mu_+(x)|<1$ so that if $\mu=\begin{pmatrix}0 & \mu_+\\ \mu_+^* & 0\end{pmatrix}$ then $\gamma_2=(1+\mu)\gamma_1(1+\mu)^{-1}$ [12, Section 4α, 14, Section 2(i)]. For $t\in[0,1]$, put $\gamma(t)=(1+t\mu)\gamma_1(1+t\mu)^{-1}$.

The corresponding inner product space has

$$\langle\omega_1,\omega_2\rangle(t) = \langle\omega_1,(1-t\mu)(1+t\mu)^{-1}\omega_2\rangle(0). \tag{7.23}$$

The operator $F(t)$ is one on $\overline{\mathrm{Im}(d)}$, minus one on $\gamma(t)\overline{\mathrm{Im}(d)}$, and zero on $\left(\overline{\mathrm{Im}(d)}\oplus\gamma(t)\overline{\mathrm{Im}(d)}\right)^{\perp}$.

The Hilbert spaces $\{H(t)\}_{t\in[0,1]}$ form a Hilbert $C([0,1])$-module. They all have the same underlying topological vector space. We claim that the operators $\{F(t)\}_{t\in[0,1]}$ are norm-continuous in t. For this, it suffices to show that the projection operators $P_{\overline{\mathrm{Im}(d)}}$ and $P_{\overline{\mathrm{Im}(d^*)}}$ are norm-continuous in t. As $\overline{\mathrm{Im}(d)}$ is independent of t, [19, Lemma 6.2] implies that $P_{\overline{\mathrm{Im}(d)}}$ is norm-continuous in t. As $\mathrm{Ker}(d)=\overline{\mathrm{Im}(d)}\oplus\mathcal{H}$ is independent of t, it also follows from [19, Lemma 6.2] that $P_{\overline{\mathrm{Im}(d)}}+P_{\mathcal{H}}$ is norm-continuous in t. Then $P_{\overline{\mathrm{Im}(d^*)}} = I - P_{\overline{\mathrm{Im}(d)}} - P_{\mathcal{H}}$ is norm-continuous in t.

The operators $\gamma(t)$ are also norm-continuous in t. In order to show that $\{(H(t),\gamma(t),F(t))\}_{t\in[0,1]}$ is a homotopy of K-cycles, it now suffices to show that for all $a\in C_0(X)$, $[F(t),a]$, and $a(F(t)^2-1)$ are compact operators. We may assume that $a\in C_c^{\infty}(X)$. From Propositions 7.13 and 7.15, $[F(0),a]$ and $a(F(0)^2-1)$ are compact. Using the fact that $\frac{\mathrm{d}}{\mathrm{d}t}d^* = [\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1},d^*]$, one can compute that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}P_{\overline{\mathrm{Im}(d)}} &= -P_{\overline{\mathrm{Im}(d)}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}(I-P_{\overline{\mathrm{Im}(d)}}),\\
\frac{\mathrm{d}}{\mathrm{d}t}P_{\overline{\mathrm{Im}(d^*)}} &= (I-P_{\overline{\mathrm{Im}(d^*)}})\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}P_{\overline{\mathrm{Im}(d)^*}},\\
\frac{\mathrm{d}}{\mathrm{d}t}P_{\mathcal{H}} &= -P_{\mathcal{H}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}P_{\overline{\mathrm{Im}(d^*)}} + P_{\overline{\mathrm{Im}(d)}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}P_{\mathcal{H}}.
\end{aligned} \tag{7.24}$$

To compute $\frac{\mathrm{d}}{\mathrm{d}t}[F(t),a]$, it suffices to compute $\frac{\mathrm{d}}{\mathrm{d}t}[P_{\overline{\mathrm{Im}(d)}},a]$ and $\frac{\mathrm{d}}{\mathrm{d}t}[P_{\overline{\mathrm{Im}(d^*)}},a]$. Now

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}\left[P_{\overline{\mathrm{Im}(d)}},a\right] &= -\left[P_{\overline{\mathrm{Im}(d)}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}(I-P_{\overline{\mathrm{Im}(d)}}),a\right]\\
&= -\left[P_{\overline{\mathrm{Im}(d)}},a\right]\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}(I-P_{\overline{\mathrm{Im}(d)}})-\\
&\quad -P_{\overline{\mathrm{Im}(d)}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}\left[(I-P_{\overline{\mathrm{Im}(d)}}),a\right]\\
&= -\left[P_{\overline{\mathrm{Im}(d)}},a\right]\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}(I-P_{\overline{\mathrm{Im}(d)}})+\\
&\quad +P_{\overline{\mathrm{Im}(d)}}\frac{\mathrm{d}\gamma}{\mathrm{d}t}\gamma^{-1}\left[P_{\overline{\mathrm{Im}(d)}},a\right].
\end{aligned} \tag{7.25}$$

From the proof of Proposition 7.15, at $t=0$, $\left[P_{\overline{\mathrm{Im}(d)}}(0),a\right]$ is compact. From (7.25), we can write $[P_{\overline{\mathrm{Im}(d)}}(t),a] = U(t)\left[P_{\overline{\mathrm{Im}(d)}}(0),a\right]V(t)$, where $U(0)=V(0)=I$ and

$$\begin{aligned}\frac{dU}{dt} &= P_{\overline{\mathrm{Im}(d)}}(t)\,\frac{d\gamma}{dt}\,\gamma^{-1}\,U(t),\\ \frac{dV}{dt} &= -\,V(t)\,\frac{d\gamma}{dt}\,\gamma^{-1}\,(I - P_{\overline{\mathrm{Im}(d)}}(t)).\end{aligned} \tag{7.26}$$

The solution of the first equation in (7.26), for example, is given by

$$\begin{aligned}U(t) = I + \int_0^t P_{\overline{\mathrm{Im}(d)}}(s)\,\frac{d\gamma}{ds}\,\gamma^{-1}(s)\,ds + \\ + \int_{t\geqslant s_1\geqslant s_2\geqslant 0} P_{\overline{\mathrm{Im}(d)}}(s_1)\frac{d\gamma}{ds_1}\gamma^{-1}(s_1)\,P_{\overline{\mathrm{Im}(d)}}(s_2)\frac{d\gamma}{ds_2}\,\gamma^{-1}(s_2)\,ds_1\,ds_2 + \cdots\end{aligned} \tag{7.27}$$

The series in (7.27) is convergent because $\frac{d\gamma}{ds}\,\gamma^{-1}(s)$ is uniformly bounded for $s\in[0,t]$. One can write a similar series for $U(t)^{-1}$, showing that $U(t)$ is invertible.

Hence $\left[P_{\overline{\mathrm{Im}(d)}}(t), a\right]$ is compact for all $t \in [0,1]$. A similar argument shows that $\left[P_{\overline{\mathrm{Im}(d^*)}}(t), a\right]$ is compact for all $t\in[0,1]$. Thus $[F(t),a]$ is compact for all $t\in[0,1]$.

Next, $a(F(t)^2-1) = -\,aP_{\mathcal{H}}$, and

$$\begin{aligned}\frac{d}{dt}\,aP_{\mathcal{H}} &= a\left(-\,P_{\mathcal{H}}\,\frac{d\gamma}{dt}\,\gamma^{-1}\,P_{\overline{\mathrm{Im}(d^*)}} + P_{\overline{\mathrm{Im}(d)}}\,\frac{d\gamma}{dt}\,\gamma^{-1}\,P_{\mathcal{H}}\right)\\ &= -\,a\,P_{\mathcal{H}}\,\frac{d\gamma}{dt}\,\gamma^{-1}\,P_{\overline{\mathrm{Im}(d^*)}} + [a,\,P_{\overline{\mathrm{Im}(d)}}]\,\frac{d\gamma}{dt}\,\gamma^{-1}\,P_{\mathcal{H}} + \\ &\quad + P_{\overline{\mathrm{Im}(d)}}\,\frac{d\gamma}{dt}\,\gamma^{-1}\,a\,P_{\mathcal{H}}.\end{aligned} \tag{7.28}$$

Putting $M(0) = N(0) = I$ and solving

$$\begin{aligned}\frac{dM}{dt} &= -\,M(t)\,P_{\overline{\mathrm{Im}(d)}}(t)\,\frac{d\gamma}{dt}\,\gamma^{-1},\\ \frac{dN}{dt} &= \frac{d\gamma}{dt}\,\gamma^{-1}\,P_{\overline{\mathrm{Im}(d^*)}}(t)\,N(t),\end{aligned} \tag{7.29}$$

we can write

$$M(t)\,a\,P_{\mathcal{H}}(t)\,N(t) - a\,P_{\mathcal{H}}(0) = \int_0^t M(s)\,[a, P_{\overline{\mathrm{Im}(d)}}(s)]\,\frac{d\gamma}{ds}\,\gamma^{-1}P_{\mathcal{H}}(s)\,N(s)\,ds. \tag{7.30}$$

As $M(t)$ and $N(t)$ are invertible and $aP_{\mathcal{H}}(0)$ is compact, it follows that $aP_{\mathcal{H}}(t)$ is compact for all $t\in[0,1]$. □

COROLLARY 7.31. *If* $\phi: X_1 \to X_2$ *is an orientation-preserving* K-*quasiconformal homeomorphism, for some* $\mathrm{K} < \infty$, *and* X_1 *satisfies Assumption* 7.11, *then* (H_2, γ_2, F_2) *defines a cycle for* $\mathrm{KK}_{2k}(C_0(X_2); \mathbb{C})$.

Proof. This follows from the proof of Proposition 7.22. □

COROLLARY 7.32 [19, Theorem 1.1, 12, p. 678]. *If* $\phi: X_1 \to X_2$ *is an orientation-preserving homeomorphism between compact oriented smooth manifolds then the pushforward of the signature class of* X_1 *coincides with the signature class of* X_2, *in* $\mathrm{KK}_{2k}(C(X_2); \mathbb{C})$.

Proof. If $\dim(X) \neq 4$ then there is an orientation-preserving quasiconformal homeomorphism from X_1 to X_2 that is isotopic to ϕ [41], and the corollary follows from Proposition 7.22. If $\dim(X) = 4$ then one can instead consider $X \times S^2$. □

Remark. If $X' = X - Z$, where Z has Hausdorff dimension at most $2k - 2$, then the cycle (H, γ, F) for $\mathrm{KK}_{2k}(C_0(X); \mathbb{C})$ also defines a signature cycle for $\mathrm{KK}_{2k}(C_0(X'); \mathbb{C})$. This is because the triple (H, γ, F) is the same as the corresponding triple for X', and an element $a \in C_0(X')$ extends by zero to an element of $C_0(X)$. For example, writing $\mathbb{R}^{2k} = S^{2k} - \mathrm{pt}$, we obtain a cycle (H, γ, F) for $\mathrm{KK}_{2k}(C_0(\mathbb{R}^{2k}); \mathbb{C})$.

7.4. WHEN THE LIMIT SET IS THE ENTIRE SPHERE, EVEN-DIMENSIONAL

In this section we use F to construct an equivariant K-cycle for $C(\Lambda)$ when $\Lambda = S^{2k}$.

Suppose that $\Lambda = S^{2k}$. The triple (H, γ, F) of Section 7.1 is Γ-equivariant and so gives a cycle for a class $[(H, \gamma, F)] \in \mathrm{KK}^{\Gamma}_{2k}(C(S^{2k}); \mathbb{C})$. As the non-equivariant K-homology class represented by (H, γ, F) is the signature class, it follows from the discussion of Section 6.1 that $[(H, \gamma, F)]$ is a nontorsion element of $\mathrm{KK}^{\Gamma}_{2k}(C(S^{2k}); \mathbb{C})$.

8. From Even Cycles to Odd Cycles

In this section we consider a manifold X as in Section 7 equipped with a partial compactification $\overline{X}$. Putting $\partial\overline{X} = \overline{X} - X$, we give a sufficient condition for the triple (H, γ, F) to extend to a cycle for $\mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C})$. We then consider the boundary map $\mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C}) \to \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$. We describe the image of the cycle (H, γ, F) as an element of $\mathrm{Ext}(C(\partial\overline{X}))$. If $\partial\overline{X}$ is a manifold then the relevant Hilbert space turns out to be the exact k-forms on $\partial\overline{X}$ of a certain regularity. In the special case when $\partial\overline{X} = S^{2k-1}$, we show that the Hilbert space of such $H^{-1/2}$-regular forms is Möbius-invariant, along with the Ext element.

A second technical assumption arises in this section, which will again be satisfied in the cases that are relevant for limit sets.

8.1. A RELATIVE K-CYCLE

In this subsection we start with a partial compactification $\overline{X}$ of X. Applying the boundary map to the K-cycle (H, γ, F) for $C_0(X)$ gives a class in $\mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$. We show the compatibility of this map with quasiconformal homeomorphisms. If X is the domain of discontinuity Ω for Γ then we discuss the twisting of this construction by the pullback of a vector bundle on Ω/Γ.

Let $\overline{X}$ be a locally compact Hausdorff space that contains X as an open dense subset. Put $\partial\overline{X} = \overline{X} - X$, which we assume to be compact. There is a short exact sequence of C^*-algebras

$$0 \longrightarrow C_0(X) \longrightarrow C_0(\overline{X}) \longrightarrow C(\partial\overline{X}) \longrightarrow 0. \tag{8.1}$$

From [6, Theorem 14.24, 25 or 18, Theorem 5.4.5], there is an isomorphism $\mathrm{KK}_{2k}(C_0(X); \mathbb{C}) \cong \mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C})$. Furthermore, there is a boundary map $\partial : \mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C}) \to \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$.

Let $e \in M_N(C^\infty(X))$ be a projection. If (H, γ, F) is a K-cycle for $C_0(X)$ then there is a new K-cycle (H_e, γ_e, F_e), where $H_e = H^N e$, $\gamma_e = e\gamma e$ and $F_e = eFe$. In this way, we obtain a map $\mathrm{K}^0(X) \to \mathrm{KK}_{2k}(C_0(X); \mathbb{C})$. Composing with the boundary map gives a map $\mathrm{K}^0(X) \to \mathrm{KK}_{2k}(C_0(X); \mathbb{C}) \xrightarrow{\partial} \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$.

In this paragraph we take $X = \Omega \neq \emptyset$ and $\overline{X} = S^{2k}$, so $\partial\overline{X} = \Lambda$. If X satisfies Assumption 7.11 then we have the K-cycle (H, γ, F) of Section 7.2. Let $p \in M_N(C^\infty(\Omega/\Gamma))$ be a projection. If $\pi : \Omega \to \Omega/\Gamma$ is the quotient map then $e = \pi^* p$ is a projection in $M_N(C^\infty(\Omega))$. Applying the preceding construction and taking into account the Γ-equivariance, we obtain maps

$$\mathrm{K}^0(\Omega) \to \mathrm{KK}_{2k-1}(C(\Lambda); \mathbb{C}) \tag{8.2}$$

and

$$\mathrm{K}^0(\Omega/\Gamma) \to \mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda); \mathbb{C}). \tag{8.3}$$

With reference to Proposition 6.1, the maps (8.2) and (8.3) are rationally the same as the connecting maps

$$\mathrm{K}^0(\Omega) \to \mathrm{K}^1(S^{2k}, \Omega) \cong \mathrm{KK}_{2k-1}(C(\Lambda); \mathbb{C}) \tag{8.4}$$

and

$$\mathrm{K}^0(\Omega/\Gamma) \cong \mathrm{K}^0_\Gamma(\Omega) \to \mathrm{K}^1_\Gamma(S^{2k}, \Omega) \cong \mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda); \mathbb{C}). \tag{8.5}$$

We obtain a rational instead of integral statement because the K-homology classes defined by the signature and Dirac operator on S^{2k}, the latter being the fundamental class, are only rationally equivalent.

Returning to general X, let X' be another manifold as in Section 7.1, with partial compactification $\overline{X}'$ and boundary $\partial\overline{X}'$. Let $\phi : \overline{X}' \to \overline{X}$ be a homeomorphism that restricts to a K-quasiconformal homeomorphism from X' to X. By naturality, there is an isomorphism $\left(\phi\big|_{\partial\overline{X}'}\right)_* : \mathrm{KK}_{2k-1}(C(\partial\overline{X}'); \mathbb{C}) \to \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$. Suppose that X' satisfies Assumption 7.11. By Propositions 7.13, 7.15, and Corollary 7.31, there are well-defined signature classes $[(H', \gamma', F')] \in \mathrm{KK}_{2k}(C_0(X'); \mathbb{C}) \cong \mathrm{KK}_{2k}(C_0(\overline{X'}), C(\partial\overline{X}'); \mathbb{C})$ and $[(H, \gamma, F)] \in \mathrm{KK}_{2k}(C_0(X); \mathbb{C}) \cong \mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C})$.

PROPOSITION 8.6. $\left(\phi\big|_{\partial\overline{X}'}\right)_*(\partial[(H', \gamma', F')]) = \partial[(H, \gamma, F)]$ *in* $\mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$.

Proof. There is a commutative diagram

$$\begin{array}{ccc} \mathrm{KK}_{2k}(C_0(\overline{X}'), C(\partial\overline{X}')); \mathbb{C}) & \xrightarrow{\phi*} & \mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C}) \\ \partial\Big\downarrow & & \partial\Big\downarrow \\ \mathrm{KK}_{2k-1}(C(\partial\overline{X}'); \mathbb{C}) & \xrightarrow{\left(\phi|_{\partial\overline{X}'}\right)_*} & \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C}), \end{array} \tag{8.7}$$

where the horizontal arrows are isomorphisms. From Proposition 7.22, $\phi_*([(H', \gamma', F')]) = [(H, \gamma, F)]$. The claim follows from the commutativity of the diagram. □

8.2. THE INDUCED STRUCTURE ON THE BOUNDARY

In this section we consider a manifold X as before with a compactification $\overline{X}$. With an assumption on $\overline{X}$, related to the Higson corona of X, we show that the K-cycle (H, γ, F) for $C_0(X)$ extends to a K-cycle for $(C_0(\overline{X}), C(\partial\overline{X}))$. We describe the Baum–Douglas boundary map in this case.

Let X be a manifold as in Section 8.1 satisfying Assumption 7.11, with a partial compactification $\overline{X}$. We recall that a relative K-cycle for the pair $(C_0(\overline{X}), C(\partial\overline{X}))$ is given by a K-cycle (H, γ, F) for the ideal $C_0(X)$ so that the action of $C_0(X)$ on H extends to an action of $C_0(\overline{X})$, and for all $a \in C_0(\overline{X})$, $[F, a] \in K(H)$.

We wish to extend the K-cycle of Section 7.2 for $C_0(X)$ to a K-cycle for $(C_0(\overline{X}), C(\partial\overline{X}))$. There is an evident action of $C_0(\overline{X})$ on H. We will need an additional condition on $\overline{X}$.

ASSUMPTION 8.8. *With respect to a Riemannian metric on X satisfying Assumption* 7.11, *for each* $a \in C_0(\overline{X})$, $a\big|_X$ *is the norm limit of a sequence* $\{a_i\}_{i=1}^{\infty}$ *of bounded elements of* $C^{\infty}(X)$ *satisfying* $|da_i| \in C_0(X)$.

If $\overline{X}$ is compact then Assumption 8.8 is equivalent to saying that $\partial\overline{X}$ is a quotient of the Higson corona, the latter being defined using the given Riemannian metric on X.

EXAMPLE 1′. With reference to Example 1, Assumption 8.8 is satisfied by an asymptotically hyperbolic metric on X.

EXAMPLE 2′. With reference to Example 2, Assumption 8.8 is satisfied when $\overline{X} = S^{2k}$ is the one-point-compactification of X.

PROPOSITION 8.9. *If Assumption* 8.8 *is satisfied and* (H, γ, F) *is the cycle for* $\mathrm{KK}_{2k}(C_0(X); \mathbb{C})$ *from Section* 7.2 *then* (H, γ, F) *is also a cycle for* $\mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C})$.

Proof. We must show that for all $a \in C_0(\overline{X})$, $[F, a]$ is compact. We may assume that $a\big|_X$ is smooth and $|da| \in C_0(X)$. Then the proof of Proposition 7.15 applies. □

The boundary map $\partial\colon \mathrm{KK}_{2k}(C_0(\overline{X}), C(\partial\overline{X}); \mathbb{C}) \to \mathrm{KK}_{2k-1}(C(\partial\overline{X}); \mathbb{C})$ can be explicitly described as follows. Given $a \in C(\partial\overline{X})$, let a' be an extension of it to $C_0(\overline{X})$. Then $P_{\mathcal{H}_\pm} a' P_{\mathcal{H}_\pm}$ is an element of $B(\mathcal{H}_\pm)$. The corresponding element $[P_{\mathcal{H}_\pm} a' P_{\mathcal{H}_\pm}]$ of the Calkin algebra $Q(\mathcal{H}_\pm)$ is independent of the choice of extension and defines an algebra homomorphism $\sigma_\pm\colon C_0(\partial\overline{X}) \to Q(\mathcal{H}_\pm)$. Then $\partial[(H, \gamma, F)]$ is represented by the Ext class $[\sigma_+] - [\sigma_-]$ [Definition 4.6, Theorems 14.23 and 14.24, 18, Remark 8.5.7].

8.3. THE CASE OF A SMOOTH MANIFOLD-WITH-BOUNDARY

In this section we consider the case when $\overline{X}$ is a smooth manifold-with-boundary. We construct a Hilbert space $H_{\partial\overline{X}}$ of exact k-forms on $\partial\overline{X}$ as boundary values of L^2-harmonic k-forms on X. There is a natural $\mathbb{Z}_2$-grading on the Hilbert space coming from a diffeomorphism-invariant Hermitian form. In the case when $\overline{X} = [0, \infty) \times \partial\overline{X}$, we show that the inner product on $H_{\partial\overline{X}}$ is the $H^{-1/2}$ inner product.

Suppose that $\overline{X}^{2k}$ is a smooth oriented manifold-with-boundary with compact boundary $\partial\overline{X}$. Let g_0 be a smooth Riemannian metric on $\overline{X}$ and consider the corresponding conformal class on X. We assume that the reduced L^2-cohomology group $\mathrm{H}^k_{(2)}(\overline{X}; \mathbb{R}) \cong \mathrm{H}^k_{(2)}(\overline{X}, \partial\overline{X}; \mathbb{R})$ vanishes. (Note that $\mathrm{H}^k_{(2)}(\overline{X}; \mathbb{R})$ and $\mathrm{H}^k_{(2)}(\overline{X}, \partial\overline{X}; \mathbb{R})$ have harmonic representatives defined using boundary conditions, and are generally much smaller than $\mathcal{H}$.)

Let $i\colon \partial\overline{X} \to \overline{X}$ be the boundary inclusion. We note that by conformal invariance, the L^2-harmonic k-forms on X can be computed using the metric g_0 which is smooth up to the boundary $\partial\overline{X}$. It follows that $i^*: \mathcal{H} \to H^{-1/2}(\partial\overline{X}; \Lambda^k)$ is well-defined [20, B.2.7–B.2.9].

PROPOSITION 8.10. *Given* $\omega \in \mathrm{Im}\left(d: C^\infty(\partial\overline{X}; \Lambda^{k-1}) \to C^\infty(\partial\overline{X}; \Lambda^k)\right)$*, there is a unique* $\omega' \in \mathcal{H}$ *so that* $i^*\omega' = \omega$.

Proof. Write $\omega = d\eta$ for some $\eta \in C^\infty(\partial\overline{X}; \Lambda^{k-1})$. Let $\eta' \in C_c^\infty(\overline{X}; \Lambda^{k-1})$ satisfy $i^*\eta' = \eta$. Let G be the Green's operator for the Laplacian on $\overline{X}$, as defined using g_0, with relative boundary conditions. In particular, $i^* \circ G = 0$. If ω' exists then it satisfies $d(\omega' - d\eta') = 0$, $d^*(\omega' - d\eta') = -d^*d\eta'$ and $i^*(\omega' - d\eta') = 0$. These equations would imply $\triangle(\omega' - d\eta') = -dd^*d\eta'$, which has the solution $\omega' - d\eta' = -Gdd^*d\eta'$. This motivates putting $\omega' = d\left(\eta' - Gd^*d\eta'\right)$, which works. Note that ω' is square-integrable with respect to g_0, and hence lies in $L^2(X; \Lambda^k)$.

If ω'_1 and ω'_2 both satisfy the conclusion of the proposition then $d(\omega'_1 - \omega'_2) = d^*(\omega'_1 - \omega'_2) = i^*(\omega'_1 - \omega'_2) = 0$. The cohomology assumption then implies that $\omega'_1 = \omega'_2$. □

DEFINITION 8.11. *The Hilbert space* $H_{\partial\overline{X}}$ *is the completion of* $\mathrm{Im}\,(d: C^\infty(\partial\overline{X}; \Lambda^{k-1}) \to C^\infty(\partial\overline{X}; \Lambda^k))$ *with respect to the norm* $\omega \to \| \omega' \|_{\mathcal{H}}$.

COROLLARY 8.12. *If* $i^*: \mathrm{H}^k(\overline{X}; \mathbb{C}) \to \mathrm{H}^k(\partial\overline{X}; \mathbb{C})$ *is the zero map then pullback gives an isometric isomorphism* $i^*: \mathcal{H} \to H_{\partial\overline{X}}$.

Proof. Given $\omega \in \mathcal{H}$, it represents a class $[\omega] \in \mathrm{H}^k(\overline{X})$. By assumption, $[i^*\omega]$ vanishes in $\mathrm{H}^k(\partial\overline{X})$. Hence $i^*\omega \in \mathrm{Im}(d)$. The lemma now follows from Proposition 8.10. □

DEFINITION 8.13. *The operator* $T \in B\left(H_{\partial\overline{X}}\right)$ *is given by*

$$T\omega = \begin{cases} \omega & \text{if } \omega \in i^*\mathcal{H}_+, \\ -\omega & \text{if } \omega \in i^*\mathcal{H}_-. \end{cases} \tag{8.14}$$

PROPOSITION 8.15. *For all* $\omega_1, \omega_2 \in \mathrm{Im}\left(d: C^\infty(\partial\overline{X}; \Lambda^{k-1}) \to C^\infty(\partial\overline{X}; \Lambda^k)\right)$,

$$\langle T\omega_1, \omega_2 \rangle = i^k \int_{\partial\overline{X}} \eta_1 \wedge \overline{\omega_2}, \tag{8.16}$$

where $\eta_1 \in C^\infty(\partial\overline{X}; \Lambda^{k-1})$ *is an arbitrary solution of* $d\eta_1 = \omega_1$.

Proof. Suppose that $\omega_1 = i^*\omega'_1$ and $\omega_2 = i^*\omega'_2$, with $\omega'_1, \omega'_2 \in \mathcal{H}$ being uniquely determined. Let $\eta'_1 \in C_c^\infty(\overline{X}; \Lambda^{k-1})$ satisfy $i^*\eta'_1 = \eta_1$. Then as in the proof of Proposition 8.10, $\omega'_1 = d\left(\eta'_1 - Gd^*d\eta'_1\right)$.

Suppose that $\omega_2' \in \mathcal{H}_\pm$. Then $*\omega_2' = \pm i^{-k}\omega_2'$ and so

$$\begin{aligned}\langle T\omega_1, \omega_2\rangle = \langle \omega_1, T\omega_2\rangle &= \pm \int_{\overline{X}} \omega_1' \wedge \overline{*\omega_2'} = i^k \int_{\overline{X}} \omega_1' \wedge \overline{\omega_2'} \\ &= i^k \int_{\overline{X}} d\left(\eta_1' - Gd^* d\eta_1'\right) \wedge \overline{\omega_2'} = i^k \int_{\partial\overline{X}} i^*\left(\eta_1' - Gd^* d\eta_1'\right) \wedge i^* \overline{\omega_2'} \\ &= i^k \int_{\partial\overline{X}} \eta_1 \wedge \overline{\omega_2}. \end{aligned} \tag{8.17}$$

To see directly that (8.16) is independent of the choice of η_1, suppose that η_1 and $\widetilde{\eta}_1$ satisfy $d\eta_1 = d\widetilde{\eta}_1 = \omega_1$. Write $\omega_2 = d\eta_2$. Then

$$\begin{aligned}\int_{\partial\overline{X}} (\eta_1 - \widetilde{\eta}_1) \wedge \overline{\omega_2} &= \int_{\partial\overline{X}} (\eta_1 - \widetilde{\eta}_1) \wedge d\overline{\eta_2} \\ &= (-1)^k \int_{\partial\overline{X}} d(\eta_1 - \widetilde{\eta}_1) \wedge \overline{\eta_2} = 0. \end{aligned} \tag{8.18}$$

□

PROPOSITION 8.19. *Let $\partial\overline{X}$ be a closed oriented $(2k-1)$-dimensional Riemannian manifold. If $\overline{X} = [0,\infty) \times \partial\overline{X}$ then*

$$H_{\partial\overline{X}} = \operatorname{Im}\left(d : H^{1/2}(\partial\overline{X}; \Lambda^{k-1}) \to H^{-1/2}(\partial\overline{X}; \Lambda^k)\right). \tag{8.20}$$

Proof. The Künneth formula for reduced L^2-cohomology, along with the fact that $[0,\infty)$ has vanishing absolute and relative reduced L^2-cohomology, implies that $\overline{X}$ has vanishing absolute and relative reduced L^2-cohomology. Hence the hypotheses of Proposition 8.10 are satisfied.

If $p : \overline{X} \to \partial\overline{X}$ is projection and $\omega \in C^\infty(\partial\overline{X}; \Lambda^k)$ then we will abuse notation to also write ω for $p^*\omega$. Let $\widehat{d}$ be the exterior derivative on $\partial\overline{X}$ and let $\widehat{*}$ be the Hodge duality operator on $\partial\overline{X}$. Let t be the coordinate on $[0,\infty)$. Then

$$|\omega|^2 \, d\,\mathrm{vol}_{\overline{X}} = \omega \wedge \widehat{*}\omega \wedge \mathrm{d}t = (-1)^{k-1}\, \omega \wedge \mathrm{d}t \wedge \widehat{*}\omega. \tag{8.21}$$

Hence $*\omega = (-1)^{k-1}\, \mathrm{d}t \wedge \widehat{*}\omega$.

Suppose that $\omega \in C^\infty(\partial\overline{X}; \Lambda^k)$ satisfies $\widehat{d}\,\omega = 0$ and

$$(-i)^k\, \widehat{d}\,\widehat{*}\omega = \lambda\, \omega \tag{8.22}$$

with $\lambda \in \mathbb{R}$. If $\lambda > 0$ then $\mathrm{e}^{-\lambda t}\left(\omega - (-i)^k\, \mathrm{d}t \wedge \widehat{*}\omega\right) \in H_+$ and

$$d\left(\mathrm{e}^{-\lambda t}\left(\omega - (-i)^k\, dt \wedge \widehat{*}\omega\right)\right) = 0. \tag{8.23}$$

From the self-duality of $\mathrm{e}^{-\lambda t}\left(\omega - (-i)^k\, \mathrm{d}t \wedge \widehat{*}\omega\right)$, we also have

$$d^*\left(\mathrm{e}^{-\lambda t}\left(\omega - (-i)^k\, \mathrm{d}t \wedge \widehat{*}\omega\right)\right) = 0. \tag{8.24}$$

Thus $\omega \in i^*\mathcal{H}_+$. Furthermore, from (8.22),

$$\widehat{d}\left(\frac{1}{\lambda}(-i)^k \widehat{*}\omega\right) = \omega. \tag{8.25}$$

Then from Proposition 8.15,

$$\begin{aligned}\langle \omega, \omega \rangle &= i^k \int_{\partial \overline{X}} \left(\frac{1}{\lambda}(-i)^k \widehat{*}\omega\right) \wedge \overline{\omega} = \frac{1}{\lambda} \int_{\partial \overline{X}} \widehat{*}\omega \wedge \overline{\omega} = \frac{1}{\lambda} \int_{\partial \overline{X}} \overline{\omega} \wedge \widehat{*}\omega \\ &= \overline{\frac{1}{\lambda} \int_{\partial \overline{X}} \omega \wedge \overline{\widehat{*}\omega}} = \frac{1}{\lambda} \int_{\partial \overline{X}} \omega \wedge \overline{\widehat{*}\omega}.\end{aligned} \tag{8.26}$$

If $\lambda < 0$ then $e^{\lambda t}\left(\omega + (-i)^k \mathrm{d}t \wedge \widehat{*}\omega\right) \in H_-$ and

$$d\left(e^{-\lambda t}\left(\omega + (-i)^k \mathrm{d}t \wedge \widehat{*}\omega\right)\right) = 0, \tag{8.27}$$

so $\omega \in i^*\mathcal{H}_-$. A similar calculation gives $\langle \omega, \omega \rangle = -\frac{1}{\lambda} \int_{\partial \overline{X}} \omega \wedge \overline{\widehat{*}\omega}$. Thus in either case,

$$\langle \omega, \omega \rangle = \frac{1}{|\lambda|} \int_{\partial \overline{X}} \omega \wedge \overline{\widehat{*}\omega}. \tag{8.28}$$

As the closure of $\mathrm{Im}\left(d : C^\infty(\partial \overline{X}; \Lambda^{k-1}) \to C^\infty(\partial \overline{X}; \Lambda^k)\right)$ has an orthonormal basis given by such eigenforms, the proposition follows. □

8.4. MÖBIUS-INVARIANT ANALYSIS ON ODD-DIMENSIONAL SPHERES

In this section we specialize Section 8.3 to the case $X = B^{2k}$. We show that the Hilbert space $H_{\partial \overline{X}}$ is the $H^{-1/2}$ space of exact k-forms on S^{2k-1}. We show that Möbius transformations of S^{2k-1} act by isometries on $H_{\partial \overline{X}}$, and quasiconformal homeomorphisms of S^{2k-1} act boundedly on $H_{\partial \overline{X}}$.

Take $X = H^{2k}$, the upper hemisphere in S^{2k}, and $\overline{X} = \overline{H^{2k}}$. Then $\mathrm{H}^k_{(2)}(\overline{X}; \mathbb{R}) = \mathrm{H}^k_{(2)}(\overline{X}, \partial \overline{X}; \mathbb{R}) = 0$ and $i^* : \mathrm{H}^k(\overline{X}; \mathbb{C}) \to \mathrm{H}^k(\partial \overline{X}; \mathbb{C})$ is the zero map, so we can apply Proposition 8.10 and Corollary 8.12.

COROLLARY 8.29 (c.f. [9, Proposition 3.2]). *The group* $\mathrm{Isom}^+(H^{2k})$ *acts isometrically on*

$$H_{S^{2k-1}} = \mathrm{Im}\left(d : H^{1/2}(S^{2k-1}; \Lambda^{k-1}) \to H^{-1/2}(S^{2k-1}; \Lambda^k)\right) \tag{8.30}$$

preserving T.

Proof. If $x_0 \in H^{2k}$ is a basepoint then $\overline{H^{2k}} - x_0$ is conformally equivalent to $[0, \infty) \times S^{2k-1}$. The same calculations as in the proof of Proposition 8.19 show that

$$H_{S^{2k-1}} = \mathrm{Im}\left(d : H^{1/2}(S^{2k-1}; \Lambda^{k-1}) \to H^{-1/2}(S^{2k-1}; \Lambda^k)\right). \tag{8.31}$$

As $\mathrm{Isom}^+(H^{2k})$ acts isometrically on $\mathcal{H}$, it acts isometrically on $H_{S^{2k-1}}$. The Hermitian form (8.16) is preserved by all orientation-preserving diffeomorphisms of $\partial\overline{X}$. □

COROLLARY 8.32. *The group* $\mathrm{Isom}^+(H^{2k})$ *acts isometrically on* $H^{1/2}(S^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$*, preserving the Hermitian form*

$$S(\omega_1,\omega_2)=i^k\int_{S^{2k-1}}\omega_1\wedge \mathrm{d}\overline{\omega_2}. \tag{8.33}$$

Proof. The dual space to $\mathrm{Im}\left(d: H^{1/2}(S^{2k-1};\Lambda^{k-1})\to H^{-1/2}(S^{2k-1};\Lambda^k)\right)$ is $H^{1/2}(S^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$, which inherits an isometric action of $\mathrm{Isom}^+(H^{2k})$. The inner product on $H^{1/2}(S^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$ is given by $\omega\to\langle \mathrm{d}\omega, G^{1/2}\mathrm{d}\omega\rangle_{L^2}$. The Hermitian form S is preserved because of its diffeomorphism invariance. □

We do not claim that the inner product on $H^{1/2}(S^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$ is conformally invariant, i.e. invariant with respect to a conformal change of the metric.

We remark that in the case $k=2$, $S(\omega,\omega)$ can be identified (up to a sign) with the helicity, or asymptotic self-linking number, of a vector field ξ satisfying $i_\xi\, d\,\mathrm{vol}=\mathrm{d}\omega$ [4, Definition III.1.14, Theorem II.4.4].

PROPOSITION 8.34. *An orientation-preserving quasiconformal homeomorphism* $\phi: S^{2k-1}\to S^{2k-1}$ *acts boundedly by pullback on* $H^{1/2}(S^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$*, preserving the Hermitian form* S.

Proof. The method of proof is that of [32, Corollary 3.2], which proves the proposition in the (quasisymmetric) case $k=1$. By composing ϕ with a Möbius transformation, we may assume that ϕ has a fixed point $x_\infty\in S^{2k-1}$. Performing a linear fractional transformation to send x_∞ to infinity, we may replace S^{2k-1} by $\mathbb{R}^{2k-1}$. Given $\omega\in H^{1/2}(\mathbb{R}^{2k-1};\Lambda^{k-1})/\mathrm{Ker}(d)$, consider its extensions $\omega'\in H^1(\mathbb{R}^{2k}_+;\Lambda^{k-1})/\mathrm{Ker}(d)$. Then

$$\|\omega\|=\inf_{\omega':i^*\omega'=\omega}\|d\omega'\|_{L^2}. \tag{8.35}$$

There is an extension ϕ' of ϕ to a K-quasiconformal homeomorphism of $\mathbb{R}^{2k}_+$, for some $\mathrm{K}<\infty$ [46]. The proposition now follows from the fact that ϕ' acts boundedly by pullback on $L^2(\mathbb{R}^{2k}_+;\Lambda^k)$. □

8.5. THE BOUNDARY SIGNATURE OPERATOR AS AN EXT CLASS

With $\overline{X}$ as in Section 8.3, we show that the image of the cycle (H,γ,F) under the Baum-Douglas boundary map can be described intrinsically in terms of $\partial\overline{X}$. It is given by certain homomorphisms from $C(\partial\overline{X})$ to the

Calkin algebra of $H_{\partial\overline{X}}$. If $\partial\overline{X} = S^{2k-1}$ then we show that the homomorphisms are equivariant with respect to Möbius transformations of S^{2k-1}.

Suppose that $\overline{X}$ is a partial compactification as in Section 8.3, satisfying Assumption 8.8, and the hypothesis of Corollary 8.12. With reference to Definition 8.13, there is a $\mathbb{Z}_2$-grading $H_{\partial\overline{X}} = H_{\partial\overline{X},+} \oplus H_{\partial\overline{X},-}$ coming from T. We put a smooth Riemannian metric g_0 on the manifold-with-boundary $\overline{X}$ in the given conformal class. We define $H^{-1/2}(\partial\overline{X}; \Lambda^k)$ using the induced metric on $\partial\overline{X}$. Let $P_{H_{\partial\overline{X},\pm}}$ denote orthogonal projection from $H^{-1/2}(\partial\overline{X}; \Lambda^k)$ to $H_{\partial\overline{X},\pm}$. From elliptic theory, for all $a \in C(\partial\overline{X})$, $\left[P_{H_{\partial\overline{X},\pm}}, (1+\triangle)^{1/4} a\,(1+\triangle)^{-1/4}\right]$ is compact. Hence one obtains homomorphisms $\tau_\pm : C(\partial\overline{X}) \to Q\left(H_{\partial\overline{X},\pm}\right)$ by $\tau_\pm(a) = \left[P_{H_{\partial\overline{X},\pm}} (1+\triangle)^{1/4} a\,(1+\triangle)^{-1/4} P_{H_{\partial\overline{X},\pm}}\right]$.

PROPOSITION 8.36. $\partial[(H, \gamma, F)]$ *equals* $[\tau_+] - [\tau_-]$ *in* $\mathrm{Ext}(C(\partial\overline{X})) \cong \mathrm{KK}_{2k-1}\,(C(\partial\overline{X}); \mathbb{C})$.

Proof. We wish to show that $[\sigma_\pm] = [\tau_\pm]$. The method of proof is similar to that of [7, Proposition 4.3]. The subspace $H_{\partial\overline{X}}$ of $H^{-1/2}(\partial\overline{X}; \Lambda^k)$ has an induced inner product that is boundedly equivalent to the inner product of Definition 8.11. To prove the proposition, it is sufficient to use the new inner product on $H_{\partial\overline{X}}$. Suppose first that $a \in C^\infty(\partial\overline{X})$. We will show that $[\sigma_\pm](a)$ equals the class of $\left[P_{H_{\partial\overline{X},\pm}}\, a\, P_{H_{\partial\overline{X},\pm}}\right]$ in $Q\left(H_{\partial\overline{X},\pm}\right)$. From elliptic theory, this in turn equals the class of $\left[P_{H_{\partial\overline{X},\pm}} (1+\triangle)^{1/4} a\,(1+\triangle)^{-1/4} P_{H_{\partial\overline{X},\pm}}\right]$.

Let $a' \in C_c^\infty(\overline{X})$ be an extension of a. Using the isomorphism $i^*: \mathcal{H} \to H_{\partial\overline{X}}$, it suffices to show that $i^* P_{\mathcal{H}} a' - P_{H_{\partial\overline{X}}} a i^*$ is compact from $\mathcal{H}$ to $H_{\partial\overline{X}}$. As $i^* a' P_{\mathcal{H}} - a P_{H_{\partial\overline{X}}} i^*$ vanishes on $\mathcal{H}$, it suffices to show that $i^*[P_{\mathcal{H}}, a'] - [P_{H_{\partial\overline{X}}}, a] i^*$ is compact.

As $P_{H_{\partial\overline{X}}}$ is a zeroth order pseudodifferential operator, $[P_{H_{\partial\overline{X}}}, a]$ is compact on $H^{-1/2}(\partial\overline{X}; \Lambda^k)$, so $[P_{H_{\partial\overline{X}}}, a] i^*$ is compact from $\mathcal{H}$ to $H^{-1/2}(\partial\overline{X}; \Lambda^k)$.

From Proposition 8.9, $[P_{\mathcal{H}}, a']$ is compact from $L^2(X; \Lambda^k)$ to $L^2(X; \Lambda^k)$. Let D be the operator $d + d^*$ on X, where d^* is defined using g_0. Its maximal domain is $\mathrm{Dom}(D_{\max}) = \{\omega \in L^2(X; \Lambda^*) : (d + d^*)\,\omega \in L^2(X; \Lambda^*)\}$. Clearly $\mathcal{H} \subset \mathrm{Dom}(D_{\max})$. Applying [7, Lemma 3.2], we conclude that $i^*[P_{\mathcal{H}}, a']$ is compact from $\mathcal{H}$ to $H^{-1/2}(\partial\overline{X}; \Lambda^k)$.

If a is merely continuous then multiplication by a may not be defined on $H^{-1/2}(\partial\overline{X}; \Lambda^k)$. However, the operator $(1+\triangle)^{1/4} a\,(1+\triangle)^{-1/4}$ is well-defined and gives a homomorphism $C(\partial\overline{X}) \to B(H^{-1/2}(\partial\overline{X}; \Lambda^k))$. The proposition now follows from the norm density of $C^\infty(\partial\overline{X})$ in $C(\partial\overline{X})$. □

Taking $X \subset S^{2k}$ to be the upper hemisphere H^{2k}, it follows that $[\tau_+] - [\tau_-] \in \mathrm{Ext}(C(S^{2k-1})) \cong \mathrm{KK}_{2k-1}(C(S^{2k-1}); \mathbb{C})$ is the signature class of S^{2k-1}.

COROLLARY 8.37. *The map* $\tau_{\pm}: C(S^{2k-1}) \to Q(H_{S^{2k-1},\pm})$ *is* $\mathrm{Isom}^+(H^{2k})$-*equivariant.*

Proof. This follows from the fact that the proof of Proposition 8.36 is essentially $\mathrm{Isom}^+(H^{2k})$-equivariant. We give an alternative direct argument.

The group $\mathrm{Isom}^+(H^{2k})$ acts on $H^{-1/2}(S^{2k-1};\Lambda^k)$ through its action on S^{2k-1}, although not isometrically. For $g \in \mathrm{Isom}^+(H^{2k})$, we have $g\,P_{H_{S^{2k-1}}} = P_{H_{S^{2k-1}}}\,g\,P_{H_{S^{2k-1}}}$. Then $\mathrm{Isom}^+(H^{2k})$ acts by automorphisms on $B(H_{S^{2k-1}})$, with $g \in \mathrm{Isom}^+(H^{2k})$ sending $T \in B(H_{S^{2k-1}})$ to $P_{H_{S^{2k-1}}} g T g^{-1} P_{H_{S^{2k-1}}} = P_{H_{S^{2k-1}}} g P_{H_{S^{2k-1}}} T P_{H_{S^{2k-1}}} g^{-1} P_{H_{S^{2k-1}}}$. There is an induced action on $Q(H_{S^{2k-1}})$.

Suppose that $a \in C^\infty(S^{2k-1})$ and $g \in \mathrm{Isom}^+(H^{2k})$. Then

$$\begin{aligned} P_{H_{S^{2k-1}}} g a g^{-1} P_{H_{S^{2k-1}}} &= P_{H_{S^{2k-1}}} g a P_{H_{S^{2k-1}}} g^{-1} P_{H_{S^{2k-1}}} \\ &= P_{H_{S^{2k-1}}} g a P^2_{H_{S^{2k-1}}} g^{-1} P_{H_{S^{2k-1}}} \\ &= P_{H_{S^{2k-1}}} g \left(a P_{H_{S^{2k-1}}} - P_{H_{S^{2k-1}}} a\right) P_{H_{S^{2k-1}}} g^{-1} P_{H_{S^{2k-1}}} + \\ &\quad + P_{H_{S^{2k-1}}} g P_{H_{S^{2k-1}}} a P_{H_{S^{2k-1}}} g^{-1} P_{H_{S^{2k-1}}}. \end{aligned} \tag{8.38}$$

From elliptic theory, $a P_{H_{S^{2k-1}}} - P_{H_{S^{2k-1}}} a$ is compact. It follows that the homomorphism $C^\infty(S^{2k-1}) \to Q(H_{S^{2k-1},\pm})$ is $\mathrm{Isom}^+(H^{2k})$-equivariant. The corollary now follows by continuity. □

9. Odd Cycles on Limit Sets

In this section we construct Γ-equivariant Ext cycles on limit sets. If the limit set is the entire sphere-at-infinity S^{2k-1} then we use the Ext cycle of Section 8.5. If the limit set is a proper subset of the sphere-at-infinity S^{2k} then we take X to be a Γ-invariant union of connected components of the domain-of-discontinuity Ω. We apply the boundary construction of Section 8.2 to get an Ext cycle on Λ. We show that the resulting K-homology class is invariant under quasiconformal deformation. We use Section 8.5 to describe an explicit Ext cycle for the K-homology class in the quasiFuchsian case, and in the case of an acylindrical convex-cocompact hyperbolic three-manifold with incompressible boundary.

9.1. WHEN THE LIMIT SET IS THE ENTIRE SPHERE, ODD-DIMENSIONAL

In this section we suppose that $n = 2k-1$ and $\Lambda = S^{2k-1}$.

From Corollary 8.37, we have Γ-equivariant homomorphisms $\tau_{\pm}: C(S^{2k-1}) \to Q(H_{S^{2k-1},\pm})$. In the nonequivariant case the difference of such homomorphisms defines an Ext class and hence an odd KK-class, as the relevant algebra $C(S^{2k-1})$ is nuclear [18, Corollary 5.2.11 and Theorem 8.4.3]. In the equivariant case an odd KK-class gives rise to a Γ-equivariant Ext class, but the converse is not automatic (see [44]). However, it is true in our case, where the relevant KK-class is the image of the signature class of B^{2k} under the maps $\mathrm{KK}^{\Gamma}_{2k}(C_0(B^{2k}); \mathbb{C}) \cong \mathrm{KK}^{\Gamma}_{2k}(C(\overline{B^{2k}}), C(S^{2k-1}); \mathbb{C}) \xrightarrow{\partial} \mathrm{KK}^{\Gamma}_{2k-1}(C(S^{2k-1}); \mathbb{C})$. From the discussion of Section 6, this is a nontorsion class.

9.2. QUASICONFORMAL INVARIANCE II

In this section we take X to be a Γ-invariant union of connected components of the domain-of-discontinuity Ω. We give sufficient conditions for Assumption 8.8 to be satisfied. We show that the K-homology class arising from the boundary construction of Section 8.2 is invariant under quasiconformal deformation.

Let Γ' be a discrete torsion-free subgroup of $\mathrm{Isom}^+(H^{2k+1})$, with limit set Λ' and domain of discontinuity $X' = \Omega'$. We take the compactification $\overline{X}' = S^{2k}$.

PROPOSITION 9.1.

(1) *If* $\Lambda' = S^{2k-l}$ *and* $l \neq 2$ *then the compactification satisfies Assumption 8.8.*
(2) *If* Γ' *is convex-cocompact but not cocompact, and the convex core has totally geodesic boundary, then the compactification satisfies Assumption 8.8.*

Proof.

(1) If $\Lambda' = S^{2k-l}$ then Ω' is conformally equivalent to $H^{2k-l+1} \times S^{l-1}$. Consider the metric on $H^{2k-l+1} \times S^{l-1}$ that is a product of constant-curvature metrics. If l is odd then the differential form Laplacian on H^{2k-l+1} has a gap away from zero in its spectrum. It follows that Assumption 7.11 is satisfied in this case. If l is even then the p-form Laplacian on H^{2k-l+1} is strictly positive if $p \neq k - \frac{l}{2}, k - \frac{l}{2} + 1$. From this, the p-form Laplacian on $H^{2k-l+1} \times S^{l-1}$ is strictly positive if $p \neq k - \frac{l}{2}, k - \frac{l}{2} + 1, k + \frac{l}{2} - 1, k + \frac{l}{2}$. It follows that the k-form Laplacian on $H^{2k-l+1} \times S^{l-1}$ is strictly positive if $l \neq 2$. As the inclusion $\Omega' \to S^{2k}$ factors through continuous maps $\Omega' \to \overline{H^{2k-l+1}} \times S^{l-1} \to S^{2k}$, it follows that Assumption 8.8 is satisfied.
(2) In this case Ω' is a union of round balls in S^{2k} with disjoint closures. Putting the hyperbolic metric on each of these balls, Assumption 8.8 is satisfied. □

There is an evident extension of Proposition 9.1.2 to the case when rank-$2k$ cusps are allowed.

Let Γ and Γ' be discrete torsion-free subgroups of $\mathrm{Isom}^+(H^{2k+1})$. They are said to be quasiconformally related if there are an isomorphism $i : \Gamma' \to \Gamma$ and a quasiconformal homeomorphism $\phi : S^{2k} \to S^{2k}$ satisfying

$$\phi \circ \gamma' \circ \phi^{-1} = i(\gamma') \tag{9.2}$$

for all $\gamma' \in \Gamma'$. It follows that the limit sets Λ' and Λ are related by $\phi(\Lambda') = \Lambda$.

Let X' be a Γ'-invariant union of connected components of Ω'. Suppose that X' satisfies Assumption 8.8. Then the construction described in Section 8.2 gives Γ'-equivariant homomorphisms $\sigma_{\pm} : C(\Lambda') \to Q(H_{\partial \overline{X}', \pm})$. As in the previous section, the equivariant Ext class $[\sigma_+] - [\sigma_-]$ arises from a class in $\mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda'); \mathbb{C})$.

Suppose that Γ and Γ' are quasiconformally related. By naturality, there is an isomorphism $\left(\phi\big|_{\Lambda'}\right)_* : \mathrm{KK}^{\Gamma'}_{2k-1}(C(\Lambda'); \mathbb{C}) \to \mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda); \mathbb{C})$. Put $X = \phi(X')$. Then $\partial \overline{X'} = \Lambda'$ and $\partial \overline{X} = \Lambda$. Suppose that X' satisfies Assumption 7.11. By Propositions 7.13, 7.15, and Corollary 7.31, there are well-defined signature classes $[(H', \gamma', F')] \in \mathrm{KK}^{\Gamma'}_{2k}(C(X'); \mathbb{C}) \cong \mathrm{KK}^{\Gamma'}_{2k}(C(\overline{X'}), C(\Lambda'); \mathbb{C})$ and $[(H, \gamma, F)] \in \mathrm{KK}^{\Gamma}_{2k}(C(X); \mathbb{C}) \cong \mathrm{KK}^{\Gamma}_{2k}(C(\overline{X}), C(\Lambda); \mathbb{C})$.

PROPOSITION 9.3. $\left(\phi\big|_{\Lambda'}\right)_*(\partial[(H', \gamma', F')]) = \partial[(H, \gamma, F)]$ *in* $\mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda); \mathbb{C})$.

Proof. The proof is the same as that of Proposition 8.6, extended to the equivariant setting. □

Given a discrete group G, it follows that quasiconformally equivalent embeddings $G \to \mathrm{Isom}^+(H^{n+1})$ give rise to the same KK-class. We note that if Γ is a convex-cocompact representation of G then G is Gromov-hyperbolic and Λ is homeomorphic to ∂G. In principle the K-cycle that we have constructed for $\mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda); \mathbb{C})$ can be expressed entirely in terms of G.

9.3. ODD-DIMENSIONAL QUASIFUCHSIAN MANIFOLDS

In this section we give an explicit Γ-equivariant Ext cycle for the K-homology class in the quasiFuchsian case, as a pushforward of the Fuchsian cycle.

Let Γ' be a discrete torsion-free subgroup of $\mathrm{Isom}^+(H^{2k})$ whose limit set is S^{2k-1}. There is a natural Fuchsian embedding $\Gamma' \subset \mathrm{Isom}^+(H^{2k+1})$. Take $X' = B^{2k}$, the upper hemisphere. By Proposition 9.1.1, Assumption 8.8 is satisfied. A group $\Gamma \subset \mathrm{Isom}^+(H^{2k+1})$ that is quasiconformally related to Γ' is said to be a quasiFuchsian deformation of Γ'.

COROLLARY 9.4. $\partial[(H, \gamma, F)]$ *is the pushforward under* $\phi\big|_{S^{2k-1}}$ *of the signature class of* S^{2k-1} *in* $\mathrm{KK}^{\Gamma}_{2k-1}(C(S^{2k-1}); \mathbb{C})$.

Proof. This follows from Proposition 9.3. □

The Ext cycle for the signature class of S^{2k-1} in $\mathrm{KK}^{\Gamma}_{2k-1}(C(S^{2k-1});\mathbb{C})$ was described in Section 9.1. Given the quasiFuchsian group Γ, suppose that ϕ_1 and ϕ_2 are two quasiconformal maps satisfying (9.2). Then $\phi_1^{-1}\circ\phi_2\big|_{S^{2k-1}}: S^{2k-1}\to S^{2k-1}$ commutes with each element of Γ'. As the fixed points of the hyperbolic elements of Γ' are dense in its limit set S^{2k-1}, it follows that $\phi_1^{-1}\circ\phi_2\big|_{S^{2k-1}}=\mathrm{Id}_{S^{2k-1}}$, so $\phi_1\big|_{S^{2k-1}}=\phi_2\big|_{S^{2k-1}}$. Next, suppose that Γ'' is another Fuchsian group such that H^{2k}/Γ' is orientation-preserving isometric to H^{2k}/Γ''. Then there is some $g\in\mathrm{Isom}^+(H^{2k})$ so that $g\Gamma' g^{-1}=\Gamma''$. As g acts conformally on S^{2k-1}, we can *define* a conformal structure on Λ to be the standard conformal structure on the homeomorphic set $\phi^{-1}(\Lambda)=S^{2k-1}$. This is independent of the choices made.

The upshot is that there is a Γ-equivariant Ext cycle for the K-homology class in $\mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda);\mathbb{C})$, given by the pushforward of the signature Ext cycle for S^{2k-1} under the homeomorphism $\phi\big|_{S^{2k-1}}: S^{2k-1}\to\Lambda$. From Section 9.1, the signature Ext class for S^{2k-1} is nontorsion in $\mathrm{KK}^{\Gamma'}_{2k-1}(C(S^{2k-1});\mathbb{C})$. As $\left(\phi\big|_{S^{2k-1}}\right)_*$ is an isomorphism, it follows that the class in $\mathrm{KK}^{\Gamma}_{2k-1}(C(\Lambda);\mathbb{C})$ is also nontorsion.

9.4. THE CASE OF A QUASICIRCLE

Applying the construction of Section 9.3 in the case $k=1$, we show that we recover the K-homology class on a quasicircle considered by Connes and Sullivan.

Suppose that $k=1$ and $\Gamma\subset\mathrm{Isom}^+(H^3)$ is a quasiFuchsian group. Let B^2 be the open upper hemisphere in S^2 and put $X=\phi(B^2)$. If D^2 is the closed disk in $\mathbb{C}$, let $Z:\mathrm{int}(D^2)\to X$ be a uniformization, i.e. a holomorphic isomorphism. The pullback $Z^*: L^2(X;\Lambda^1)\to L^2(\mathrm{int}(D^2);\Lambda^1)$ is an isometry. Because Z is a conformal diffeomorphism, Z^* sends $\mathcal{H}_X$ isometrically to $\mathcal{H}_{\mathrm{int}(D^2)}$. More explicitly, the elements of $\mathcal{H}_{\mathrm{int}(D^2)}$ are square-integrable forms $f_1(z)\mathrm{d}z+\overline{f_2(z)}\mathrm{d}\overline{z}$ on $\mathrm{int}(D^2)$, where f_1 and f_2 are holomorphic functions on $\mathrm{int}(D^2)$.

By Carathéodory's theorem, Z extends to a homeomorphism $Z: D^2\to\overline{X}$ [38, Theorem 14.19]. Then $Z^*H_{\partial\overline{X}}$ is isometric to $\mathrm{Im}\left(d: H^{1/2}(S^1;\Lambda^0)\to H^{-1/2}(S^1;\Lambda^1)\right)$, with the operator T acting by

$$T\left(\mathrm{e}^{ik\theta}\mathrm{d}\theta\right)=\begin{cases}\mathrm{e}^{ik\theta}\mathrm{d}\theta & \text{if } k>0,\\ -\mathrm{e}^{ik\theta}\mathrm{d}\theta & \text{if } k<0.\end{cases}\tag{9.5}$$

Unequivariantly, the homomorphisms $\sigma_\pm: C(S^1)\to Q(H_{S^1,\pm})$ are essentially the same as the standard Toeplitz homomorphisms.

We remark that the dual space to $Z^* H_{\partial \overline{X}}$ is $H^{1/2}(S^1; \Lambda^0)/\mathbb{C}$. The Hermitian form $S(f_1, f_2) = \int_{S^1} f_1 \wedge \mathrm{d}\overline{f_2}$ on $H^{1/2}(S^1; \Lambda^0)/\mathbb{C}$ is the Hermitian form of the Hilbert transform.

Let us compare the equivariant Ext class $[\sigma_+] - [\sigma_-]$ with that considered by Connes [1, Section IV.3.γ]. The latter is based on the Hilbert space $H_0 = L^2(S^1)$. The obvious Γ-action on H_0 is not unitary, but one can make it unitary by adding compensating weights. Then there is a Γ-invariant operator T_0 on H_0, which is essentially the Hilbert transform, and satisfies $T_0^2 = 1$. Decomposing H_0 with respect to T_0 as $H_0 = H_{0,+} \oplus H_{0,-}$, one obtains Γ-invariant homomorphisms $\sigma_{0,\pm} : C(S^1) \to Q(H_{0,\pm})$ given by $\sigma_{0,\pm}(f) = \frac{1 \pm T_0}{2} f \frac{1 \pm T_0}{2}$, modulo $K(H_{0,\pm})$.

Although there is a formal similarity between $H_{S^1,\pm}$ and $H_{0,\pm}$, they carry distinct representations of Γ. Nevertheless, the ensuing classes in $\mathrm{K}_1^\Gamma(C(S^1); \mathbb{C})$ are the same. To see this, consider the E_2-term $E_2^{0,0} = H^0(\Gamma; K_1(S^1))$ in the proof of Proposition 6.1. This term is unaffected by the differentials of the spectral sequence and passes to the limit to give a contribution to $\mathrm{K}_1^\Gamma(C(S^1); \mathbb{C})$. It corresponds to Γ-invariant elements of $K_1(S^1)$. Unequivariantly, $[\sigma_+] - [\sigma_-] = [\sigma_{0,+}] - [\sigma_{0,-}]$ in $K_1(S^1)$. As both sides are Γ-invariant, it follows that they give rise to the same class in $\mathrm{K}_1^\Gamma(C(S^1); \mathbb{C})$.

We note that the main use of the Connes–Sullivan cycle is to define certain operators on H_0 for which one wants to compute the trace. As the trace is formally independent of the choice of inner product, one can consider the same operators on H_{S^1}. See the remark after Proposition 11.4 for further discussion.

9.5. ODD-DIMENSIONAL CONVEX-COCOMPACT MANIFOLDS

In this section we give an explicit Γ-equivariant Ext cycle in the case of an odd-dimensional convex-cocompact hyperbolic manifold whose convex core has totally geodesic boundary. We use this to give an explicit cycle in the case of an arbitrary acylindrical convex-cocompact hyperbolic three-manifold with incompressible boundary.

Let M^{2k+1} be a noncompact convex-cocompact hyperbolic manifold with a convex core $Z \subset M$ whose boundary is totally geodesic. Let C be a boundary component of $\partial \overline{M}$. Then the preimage X of C in Ω is a union $\bigcup_{i=1}^\infty B_i$ of round balls in S^{2k} with disjoint closures. Put $Y_i = \partial \overline{B_i}$. Then Λ is the closure of $\bigcup_{i=1}^\infty Y_i$. By Proposition 9.1.2, Assumption 8.8 is satisfied. We now describe the Ext cycle on Λ coming from Section 8.2. From Section 8.4, the Hilbert space will be $H = \bigoplus_{i=1}^\infty \mathrm{Im}\left(d : H^{1/2}(Y_i; \Lambda^{k-1}) \to H^{-1/2}(Y_i; \Lambda^k)\right)$. It is $\mathbb{Z}_2$-graded by the operator T of Definition 8.13, applied separately to each Y_i. The Ext class will be $[\sigma_+] - [\sigma_-]$, where the homomorphisms $\sigma_\pm : C(\Lambda) \to$

$Q(H_{\pm})$ come from restricting $f \in C(\Lambda)$ to each Y_i and applying the map $\tau_{\pm}$ of Corollary 8.37.

Now let M be a noncompact acylindrical convex-cocompact hyperbolic three-manifold with incompressible boundary. Let Z be a compact core for M. There is a hyperbolic three-manifold M', homeomorphic to M, whose convex core has totally geodesic boundary (one applies Thurston's hyperbolization theorem for Haken manifolds to get an involution-invariant hyperbolic metric on the double DZ). Furthermore, it follows from [29, Theorem 8.1] that the groups $\Gamma' = \pi_1(M')$ and $\Gamma = \pi_1(M)$ are quasiconformally related. The K-homology class on Λ' is represented by the Ext cycle of the preceding paragraph. From Proposition 9.3, the K-homology class on Λ is represented by the pushforward of this Ext cycle by $\phi\big|_{\Lambda'}$. From the discussion of Section 6, if $\partial\overline{M}$ has more than one connected component then one gets nontorsion K-homology classes from this construction. Topologically, Λ is a Sierpinski curve.

There is an evident extension to the case when M is allowed to have rank-two cusps.

10. From Odd Cycles to Even Cycles

In Section 9 we considered the case when Λ is a proper subset of S^{2k} and showed how to pass from an even K-cycle on Ω to an Ext cycle on Λ. In this section we consider the case when Λ is a proper subset of S^{2k-1}. We then want to start with an odd cycle on Ω and construct an even K-cycle on Λ.

In the closed case, the relevant Hilbert space for an Ext cycle is the dual space to that of Section 8.3, namely $H^{1/2}(X, \Lambda^{k-1})/\operatorname{Ker}(d)$. If X instead has a compactification $\overline{X}$ then there are different choices for $H^{1/2}(X, \Lambda^{k-1})/\operatorname{Ker}(d)$, depending on the particular metric (complete or incomplete) taken in the given conformal class. This point deserves further study. A related problem is to develop a good notion of a relative version of Ext and the corresponding boundary map, as mentioned in [6, p. 3]. Of course there is a boundary map in odd relative K-homology [18, Proposition 8.5.6(b)], but in our case the natural cycles are Ext cycles. In this section we will just illustrate using smooth forms how to go from the odd cycle on X to an even K-cycle on ∂X. We describe the resulting K-cycle in the quasiFuchsian case, and in the case of a quasiconformal deformation of a convex-cocompact hyperbolic manifold whose convex core has totally geodesic boundary. In the case $k = 1$ we recover the K-cycle on a Cantor set considered by Connes and Sullivan.

10.1. THE BOUNDARY MAP IN THE ODD CASE

In this section we describe a formalism to go from the Ext cycle of Section 8.3, considered on an odd-dimensional manifold-with-boundary, to an even K-cycle on the boundary.

Let X^{2k-1} be an odd-dimensional compact oriented manifold-with-boundary. Let $i:\partial X \to X$ be the boundary inclusion. We write

$$\mathrm{Ker}(d) = \mathrm{Ker}\left(d : C^\infty(X;\Lambda^{k-1}) \to C^\infty(X;\Lambda^k)\right) \tag{10.1}$$

and

$$\mathrm{Ker}(d)_0 = \{\omega \in \mathrm{Ker}(d) : i^*\omega = 0\}. \tag{10.2}$$

The form

$$S(\omega_1,\omega_2) = i^k \int_X \omega_1 \wedge \mathrm{d}\overline{\omega_2} \tag{10.3}$$

is well-defined on $C^\infty(X;\Lambda^{k-1})/\mathrm{Ker}(d)_0$ and satisfies

$$S(\omega_1,\omega_2) - \overline{S(\omega_2,\omega_1)} = -(-i)^k \int_{\partial X} i^*\omega_1 \wedge i^*\overline{\omega_2}. \tag{10.4}$$

The map $i^* : C^\infty(X;\Lambda^{k-1}) \to C^\infty(\partial X;\Lambda^{k-1})$ restricts to a map on $\mathrm{Ker}(d)/\,\mathrm{Ker}(d)_0$, with image $i^*\,\mathrm{Ker}(d) \subset C^\infty(\partial X;\Lambda^{k-1})$.

We now assume that ∂X has a conformal structure. Then we have the Hilbert space $H_{\partial X} = L^2(\partial X;\Lambda^{k-1})$, with $\mathbb{Z}_2$-grading operator γ as in (7.2). From (10.4),

$$S(\omega_1,\omega_2) - \overline{S(\omega_2,\omega_1)} = (-1)^{k+1}\, i\, \langle i^*\omega_1, \gamma\, i^*\omega_2\rangle_{\partial X}. \tag{10.5}$$

This is a compatibility between the form S on X and the inner product on ∂X.

PROPOSITION 10.6. *There is an orthogonal decomposition*

$$H_{\partial X} = \overline{i^*\,\mathrm{Ker}(d)} \oplus \gamma\, \overline{i^*\,\mathrm{Ker}(d)}. \tag{10.7}$$

Proof. Suppose that $\omega_1', \omega_2' \in \mathrm{Ker}(d) \subset C^\infty(X;\Lambda^{k-1})$. Then

$$\int_{\partial X} \omega_1' \wedge \overline{\omega_2'} = \int_X \mathrm{d}(\omega_1' \wedge \overline{\omega_2'}) = 0. \tag{10.8}$$

This implies that $\overline{i^*\,\mathrm{Ker}(d)}$ and $\gamma\, \overline{i^*\,\mathrm{Ker}(d)}$ are perpendicular.

If $\omega = d\eta$ with $\eta \in C^\infty(\partial X;\Lambda^{k-2})$, and $\eta' \in C^\infty(X;\Lambda^{k-2})$ satisfies $i^*\eta' = \eta$, then $\omega = i^* d\eta'$. Thus $\mathrm{Im}(d : C^\infty(\partial X;\Lambda^{k-2}) \to C^\infty(\partial X;\Lambda^{k-1}))$ is contained in $i^*\,\mathrm{Ker}(d)$, and similarly $\mathrm{Im}(d^* : C^\infty(\partial X;\Lambda^k) \to C^\infty(X;\Lambda^{k-1}))$ is contained in $\gamma\, i^*\,\mathrm{Ker}(d)$.

Suppose that $\omega \in H_{\partial X}$ is orthogonal to $\overline{i^* \mathrm{Ker}(\mathrm{d})}$ and $\gamma \, \overline{i^* \mathrm{Ker}(d)}$. It follows that $\mathrm{d}\omega = d^*\omega = 0$. Without loss of generality, we can take ω to be real. Let $[\omega] \in \mathrm{H}^{k-1}(\partial X; \mathbb{R})$ denote the corresponding cohomology class. From the cohomology exact sequence

$$\cdots \to \mathrm{H}^{k-1}(X;\mathbb{R}) \xrightarrow{i^*} \mathrm{H}^{k-1}(\partial X;\mathbb{R}) \xrightarrow{(i^*)^*} \mathrm{H}^{k}(X, \partial X;\mathbb{R}) \to \cdots, \tag{10.9}$$

$i^* \mathrm{H}^{k-1}(X;\mathbb{R})$ is a maximal isotropic subspace of $\mathrm{H}^{k-1}(\partial X;\mathbb{R})$. Representing $\mathrm{H}^{k-1}(\partial X;\mathbb{R})$ by harmonic forms, $\gamma \, i^* \mathrm{H}^{k-1}(X;\mathbb{R})$ is orthogonal to $i^* \mathrm{H}^{k-1}(X;\mathbb{R})$. By assumption, ω is orthogonal to $i^* \mathrm{H}^{k-1}(X;\mathbb{R})$ and $\gamma \, i^* \mathrm{H}^{k-1}(X;\mathbb{R})$. Thus $\omega = 0$. □

Define $F'_{\partial X} \in B(H_{\partial X})$ by

$$F'_{\partial X}(\omega) = \begin{cases} \omega & \text{if } \omega \in \overline{i^* \mathrm{Ker}(d)}, \\ -\omega & \text{if } \omega \in \gamma \, \overline{i^* \mathrm{Ker}(d)}. \end{cases} \tag{10.10}$$

PROPOSITION 10.11. *The triple* $(H_{\partial X}, \gamma, F'_{\partial X})$ *represents the same class in* $\mathrm{KK}_{2k-2}(C(\partial X);\mathbb{C})$ *as the triple* $(H_{\partial X}, \gamma, F)$ *of Section* 7.1.

Proof. As $\mathrm{H}^{k-1}(\partial X;\mathbb{C})$ is finite-dimensional, $F'_{\partial X} - F$ is compact. □

Proposition 10.11 shows the K-cycle on ∂X constructed from X, namely $(H_{\partial X}, \gamma, F'_{\partial X})$, represents the desired K-homology class on ∂X.

10.2. EVEN-DIMENSIONAL QUASIFUCHSIAN MANIFOLDS

In this section we apply the formalism of Section 10.1 to describe an equivariant K-cycle on the limit set of an even-dimensional quasiFuchsian manifold, in analogy with Section 9.3.

We first consider the case of a Fuchsian manifold. Let Γ' be a discrete torsion-free subgroup of $\mathrm{Isom}^+(H^{2k-1})$ whose limit set is S^{2k-2}. There is a natural embedding $\Gamma' \subset \mathrm{Isom}^+(H^{2k})$, with limit set $\Lambda' = S^{2k-2} \subset S^{2k-1}$. Applying Section 10.1 with X being the upper hemisphere $H^{2k-1} \subset S^{2k-1}$ gives the K-cycle for $\mathrm{KK}^{\Gamma'}_{2k-2}(C(S^{2k-2});\mathbb{C})$ of Section 7.4.

A group $\Gamma \subset \mathrm{Isom}^+(H^{2k})$ that is quasiconformally related to Γ' is said to be a quasiFuchsian deformation of Γ'. Motivated by Section 9.3, we can *define* a cycle for $\mathrm{KK}^{\Gamma}_{2k-2}(C(\Lambda);\mathbb{C})$ by the pushforward under $\phi|_{S^{2k-2}}$ of the K-cycle for $\mathrm{KK}^{\Gamma'}_{2k-2}(C(S^{2k-2});\mathbb{C})$. As in Section 9.3, this is independent of the choice of ϕ. From Section 7.4, the signature class for S^{2k-2} is nontorsion in $\mathrm{KK}^{\Gamma'}_{2k-2}(C(S^{2k-2});\mathbb{C})$. As $\left(\phi|_{S^{2k-2}}\right)_*$ is an isomorphism, it follows that the class in $\mathrm{KK}^{\Gamma}_{2k-2}(C(\Lambda);\mathbb{C})$ is also nontorsion.

10.3. EVEN-DIMENSIONAL CONVEX-COCOMPACT MANIFOLDS

In this section we apply the formalism of Section 10.1 to describe an equivariant K-cycle on the limit set of a quasiconformal deformation of an even-dimensional convex-cocompact hyperbolic manifold whose convex core has totally geodesic boundary.

Let Γ' be a convex-cocompact subgroup of $\mathrm{Isom}^+(H^{2k})$ whose convex core has totally geodesic boundary. Let C be a connected component of $\partial\overline{M}$. Then the preimage X of C in Ω is a union $\bigcup_{i=1}^{\infty} B_i$ of round balls in S^{2k-1} with disjoint closures. Put $Y_i = \partial\overline{B_i}$. Then the limit set Λ' is the closure of $\bigcup_{i=1}^{\infty} Y_i$.

The Hilbert space of Section 10.1 becomes $H = \bigoplus_{i=1}^{\infty} L^2(Y_i; \Lambda^{k-1})$. Define $\gamma_i \in B(L^2(Y_i; \Lambda^{k-1}))$ as in (7.2). Put $\gamma = \bigoplus_{i=1}^{\infty} \gamma_i$. The operator F of (10.10) becomes a direct sum $F = \bigoplus_{i=1}^{\infty} F_i$ where $F_i \in B(L^2(Y_i; \Lambda^{k-1}))$ is as in (7.10). An element $a \in C(\Lambda')$ acts diagonally on H as multiplication by $a_i = a\big|_{Y_i}$ on $L^2(Y_i; \Lambda^{k-1})$.

PROPOSITION 10.12. *(H, γ, F) is a cycle for* $\mathrm{KK}^{\Gamma'}_{2k-2}(C(\Lambda'); \mathbb{C})$.

Proof. Given $a \in C(\Lambda')$, we must show that $[F, a]$ is compact. Extending a to $a' \in C(S^{2k-1})$ and approximating the latter by smooth functions, we may assume that a' is smooth.

We know that for each i, $[F_i, a_i]$ is compact. It suffices to show that $\lim_{i\to\infty} \| [F_i, a_i] \| = 0$. Fixing a round metric on S^{2k-1}, let $\overline{a}_i$ be the average value of a_i on Y_i. Then $[F_i, a_i] = [F_i, a_i - \overline{a}_i]$ and $\lim_{i\to\infty} \| a_i - \overline{a}_i \| = 0$, from which the proposition follows. □

Now let Γ be a quasiconformal deformation of Γ'. We can construct a cycle for $\mathrm{KK}^{\Gamma}_{2k-2}(C(\Lambda); \mathbb{C})$ as the pushforward of the preceding K-cycle by $\phi\big|_{\Lambda'}$. As in Section 9.3, this is independent of the choice of ϕ.

10.4. THE CASE OF A CANTOR SET

In this section we specialize Section 10.3 to the case $k = 1$.

Let $\Gamma \subset \mathrm{Isom}^+(H^2)$ be a convex-cocompact subgroup. If $M = H^2/\Gamma$ is noncompact then it has a convex core with totally geodesic boundary, and Λ is a Cantor set. Let C be a connected component of Ω/Γ. Then its preimage X in Ω is a countable disjoint union of open intervals (b_i, c_i) in S^1, and Λ is the closure of the endpoints $\{b_i, c_i\}_{i=1}^{\infty}$. We have $H = l^2\left(\{b_i, c_i\}_{i=1}^{\infty}\right)$. Define $\gamma \in B(H)$ by saying that for each $\omega \in H$ and each i, $(\gamma\omega)(b_i) = -\omega(b_i)$ and $(\gamma\omega)(c_i) = \omega(c_i)$. As $\mathrm{Ker}(d)$ consists of locally constant functions on X, we obtain $(F\omega)(b_i) = \omega(c_i)$ and $(F\omega)(c_i) = \omega(b_i)$.

Taking a direct sum over the connected components C gives the K-cycle (H, γ, F) considered in [11, Proposition 21, Section IV.3.ϵ]. The cited reference discusses (H, F) as an ungraded K-cycle.)

11. *p*-Summability

In this section we show the p-summability of a certain Fredholm module $(\mathcal{A}, H, F)$ for sufficiently large p.

With reference to Section 10.2, let $\mathcal{A}$ be the restriction of $\phi^* C^\infty(S^{2k-1})$ to S^{2k-2}, a subalgebra of $C(S^{2k-2})$. Then we have an even Fredholm module $(\mathcal{A}, L^2(S^{2k-2}; \Lambda^{k-1}), F)$ in the sense of Connes [11, Chapter IV, Definition 1].

PROPOSITION 11.1. *For sufficiently large p,* $(\mathcal{A}, L^2(S^{2k-2}; \Lambda^{k-1}), F)$ *is p-summable in the sense of Connes* [114*, Chapter IV, Definition* 3].

Proof. We claim that for p large, $[F, a]$ is in the p-Schatten ideal for all $a \in \mathcal{A}$. Given $x, y \in S^{2k-2} \subset \mathbb{R}^{2k-1}$, let $|x-y|$ denote the chordal distance between them. From Janson-Wolff (1982), it suffices to show that

$$\int_{S^{2k-2}\times S^{2k-2}} \frac{|a(x)-a(y)|^p}{|x-y|^{4k-4}} \,\mathrm{d}x\,\mathrm{d}y < \infty. \tag{11.2}$$

(The statement of Janson and Wolff [21] is for operators on $\mathbb{R}^{2k-2}$ instead of S^{2k-2}. We can go from one to the other by stereographic projection, using the conformally-invariant measure $\frac{\mathrm{d}x\,\mathrm{d}y}{|x-y|^{4k-4}}$.) As ϕ is a quasiconformal homeomorphism, it lies in the Hölder space $C^{0,\alpha}$ for some $\alpha \in (0,1)$. Then there is a constant $C > 0$ such that $|a(x)-a(y)|^p \leqslant C\,|x-y|^{\alpha p}$ for all $x, y \in S^{2k-2}$. The claim follows for $p > \frac{2k-2}{\alpha}$. □

With reference to Section 9.3, let $\mathcal{A}$ be the restriction of $\phi^* C^\infty(S^{2k})$ to S^{2k-1}, a subalgebra of $C(S^{2k-1})$. Let $E_\pm$ be the projection from $L^2(S^{2k-1}; \Lambda^k)$ to the ± 1-eigenspace of $\mathrm{sign}((-i)^k\, d*)$ acting on $\mathrm{Im}(d) \subset L^2(S^{2k-1}; \Lambda^k)$. Explicitly,

$$E_\pm = \frac{1}{2}\left(I \pm \frac{(-i)^k\, d*}{\triangle \mathrm{e}^{1/2}}\right)\frac{dd^*}{\triangle}. \tag{11.3}$$

For the motivation for Proposition 11.4, we refer to Connes [10, Section 7].

PROPOSITION 11.4. *For sufficiently large p,* $[E_\pm, a]$ *is in the p-Schatten ideal of operators on* $L^2(S^{2k-1}; \Lambda^k)$ *for all* $a \in \mathcal{A}$.

Proof. The proof is the same as that of Proposition 11.1. □

We note that Proposition 11.4 refers to $L^2(S^{2k-1}; \Lambda^k)$, whereas it is the $H^{-1/2}$-space $\mathrm{Im}(d) \subset H^{-1/2}(S^{2k-1}; \Lambda^k)$ that is Möbius invariant. We can consider $L^2(S^{2k-1}; \Lambda^k)$ to be a dense subspace of $H^{-1/2}(S^{2k-1}; \Lambda^k)$. The orthogonal projection $E'_\pm$ from $H^{-1/2}(S^{2k-1}; \Lambda^k)$ to the ± 1-eigenspace of $\mathrm{sign}((-i)^k\, d*)$ acting on $\mathrm{Im}(d) \subset H^{-1/2}(S^{2k-1}; \Lambda^k)$, i.e. to $\mathrm{Im}\left(\frac{I \pm T}{2}\right)$, is again given by the formula in (11.3). Although we do not show the p-summability of the ungraded Fredholm module $(\mathcal{A}, H^{-1/2}(S^{2k-1}; \Lambda^k), E'_+ - E'_-)$, Proposition 11.4 suffices for making sense of the cyclic cocycles of Connes [10, Section 7] in our case.

In the case $k=1$ of Proposition 11.4, [11, Section IV.3.γ, Proposition 14] has the stronger statement that

$$\delta(\Gamma) = \inf\{p : [E_\pm, a] \text{ is in the } p\text{-Schatten ideal for all } a \in \mathcal{A}\}. \quad (11.5)$$

We do not know if a similar statement holds for all k. Using [21], it reduces to a question about the Besov regularity of $\phi\big|_{S^{2k-1}}$. The proof in [11, Section IV.3.γ, Proposition 14] uses facts about holomorphic functions that are special to the case $k=1$. One can ask the same question in the setup of Proposition 11.1.

Again in the case $k=1$, [11, Section IV.3.γ, Theorem 17] expresses the Patterson–Sullivan measure on the limit set in terms of the Dixmier trace.

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