Bridging scales
from microscopic dynamics to macroscopic laws

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Guiding principles in probability

Symmetry
If different outcomes are equivalent (from the perspective of the mechanism causing them), they should have the same probability.

Universality
In many instances, if a random outcome is a consequence of many different sources of randomness, the details of its description should not matter much. (Outcomes of successive coin tosses: de Moivre 1733, Laplace 1812, ...)

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Independently in 1905-1906 give a theory of Brownian motion, building on earlier work by Lord Rayleigh.

1. **Physics**: Random motion caused by collision with fluid molecules.

2. **Mathematics**: Probability distribution of the position of the particle is described by the heat equation.

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Two dimensional analogue of random walk:

Random function $h: \text{Grid} \rightarrow \mathbb{Z}$ such that $|h(x) - h(y)| = 1$ for $x \sim y$. What does $h$ look like at very large scales?
Large scale behaviour should be described by “free field”, Gaussian generalised function with $\mathbb{E} h(x) h(y) = -\log |x - y|$. No proof yet! (But for similar models, see Borodin, Johansson, Kenyon, Okounkov, Peled, Toninelli, etc.)

Formally, $P(dh) \propto \exp(-\int |\nabla h|^2 \, dx) \, “dh”.}$
Beyond “free” systems

Ising model: state space $\sigma: \Lambda \to \{\pm 1\}$. Probability to see $\sigma$ proportional to $\exp(\beta \sum_{x \sim y} \sigma_x \sigma_y)$.

At critical temperature, one has a non-Gaussian scaling limit (rigorous proofs only over last few years), conjectured to be universal for many phase transition models.
Context: Two “phases” of differing stability. Stable phase invades unstable phase.

Many simple mathematical models exhibit similar features, appear to have same scaling limit. (See Borodin, Corwin, Ferrari, Johansson, Quastel, Sasamoto, Spohn, . . . )
Interface growth models

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Some properties of these objects

In general: Gaussianity not expected when interactions are present.

Scale invariance holds for such scaling limits essentially by definition. Markov property in space(-time) natural for systems with local interactions. Translation invariance and Rotation invariance holds as soon as limit is canonical in some sense. Leap of faith: conformal invariance.

Two dimensions: conformal invariance gives infinite-dimensional symmetry group. Consequence: a lot is known explicitly for a one-parameter family of conformally invariant / covariant objects called conformal field theories. (From probability perspective, see SLE, QLE, CLE, ...)
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Crossover regimes

Consider models that converge to a Gaussian fixed point when “zooming in” and a non-Gaussian FP when “zooming out”. Described by simple “normal form” equations:

\[
\partial_t h = \partial_{xx}^2 h + (\partial_x h)^2 + \xi - C, \quad (\text{KPZ; } d = 1)
\]

\[
\partial_t \Phi = -\Delta (\Delta \Phi + C \Phi - \Phi^3) + \nabla \xi. \quad (\Phi^4; \ d = 2, 3)
\]

Here $\xi$ is space-time white noise (think of independent random variables at every space-time point).

KPZ: universal model for weakly asymmetric interface growth.

$\Phi^4$: universal model for phase coexistence near mean-field.

Problem: red terms ill-posed, requires $C = \infty$!!
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A general theorem

Joint with Y. Bruned, A. Chandra, I. Chevyrev, L. Zambotti.

Consider a system of semilinear stochastic PDEs of the form

$$\partial_t u_i = \mathcal{L}_i u_i + G_i(u, \nabla u, \ldots) + F_{ij}(u)\xi_j,$$

with elliptic $\mathcal{L}_i$ and stationary random (generalised) fields $\xi_j$ that are scale invariant with exponents for which $(\star)$ is subcritical.

Then, there exists a canonical family $\Phi_g : (u_0, \xi) \mapsto u$ of “solutions” parametrised by $g \in \mathcal{R}$, a finite-dimensional nilpotent Lie group built from $(\star)$. Furthermore, the maps $\Phi_g$ are continuous in both of their arguments.
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Then, there exists a canonical family $\Phi_g: (u_0, \xi) \mapsto u$ of “solutions” parametrised by $g \in \mathcal{K}$, a finite-dimensional nilpotent Lie group built from $(\star)$. Furthermore, the maps $\Phi_g$ are continuous in both of their arguments.
Some clarifications

**Canonicity**
Family \( \{ \Phi_g : g \in \mathcal{R} \} \) is canonical, but parametrisation only canonical modulo shifts: action of \( \mathcal{R} \) on \((F,G)\) such that

\[
\Phi^{(F,G)}_{g\tilde{g}} = \Phi^{\tilde{g}(F,G)}_g.
\]

For smooth \( \xi \), one has a classical solution map \( \Phi^{(F,G)} \) and
\[
\Phi^{(F,G)}_g = \Phi(g \circ \hat{g} (\xi))(F,G).
\]

**Continuity**
Measure \( S \) of “size” of noise. Take \( \xi_n \) with \( \sup_n S(\xi_n) < \infty \) and \( \xi_n \rightarrow \xi \) weakly in probability. Then \( \Phi_g(\cdot, \xi_n) \rightarrow \Phi_g(\cdot, \xi) \) in some \( C^\alpha \), locally uniformly in time and initial condition, in probability. However, \( \xi \mapsto \hat{g}(\xi) \) not continuous, not even defined!
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For smooth $\xi$, one has a classical solution map $\Phi^{(F,G)}$ and $\Phi_g^{(F,G)} = \Phi(g \circ \hat{g}(\xi))(F,G)$.

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Construction of $\Phi_g$

Crucial remark: **Locally**, near any space-time point $z$, solution looks like a linear combination of functions / distributions $\Pi_z \tau$ such that, for each index $\tau$, $\Pi_z \tau$ is scale-invariant with exponent $\deg \tau$.

Deterministic analogue: solutions to parabolic PDEs are smooth, so are locally a linear combination of $(\Pi_z X^k)(\tilde{z}) = (\tilde{z} - z)^k$, scale-invariant with exponent $|k|$.

Methodology: Work in spaces of distributions locally described by $\Pi_z H$ for continuous coefficient-valued functions $H$ and look for a fixed point problem for $H$. 
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Example / problem

Solution to

$$\partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi,$$

locally given by $h(\tilde{z}) \approx (\Pi_z H(z))(\tilde{z})$ with

$$H = h \mathbf{1} + \sigma(h) \circ \circ + (\sigma \sigma')(h) \circ \circ + (f \sigma^2)(h) \circ \circ + h' X$$

$$+ 2(f \sigma^2 \sigma')(h) \circ \circ + 2(f^2 \sigma^3)(h) \circ \circ + (f' \sigma^3)(h) \circ \circ$$

$$+ \frac{1}{2}(\sigma^2 \sigma'')(h) \circ \circ + (\sigma (\sigma')^2)(h) \circ \circ + (f \sigma^2 \sigma')(h) \circ \circ$$

$$+ (f' \sigma)(h)h' \circ \circ + 2(f \sigma)(h)h' \circ \circ + \ldots$$

Problem: $\Pi_z \circ \circ = G \star (\partial_x G \star \xi)^2$ is divergent!
Example / problem

Solution to
\[ \partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi, \]
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\[ h(\tilde{z}) \approx (\Pi_z H(z))(\tilde{z}) \]
with
\[ H = h 1 + \sigma(h) \dot{\sigma} + (\sigma\sigma')(h) \ddot{\sigma} + (f\sigma^2)(h) \sigma' + h' X \]
\[ + 2(f\sigma^2\sigma')(h) \sigma'' + 2(f^2\sigma^3)(h) \gamma + (f'\sigma^3)(h) \gamma \]
\[ + \frac{1}{2}(\sigma^2\sigma'')(h) \gamma + (\sigma(\sigma')^2)(h) \sigma' + (f\sigma^2\sigma')(h) \gamma \]
\[ + (f'\sigma)(h)h' \dot{\sigma} + 2(f\sigma)(h)h' \sigma' + \ldots. \]

Problem: \( \Pi_z \sigma' = G \ast (\partial_x G \ast \xi)^2 \) is divergent!
Example / problem

Solution to
\[ \partial_t h = \partial_x^2 h + f(h)(\partial_x h)^2 + \sigma(h)\xi, \]
locally given by \( h(\tilde{z}) \approx (\Pi_{\tilde{z}} H(z))(\tilde{z}) \) with
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H = h \mathbf{1} + \sigma(h) \bullet + (\sigma\sigma')(h) \circ + (f\sigma^2)(h) \bigcirc + h' X \\
+ 2(f\sigma^2\sigma')(h) \bigtriangleup + 2(f^2\sigma^3)(h) \bigtriangledown + (f'\sigma^3)(h) \bigstar \\
+ \frac{1}{2}(\sigma^2\sigma'')(h) \bullet + (\sigma(\sigma')^2)(h) \circ + (f\sigma^2\sigma')(h) \bigstar \\
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H = h \, 1 + \sigma(h) \, \circ + (\sigma \sigma')(h) \, \bullet + (f \sigma^2)(h) \, \bigtriangledown + h' \, X
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\]
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Problem: \( \Pi_z G = G \ast (\partial_x G \ast \xi)^2 \) is divergent!
Example / problem

Solution to

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Problem: \( \Pi_z \times = G \ast (\partial_x G \ast \xi)^2 \) is divergent!
Steps of proof

1. Show that \((h, h')\) depend \textit{continuously} on the data \(\{\Pi_{z\tau} : z, \tau\}\) in suitable topology enforcing some natural algebraic relations.

2. Replace \(\Pi_{z\tau}\) by “renormalised version” \(\Pi_{z\tau}^g\) such that algebraic relations between the \(\Pi_{z\tau}^g\)’s remain unchanged. (Determines the group \(\mathcal{R}\).)

3. Choose \(g\) such that \(E\Pi_{z\tau}^g = 0\) for \(\deg \tau \leq 0\). (Determines the element \(g\) from the law of \(\xi\).)

4. Show stability / continuity of \(\xi \mapsto \Pi_{z\tau}^g(\xi)\).

5. Show that the substitution \(\Pi \mapsto \Pi^g\) is equivalent to replacing \((f, \sigma)\) by \(g(f, \sigma)\) for suitable action of \(\mathcal{R}\).
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Steps of proof

1. Show that \((h, h')\) depend continuously on the data \(\{\Pi_{z\tau} : z, \tau\}\) in suitable topology enforcing some natural algebraic relations.

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Thanks

Thank you very much for your attention

(See you tomorrow...)
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