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Local index theory over foliation groupoids

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Abstract

We give a local proof of an index theorem for a Dirac-type operator that is invariant with respect to the action of a foliation groupoid G. If M denotes the space of units of G then the input is a G-equivariant fiber bundle $P \to M$ along with a G-invariant fiberwise Dirac-type operator D on P. The index theorem is a formula for the pairing of the index of D, as an element of a certain K-theory group, with a closed graded trace on a certain noncommutative de Rham algebra $\Omega^*\mathscr{B}$ associated to G. The proof is by means of superconnections in the framework of noncommutative geometry.

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1. Introduction

It has been clear for some time, especially since the work of Connes [9] and Renault [27], that many interesting spaces in noncommutative geometry arise from groupoids. For background information, we refer to Connes' book [11, Chapter II]. In particular, to

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a smooth groupoid G one can assign its convolution algebra $C_c^\infty(G)$, which represents a class of smooth functions on the noncommutative space specified by G.

An important motivation for noncommutative geometry comes from index theory. The notion of groupoid allows one to unify various index theorems that arise in the literature, such as the Atiyah–Singer families index theorem [2], the Connes–Skandalis foliation index theorem [13] and the Connes–Moscovici covering space index theorem [12]. All of these theorems can be placed in the setting of a proper cocompact action of a smooth groupoid G on a manifold P. Given a G-invariant Dirac-type operator D on P, the construction of [12] allows one to form its analytic index, Ind_a, as an element of the K-theory of the algebra $C_c^\infty(G) \otimes \mathcal{R}$, where $\mathcal{R}$ is an algebra of infinite matrices whose entries decay rapidly [11, Sections III.4, III.7.γ]. When composed with the trace on $\mathcal{R}$, the Chern character $\mathrm{ch}(\mathrm{Ind}_a)$ lies in the periodic cyclic homology group $\mathrm{PHC}_*(C_c^\infty(G))$. The index theorem, at the level of Chern characters, equates $\mathrm{ch}(\mathrm{Ind}_a)$ with a topological expression $\mathrm{ch}(\mathrm{Ind}_t)$.

We remark that in the literature, one often sees the analytic index defined as an element of K-theory of the groupoid C^*-algebra $C_r^*(G)$. The index in $\mathrm{K}_*(C_c^\infty(G) \otimes \mathcal{R})$ is a more refined object. However, to obtain geometric and topological consequences from the index theorem, it appears that one has to pass to $C_r^*(G)$; we refer to [11, Chapter III] for discussion. In this paper we will work with $C_c^\infty(G)$.

We prove a local index theorem for a Dirac-type operator that is invariant with respect to the action of a foliation groupoid. In the terminology of Crainic–Moerdijk [15], a foliation groupoid is a smooth groupoid G with discrete isotropy groups, or equivalently, which is Morita equivalent to a smooth étale groupoid.

A motivation for our work comes from the Connes–Skandalis index theorem for a compact foliated manifold $(M, \mathcal{F})$ with a longitudinal Dirac-type operator [13]. To a foliated manifold $(M, \mathcal{F})$ one can associate its holonomy groupoid G_{hol}, which is an example of a foliation groupoid. The general foliation index theorem equates Ind_a with a topological index Ind_t. For details, we refer to [11, Sections I.5, II.8-9, III.6-7].

We now state the index theorem that we prove. Let M be the space of units of a foliation groupoid G. It carries a foliation $\mathcal{F}$. Let ρ be a closed holonomy-invariant transverse current on M. There is a corresponding universal class $\omega_\rho \in \mathrm{H}^*(BG; o)$, where o is a certain orientation character on the classifying space BG. Suppose that G acts freely, properly and cocompactly on a manifold P. In particular, there is a submersion $\pi : P \to M$. There is an induced foliation $\pi^*\mathcal{F}$ of P with the same codimension as $\mathcal{F}$, satisfying $T\pi^*\mathcal{F} = (d\pi)^{-1}T\mathcal{F}$. Let g^{TZ} be a smooth G-invariant vertical Riemannian metric on P. Suppose that the vertical tangent bundle TZ is even-dimensional and has a G-invariant spin structure. Let S^Z be the corresponding vertical spinor bundle. Let $\widetilde{V}$ be an auxiliary G-invariant Hermitian vector bundle on P with a G-invariant Hermitian connection. Put $E = S^Z \widehat{\otimes} \widetilde{V}$, a G-invariant $\mathbb{Z}_2$-graded Clifford bundle on P which has a G-invariant connection. The Dirac-type operator Q acts fiberwise on sections of E. Let D be its restriction to the sections of positive parity. (The case of general G-invariant Clifford bundles E is completely analogous.) Let $\mu : P \to P/G$ be the quotient map. Then, P/G is a smooth compact manifold with a foliation $F = (\pi^*\mathcal{F})/G$ satisfying $(d\mu)^{-1}TF = T\pi^*\mathcal{F}$. Put $V = \widetilde{V}/G$, a Hermitian vector bundle on P/G with a Hermitian connection ∇^V. The G-action on P is classified by a map $\nu : P/G \to BG$, defined up to homotopy.

The main point of this paper is to give a local proof of the following theorem.

Theorem 1.

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho\rangle = \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\ \nu^* \omega_\rho. \tag{1}$$

Here $\mathrm{Ind}\, D$ lies in $\mathrm{K}_*(C_c^\infty(G) \otimes \mathcal{R})$. If M is a compact foliated manifold and one takes $P = G = G_{\mathrm{hol}}$ then one recovers the result of pairing the Connes–Skandalis theorem with ρ; see also Nistor [24].

In saying that we give a local proof of Theorem 1, the word "local" is in the sense of Bismut's proof of the Atiyah–Singer family index theorem [6]. In our previous paper [16], we gave a local proof of such a theorem in the étale case. One can reduce Theorem 1 to the étale case by choosing a complete transversal T, i.e. a submanifold of M, possibly disconnected, with $\dim(T) = \mathrm{codim}(\mathcal{F})$ and which intersects each leaf of the foliation. Using T, one can reduce the holonomy groupoid G to a Morita-equivalent étale groupoid G_{et}. We gave a local proof of Connes' index theorem concerning an étale groupoid G_{et} acting freely, properly and cocompactly on a manifold P, preserving a fiberwise Dirac-type operator Q on P. Our local proof has since been used by Leichtnam and Piazza to prove an index theorem for foliated manifolds-with-boundary [21].

In the present paper, we give a local proof of Theorem 1 working directly with foliation groupoids. In particular, the new proof avoids the noncanonical choice of a complete transversal T.

The overall method of proof is by means of superconnections in the context of noncommutative geometry, as in [16]. However, there are conceptual differences with respect to [16]. As in [16], we first establish an appropriate differential calculus on the noncommutative space determined by a foliation groupoid G. The notion of "smooth functions" on the noncommutative space is clear, and is given by the elements of the convolution algebra $\mathcal{B} = C_c^\infty(G)$. We define a certain graded algebra, $\Omega^*\mathcal{B}$, which plays the role of the differential forms on the noncommutative space. The algebra $\Omega^*\mathcal{B}$ is equipped with a degree-1 derivation d, which is the analog of the de Rham differential. Unlike in the étale case, it turns out that in general, $d^2 \neq 0$. The reason for this is that to define d, we must choose a horizontal distribution $T^{\mathrm{H}}M$ on M, where "horizontal" means transverse to $\mathcal{F}$. In general $T^{\mathrm{H}}M$ is not integrable, which leads to the nonvanishing of d^2. This issue does not arise in the étale case.

As we wish to deal with superconnections in such a context, we must first understand how to do Chern–Weil theory when $d^2 \neq 0$. If d^2 is given by commutation with a two-form then a trick of Connes [11, Chapter III.3, Lemma 9] allows one to construct a new complex with $d^2 = 0$, thereby reducing to the usual case. We give a somewhat more general formalism that may be useful in other contexts. It assumes that for the relevant $\mathcal{B}$-module $\mathcal{E}$ and connection $\nabla : \mathcal{E} \to \Omega^1\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$, there is a linear map $l : \mathcal{E} \to \Omega^2\mathcal{B} \otimes_{\mathcal{B}} \mathcal{E}$ such that

$$l(b\xi) - b\, l(\xi) = d^2(b)\, \xi \tag{2}$$

and

$$l(\nabla \xi) = \nabla l(\xi) \tag{3}$$

for $b \in \mathcal{B}$, $\xi \in \mathcal{E}$. With this additional structure, we show in Section 2 how to do Chern–Weil theory, both for connections and superconnections on a $\mathcal{B}$-module $\mathcal{E}$. In the case when d^2 is a commutator, one recovers Connes' construction of Chern classes.

Next, we consider certain "homology classes" of the noncommutative space. A graded trace on $\Omega^*\mathcal{B}$ is said to be closed if it annihilates $\mathrm{Im}(d)$. A closed holonomy-invariant transverse current ρ on the space of units M gives a closed graded trace on $\Omega^*\mathcal{B}$.

The action of G on P gives rise to a left $\mathcal{B}$-module $\mathcal{E}$, which essentially consists of compactly supported sections of E coupled to a vertical density. We extend $\mathcal{E}$ to a left-$\Omega^*\mathcal{B}$ module $\Omega^*\mathcal{E}$ of "$\mathcal{E}$-valued differential forms". There is a natural linear map $l : \mathcal{E} \to \Omega^2\mathcal{E}$ satisfying (2) and (3).

We then consider the Bismut superconnection A_s on $\mathcal{E}$. The formal expression for its Chern character involves $e^{-A_s^2 + l}$. The latter is well-defined in $\mathrm{Hom}^\omega(\mathcal{E}, \Omega^*\mathcal{E})$, an algebra consisting of rapid-decay kernels. We construct a graded trace $\tau : \mathrm{Hom}^\omega(\mathcal{E}, \Omega^*\mathcal{E}) \to \Omega^*\mathcal{B}$. This allows us to define the Chern character of the superconnection by

$$\mathrm{ch}(A_s) = \mathcal{R}\left(\tau e^{-A_s^2 + l}\right). \tag{4}$$

Here, $\mathcal{R}$ is the rescaling operator which, for p even, multiplies a p-form by $(2\pi i)^{-\frac{p}{2}}$.

Now, let ρ be a closed holonomy-invariant transverse current on M as above. Then $\rho(\mathrm{ch}(A_s))$ is defined and we compute its limit when $s \to 0$, to obtain a differential form version of the right-hand side of (1). (In the case when $P = G = G_{\mathrm{hol}}$ an analogous computation was done by Heitsch [18, Theorem 2.1].)

Next, we use the argument of [16, Section 5] to show that for all $s > 0$, $\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle = \rho(\mathrm{ch}(A_s))$. (In the case when $P = G = G_{\mathrm{hol}}$, this was shown under some further restrictions by Heitsch [18, Theorem 4.6] and Heitsch–Lazarov [19, Theorem 5].) This proves Theorem 1.

We note that our extension of [16] from étale groupoids to foliation groupoids is only partial. The local index theorem of [16] allows for pairing with more general objects than transverse currents, such as the Godbillon–Vey class. The paper [16] used a bicomplex $\Omega^{*,*}\mathcal{B}$ of forms, in which the second component consists of forms in the "noncommutative" direction. There was also a connection ∇ on $\mathcal{E}$ which involved a differentiation in the noncommutative direction. In the setting of a foliation groupoid, one again has a bicomplex $\Omega^{*,*}\mathcal{B}$ and a connection ∇. However, (3) is not satisfied. Because of this we work instead with the smaller complex of forms $\Omega^{*,0}\mathcal{B}$, where this problem does not arise.

The paper is organized as follows. In Section 2, we discuss Chern–Weil theory in the context of a graded algebra with derivation whose square is nonzero. In Section 3, we describe the differential algebra $\Omega^*\mathcal{B}$ associated to a foliation groupoid G. In Section 4, we add a manifold P on which G acts properly. We define a certain left-$\mathcal{B}$ module $\mathcal{E}$ and superconnection A_s on $\mathcal{E}$. We compute the $s \to 0$ limit of $\rho(\mathrm{ch}(A_s))$. In Section 5, we explain the relation between the superconnection computations and the

K-theoretic index, construct the cohomology class $\omega_\rho \in \mathrm{H}^*(BG; o)$ and prove Theorem 1. We show that Theorem 1 implies some well-known index theorems.

In an Appendix to this paper, we give a technical improvement to our previous paper [16]. The index theorem in [16] assumed that the closed graded trace η on $\Omega^*(B, \mathbb{C}\Gamma)$ extended to an algebra of rapidly decaying forms $\Omega^*(B, \mathcal{B}^\omega)$. The appearance of $\Omega^*(B, \mathcal{B}^\omega)$ was due to the noncompact support of the heat kernel, which affects the trace of the superconnection Chern character. In Appendix, we show how to replace $\Omega^*(B, \mathcal{B}^\omega)$ by $\Omega^*(B, \mathbb{C}\Gamma)$, by using finite propagation speed methods. Let $f \in C_c^\infty(\mathbb{R})$ be a smooth even function with support in $[-\varepsilon, \varepsilon]$. Let $\widehat{f}$ be its Fourier transform. We can define $\widehat{f}(A_s)$ and show that $\eta\left(\mathcal{R}\,\tau \widehat{f}(A_s)\right)$ is defined for graded traces η on $\Omega^*(B, \mathbb{C}\Gamma)$. We prove the corresponding analog of [16, Theorem 3], with the Gaussian function in the definition of the Chern character replaced by an appropriate function $\widehat{f}$. This then implies the result stated in [16, Theorem 3] without the condition of η being extendible to $\Omega^*(B, \mathcal{B}^\omega)$. We remark that this issue of replacing $\Omega^*(B, \mathcal{B}^\omega)$ by $\Omega^*(B, \mathbb{C}\Gamma)$ does not arise in the present paper.

More detailed summaries are given at the beginning of the sections.

2. The Chern character

In this section, we collect some algebraic facts needed to define the Chern character of a superconnection in our setting. We consider an algebra $\mathcal{B}$ and a graded algebra Ω^* with $\Omega^0 = \mathcal{B}$. We assume that Ω^* is equipped with a degree-1 derivation d whose square may be nonzero. If $\mathcal{E}$ is a left $\mathcal{B}$-module then the notion of a connection ∇ on $\mathcal{E}$ is the usual one from noncommutative geometry; see Connes [11, Section III.3, Definition 5] and Karoubi [20, Chapitre 1]. We assume the additional structure of a map l satisfying (2) and (3). We show that $\nabla^2 - l$ is then the right notion of curvature. If $\mathcal{E}$ is a finitely generated projective $\mathcal{B}$-module then we carry out Chern–Weil theory for the connection ∇, and show how it extends to the case of a superconnection A. Many of the lemmas in this section are standard in the case when $d^2 = 0$ and $l = 0$, but we present them in detail in order to make clear what goes through to the case when $d^2 \neq 0$. In the case when d^2 is given by a commutator, the Chern character turns out to be the same as what one would get using Connes' X-trick [11, Section III.3, Lemma 9].

Let $\mathcal{B}$ be an algebra over $\mathbb{C}$, possibly nonunital. Let $\Omega = \bigoplus_{i=1}^{\infty} \Omega^i$ be a graded algebra with $\Omega^0 = \mathcal{B}$. Let $d : \Omega^* \to \Omega^{*+1}$ be a graded derivation of Ω^*. Define $\alpha : \Omega^* \to \Omega^{*+2}$ by $\alpha = d^2$; then for all $\omega, \omega' \in \Omega^*$,

$$\alpha(d\omega) = d\alpha(\omega), \quad \alpha(\omega\omega') = \alpha(\omega)\,\omega' + \omega\,\alpha(\omega'). \tag{5}$$

By a graded trace, we will mean a linear functional $\eta : \Omega^* \to \mathbb{C}$ such that

$$\eta(\alpha(\omega)) = 0, \quad \eta([\omega, \omega']) = 0 \tag{6}$$

for all $\omega, \omega' \in \Omega^*$. Define $d^t\eta$ by $(d^t\eta)(\omega) = \eta(d\omega)$. Then the graded traces on Ω^* form a complex with differential d^t. A graded trace η will be said to be closed if $d^t\eta = 0$, i.e. for all $\omega \in \Omega^*$, $\eta(d\omega) = 0$.

Example 1. Let E be a complex vector bundle over a smooth manifold M. Let ∇^E be a connection on E, with curvature $\theta^E \in \Omega^2(M; \mathrm{End}(E))$. Put $\mathcal{B} = C^\infty(M; \mathrm{End}(E))$ and $\Omega^* = \Omega^*(M; \mathrm{End}(E))$. Let d be the extension of the connection ∇^E to $\Omega^*(M; \mathrm{End}(E))$. Then $\alpha(\omega) = \theta^E\,\omega - \omega\,\theta^E$. If c is a closed current on M then we obtain a closed graded trace η on Ω^* by $\eta(\omega) = \int_c \mathrm{tr}(\omega)$.

Let $\mathcal{E}$ be a left $\mathcal{B}$-module. We assume that there is a $\mathbb{C}$-linear map $l : \mathcal{E} \to \Omega^2 \otimes_{\mathcal{B}} \mathcal{E}$, such that for all $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$,

$$l(b\xi) = \alpha(b)\,\xi + b\,l(\xi). \tag{7}$$

Example 2. Suppose that for some $\theta \in \Omega^2$, $\alpha(\omega) = \theta\,\omega - \omega\theta$. Then we can take $l(\xi) = \theta\xi$.

Lemma 1. *There is an extension of l to a linear map $l : \Omega^* \otimes_{\mathcal{B}} \mathcal{E} \to \Omega^{*+2} \otimes_{\mathcal{B}} \mathcal{E}$ so that for $\omega \in \Omega^*$ and $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,*

$$l(\omega\mu) = \alpha(\omega)\,\mu + \omega\,l(\mu). \tag{8}$$

Proof. We define $l : \Omega^* \otimes_{\mathbb{C}} \mathcal{E} \to \Omega^{*+2} \otimes_{\mathcal{B}} \mathcal{E}$ by

$$l(\omega \otimes \xi) = \alpha(\omega)\,\xi + \omega\,l(\xi). \tag{9}$$

Then for $b \in \mathcal{B}$,

$$\begin{aligned} l(\omega b \otimes \xi) &= \alpha(\omega b)\,\xi + \omega b\,l(\xi) = \alpha(\omega)\,b\,\xi + \omega\,\alpha(b)\,\xi + \omega\,b\,l(\xi) \\ &= \alpha(\omega)\,b\xi + \omega\,l(b\xi) = l(\omega \otimes b\xi). \end{aligned} \tag{10}$$

Thus, l is defined on $\Omega^* \otimes_{\mathcal{B}} \mathcal{E}$. Next, for $\omega, \omega' \in \Omega^*$ and $\xi \in \mathcal{E}$,

$$\begin{aligned} l(\omega(\omega'\xi)) &= \alpha(\omega\omega')\,\xi + \omega\omega'\,l(\xi) = \alpha(\omega)\,\omega'\,\xi + \omega\,\alpha(\omega')\,\xi + \omega\omega'\,l(\xi) \\ &= \alpha(\omega)\,\omega'\,\xi + \omega\,l(\omega'\,\xi). \end{aligned} \tag{11}$$

This proves the lemma. □

Let $\nabla : \mathcal{E} \to \Omega^1 \otimes_{\mathcal{B}} \mathcal{E}$ be a connection, i.e. a $\mathbb{C}$-linear map satisfying

$$\nabla(b\xi) = db \otimes \xi + b\nabla\xi \tag{12}$$

for all $b \in \mathcal{B}$, $\xi \in \mathcal{E}$. Extend ∇ to a $\mathbb{C}$-linear map $\nabla : \Omega^* \otimes_{\mathcal{B}} \mathcal{E} \to \Omega^{*+1} \otimes_{\mathcal{B}} \mathcal{E}$ so that for all $\omega \in \Omega^*$ and $\xi \in \mathcal{E}$,

$$\nabla(\omega\xi) = d\omega \otimes \xi + (-1)^{|\omega|} \omega\nabla\xi. \tag{13}$$

We assume that for all $\xi \in \mathcal{E}$,

$$l(\nabla\xi) = \nabla l(\xi). \tag{14}$$

Lemma 2. *$\nabla^2 - l : \mathcal{E} \to \Omega^2 \otimes_{\mathcal{B}} \mathcal{E}$ is left-$\mathcal{B}$-linear.*

Proof. For $b \in \mathcal{B}$ and $\xi \in \mathcal{E}$,

$$\begin{aligned}(\nabla^2 - l)(b\xi) &= \nabla(db \otimes \xi + b\nabla\xi) - l(b\xi) = d^2 b \otimes \xi + b\nabla^2\xi - l(b\xi)\\ &= \alpha(b)\xi + b\nabla^2\xi - l(b\xi) = b\,(\nabla^2 - l)(\xi).\end{aligned} \tag{15}$$

This proves the lemma. □

Put $\Omega^*_{ab} = \Omega^*/[\Omega^*, \Omega^*]$, the quotient by the graded commutator, with the induced d. For simplicity, in the rest of this section, we assume that $\mathcal{B}$ is unital and $\mathcal{E}$ is a finitely generated projective left $\mathcal{B}$-module. Consider the graded algebra $\mathrm{Hom}_{\mathcal{B}}\left(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right) \cong \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$. There is a graded trace on $\mathrm{Hom}_{\mathcal{B}}\left(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$, with value in Ω^*_{ab}, defined as follows. Write $\mathcal{E}$ as $\mathcal{B}^N e$ for some idempotent $e \in M_N(\mathcal{B})$. Then any $T \in \mathrm{Hom}_{\mathcal{B}}\left(\mathcal{E}, \Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$ can be represented as right-multiplication on $\mathcal{B}^N e$ by a matrix $T \in M_N(\Omega^*)$ satisfying $T = eT = Te$. By definition $\mathrm{tr}(T) = \sum_{i=1}^N T_{ii} \bmod [\Omega^*, \Omega^*]$. It is independent of the representation of $\mathcal{E}$ as $\mathcal{B}^N e$.

Given $T_1, T_2 \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$, define their (graded) commutator by

$$[T_1, T_2] = T_1 \circ T_2 - (-1)^{|T_1||T_2|} T_2 \circ T_1. \tag{16}$$

For $T \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$, define $[\nabla, T] \in \mathrm{End}_{\mathbb{C}}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$ by

$$[\nabla, T](\mu) = (-1)^{|\mu|} (\nabla(T(\mu)) - T(\nabla\mu)) \tag{17}$$

for $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$.

Lemma 3. $[\nabla, T] \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$.

Proof. Given $\omega \in \Omega^*$ and $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,

$$\begin{aligned}[\nabla, T](\omega\mu) &= (-1)^{|\omega|+|\mu|} (\nabla(T(\omega\mu)) - T(\nabla(\omega\mu))) \\ &= (-1)^{|\omega|+|\mu|} \big(\nabla(\omega T(\mu)) - T((d\omega)\mu + (-1)^{\omega}\omega\nabla\mu)\big) \\ &= (-1)^{|\omega|+|\mu|} \big((d\omega)T(\mu) + (-1)^{\omega}\omega\nabla(T(\mu)) - (d\omega)T(\mu) \\ &\quad - (-1)^{\omega}\, \omega T(\nabla\mu)\big) \\ &= \omega\,[\nabla, T](\mu). \end{aligned} \tag{18}$$

This proves the lemma. □

Lemma 4. *Given* $T_1, T_2 \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$,

$$[\nabla, T_1 \circ T_2] = T_1 \circ [\nabla, T_2] + (-1)^{|T_2|} [\nabla, T_1] \circ T_2. \tag{19}$$

Proof. Given $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$,

$$[\nabla, T_1 \circ T_2](\mu) = (-1)^{|\mu|} \{\nabla(T_1(T_2(\mu))) - T_1(T_2(\nabla(\mu)))\}, \tag{20}$$

$$(T_1 \circ [\nabla, T_2])(\mu) = (-1)^{|\mu|}\, T_1 (\nabla(T_2(\mu)) - T_2(\nabla(\mu))) \tag{21}$$

and

$$([\nabla, T_1] \circ T_2)(\mu) = [\nabla, T_1](T_2(\mu)) = (-1)^{|T_2(\mu)|} \{\nabla(T_1(T_2(\mu))) - T_1(\nabla(T_2(\mu)))\}. \tag{22}$$

The lemma follows. □

Lemma 5. *Given* $T_1, T_2 \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$,

$$[\nabla, [T_1, T_2]] = [T_1, [\nabla, T_2]] + (-1)^{|T_2|} [[\nabla, T_1], T_2]. \tag{23}$$

Proof. This follows from (16) and (19). We omit the details. □

Lemma 6. *For* $T \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$,

$$\mathrm{tr}([\nabla, T]) = d\,\mathrm{tr}(T) \in \Omega^*_{ab}. \tag{24}$$

Proof. Let us write $\mathcal{E} = \mathcal{B}^N e$ for an idempotent $e \in M_N(\mathcal{B})$. Given $A \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$, it acts on $\mathcal{B}^N e$ on the right by a matrix $A \in M_N(\Omega^1)$ with $A = eA = Ae$. Then, there is some $A \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$ so that for $\mu \in \Omega^* \otimes_{\mathcal{B}} \mathcal{E} = (\Omega^*)^N e$,

$$\nabla(\mu) = (d\mu)\, e + (-1)^{|\mu|}\, \mu\, A; \tag{25}$$

in fact, this equation defines A.

An element $T \in \mathrm{End}_{\Omega^*}\left(\Omega^* \otimes_{\mathcal{B}} \mathcal{E}\right)$ acts by right multiplication on $\Omega^* \otimes_{\mathcal{B}} \mathcal{E} = (\Omega^*)^N e$ by a matrix $T \in M_N(\Omega^*)$ satisfying $T = eT = Te$. Then for $\xi \in \mathcal{E} = \mathcal{B}^N e$,

$$[\nabla, T](\xi) = \nabla(\xi T) - (\nabla(\xi))T = \left\{d(\xi T)\, e + (-1)^{|T|}\, \xi TA\right\} - \{(d\xi)\, e + \xi A\}\, T$$
$$= \xi\left((dT)\, e + (-1)^{|T|}\, TA - AT\right). \tag{26}$$

Thus, $[\nabla, T]$ acts as right multiplication by the matrix

$$e(dT)e + (-1)^{|T|}\, TA - AT, \tag{27}$$

and so $\mathrm{tr}([\nabla, T]) \equiv \mathrm{tr}(e(dT)e)$. On the other hand, using the identity $e(de)e = 0$ and taking the trace of $N \times N$ matrices, we obtain

$$d\,\mathrm{tr}(T) = d\,\mathrm{tr}(eTe) = \mathrm{tr}\left((de)Te + e(dT)e + (-1)^{|T|}\, eT(de)\right)$$
$$= \mathrm{tr}\left((de)eTe + e(dT)e + (-1)^{|T|}\, eTe(de)\right)$$
$$\equiv \mathrm{tr}\left(e(de)eT + e(dT)e + (-1)^{|T|}\, Te(de)e\right) = \mathrm{tr}(e(dT)e). \tag{28}$$

This proves the lemma. □

Lemma 7. $[\nabla, \nabla^2 - l] = 0$.

Proof. This follows from (14). □

Definition 1. The Chern character form of ∇ is

$$\mathrm{ch}(\nabla) = \mathrm{tr}\left(e^{-\frac{\nabla^2 - l}{2\pi i}}\right) \in \Omega^*_{ab}. \tag{29}$$

Lemma 8. *Given $\mathcal{E}$, if η is a closed graded trace on Ω^* then $\eta(\mathrm{ch}(\nabla))$ is independent of the choice of ∇. If η_1 and η_2 are homologous closed graded traces then $\eta_1(\mathrm{ch}(\nabla)) = \eta_2(\mathrm{ch}(\nabla))$.*

Proof. Let ∇_1 and ∇_2 be two connections on $\mathcal{E}$. For $t \in [0, 1]$, define a connection by $\nabla(t) = t\nabla_2 + (1-t)\nabla_1$. Then $\frac{d\nabla}{dt} = \nabla_2 - \nabla_1 \in \mathrm{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^1 \otimes_{\mathcal{B}} \mathcal{E})$. We claim that $\eta(\mathrm{ch}(\nabla(t)))$ is independent of t. As $\frac{d(\nabla^2 - l)}{dt} = \nabla \frac{d\nabla}{dt} + \frac{d\nabla}{dt}\nabla$, we have

$$\frac{d\,\mathrm{ch}(\nabla)}{dt} = -\frac{1}{2\pi i}\,\mathrm{tr}\left(\left(\nabla\frac{d\nabla}{dt} + \frac{d\nabla}{dt}\nabla\right) e^{-\frac{\nabla^2 - l}{2\pi i}}\right) = -\frac{1}{2\pi i}\,\mathrm{tr}\left(\left[\nabla, \frac{d\nabla}{dt}\, e^{-\frac{\nabla^2 - l}{2\pi i}}\right]\right)$$
$$= -\frac{1}{2\pi i}\, d\,\mathrm{tr}\left(\frac{d\nabla}{dt}\, e^{-\frac{\nabla^2 - l}{2\pi i}}\right). \tag{30}$$

Then

$$\mathrm{ch}(\nabla_2) - \mathrm{ch}(\nabla_1) = -\frac{1}{2\pi i}\, d \int_0^1 \mathrm{tr}\left((\nabla_2 - \nabla_1)\, e^{-\frac{\nabla(t)^2 - l}{2\pi i}} \right) dt, \tag{31}$$

from which the claim follows. We note after expanding the exponential in (31), the integral gives an expression, that is, purely algebraic in ∇_1 and ∇_2.

If η_1 and η_2 are homologous then there is a graded trace η' such that $\eta_1 - \eta_2 = d^t \eta'$. Thus,

$$\eta_1(\mathrm{ch}(\nabla)) - \eta_2(\mathrm{ch}(\nabla)) = \eta'(d\,\mathrm{ch}(\nabla)). \tag{32}$$

However,

$$d\,\mathrm{ch}(\nabla) = d\,\mathrm{tr}\left(e^{-\frac{\nabla^2 - l}{2\pi i}} \right) = \mathrm{tr}\left(\left[\nabla, e^{-\frac{\nabla^2 - l}{2\pi i}} \right] \right) = 0. \tag{33}$$

This proves the lemma. □

Example 3. With the notation of Example 1, let F be another complex vector bundle on M, with connection ∇^F. Put $\mathcal{E} = C^\infty(M; E \otimes F)$, with $l(\xi) = (\theta^E \otimes I)\,\xi$ for $\xi \in \mathcal{E}$. Let ∇ be the tensor product of ∇^E and ∇^F. Then one finds that $\eta(\mathrm{ch}(\nabla)) = \int_c \mathrm{ch}(\nabla^F)$.

If $\mathcal{E}$ is $\mathbb{Z}_2$-graded, let $A : \mathcal{E} \to \Omega^* \otimes_{\mathcal{B}} \mathcal{E}$ be a superconnection. Then there are obvious extensions of the results of this section. In particular, let $\mathcal{R}$ be the rescaling operator on $\Omega^{\mathrm{even}}_{ab}$ which multiplies an element of Ω^{2k}_{ab} by $(2\pi i)^{-k}$.

Definition 2. The Chern character form of A is

$$\mathrm{ch}(A) = \mathcal{R}\,\mathrm{tr}_s\left(e^{-(A^2 - l)} \right) \in \Omega^*_{ab}. \tag{34}$$

We have the following analog of Lemma 8.

Lemma 9. *Given $\mathcal{E}$, if η is a closed graded trace on Ω^* then $\eta(\mathrm{ch}(A))$ is independent of the choice of A. If η_1 and η_2 are homologous closed graded traces then $\eta_1(\mathrm{ch}(A)) = \eta_2(\mathrm{ch}(A))$.*

3. Differential calculus for foliation groupoids

In this section, given a foliation groupoid G, we construct a graded algebra $\Omega^*\mathcal{B}$ whose degree-0 component $\mathcal{B}$ is the convolution algebra of G. We then construct a degree-1 derivation $d = d^{\mathrm{H}}$ of $\Omega^*\mathcal{B}$. Finally, we compute d^2.

3.1. The differential forms

Let G be a groupoid. We use the groupoid notation of [11, Section II.5]. The units of G are denoted $G^{(0)}$ and the range and source maps are denoted $r, s : G \to G^{(0)}$. To construct the product of $g_0, g_1 \in G$, we must have $s(g_0) = r(g_1)$. Then $r(g_0 g_1) = r(g_0)$ and $s(g_0 g_1) = s(g_1)$. Given $m \in G^{(0)}$, put $G^m = r^{-1}(m)$, $G_m = s^{-1}(m)$ and $G^m_m = G^m \cap G_m$.

We assume that G is a Lie groupoid, meaning that G and $G^{(0)}$ are smooth manifolds, and r and s are smooth submersions. For simplicity, we will assume that G is Hausdorff. The results of the paper extend to the non-Hausdorff case, using the notion of differential forms on a non-Hausdorff manifold given by Crainic and Moerdijk [14, Section 2.2.5]. (The paper [14] is an extension of work by Brylinski and Nistor [8].)

The Lie algebroid $\mathfrak{g}$ of G is a vector bundle over $G^{(0)}$ with fibers $\mathfrak{g}_m = T_m G^m = \mathrm{Ker}(dr_m : T_m G \to T_m G^{(0)})$. The anchor map $\mathfrak{g} \to TG^{(0)}$, a map of vector bundles, is the restriction of $ds_m : T_m G \to T_m G^{(0)}$ to $\mathfrak{g}_m$. In general, the image of the anchor map need not be of constant rank.

We now assume that G is a foliation groupoid in the sense of [15], i.e. that G satisfies one of the three following equivalent conditions [15, Theorem 1]:

1. G is Morita equivalent to a smooth étale groupoid.
2. The anchor map of G is injective.
3. All isotropy Lie groups G^m_m of G are discrete.

Example 4. If G is an smooth étale groupoid then G is a foliation groupoid. If $(M, \mathcal{F})$ is a smooth foliated manifold then its holonomy groupoid (see Connes [11, Section II.8.α]) and its monodromy (= homotopy) groupoid (see Baum–Connes [3] and Phillips [26]) are foliation groupoids. In this case, the anchor map is the inclusion map $T\mathcal{F} \to TM$. If a Lie group L acts smoothly on a manifold M and the isotropy groups $L_m = \{l \in L : ml = m\}$ are discrete then the cross-product groupoid $M \rtimes L$ is a foliation groupoid.

Put $M = G^{(0)}$. It inherits a foliation $\mathcal{F}$, with the leafwise tangent bundle $T\mathcal{F}$ being the image of the anchor map.

Note that the foliated manifold $(M, \mathcal{F})$ has a holonomy groupoid Hol which is itself a foliation groupoid. However, Hol may not be the same as G. If G is a foliation groupoid with the property that G_m is connected for all m then G lies between the holonomy groupoid of $\mathcal{F}$ and the monodromy groupoid of $\mathcal{F}$; see [15, Proposition 1] for further discussion. The reader may just want to keep in mind the case when G is actually the holonomy groupoid of a foliated manifold $(M, \mathcal{F})$.

Let $\tau = TM/T\mathcal{F}$ be the normal bundle to the foliation. Given $g \in G$, let $U \subset M$ be a sufficiently small neighborhood of $s(g)$ and let $c : U \to G$ be a smooth map such that $c(s(g)) = g$ and $s \circ c = \mathrm{Id}_U$. Then $d(r \circ c)_{s(g)} : T_{s(g)}M \to T_{r(g)}M$ sends $T_{s(g)}\mathcal{F}$ to $T_{r(g)}\mathcal{F}$. The induced map from $\tau_{s(g)}$ to $\tau_{r(g)}$ has an inverse $g_* : \tau_{r(g)} \to \tau_{s(g)}$ called the holonomy of the element $g \in G$. It is independent of the choices of U and c.

Let $\mathcal{D}$ denote the real line bundle on M formed by leafwise densities. We define a graded algebra $\Omega^*\mathcal{B}$ whose components, as vector spaces, are given by

$$\Omega^n\mathcal{B} = C_c^\infty\left(G; \Lambda^n(r^*\tau^*) \otimes s^*\mathcal{D}\right). \tag{35}$$

In particular,

$$\mathcal{B} = \Omega^0\mathcal{B} = C_c^\infty(G; s^*\mathcal{D}) \tag{36}$$

is the groupoid algebra. (Instead of using half-densities, we have placed a full density at the source.) The product of $\phi_1 \in \Omega^{n_1}\mathcal{B}$ and $\phi_2 \in \Omega^{n_2}\mathcal{B}$ is given by

$$(\phi_1\phi_2)(g) = \int_{g'g''=g} \phi_1(g') \wedge \phi_2(g''). \tag{37}$$

In forming the wedge product, the holonomy of g' is used to identify conormal spaces.

Let $T^{\mathrm{H}}M$ be a horizontal distribution on M, i.e. a splitting of the short exact sequence $0 \to T\mathcal{F} \to TM \to \tau \to 0$. Then there is a horizontal differentiation $d^{\mathrm{H}} : \Omega^n\mathcal{B} \to \Omega^{n+1}\mathcal{B}$, which we now define. The definition will proceed by building up d^{H} from smaller pieces (cf. [11, Section II.7.α, Proposition 3]).

First, the choice of horizontal distribution allows us to define a horizontal differential $d^{\mathrm{H}} : \Omega^*(M) \to \Omega^{*+1}(M)$ as in Bismut–Lott [7, Definition 3.2] and Connes [11, Section III.7.α]. Using the local description of an element of $C^\infty(M;\mathcal{D})$ as a vertical $\dim(\mathcal{F})$-form on M, we also obtain a horizontal differential $d^{\mathrm{H}} : C^\infty(M;\mathcal{D}) \to C^\infty(M;\tau^*\otimes\mathcal{D})$ [11, Section III.7.α] and a horizontal differential $d^{\mathrm{H}} : C^\infty\left(M;\Lambda^n\tau^*\right) \to C^\infty\left(M;\Lambda^{n+1}\tau^*\right)$.

Given $f \in C_c^\infty(G)$, we now define its horizontal differential $d^{\mathrm{H}}f \in C_c^\infty(G; r^*\tau^*)$ by simultaneously differentiating f with respect to its arguments, in a horizontal direction. That is, consider a point $g \in G$ and a vector $X_0 \in \tau_{r(g)}$. Put $X_1 = g_*(X_0)$. Next, use the horizontal distribution $T^{\mathrm{H}}M$ to construct the corresponding horizontal vectors $\widetilde{X}_0$ and $\widetilde{X}_1$. We now have a vector $\widetilde{X} = \left(\widetilde{X}_0, \widetilde{X}_1\right) \in T_{(r(g),s(g))}(M\times M)$. It is the image of a unique vector $X \in T_gG$ under the immersion

$$(r,s) : G \to M \times M. \tag{38}$$

We define $d^{\mathrm{H}}f$ by putting $\left((d^{\mathrm{H}}f)(X_0)\right)(g) = Xf$.

Next, to horizontally differentiate an element of $C_c^\infty\left(G;\Lambda^n(r^*\tau^*)\otimes s^*\mathcal{D}\right)$, we write it as a finite sum of terms of the form $f\, r^*(\omega)\, s^*(\beta)$, with $f \in C_c^\infty(G)$, $\omega \in C^\infty(M;\Lambda^n\tau^*)$, and $\beta \in C^\infty(M;\mathcal{D})$. For an element of this form, put

$$\begin{aligned} d^{\mathrm{H}}\left(f\, r^*(\omega)\, s^*(\beta)\right) &= (d^{\mathrm{H}}f)\, r^*(\omega)\, s^*(\beta) + f\, r^*(d^{\mathrm{H}}\omega)\, s^*(\beta) \\ &\quad + (-1)^n\, f\, r^*(\omega)\, s^*(d^{\mathrm{H}}\beta), \end{aligned} \tag{39}$$

where the holonomy is used in defining products.

Lemma 10. *The operator* d^{H} *is a graded derivation of* $\Omega^*\mathcal{B}$.

Proof. This follows from a straightforward computation, which we omit. □

Put $d = d^{\mathrm{H}}$. We now describe $\alpha = d^2$. Let $T \in \Omega^2(M; T\mathcal{F})$ be the curvature of the horizontal distribution $T^{\mathrm{H}}M$ [7, (3.11)]. It is a horizontal 2-form on M with values in $T\mathcal{F}$, defined by $T(X_1, X_2) = -P^{\mathrm{vert}}[X_1^{\mathrm{H}}, X_2^{\mathrm{H}}]$. One can define the Lie derivative $\mathcal{L}_T : \Omega^*(M) \to \Omega^{*+2}(M)$, an operation which increases the horizontal grading by two, as in [7, (3.14)]. Then one can define $\mathcal{L}_T : C^\infty(M; \mathcal{D}) \to C^\infty(M; \Lambda^2\tau^* \otimes \mathcal{D})$ and $\mathcal{L}_T : C^\infty(M; \Lambda^n\tau^*) \to C^\infty(M; \Lambda^{n+2}\tau^*)$ in obvious ways.

Given $f \in C_c^\infty(G)$, we define its Lie derivative $\mathcal{L}_T f \in C_c^\infty\left(G; \Lambda^2(r^*\tau^*)\right)$ by simultaneously differentiating f with respect to its arguments, in the vertical direction. That is, consider a point $g \in G$ and $X_0, Y_0 \in \tau_{r(g)}$. Put $X_1 = g_*(X_0)$ and $Y_1 = g_*(Y_0)$. Next, use the horizontal distribution $T^{\mathrm{H}}M$ to construct the corresponding horizontal vectors $\widetilde{X}_0$, $\widetilde{X}_1$, $\widetilde{Y}_0$ and $\widetilde{Y}_1$. Consider the vertical vectors $T(\widetilde{X}_0, \widetilde{Y}_0) \in T_{r(g)}\mathcal{F}$ and $T(\widetilde{X}_1, \widetilde{Y}_1) \in T_{s(g)}\mathcal{F}$. We now have a total vector $\widetilde{V} = \left(T(\widetilde{X}_0, \widetilde{Y}_0), T(\widetilde{X}_1, \widetilde{Y}_1)\right) \in T_{(r(g),s(g))}(M \times M)$. It is the image of a unique vector $V \in T_gG$ under the immersion (38). We define $\mathcal{L}_T f$ by putting $((\mathcal{L}_T f)(X_0, Y_0))(g) = Vf$.

Now for $f\, r^*(\omega)\, s^*(\beta)$ as before, we put

$$\mathcal{L}_T\left(f\, r^*(\omega)\, s^*(\beta)\right) = (\mathcal{L}_T f)\, r^*(\omega)\, s^*(\beta) + f\, r^*(\mathcal{L}_T\omega)\, s^*(\beta) + f\, r^*(\omega)\, s^*(\mathcal{L}_T\alpha_1), \quad (40)$$

where the holonomy is used in defining products.

Lemma 11. *We have*

$$\alpha = -\mathcal{L}_T. \quad (41)$$

Proof. This follows from the method of proof of [7, (3.13)] or [11, Section III.7.α]. □

Remark. One can consider α to be commutation with a (distributional) element of the multiplier algebra $C^{-\infty}\left(G; \Lambda^2(p_0^*\tau^*) \otimes p_1^*\mathcal{D}\right)$, namely the one that implements the Lie differentiation [11, Section III.7.α, Lemma 4].

4. Superconnection and Chern character

In this section, we consider a smooth manifold P on which G acts freely, properly and cocompactly, along with a G-invariant $\mathbb{Z}_2$-graded vector bundle E on P. We construct a corresponding left-$\mathcal{B}$-module $\mathcal{E}$. Given a G-invariant Dirac-type operator which acts on sections of E, we consider the Bismut superconnections $\{A_s\}_{s>0}$. We compute the $s \to 0$ limit of the pairing between the Chern character of A_s and a closed graded trace on $\Omega^*\mathcal{B}$, that is, concentrated on the units M. More detailed summaries appear at the beginning of the subsections.

4.1. Module and connection

In this subsection, we consider a left $\mathcal{B}$-module $\mathcal{E}$ consisting of sections of E, and its extension to a left $\Omega^*\mathcal{B}$-module $\Omega^*\mathcal{E}$. We construct a map $l : \mathcal{E} \to \Omega^2\mathcal{E}$ satisfying (2). Given a lift $T^{\mathrm{H}}P$ of $T^{\mathrm{H}}M$, we construct a connection $\nabla^{\mathcal{E}}$ on $\mathcal{E}$.

Let P be a smooth G-manifold [11, Section II.10.α, Definition 1]. That is, first of all, there is a submersion $\pi : P \to M$. Given $m \in M$, we write $Z_m = \pi^{-1}(m)$. Putting

$$P \times_r G = \{(p, g) \in P \times G : p \in Z_{r(g)}\}, \tag{42}$$

we must also have a smooth map $P \times_r G \to P$, denoted $(p, g) \to pg$, such that $pg \in Z_{s(g)}$ and $(pg_1)g_2 = p(g_1g_2)$ for all $(g_1, g_2) \in G^{(2)}$. It follows that for each $g \in G$, the map $p \to pg$ gives a diffeomorphism from $Z_{r(g)}$ to $Z_{s(g)}$. Let $\mathcal{D}_Z$ denote the real line bundle on P formed by the fiberwise densities.

Hereafter, we assume that P is a proper G-manifold [11, Section II.10.α, Definition 2], i.e. that the map $P \times_r G \to P \times P$ given by $(p, g) \to (p, pg)$ is proper. We also assume that G acts cocompactly on P, i.e. that the quotient of P by the equivalence relation ($p \sim p'$ if $p = p'g$ for some $g \in G$) is compact. And we assume that G acts freely on P, i.e. that $pg = p$ implies that $g \in M$. Then P/G is a smooth compact manifold.

Example 5. Take $P = G$, with $\pi = s$. Then G acts properly, freely, and, if M is compact, cocompactly on P.

We will say that a covariant object (vector bundle, connection, metric, etc.) on P is G-invariant if it is the pullback of a similar object from P/G. Let E be a G-invariant $\mathbb{Z}_2$-graded vector bundle on P, with supertrace tr_s on $\mathrm{End}(E)$. Put $\mathcal{E} = C_c^\infty(P; E)$. It is a left-$\mathcal{B}$-module, with the action of $b \in \mathcal{B}$ on $\xi \in \mathcal{E}$ given by

$$(b\xi)(p) = \int_{G^{\pi(p)}} b(g)\, \xi(pg). \tag{43}$$

In writing (43), we have used the g-action to identify E_p and E_{pg}.

Put

$$\Omega^n\mathcal{E} = C_c^\infty\left(P; \Lambda^n(\pi^*\tau^*) \otimes E\right). \tag{44}$$

Then $\Omega^*\mathcal{E}$ is a left-$\Omega^*\mathcal{B}$-module with the action of $\Omega^*\mathcal{B}$ on $\Omega^*\mathcal{E}$ given by

$$(\phi\, \omega)(p) = \int_{G^{\pi(p)}} \phi(g) \wedge \omega(pg). \tag{45}$$

Let $\widetilde{\mathcal{F}}$ be the foliation on P whose leaf through $p \in P$ consists of the elements pg where g runs through the connected component of $G^{\pi(p)}$ that contains the unit $\pi(p)$. Note that $\dim(\widetilde{\mathcal{F}}) = \dim(\mathcal{F})$. Given $p \in P$ and $X, Y \in \tau_{\pi(p)}$, let $\widetilde{T}(X, Y) \in T_p\widetilde{\mathcal{F}}$ be the lift of $T(X, Y) \in T_{\pi(p)}\mathcal{F}$. Define $l : \mathcal{E} \to \Omega^2\mathcal{E}$ by saying that for $X, Y \in \tau_{\pi(p)}$ and $\xi \in \mathcal{E}$,

$$(l(\xi)(X, Y))(p) = -\widetilde{T}(X, Y)\xi. \tag{46}$$

Here, we have used the G-invariance of E to define the action of $\widetilde{T}(X, Y)$ on ξ.

Lemma 12. *For all* $X, Y \in \tau_{\pi(p)}$, $b \in \mathcal{B}$ *and* $\xi \in \mathcal{E}$,

$$l(b\xi) = \alpha(b)\,\xi + b\,l(\xi). \tag{47}$$

Proof. We have

$$(l(b\xi)(X, Y))(p) = -\widetilde{T}(X, Y)\int_{G^{\pi(p)}} b(g)\,\xi(pg) = -\int_{G^{\pi(p)}} T(X, Y)b(g)\,\xi(pg), \tag{48}$$

$$(\alpha(b)(X, Y)\xi)(p) = -\int_{G^{\pi(p)}} (T(X, Y)b + T(g_*X, g_*Y)b)\,(g)\,\xi(pg) \tag{49}$$

and

$$(bl(\xi)(X, Y))(p) = -\int_{G^{\pi(p)}} b(g)\,\widetilde{T}(g_*X, g_*Y)\xi(pg). \tag{50}$$

Then

$$\begin{aligned}&(l(b\xi)(X, Y))(p) - (\alpha(b)(X, Y)\xi)(p) - (bl(\xi)(X, Y))(p)\\ &\quad = \int_{G^{\pi(p)}} \left(T(g_*X, g_*Y)b(g)\,\xi(pg) + b(g)\,\widetilde{T}(g_*X, g_*Y)\xi(pg)\right).\end{aligned} \tag{51}$$

We can write (51) more succinctly as

$$l(b\xi) - \alpha(b)\xi - bl(\xi) = \int_{G^{\pi(p)}} \mathcal{L}_{\widetilde{T}}(b(g)\xi(pg)), \tag{52}$$

where the Lie differentiation is at pg. The right-hand side of (52) vanishes, being the integral of a Lie derivative of a compactly-supported density. □

We extend l to a linear map $l : \Omega^n\mathcal{E} \to \Omega^{n+2}\mathcal{E}$ as Lie differentiation in the $\widetilde{T}$-direction with respect to P.

Lemma 13. *For all* $\omega \in \Omega^* \mathcal{B}$ *and* $\mu \in \Omega^* \mathcal{E}$,

$$l(\omega\mu) = \alpha(\omega)\, \mu + \omega\, l(\mu). \tag{53}$$

Proof. The proof is similar to that of Lemma 12. We omit the details. □

There is a pullback foliation $\pi^*\mathcal{F}$ on P with the same codimension as $\mathcal{F}$, satisfying $T\pi^*\mathcal{F} = (d\pi)^{-1}T\mathcal{F}$. Let $\mu : P \to P/G$ be the quotient map. Then P/G is a smooth compact manifold with a foliation $F = (\pi^*\mathcal{F})/G$ satisfying $(d\mu)^{-1}TF = T\pi^*\mathcal{F}$. We note that the normal bundle NF to F satisfies $\mu^*NF = \pi^*\tau$.

Let $T^{\mathrm{H}}(P/G)$ be a horizontal distribution on P/G, transverse to F. Then $(d\mu)^{-1}(T^{\mathrm{H}}(P/G))$ is a G-invariant distribution on P that is transverse to the vertical tangent bundle TZ. Put $T^{\mathrm{H}}P = (d\mu)^{-1}(T^{\mathrm{H}}(P/G)) \cap (d\pi)^{-1}(T^{\mathrm{H}}M)$, a distribution on P that is transverse to $\pi^*\mathcal{F}$ and that projects isomorphically under π to $T^{\mathrm{H}}M$.

Let $\nabla^{\mathcal{E}} : \mathcal{E} \to \Omega^1\mathcal{E}$ be covariant differentiation on $\mathcal{E} = C_c^\infty(P; E)$ with respect to $T^{\mathrm{H}}P$.

Lemma 14. $\nabla^{\mathcal{E}}$ *is a connection.*

Proof. We wish to show that

$$\nabla^{\mathcal{E}}(b\xi) = b\nabla^{\mathcal{E}}\xi + (d^{\mathrm{H}}b)\xi. \tag{54}$$

As the claim of the lemma is local on P, consider first the case when $T^{\mathrm{H}}(P/G)$ is integrable. Let $T^{\mathrm{H}}P_1$ and $\nabla_1^{\mathcal{E}}$ denote the corresponding objects on P. Then one is geometrically in a product situation and one can reduce to the case $P = M$, where one can check that (54) holds. If $T^{\mathrm{H}}(P/G)$ is not integrable then $T^{\mathrm{H}}P - T^{\mathrm{H}}P_1 \in \mathrm{Hom}(\pi^*\tau, TZ)$ is the pullback under μ of an element of $\mathrm{Hom}(NF, TF)$. Hence, $T^{\mathrm{H}}P - T^{\mathrm{H}}P_1$ is G-invariant and it follows that $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ commutes with $\mathcal{B}$, which proves the lemma. □

We extend $\nabla^{\mathcal{E}}$ to act on $\Omega^*\mathcal{E}$ so as to satisfy Leibnitz' rule.

Lemma 15. *For all* $\xi \in \mathcal{E}$,

$$l(\nabla^{\mathcal{E}}\xi) = \nabla^{\mathcal{E}} l(\xi). \tag{55}$$

Proof. As d^{H} commutes with $(d^{\mathrm{H}})^2$, it follows that d^{H} commutes with $\mathcal{L}_T$. As the claim of the lemma is local on P, consider first the case when $T^{\mathrm{H}}(P/G)$ is integrable. Let $T^{\mathrm{H}}P_1$ and $\nabla_1^{\mathcal{E}}$ denote the corresponding objects on P. Then, one is in a local product situation and the lemma follows from the fact that d^{H} commutes with $\mathcal{L}_T$. If $T^{\mathrm{H}}(P/G)$ is not integrable then $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ is given by covariant differentiation in the TZ direction, with respect to $T^{\mathrm{H}}P - T^{\mathrm{H}}P_1 \in \mathrm{Hom}(\pi^*\tau, TZ)$. As $\widetilde{T}$ pulls back from M, $\nabla^{\mathcal{E}} - \nabla_1^{\mathcal{E}}$ commutes with l. The lemma follows. □

4.2. Supertraces

In this subsection, we consider a certain algebra $\mathrm{End}^{\infty}_{\mathcal{B}}(\mathcal{E})$ of operators with smooth kernel on P. We show that a trace on $\mathcal{B}$, concentrated on the units M, gives a supertrace on $\mathrm{End}^{\infty}_{\mathcal{B}}(\mathcal{E})$. We then consider an algebra $\mathrm{Hom}^{\infty}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$ of form-valued operators. We show that a closed graded trace on $\Omega^*\mathcal{B}$, concentrated on M, gives rise to a closed graded trace on $\mathrm{Hom}^{\infty}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$.

An operator $K \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ has a Schwartz kernel $K(p'|p)$ so that

$$(K\xi)(p) = \int_{Z_{\pi(p)}} \xi(p')\, K(p'|p). \tag{56}$$

Define $q', q : P \times_M P \to P$ by $q'(p', p) = p'$ and $q(p', p) = p$. Let $\mathrm{End}^{\infty}_{\mathcal{B}}(\mathcal{E})$ denote the subalgebra of $\mathrm{End}_{\mathcal{B}}(\mathcal{E})$ consisting of operators whose Schwartz kernel lies in $C^{\infty}_c(P \times_M P; (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E))$.

Choose $\Phi \in C^{\infty}_c(P; \pi^*\mathcal{D})$ so that

$$\int_{G^{\pi(p)}} \Phi(pg) = 1 \tag{57}$$

for all $p \in P$; that such a Φ exists was shown by Tu [30, Proposition 6.11]. Define $\tau K \in C^{\infty}_c(M; \mathcal{D})$ by

$$(\tau K)(m) = \int_{Z_m} \Phi(p)\, \mathrm{tr}_s\, K(p|p). \tag{58}$$

Proposition 1. *Let ρ be a linear functional on $C^{\infty}_c(M; \mathcal{D})$. Suppose that the linear functional η on $\mathcal{B}$, defined by*

$$\eta(b) = \rho(b\big|_M), \tag{59}$$

is a trace on $\mathcal{B}$. Then $\rho \circ \tau$ is a supertrace on $\mathrm{End}^{\infty}_{\mathcal{B}}(\mathcal{E})$.

Proof. Consider the algebra $\mathrm{End}_{C^{\infty}_c(M)}(\mathcal{E})$. An operator $K \in \mathrm{End}_{C^{\infty}_c(M)}(\mathcal{E})$ has a Schwartz kernel $K(p|p')$ so that

$$(K\xi)(p) = \int_{Z_{\pi(p)}} K(p|p')\, \xi(p'). \tag{60}$$

(Note the difference in ordering as compared to (56).) For this proof, define $q, q' : P \times_M P \to P$ by $q(p, p') = p$ and $q'(p, p') = p'$. Let $\mathrm{End}^{\infty}_{C^{\infty}_c(M)}(\mathcal{E})$ denote the subalgebra of $\mathrm{End}_{C_c^\infty(M)}(\mathcal{E})$ consisting of operators whose Schwartz kernel lies in $C_c^\infty(P \times_M P; q^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E))$. The product in $\mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$ is given by

$$(KK')(p|p') = \int_{p''} K(p|p'')\, K'(p''|p'). \tag{61}$$

Note that an element of $\mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$ is not necessarily G-invariant. Note also that there is an injective homomorphism $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E}) \to \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})^{\mathrm{op}}$, where op denotes the opposite algebra, i.e. with the transpose multiplication. There is a fiberwise G-invariant supertrace $\mathrm{Tr}_s : \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E}) \to C_c^\infty(M)$ given by

$$(\mathrm{Tr}_s K)(m) = \int_{Z_m} \mathrm{tr}_s\, K(p|p). \tag{62}$$

Consider the algebra $\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$. The product in the algebra takes into account the action of $\mathcal{B}$ on $\mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$, which derives from the G-action on P. An element of the algebra has a kernel $K(g, p|p')$, where $p, p' \in Z_{s(g)}$. The product is given by

$$(K_1K_2)(g, p|p') = \int_{g'g''=g} \int_{p'' \in Z_{s(g')}} K_1(g', p(g'')^{-1}|p'')\, K_2(g'', p''g''|p'). \tag{63}$$

The supertrace (62) induces a map $\mathrm{Tr}_s : \mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E}) \to \mathcal{B}$ by

$$(\mathrm{Tr}_s K)(g) = \int_{Z_{s(g)}} \mathrm{tr}_s\, K(g, p|p). \tag{64}$$

Lemma 16. *$\eta \circ \mathrm{Tr}_s$ is a supertrace on $\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$.*

Proof. We can formally write

$$(\eta \circ \mathrm{Tr}_s)(K) = \int_M \rho(m) \int_{Z_m} \mathrm{tr}_s\, K(m, p|p), \tag{65}$$

keeping in mind that ρ is actually distributional. Then

$$\begin{aligned}
&(\eta \circ \mathrm{Tr}_s)(K_1K_2)\\
&= \int_{g'\in G} \int_{p \in Z_{r(g')}} \int_{p'' \in Z_{s(g')}} \rho(r(g'))\ \mathrm{tr}_s \Big(K_1(g', pg'|p'')\, K_2((g')^{-1}, p''(g')^{-1}|p) \Big)\\
&= \int_{g'\in G} \int_{p \in Z_{r(g')}} \int_{p'' \in Z_{s(g')}} \rho(r(g'))\ \mathrm{tr}_s \Big(K_2((g')^{-1}, p''(g')^{-1}|p)\, K_1(g', pg'|p'') \Big)\\
&= \int_{g'\in G} \int_{p'' \in Z_{r(g')}} \int_{p \in Z_{s(g')}} \rho(s(g'))\ \mathrm{tr}_s \Big(K_2(g', p''g'|p)\, K_1((g')^{-1}, p(g')^{-1}|p'') \Big).
\end{aligned} \tag{66}$$

However, the fact that η is a trace on $\mathcal{B}$ translates into the fact that

$$\int_{g \in G} \rho(s(g))\, f(g) = \int_{g \in G} \rho(r(g))\, f(g) \tag{67}$$

for all $f \in C_c^\infty(G)$, from which the lemma follows. □

We define a map $i : \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E}) \to \left(\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})\right)^{\mathrm{op}}$ by

$$(i(K))(g, p|p') = \Phi(pg^{-1}) K(p|p'). \tag{68}$$

Lemma 17. *The map i is a homomorphism.*

Proof. Given $K_1, K_2 \in \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$, we have

$$\begin{aligned}
&(i(K_1)\, i(K_2))\, (g, p|p') \\
&= \int_{g'g''=g} \int_{Z_{s(g')}} i(K_1)(g', p(g'')^{-1}|p'')\, i(K_2)(g'', p''g''|p') \\
&= \int_{g'g''=g} \int_{Z_{s(g')}} \Phi(pg^{-1})\, K_1(p(g'')^{-1}|p'')\, \Phi(p'')\, K_2(p''g''|p') \\
&= \int_{g'g''=g} \int_{Z_{s(g')}} \Phi(pg^{-1})\, K_1(p|p''g'')\, \Phi(p'')\, K_2(p''g''|p') \\
&= \int_{g'g''=g} \int_{Z_{s(g'')}} \Phi(pg^{-1})\, K_1(p|p'')\, \Phi(p''(g'')^{-1})\, K_2(p''|p') \\
&= \Phi(pg^{-1}) \int_{Z_{s(g)}} K_1(p|p'')\, K_2(p''|p') \\
&= (i(K_2K_1))(g, p|p').
\end{aligned} \tag{69}$$

Thus, i gives a homomorphism from $\mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})^{\mathrm{op}}$ to $\mathcal{B} \otimes_{C_c^\infty(M)} \mathrm{End}^\infty_{C_c^\infty(M)}(\mathcal{E})$, from which the lemma follows. □

Lemma 18. *We have* $\eta \circ \mathrm{Tr}_s \circ i = \rho \circ \tau$.

Proof. Given $K \in \mathrm{End}^\infty_{\mathcal{B}}(\mathcal{E})$, we have

$$\begin{aligned}
(\eta \circ \mathrm{Tr}_s \circ i)(K) &= \int_M \rho(m) \int_{Z_m} \mathrm{tr}_s(i(K))(m, p|p) \\
&= \int_M \rho(m) \int_{Z_m} \Phi(p)\ \mathrm{tr}_s\, K(p|p) = (\rho \circ \tau)(K).
\end{aligned} \tag{70}$$

This proves the lemma. □

Proposition 1 now follows from Lemmas 16–18. □

Example 6. Let μ be a holonomy-invariant transverse measure for $\mathcal{F}$. Let $\{U_i\}_{i=1}^N$ be an open covering of M by flowboxes, with $U_i = V_i \times W_i$, $V_i \subset \mathbb{R}^{\operatorname{codim}(\mathcal{F})}$ and $W_i \subset \mathbb{R}^{\dim(\mathcal{F})}$. Let μ_i be the measure on V_i which is the restriction of μ. Let $\{\phi_i\}_{i=1}^N$ be a partition of unity that is subordinate to $\{U_i\}_{i=1}^N$. For $f \in C_c^\infty(M; \mathcal{D})$, put $\rho(f) = \sum_{i=1}^N \int_{V_i} \left(\int_{W_i} \phi_i \, f \right) d\mu_i$. Then ρ satisfies the hypotheses of Proposition 1.

An operator $K \in \operatorname{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^* \mathcal{E})$ has a Schwartz kernel $K(p'|p)$ so that

$$(K\xi)(p) = \int_{Z_{\pi(p)}} \xi(p') \, K(p'|p). \tag{71}$$

Let $\operatorname{Hom}_{\mathcal{B}}^\infty(\mathcal{E}, \Omega^n \mathcal{E})$ denote the subspace of $\operatorname{Hom}_{\mathcal{B}}(\mathcal{E}, \Omega^n \mathcal{E})$ consisting of operators whose Schwartz kernel lies in

$$C_c^\infty(P \times_M P; \Lambda^n((\pi \circ q)^* \tau^*) \otimes (q')^* \mathcal{D}_Z \otimes \operatorname{Hom}((q')^* E, q^* E)). \tag{72}$$

Define $\tau K \in C_c^\infty(M; \Lambda^n \tau^* \otimes \mathcal{D})$ by

$$(\tau K)(m) = \int_{Z_m} \Phi(p) \, \operatorname{tr}_s K(p|p). \tag{73}$$

Proposition 2. *Let ρ be a linear functional on $C_c^\infty(M; \Lambda^n \tau^* \otimes \mathcal{D})$. Suppose that the linear functional η on $\Omega^n \mathcal{B}$, defined by*

$$\eta(\phi) = \rho(\phi\big|_M), \tag{74}$$

is a graded trace on $\Omega^* \mathcal{B}$. Then $\rho \circ \tau$ is a graded trace on $\operatorname{Hom}_{\mathcal{B}}^\infty(\mathcal{E}, \Omega^* \mathcal{E})$.

Proof. The proof is similar to that of Proposition 1. We omit the details. □

Proposition 3. *Let ρ be a linear functional on $C_c^\infty(M; \Lambda^n \tau^* \otimes \mathcal{D})$. Suppose that the linear functional η on $\Omega^n \mathcal{B}$, defined by*

$$\eta(\phi) = \rho(\phi\big|_M) \tag{75}$$

is a closed graded trace on $\Omega^* \mathcal{B}$. Then $\rho \circ \tau$ annihilates $[\nabla, K]$ for all $K \in \operatorname{Hom}_{\mathcal{B}}^\infty(\mathcal{E}, \Omega^{n-1} \mathcal{E})$.

Proof. It suffices to show that

$$(\rho \circ \tau)([\nabla^{\mathcal{E}}, K]) = \eta\left(d^{\mathrm{H}}(\tau(K)) \right). \tag{76}$$

Let $\nabla^{\mathcal{E}_0} : C_c^\infty(P) \to C_c^\infty(P; \pi^*\tau^*)$ be differentiation in the $T^{\mathrm{H}}P$-direction. It follows from (73) that

$$(d^{\mathrm{H}}(\tau K))(m) = \int_{Z_m} \Phi(p)\, \mathrm{tr}_s[\nabla^{\mathcal{E}}, K](p|p) + \int_{Z_m} \nabla^{\mathcal{E}_0}\Phi(p) \wedge \mathrm{tr}_s\, K(p|p). \tag{77}$$

Now $\eta\left(\int_{Z_m} \nabla^{\mathcal{E}_0}\Phi(p) \wedge \mathrm{tr}_s\, K(p|p)\right)$ can be written as $\int_P \nabla^{\mathcal{E}_0}\Phi \wedge \mathcal{O}$ for some G-invariant $\mathcal{O}$. From (57), $\int_{G^{\pi(p)}} \nabla^{\mathcal{E}_0}\Phi(pg) = 0$. Then decomposing the measure on P with respect to $P \to P/G$ gives that $\int_P \nabla^{\mathcal{E}_0}\Phi \wedge \mathcal{O} = 0$. Eq. (76) follows. □

Example 7. Following the notation of Example 6, let c be a closed holonomy-invariant transverse n-current for $\mathcal{F}$. Let c_i be the n-current on V_i which is the restriction of c. Let $\{\phi_i\}_{i=1}^N$ be a partition of unity that is subordinate to $\{U_i\}_{i=1}^N$. For $\omega \in C_c^\infty(M; \Lambda^n\tau^* \otimes \mathcal{D})$, put $\rho(\omega) = \sum_{i=1}^N \langle \left(\int_{W_i} \phi_i\, \omega\right), c_i \rangle$. Then ρ satisfies the hypotheses of Proposition 3.

4.3. *The $s \to 0$ limit of the superconnection Chern character*

In this subsection, we extend $\mathrm{End}^\infty(\mathcal{E})$ to an rapid-decay algebra $\mathrm{End}^\omega(\mathcal{E})$. Given a G-invariant Dirac-type operator acting on sections of E, we consider the Bismut superconnections $\{A_s\}_{s>0}$ on $\mathcal{E}$. We compute the $s \to 0$ limit of the pairing between the Chern character of A_s and a closed graded trace on $\Omega^*\mathcal{B}$, that is, concentrated on the units M.

We now choose a G-invariant vertical Riemannian metric g^{TZ} on the submersion $\pi : P \to M$ and a G-invariant horizontal distribution $T^{\mathrm{H}}P$. Given $m \in M$, let d_m denote the corresponding metric on Z_m. We note that $\{Z_m\}_{m\in M}$ has uniformly bounded geometry.

Let $\mathrm{End}^\omega_{\mathcal{B}}(\mathcal{E})$ be the algebra formed by G-invariant operators K as in (56) whose integral kernels $K(p'|p) \in C^\infty(P \times_M P; (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E))$ are such that for all $q \in \mathbb{Z}^+$,

$$\sup_{(p',p)\in P\times_M P} e^{q\, d(p',p)}\, |K(p'|p)| < \infty, \tag{78}$$

along with the analogous property for the covariant derivatives of K.

Proposition 4. *Let ρ be a linear functional on $C_c^\infty(M; \mathcal{D})$. Suppose that the linear functional η on $\mathcal{B}$, defined by*

$$\eta(b) = \rho(b\big|_M), \tag{79}$$

is a trace on $\mathcal{B}$. Then $\rho \circ \tau$ is a supertrace on $\mathrm{End}^\omega_{\mathcal{B}}(\mathcal{E})$.

Proof. The proof is formally the same as that of Proposition 1. We omit the details. □

Let $\mathrm{Hom}^{\omega}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$ be the algebra formed by G-invariant operators K as in (71) whose integral kernels

$$K(p'|p) \in C^{\infty}_c(P \times_M P; \Lambda^*((\pi \circ q)^*\tau^*) \otimes (q')^*\mathcal{D}_Z \otimes \mathrm{Hom}((q')^*E, q^*E)) \tag{80}$$

are such that for all $q \in \mathbb{Z}^+$,

$$\sup_{(p',p)\in P\times_M P} e^{q\, d(p',p)}\, |K(p'|p)| < \infty, \tag{81}$$

along with the analogous property for the covariant derivatives of K.

Proposition 5. *Let ρ be a linear functional on $C^{\infty}_c(M; \Lambda^n\tau^* \otimes \mathcal{D})$. Suppose that the linear functional η on $\Omega^n\mathcal{B}$, defined by*

$$\eta(\phi) = \rho(\phi\big|_M), \tag{82}$$

is a graded trace on $\Omega^*\mathcal{B}$. Then $\rho \circ \tau$ is a graded trace on $\mathrm{Hom}^{\omega}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$.

Proof. The proof is formally the same as that of Proposition 2. We omit the' details. □

Proposition 6. *Let ρ be a linear functional on $C^{\infty}_c(M; \Lambda^n\tau^* \otimes \mathcal{D})$. Suppose that the linear functional η on $\Omega^n\mathcal{B}$, defined by*

$$\eta(\phi) = \rho(\phi\big|_M), \tag{83}$$

is a closed graded trace on $\Omega^*\mathcal{B}$. Then $\rho \circ \tau$ annihilates $[\nabla, K]$ for all $K \in \mathrm{Hom}^{\omega}_{\mathcal{B}}(\mathcal{E}, \Omega^{n-1}\mathcal{E})$.

Proof. The proof is formally the same as that of Proposition 3. We omit the details. □

Suppose that Z is even-dimensional. Let E be a G-invariant Clifford bundle on P which is equipped with a G-invariant connection. For simplicity of notation, we assume that $E = S^Z \widehat{\otimes} \widetilde{V}$, where S^Z is a vertical spinor bundle and $\widetilde{V}$ is an auxiliary vector bundle on P. More precisely, suppose that the vertical tangent bundle TZ has a G-invariant spin structure. Let S^Z be the vertical spinor bundle, a G-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on P. Let $\widetilde{V}$ be another G-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on P which is equipped with a G-invariant Hermitian connection. That is, $\widetilde{V}$

is the pullback of a Hermitian vector bundle G on P/G with a Hermitian connection ∇^V. Then we put $E = S^Z \widehat{\otimes} \widetilde{V}$. The case of general G-invariant Clifford bundles E can be treated in a way completely analogous to what follows.

Let ∇^{TZ} be the Bismut connection on TZ, as constructed using the horizontal distribution $(d\mu)^{-1}(T^{\mathrm{H}}(P/G))$ on P; see, for example, Berline–Getzler–Vergne [5, Proposition 10.2]. The G-invariance of ∇^{TZ} and $\nabla^{\widetilde{V}}$ implies that $\widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})$ lies in $C^\infty(P; \Lambda^*(TZ)^* \otimes \Lambda^*(\pi^*\tau^*))$.

Let $Q \in \mathrm{End}_{\mathcal{B}}(\mathcal{E})$ denote the vertical Dirac-type operator. From finite-propagation-speed estimates as in Lott [22, Proof of Proposition 8], along with the bounded geometry of $\{Z_m\}_{m\in M}$, for any $s > 0$ we have

$$e^{-s^2 Q^2} \in \mathrm{End}^{\omega}_{\mathcal{B}}(\mathcal{E}). \tag{84}$$

Let $A_s : \mathcal{E} \to \Omega^*\mathcal{E}$ be the superconnection

$$A_s = s\, Q + \nabla^{\mathcal{E}} - \frac{1}{4s} c(T^P). \tag{85}$$

Here $c(T^P)$ is Clifford multiplication by the curvature 2-form T^P of $(d\mu)^{-1}(T^{\mathrm{H}}(P/G))$, restricted to the horizontal vectors $T^{\mathrm{H}}P$. We note that the analogous connection term of the Bismut superconnection [5, Proposition 10.15] has an additional term to make it Hermitian, but in our setting this term is incorporated into the horizontal differentiation of the vertical density. One can use finite-propagation-speed estimates, along with the bounded geometry of $\{Z_m\}_{m\in M}$ and the Duhamel expansion as in [5, Theorem 9.48], to show that we obtain a well-defined element $e^{-(A^s)^2 - \mathcal{L}_{\widetilde{T}}}$ of $\mathrm{Hom}^{\omega}_{\mathcal{B}}(\mathcal{E}, \Omega^*\mathcal{E})$; see [18, Theorem 3.1] for an analogous statement when $P = G = G_{\mathrm{hol}}$.

Let $\mathcal{R}$ be the rescaling operator which, for p even, multiplies a p-form by $(2\pi i)^{-\frac{p}{2}}$. Put

$$\mathrm{ch}(A_s) = \mathcal{R}\left(\tau e^{-A_s^2 - \mathcal{L}_{\widetilde{T}}}\right) \in C_c^\infty(M; \Lambda^*\tau^* \otimes \mathcal{D}). \tag{86}$$

Theorem 2. *Given a linear functional ρ which satisfies the hypotheses of Proposition* 6,

$$\lim_{s\to 0} \rho(\mathrm{ch}(A_s)) = \rho\left(\int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right). \tag{87}$$

Proof. Using Lemmas 13 and 14, $A_s^2 + \mathcal{L}_{\widetilde{T}}$ is G-invariant. Let A'_s be the corresponding Bismut superconnection on the foliated manifold P/G, a locally defined differential operator constructed using the horizontal distribution $T^{\mathrm{H}}(P/G)$. By construction, $A_s^2 + \mathcal{L}_{\widetilde{T}}$ is the pullback under μ of $(A'_s)^2$, where we use the identification $\Lambda^*(\pi^*\tau^*) = \mu^*\Lambda^*(NF)^*$. From [5, Theorem 10.23], the $s \to 0$ limit of the supertrace of the kernel of $e^{-(A'_s)^2}$, when restricted to the diagonal of $(P/G) \times (P/G)$, is $\widehat{A}(\nabla^{TF})\ \mathrm{ch}(\nabla^V)$. Then the $s \to 0$ limit of the supertrace of the kernel of $e^{-A_s^2 - \mathcal{L}_{\widetilde{T}}}$, when restricted to the diagonal of $P \times P$, is the pullback under μ of $\widehat{A}(\nabla^{TF})\,\mathrm{ch}(\nabla^{V})$, i.e. $\widehat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\widetilde{V}})$. The theorem follows. □

Remark. If $P = G = G_{\mathrm{hol}}$ then an analogue of Theorem 2 appears in [18, Theorem 2.1].

If we put

$$G' = \{(p_1, p_2) \in P \times P \,:\, \pi(p_1) = \pi(p_2)\}/G \tag{88}$$

then G' has the structure of a foliation groupoid, with units $G'^{(0)} = P/G$. In this way, we could reduce from the case of G acting on P to the case of the foliation groupoid G' acting on itself. However, doing so would not really simplify any of the constructions.

5. Index theorem

In this section, we prove the main result of the paper, Theorem 5.

5.1. The index class

In this subsection, we construct the index class $\mathrm{Ind}(D) \in \mathrm{K}_0(\mathfrak{A})$. We describe its pairing with a closed graded trace on $\mathcal{B}$. We prove that the pairing of $\mathrm{Ind}(D)$ with the closed graded trace equals the pairing of $\mathrm{ch}(A_s)$ with the closed graded trace.

Consider the algebra $\mathfrak{A} = \mathrm{End}^{\infty}_{\mathcal{B}}(\mathcal{E})$. Let $D : \mathcal{E}^{+} \to \mathcal{E}^{-}$ be the restriction of Q to the positive subspace $\mathcal{E}^{+}$ of $\mathcal{E}$. We construct an index projection following Connes–Moscovici [12] and Moscovici–Wu [23]. Let $u \in C^{\infty}(\mathbb{R})$ be an even function such that $w(x) = 1 - x^2\,u(x)$ is a Schwartz function and the Fourier transforms of u and w have compact support [23, Lemma 2.1]. Define $\overline{u} \in C^{\infty}([0,\infty))$ by $\overline{u}(x) = u(x^2)$. Put $\mathcal{P} = \overline{u}(D^*D)D^*$, which we will think of as a parametrix for D, and put $S_+ = I - \mathcal{P}D$, $S_- = I - D\mathcal{P}$. Consider the operator

$$L = \begin{pmatrix} S_+ & -(I+S_+)\mathcal{P} \\ D & S_- \end{pmatrix}, \tag{89}$$

with inverse

$$L^{-1} = \begin{pmatrix} S_+ & \mathcal{P}(I+S_-) \\ -D & S_- \end{pmatrix}. \tag{90}$$

The index projection is defined by

$$p = L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} S_+^2 & S_+(I+S_+)\mathcal{P} \\ S_-D & I - S_-^2 \end{pmatrix}. \tag{91}$$

Put

$$p_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{92}$$

By definition, the index of D is

$$\mathrm{Ind}(D) = [p - p_0] \in \mathrm{K}_0(\mathfrak{A}). \tag{93}$$

As Q is G-invariant, the operator l of (46) commutes with p, and (47) holds for $\xi \in \mathrm{Im}(p)$. If ρ is a linear functional which satisfies the hypotheses of Proposition 3, define the pairing of ρ with $\mathrm{Ind}(D)$ by

$$\begin{aligned}\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho\rangle = {} & (2\pi i)^{-\deg(\rho)/2} \\ & \times \rho\left(\tau\left(p\, e^{-(p\circ\nabla^{\mathcal{E}}\circ p)^2 - \mathcal{L}_{\widetilde{T}}}\, p - p_0\, e^{-(p_0\circ\nabla^{\mathcal{E}}\circ p_0)^2 - \mathcal{L}_{\widetilde{T}}}\, p_0\right)\right),\end{aligned} \tag{94}$$

where we have extended the ungraded trace τ in the obvious way to act on (2×2)-matrices. (See [16, Section 5] for the justification of the definition.)

Theorem 3. *For all* $s > 0$,

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho\rangle = \rho(\mathrm{ch}(A_s)). \tag{95}$$

Proof. The proof follows the lines of the proof of [16, Proposition 4 and Theorem 3], to which we refer for details. We only present the main idea. Put

$$\nabla' = \left(\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1}\circ\nabla^{\mathcal{E}}\circ L \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}\right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \nabla^{\mathcal{E}} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}\right). \tag{96}$$

Then one can show algebraically that

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \rho\rangle = \rho\left(\mathcal{R}\, \tau e^{-(\nabla')^2 - \mathcal{L}_{\widetilde{T}}}\right), \tag{97}$$

where the τ on the right-hand side is now a graded trace. Next, one shows that

$$\rho\left(\mathcal{R}\, \tau e^{-(\nabla')^2 - \mathcal{L}_{\widetilde{T}}}\right) = \rho(\mathrm{ch}(A_s)) \tag{98}$$

by performing a homotopy from ∇' to A_s, from which the theorem follows. □

5.2. Construction of ω_ρ

In this subsection, we construct the universal class $\omega_\rho \in \mathrm{H}^*(BG; o)$. We express $\rho(\mathrm{ch}(A_s))$ as an integral involving the pullback of ω_ρ.

Put $V = \widetilde{V}/G$, a Hermitian vector bundle on P/G with a compatible connection ∇^V.

Let $o(\tau)$ be the orientation bundle of τ, a flat real line bundle on M. Let ρ satisfy the hypotheses of Proposition 3. By duality, ρ corresponds to a closed distributional form $*\rho \in \Omega^{\dim(M)-n}(M; o(\tau))$.

Let EG denote the bar construction of a universal space on which G acts freely. That is, put

$$G^{(n)} = \{(g_1, \ldots, g_n) : s(g_1) = r(g_2), \ldots, s(g_{n-1}) = r(g_n)\}. \tag{99}$$

Then EG is the geometric realization of a simplicial manifold given by $E_nG = G^{(n+1)}$, with face maps

$$d_i(g_0, \ldots, g_n) = \begin{cases} (g_1, \ldots, g_n) & \text{if } i = 0, \\ (g_0, \ldots, g_{i-1}g_i, \ldots, g_n) & \text{if } 1 \leqslant i \leqslant n \end{cases} \tag{100}$$

and degeneracy maps

$$s_i(g_0, \ldots, g_n) = (g_0, \ldots, g_i, 1, g_{i+1}, \ldots g_n), \quad 0 \leqslant i \leqslant n. \tag{101}$$

Here, 1 denotes a unit. The action of G on EG is induced from the action on E_nG given by $(g_0, \ldots, g_n)\, g = (g_0, \ldots, g_n g)$. Let BG be the quotient space. Define $\pi' : EG \to M$ as the extension of $(g_0, \ldots, g_n) \to s(g_n)$. Put $\Omega^{n_1,n_2}(EG) = \Omega^{n_1}(G^{(n_2+1)})$ and $\Omega^{n_1,n_2}(BG) = \left(\Omega^{n_1,n_2}(EG)\right)^G$. Let $\Omega^*(BG)$ be the total complex of $\Omega^{*,*}(BG)$. Here, the forms on $G^{(n_2+1)}$ can be either smooth or distributional, depending on the context. We will speak correspondingly of smooth or distributional elements of $\Omega^*(BG)$. In either case, the cohomology of $\Omega^*(BG)$ equals $\mathrm{H}^*(BG; \mathbb{R})$. There is a similar discussion for twistings by a local system.

The action of G on P is classified by a continuous G-equivariant map $\widehat{\nu} : P \to EG$. Let $\nu : P/G \to BG$ be the G-quotient of $\widehat{\nu}$. There are commutative diagrams

$$\begin{array}{ccc} P & \xrightarrow{\widehat{\nu}} & EG \\ \pi \big\downarrow & & \big\downarrow \pi' \\ M & \xrightarrow{\mathrm{Id.}} & M \end{array} \tag{102}$$

and

$$\begin{array}{ccc} P & \xrightarrow{\widehat{\nu}} & EG \\ \big\downarrow & & \big\downarrow \\ P/G & \xrightarrow{\nu} & BG. \end{array} \tag{103}$$

As P/G is compact, we may assume that ν is Lipschitz.

Consider $(\pi')^*(*\rho) \in \Omega^*(EG; (\pi')^* o(\tau))$, a closed distributional form on EG. Let o be the G-quotient of $(\pi')^* o(\tau)$, a flat real line bundle on BG. Then $(\pi')^*(*\rho)$ pulls back from a closed distributional form in $\Omega^*(BG; o)$, which represents a class in $\mathrm{H}^*(BG; o)$. Let $\omega_\rho \in \Omega^*(BG; o)$ be a closed smooth form representing the same cohomology class. Let $\widehat{\omega}_\rho \in \Omega^*(EG; (\pi')^* o(\tau))$ be its pullback to EG. As v is Lipschitz, $v^* \omega_\rho$ is an L^∞-form on P/G.

Theorem 4.

$$\rho\left(\int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right) = \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\, v^* \omega_\rho. \tag{104}$$

Proof. Let $*\left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)$ be the dual of $\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})$. We will think of $*\left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)$ as a cycle on P and $(\pi')^*(*\rho)$ as a cocycle on EG. Then

$$\begin{aligned}
\rho\left(\int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right) &= \left\langle \pi_* \left(* \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), *\rho \right\rangle_M \\
&= \left\langle * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right), \pi^*(*\rho) \right\rangle_P \\
&= \left\langle \widehat{v}_* \left(* \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), (\pi')^*(*\rho) \right\rangle_{EG} \\
&= \left\langle \widehat{v}_* \left(* \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right)\right), \widehat{\omega}_\rho \right\rangle_{EG} \\
&= \left\langle * \left(\Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\right), \widehat{v}^* \widehat{\omega}_\rho \right\rangle_P \\
&= \int_P \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\, \widehat{v}^* \widehat{\omega}_\rho \\
&= \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\, v^* \omega_\rho. \qquad \square
\end{aligned} \tag{105}$$

Remark. If one were willing to work with orbifolds P/G instead of manifolds then one could extend Theorem 4 to general proper cocompact actions, with $\omega_\rho \in \mathrm{H}^*(\underline{B}G; o)$ being a cohomology class on the classifying space for proper G-actions.

5.3. Proof of index theorem

Theorem 5. *If G acts freely, properly discontinuously and cocompactly on P and ρ satisfies the hypotheses of Proposition* 6 *then*

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle = \int_{P/G} \widehat{A}(TF)\ \mathrm{ch}(V)\, v^* \omega_\rho. \tag{106}$$

Proof. If Z is even-dimensional then the claim follows from Theorems 2–4. If Z is odd-dimensional then one can reduce to the even-dimensional case by a standard trick involving taking the product with a circle. $\square$

Example 8. Suppose that $(M, \mathcal{F})$ is a closed foliated manifold. Take $P = G = G_{\mathrm{hol}}$. Let μ be a holonomy-invariant transverse measure for $\mathcal{F}$. Take ρ as in Example 6. Then Theorem 5 reduces to Connes' L^2-foliation index theorem [11, Section I.5.γ, Theorem 7]

$$\langle \mathrm{Ind}\, D, \rho \rangle = \langle \widehat{A}(TF)\ \mathrm{ch}(V), RS_{\mu} \rangle, \tag{107}$$

where RS_μ is the Ruelle–Sullivan current associated to μ [11, Section I.5.β].

Example 9. Let $(M, \mathcal{F})$ be a closed manifold equipped with a codimension-q foliation. Take $P = G = G_{\mathrm{hol}}$. Let $\mathrm{H}^*(\mathrm{Tr}\,\mathcal{F})$ denote the Haefliger cohomology of $(M, \mathcal{F})$ [17]. Recall that there is a linear map $\int_{\mathcal{F}} : \mathrm{H}^*(M) \to \mathrm{H}^{*-n+q}(\mathrm{Tr}\,\mathcal{F})$. Let c be a closed holonomy-invariant transverse current for $\mathcal{F}$. Take ρ as in Example 7. Then Theorem 5 becomes

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), \rho \rangle = \left\langle \int_{\mathcal{F}} \widehat{A}(TF)\ \mathrm{ch}(V), c \right\rangle. \tag{108}$$

This is a consequence of the Connes–Skandalis foliation index theorem, along with the result of Connes that ρ gives a higher trace on the reduced foliation C^*-algebra; see [4,10,13].

Example 10. Let M be a closed oriented n-dimensional manifold. Let $G = M$ be the groupoid that just consists of units. Let P be a closed manifold that is the total space of an oriented fiber bundle $\pi : P \to M$ with fiber Z. Let c be a closed current on M with homology class $[c] \in \mathrm{H}_*(M; \mathbb{C})$. With $* : \mathrm{H}_*(M; \mathbb{C}) \to \mathrm{H}^{n-*}(M; \mathbb{C})$ being the Poincaré isomorphism, Theorem 5 becomes

$$\langle \mathrm{ch}(\mathrm{Ind}\, D), c \rangle = \int_P \widehat{A}(TZ)\ \mathrm{ch}(V)\ \pi^*(*[c]). \tag{109}$$

This is a consequence of the Atiyah–Singer families index theorem [2], as the right-hand side equals $\langle \int_Z \widehat{A}(TZ)\ \mathrm{ch}(V), c \rangle$.

Example 11. Let G be a discrete group that acts freely, properly discontinuously and cocompactly on a manifold P. As its space of units M is a point, let ρ be the identity map $C^\infty(M) \to \mathbb{C}$. Then Theorem 5 reduces to Atiyah's L^2-index theorem [1]

$$\langle \mathrm{Ind}\, D, \rho \rangle = \int_{P/G} \widehat{A}(TP/G)\ \mathrm{ch}(V). \tag{110}$$

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Appendix A

This is an addendum to [16], in which we use finite propagation speed methods to improve [16, Theorem 3]. In the improved version we allow η to be a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$, as opposed to $\Omega^*(B, \mathcal{B}^\omega)$. There is a similar improvement of [16, Theorem 6].

We will follow the notation of [16].

A.1. Finite propagation speed

Let $f \in C_c^\infty(\mathbb{R})$ be a smooth even function with support in $[-\varepsilon, \varepsilon]$. Put

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x) \cos(xy)\, dx, \tag{A.1}$$

a smooth even function. With A_s as in [16, (4.7)], put

$$\widehat{f}(A_s) = \int_{\mathbb{R}} f(x) \cos(x A_s)\, dx. \tag{A.2}$$

Let us describe $\cos(x A_s)$ explicitly, using the fact that it satisfies

$$\left(\partial_x^2 + A_s^2\right) \cos(x A_s) = 0. \tag{A.3}$$

Write $A_s^2 = s^2 Q^2 + X$. We first consider a solution $u(\cdot, x)$ of the inhomogeneous wave equation

$$\left(\partial_x^2 + s^2 Q^2\right) u = f \tag{A.4}$$

with initial conditions $u(\cdot, 0) = u_0(\cdot)$ and $u_x(\cdot, 0) = 0$. Then $u(\cdot, x)$ is given by

$$u(x) = \cos(xsQ) u_0 + \int_0^x \frac{\sin((x - v)sQ)}{sQ} f(v)\, dv. \tag{A.5}$$

Putting $f = -Xu$ and iterating, we obtain an expansion of $\cos(x A_s)$ of the form

$$\cos(x A_s) = \cos(xsQ) - \int_0^x \frac{\sin((x - v)sQ)}{sQ} X \cos(vsQ)\, dv + \cdots . \tag{A.6}$$

Because X has positive form degree, there is no problem with the convergence of the series.

From finite propagation speed results, we know that $\cos(xsQ)$ has a Schwartz kernel $\cos(xsQ)(p'|p)$ with support on $\{(p',p) : d(p',p) \leqslant xs\}$, and similarly for $\frac{\sin(xsQ)}{sQ}$; see Taylor [29, Chapter 4.4]. Using the compactness of h, it follows that the (m,n)-component $\widehat{f}(A_s)_{(m,n)}$ lies in $\mathrm{Hom}^{\infty}_{C^\infty_c(B)\rtimes\Gamma}(C^\infty_c(\widehat{M};\widehat{E}), \Omega^{m,n}(B,\mathbb{C}\Gamma)\otimes_{C^\infty_c(B)\rtimes\Gamma} C^\infty_c(\widehat{M};\widehat{E}))$.

Finally, define $\mathrm{ch}_{\widehat{f}}(A_s) \in \Omega^*(B,\mathbb{C}\Gamma)_{ab}$ by

$$\mathrm{ch}_{\widehat{f}}(A_s) = \mathcal{R}\,\mathrm{Tr}_{s,\langle e\rangle}\,\widehat{f}(A_s). \tag{A.7}$$

A.2. Index pairing

In this subsection, we show that for all $s > 0$ and all closed graded traces η on $\Omega^*(B,\mathbb{C}\Gamma)$, $\langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle = \langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle$. The method of proof is essentially the same as that of [16, Section 5], which in turn was inspired by Nistor [25].

In analogy to [16, Section 5.3], put $\mathcal{E} = C^\infty_c(\widehat{M};\widehat{E})$ and $\widetilde{\mathfrak{A}} = \mathrm{End}^{\infty}_{C^\infty_c(B)\rtimes\Gamma}\big(C^\infty_c(\widehat{M};\widehat{E})\big)$. Let $D : \mathcal{E}^+ \to \mathcal{E}^-$ be the restriction of Q to the positive subspace $\mathcal{E}^+$ of $\mathcal{E}$. We construct an index projection following [12,23]. Let $u \in C^\infty(\mathbb{R})$ be an even function such that $w(x) = 1 - x^2 u(x)$ is a Schwartz function and the Fourier transforms of u and w have compact support [23, Lemma 2.1]. Define $\overline{u} \in C^\infty([0,\infty))$ by $\overline{u}(x) = u(x^2)$. Put $\mathcal{P} = \overline{u}(D^*D)D^*$, which we will think of as a parametrix for D, and put $S_+ = I - \mathcal{P}D$, $S_- = I - D\mathcal{P}$. Consider the operator

$$L = \begin{pmatrix} S_+ & -(I+S_+)\mathcal{P} \\ D & S_- \end{pmatrix}, \tag{A.8}$$

with inverse

$$L^{-1} = \begin{pmatrix} S_+ & \mathcal{P}(I+S_-) \\ -D & S_- \end{pmatrix}. \tag{A.9}$$

The index projection is defined by

$$p = L\begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix} L^{-1} = \begin{pmatrix} S_+^2 & S_+(I+S_+)\mathcal{P} \\ S_-D & I - S_-^2 \end{pmatrix}. \tag{A.10}$$

Put

$$p_0 = \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}. \tag{A.11}$$

By definition, the index of D is

$$\mathrm{Ind}(D) = [p - p_0] \in \mathrm{K}_0(\widetilde{\mathfrak{A}}). \tag{A.12}$$

Put $\widetilde{\Omega}^* = \mathrm{Hom}^{\infty}_{C^\infty_c(B)\rtimes\Gamma}(C^\infty_c(\widehat{M};\widehat{E}), \Omega^*(B,\mathbb{C}\Gamma) \otimes_{C^\infty_c(B)\rtimes\Gamma} C^\infty_c(\widehat{M};\widehat{E}))$, a graded algebra with derivation $\nabla = \nabla^{(1,0)} + \nabla^{(0,1)}$. If η is a closed graded trace on $\Omega^*(B,\mathbb{C}\Gamma)$, define the pairing of η with $\mathrm{Ind}(D)$ by

$$\langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle = (2\pi i)^{-\deg(\eta)/2} \left\langle \mathrm{Tr}_{\langle e\rangle} \left(\widehat{f}(p\circ\nabla\circ p) - \widehat{f}(p_0\circ\nabla\circ p_0)\right), \eta\right\rangle. \quad (A.13)$$

(See [16, Section 5] for the justification of the definition.)

Theorem 6. *For all $s > 0$,*

$$\langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle = \langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle. \quad (A.14)$$

Proof. The proof follows the lines of the proof of [16, Proposition 4 and Theorem 3], to which we refer for details. We only present the main idea. Put

$$\nabla' = \left(\begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix} L^{-1}\circ\nabla\circ L \begin{pmatrix} I & 0 \\ 0 & 0\end{pmatrix}\right) + \left(\begin{pmatrix} 0 & 0 \\ 0 & I\end{pmatrix} \nabla \begin{pmatrix} 0 & 0 \\ 0 & I\end{pmatrix}\right). \quad (A.15)$$

Then one can show algebraically that

$$\langle \widehat{f}(\mathrm{Ind}(D)), \eta\rangle = \langle \mathcal{R}\ \mathrm{Tr}_{s,\langle e\rangle}\, \widehat{f}(\nabla'), \eta\rangle. \quad (A.16)$$

Next, one shows that

$$\langle \mathcal{R}\ \mathrm{Tr}_{s,\langle e\rangle}\, \widehat{f}(\nabla'), \eta\rangle = \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle \quad (A.17)$$

by performing a homotopy from ∇' to A_s, from which the theorem follows. The argument is the same as in the proof of [16, Proposition 4]. We refer to [16], and will only indicate the necessary modifications of the equations in [16, Section 5.2].

As in [16, (5.20)], for $t \in [0,1]$ put

$$A(t) = \begin{pmatrix} (\nabla')^+ & t\, D^* \\ t\, D & (\nabla')^- \end{pmatrix}. \quad (A.18)$$

The analog of [16, (5.26)] is

$$\cos(x\, A(t)) \equiv \begin{pmatrix} \cos\left(x\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right) & \mathcal{Z} \\ 0 & D\cos\left(x\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right)\mathcal{P} \end{pmatrix}, \quad (A.19)$$

where

$$\mathcal{Z} = - \int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right)}{\sqrt{((\nabla')^+)^2 + t^2 D^*D}} \times \left(t\,[(\nabla')^-, D^*] + t((\nabla')^+ - (\nabla')^-)D^*\right) \cos\left(v\sqrt{(\nabla^-)^2 + t^2 DD^*}\right) dv \tag{A.20}$$

and the left-hand side of (A.19) is to be multiplied by f and then integrated. As in [16, (5.30)],

$$\frac{dA}{dt} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}. \tag{A.21}$$

The analog of [16, (5.31)] is

$$\begin{aligned} &\mathrm{Tr}_s\left(\frac{dA}{dt}\begin{pmatrix} \cos\left(x\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right) & \mathcal{Z} \\ 0 & D\cos\left(x\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right)\mathcal{P} \end{pmatrix}\right) \\ &= -\,\mathrm{Tr}\,(D\,\mathcal{Z}) \\ &= t\ \mathrm{Tr}\left(D\int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right)}{\sqrt{((\nabla')^+)^2 + t^2 D^*D}}\right. \\ &\quad \times \left([(\nabla')^-, D^*] + ((\nabla')^+ - (\nabla')^-)D^*\right) \\ &\quad \left.\times \cos\left(v\sqrt{(\nabla^-)^2 + t^2 DD^*}\right)\right) dv. \end{aligned} \tag{A.22}$$

The analog of [16, (5.32)] is

$$\begin{aligned} &D\int_0^x \frac{\sin\left((x-v)\sqrt{((\nabla')^+)^2 + t^2 D^*D}\right)}{\sqrt{((\nabla')^+)^2 + t^2 D^*D}} \left([(\nabla')^-, D^*] + ((\nabla')^+ - (\nabla')^-)D^*\right) \\ &\quad \times \cos\left(v\sqrt{(\nabla^-)^2 + t^2 DD^*}\right) dv \\ &\equiv \int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 + t^2 DD^*}\right)}{\sqrt{(\nabla^-)^2 + t^2 DD^*}}\, D\,\left([(\nabla')^-, D^*]\right. \\ &\quad \left. + ((\nabla')^+ - (\nabla')^-)D^*\right) \cos\left(v\sqrt{(\nabla^-)^2 + t^2 DD^*}\right) dv \\ &\equiv \int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 + t^2 DD^*}\right)}{\sqrt{(\nabla^-)^2 + t^2 DD^*}}\,[\nabla^-, DD^*] \\ &\quad \times \cos\left(v\sqrt{(\nabla^-)^2 + t^2 DD^*}\right) dv. \end{aligned} \tag{A.23}$$

The analog of [16, (5.33)] is

$$\mathrm{Tr}\left(\int_0^x \frac{\sin\left((x-v)\sqrt{(\nabla^-)^2 + t^2\, DD^*}\right)}{\sqrt{(\nabla^-)^2 + t^2\, DD^*}}\, [\nabla^-, DD^*] \right.$$
$$\left. \times \cos\left(v\, \sqrt{(\nabla^-)^2 + t^2\, DD^*}\right)\, dv\right)$$
$$= -\, t^{-2}\, d\ \mathrm{Tr}\left(\cos\left(x\, \sqrt{(\nabla^-)^2 + t^2\, DD^*}\right)\right). \tag{A.24}$$

The rest of the proof is as in [16, Proof of Proposition 4]. □

We define $\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta\rangle$ by formally taking $\widehat{f}(z) = e^{-z^2}$ in (A.13). This makes perfect sense, given that η acts on elements of a fixed degree.

Corollary 1. (a) *The left-hand side of* (A.14) *only depends on f through the derivative* $\widehat{f}^{(\deg(\eta))}(0)$.

(b) *If* $\widehat{f}^{(\deg(\eta))}(0) = \frac{d^{\deg(\eta)} e^{-z^2}}{d^{\deg(\eta)} z}\Big|_{z=0}$ *then*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta\rangle = \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle. \tag{A.25}$$

Proof. (a) From (A.13), the right-hand side of (A.14) only depends on f through the derivative $\widehat{f}^{(\deg(\eta))}(0)$. From Theorem 6, the same must be true of the left-hand side.

(b) If $\widehat{f}^{(\deg(\eta))}(0) = \frac{d^{\deg(\eta)} e^{-z^2}}{d^{\deg(\eta)} z}\Big|_{z=0}$ then $\widehat{f}$ has the same relevant term in its Taylor expansion as the function $z \to e^{-z^2}$, from which the corollary follows. □

A.3. Pairing of the Chern character of the index with general closed graded traces

In this subsection, we prove a formula for the pairing of the Chern character of the index with a closed graded trace η on $\Omega^*(B, \mathbb{C}\Gamma)$. The idea is to approximate the Gaussian function, which was previously used in forming the superconnection Chern character, by an appropriate function $\widehat{f}$.

Theorem 7. *Given a closed graded trace* η *on* $\Omega^*(B, \mathbb{C}\Gamma)$,

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta\rangle = \left\langle \int_Z \Phi\, \widehat{A}(\nabla^{TZ})\ \mathrm{ch}(\nabla^{\widetilde{V}})\, e^{-\frac{\nabla^2_{can}}{2\pi i}}, \eta \right\rangle. \tag{A.26}$$

Proof. Choose an even function $f \in C_c^\infty(\mathbb{R})$ so that $\widehat{f}$ satisfies the hypothesis of Corollary 1(b). By Corollary 1, it suffices to compute

$$\lim_{s\to 0} \langle \mathrm{ch}_{\widehat{f}}(A_s), \eta\rangle. \tag{A.27}$$

With reference to (A.2), the local supertrace $\mathrm{tr}_s \cos(x\, A_s)\,(p, p)$ exists as a distribution in x. The singularities near $x = 0$ of the distribution have coefficients that are the same, up to constants, as the leading terms in the x-expansion of $\mathrm{tr}_s\, e^{-x^2 A_s^2}(p, p)$; see, for example, Sandoval [28] for the analogous statement for $\cos(xsQ)$. As in [5, Lemma 10.22], these are the terms that enter into the local index computation. Now $\cos(x\, A_s)$ satisfies (A.3), in analogy to the fact that $e^{-t\, A_s^2}$ satisfies the heat equation

$$\left(\partial_t + A_s^2\right) e^{-t\, A_s^2} = 0. \tag{A.28}$$

We can perform a Getzler rescaling as in the proof of [16, Theorem 2], to see that for the purposes of computing the local index, we can effectively replace the A_s^2-term in the differential operator of (A.3) by [16, (4.12)]. Thus, we are reduced to considering the wave operator of the harmonic oscillator Hamiltonian. The rest of the proof of the theorem can in principle be carried out in a way similar to that of [16, Theorem 2]. However, we can shortcut the calculations by noting that Corollary 1, along with the choice of f, implies that the result of the local calculation must be the same as $\lim_{s\to 0} \langle \mathrm{ch}(A_s), \eta\rangle$, which was already calculated in [16, Theorem 2]. □

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