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Delocalized L^2-Invariants

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We define extensions of the L^2-analytic invariants of closed manifolds, called delocalized L^2-invariants. These delocalized invariants are constructed in terms of a nontrivial conjugacy class of the fundamental group. We show that in many cases, they are topological in nature. We show that the marked length spectrum of an odd-dimensional hyperbolic manifold can be recovered from its delocalized L^2-analytic torsion. There are technical convergence questions.

1. INTRODUCTION

A major theme in geometric analysis is to construct and understand analytic invariants of a Riemannian manifold which only depend on the underlying smooth structure. Similarly, one can consider analytic invariants of a Hermitian complex manifold which only depend on the underlying complex structure.

In this paper we introduce some new such invariants and compute them in certain cases. We call these invariants "delocalized L^2-invariants." Let us first give a simple but concrete example, in the complex case.

Let Σ be a closed Riemann surface of genus $\geqslant 2$. Let M denote Σ equipped with a specific Hermitian metric. The universal cover $\tilde{M}$ carries the pullback metric. Let $e^{-t\tilde{\triangle}}(x, y)$ be the Schwartz kernel of the heat operator, acting on functions on $\tilde{M}$. Let $\mathscr{F}$ be a fundamental domain in $\tilde{M}$. Given $g \in \pi_1(M)$, let $\langle g \rangle$ denote its conjugacy class.

PROPOSITION 1.1. *If $g \neq e$,*

$$\int_0^\infty \int_{\mathscr{F}} \sum_{\gamma \in \langle g \rangle} e^{-t\tilde{\triangle}}(x\gamma, x)\, d\,\mathrm{vol}(x)\, \frac{dt}{t} = \frac{e^{-l}}{2k \sinh(l)}, \tag{1.1}$$

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where l is the minimal length of a closed curve on Σ in the free-homotopy class specified by $\langle g \rangle$, as measured with the hyperbolic metric on Σ, and k is the multiplicity of the corresponding geodesic.

The interest of (1.1) is that the left hand side seems to depend on the Hermitian metric on M, whereas the right hand side only depends on the complex structure Σ. Given M as a Riemannian manifold, in principle one can compute the left hand side of (1.1) for all $g \in \pi_1(M) - \{e\}$ and thereby find the hyperbolic marked length spectrum, which in turn determines the underlying complex structure. Proposition 1 is a special case of Proposition 10, concerning the delocalized holomorphic L^2-analytic torsion of a compact complex manifold which is covered by the unit ball.

More generally, we introduce and study certain analytically defined invariants, which we call the delocalized L^2-Betti number $b_{p,\langle g \rangle}$, the delocalized L^2-analytic torsion $\mathscr{T}_{\langle g \rangle}$, the delocalized holomorphic L^2-analytic torsion $\mathscr{T}^{\mathrm{hol}}_{\langle g \rangle}$ and the delocalized L^2-eta invariant $\eta_{\langle g \rangle}$. These are defined in terms of a closed manifold M, a normal Γ-cover $\hat{M}$ of M, and a nontrivial conjugacy class $\langle g \rangle$ of Γ. The word "delocalized" comes from the fact that the invariants are defined away from the trivial element of Γ. If one were to take the trivial conjugacy class in the definitions, one would recover the usual L^2-analytic invariants [1, 5, 17, 22].

The delocalized L^2-invariants are constructed using heat kernels on $\hat{M}$. There are technical difficulties in showing that the formal expressions for the invariants are actually well defined. These difficulties involve estimating heat kernels at large distance and large time simultaneously. Some of our results are to the effect that the invariants are actually well defined. The precise definitions will be given in Section 2. The main results of the paper are the following.

PROPOSITION 2. *Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class. Then $b_{p,\langle g \rangle}(M)$ is well-defined and metric-independent. If Γ is a free abelian group then $b_{p,\langle g \rangle}(M) = 0$.*

PROPOSITION 3. *Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class. Suppose that M has positive Novikov–Shubin invariants* [18, 28]. *Then $\mathscr{T}_{\langle g \rangle}(M)$ is well defined. If M has vanishing L^2-cohomology groups then $\mathscr{T}_{\langle g \rangle}(M)$ is metric-independent.*

PROPOSITION 4. *Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class and M is a complex manifold of complex dimension d, equipped with a Hermitian metric. Suppose that M has positive holomorphic Novikov–Shubin invariants. Then $\mathscr{T}^{\mathrm{hol}}_{\langle g \rangle}(M)$ is well defined. If M has vanishing qth Dolbeault*

L^2-cohomology groups for $0 \leqslant q < d$ then $\mathcal{T}^{\mathrm{hol}}_{\langle g \rangle}(M)$ is independent of the choice of Hermitian metric.

PROPOSITION 5. *Suppose that $\langle g \rangle$ is a nontrivial finite conjugacy class and D is a Dirac-type operator. Then $\eta_{\langle g \rangle}(M)$ is well defined. If D is the (tangential) signature operator then $\eta_{\langle g \rangle}(M)$ is metric-independent. If D is the Dirac operator then $\eta_{\langle g \rangle}(M)$ is a locally constant function on the space of positive-scalar-curvature metrics on M.*

PROPOSITION 6. *Suppose that Γ is virtually nilpotent or Gromov-hyperbolic. Suppose that $0 \notin \mathrm{spec}(\hat{\triangle}_p)$ or that 0 is isolated in $\mathrm{spec}(\hat{\triangle}_p)$. Then for all nontrivial conjugacy classes $\langle g \rangle$ of Γ, $b_{p,\langle g \rangle}(M)$ is well defined and metric-independent.*

PROPOSITION 7. *Suppose that Γ is virtually nilpotent or Gromov-hyperbolic. Let M be a complex manifold of complex dimension d, equipped with a Hermitian metric. Suppose that for all $0 \leqslant q < d$, $0 \notin \mathrm{spec}(\hat{\triangle}_{0,q})$, and that either $0 \notin \mathrm{spec}(\hat{\triangle}_{0,d})$ or 0 is isolated in $\mathrm{spec}(\hat{\triangle}_{0,d})$. Then for all nontrivial conjugacy classes $\langle g \rangle$ of Γ, $\mathcal{T}^{\mathrm{hol}}_{\langle g \rangle}(M)$ is well defined and independent of the choice of Hermitian metric.*

PROPOSITION 8. *Suppose that Γ is virtually nilpotent or Gromov-hyperbolic. Suppose that $0 \notin \mathrm{spec}(\hat{D})$ or that 0 is isolated in $\mathrm{spec}(\hat{D})$. Then for all nontrivial conjugacy classes $\langle g \rangle$ of Γ, $\eta_{\langle g \rangle}(M)$ is well defined. Furthermore, if D is the Dirac operator then $\eta_{\langle g \rangle}(M)$ is a locally constant function on the space of positive-scalar-curvature metrics on M.*

PROPOSITION 9. *Let M^d be a closed oriented hyperbolic manifold. Let $\langle g \rangle$ be a nontrivial conjugacy class in $\pi_1(M)$. Then $b_{p,\langle g \rangle}(M) = 0$ for all p.*

Suppose that $d = 2n+1$. Then $\mathcal{T}_{\langle g \rangle}(M)$ and $\eta_{\langle g \rangle}(M)$ are well defined, the latter being constructed with the (tangential) signature operator. Let c be the unique closed geodesic in the free homotopy class specified by $\langle g \rangle$. Let $k \in \mathbb{Z}^+$ be the multiplicity of c, meaning the number of times that c covers a prime closed geodesic. Let l be the hyperbolic length of c and let $m \in SO(2n)$ be the parallel transport of a normal vector around c. Let $\sigma_j(m) \in SO(\Lambda^j(\mathbb{R}^{2n}))$ be the action of m on the exterior power $\Lambda^j(\mathbb{R}^{2n})$. Then

$$\mathcal{T}_{\langle g \rangle}(M) = \frac{e^{-nl}}{k \det(I - e^{-l} m^{-1})} \sum_{j=0}^{2n} (-1)^j e^{-l|n-j|} \mathrm{Tr}(\sigma_j(m)). \tag{1.2}$$

In particular, the marked length spectrum of M can be recovered from $\{\mathcal{T}_{\langle g\rangle}(M)\}_{\langle g\rangle\in\mathscr{C}}$.

If n is even then $\eta_{\langle g\rangle}(M)=0$. *If n is odd, define angles* $\{\theta_j\}_{j=1}^n$ *by saying that the diagonalization of m consists of the* 2×2 *blocks*

$$\begin{pmatrix}\cos\theta_j & -\sin\theta_j\\ \sin\theta_j & \cos\theta_j\end{pmatrix}. \tag{1.3}$$

Put $\mu_j=e^{(l+i\theta_j)/2}$. *Then*

$$\eta_{\langle g\rangle}(M)=\frac{(2i)^{n+1}}{2\pi k}\,\frac{\sin(\theta_1)\ldots\sin(\theta_n)}{|\mu_1-\mu_1^{-1}|^2\ldots|\mu_n-\mu_n^{-1}|^2}. \tag{1.4}$$

If $n=1$ *then*

$$\mathcal{T}_{\langle g\rangle}(M)-i\pi\eta_{\langle g\rangle}(M)=\frac{2}{k}\,\frac{1}{1-\mu_1^2}. \tag{1.5}$$

Proposition 10. *Let* Σ *be a closed complex manifold of complex dimension d which is a quotient of the unit ball. Let M denote* Σ *equipped with an arbitrary Hermitian metric. Let* $\langle g\rangle$ *be a nontrivial conjugacy class in* $\pi_1(M)$. *Then* $\mathcal{T}^{\mathrm{hol}}_{\langle g\rangle}(M)$ *is well-defined and independent of the choice of Hermitian metric. To describe it, give* Σ *the complex-hyperbolic metric. Let c be the unique closed geodesic on* Σ *in the free homotopy class specified by* $\langle g\rangle$. *Let* $k\in\mathbb{Z}^+$ *be the multiplicity of c, let l be the length of c and let* $m\in U(d-1)$ *be the complex parallel transport around c. Let* $\sigma_q(m)\in U(\Lambda^{0,q}(\mathbb{C}^{d-1})^*)$ *be the action of m on the exterior power* $\Lambda^{0,q}(\mathbb{C}^{d-1})^*$. *Then*

$$\mathcal{T}_{\langle g\rangle}(M)=\frac{e^{-2dl}}{k(1-e^{-2l})\,|\det(I-e^{-l}m^{-1})|^2}\sum_{q=0}^{d-1}(-1)^q\,e^{ql}\,\mathrm{Tr}(\sigma_q(m)). \tag{1.6}$$

Proposition 11. *Let* Z^n *be a smooth closed even-dimensional manifold and let* ϕ *be a diffeomorphism of Z. Let* $\phi_p^*\in\mathrm{Aut}(\mathrm{H}^p(Z;\mathbb{C}))$ *be the induced map on cohomology. Let M be the mapping torus*

$$M=(Z\times[0,1])/\{(z,0)\sim(\phi(z),1)\}. \tag{1.7}$$

Put $\Gamma=\mathbb{Z}$, *acting on the cyclic cover* $\hat{M}$ *of M. Define* $f\colon\mathbb{C}\to\mathbb{C}$ *by*

$$f(\lambda)=\begin{cases}\lambda & \text{if } |\lambda|\leqslant 1,\\ \lambda^{-1} & \text{if } |\lambda|>1.\end{cases} \tag{1.8}$$

Then

$$\mathcal{T}_{\langle k \rangle}(M) = \begin{cases} \frac{1}{k} \sum_{p=0}^{d} (-1)^p \operatorname{Tr}[f(\phi_p^*)]^k & \text{if} \quad k > 0. \\ -\frac{1}{k} \sum_{p=0}^{d} (-1)^p \operatorname{Tr}[f(\overline{\phi_p^*})]^{-k} & \text{if} \quad k < 0. \end{cases} \tag{1.9}$$

Equivalently, let $L(\phi^k)$ be the Lefschetz number of ϕ^k. For $z \in \mathbb{C}$ with $|z|$ small, put $\zeta(z) = \exp(\sum_{k>0} (z^k/k) L(\phi^k))$. Then $\zeta(z)$ has a meromorphic continuation to $z \in \mathbb{C}$, and

$$\mathcal{T}_{\langle k \rangle}(M) = \int_{S^1} e^{-ik\theta} \ln |\zeta(e^{i\theta})|^2 \frac{d\theta}{2\pi}. \tag{1.10}$$

Suppose that ϕ preserves a Dirac-type operator D_Z on Z. Let D be the suspended Dirac-type operator on M. Then $\eta_{\langle k \rangle}(M)$ is given in terms of the Atiyah–Bott indices by

$$\eta_{\langle k \rangle}(M) = \frac{i}{k\pi} \operatorname{Tr}_s(\phi^k |_{\operatorname{Ker}(D_Z)}). \tag{1.11}$$

PROPOSITION 12. *Let Z^n be a smooth closed even-dimensional manifold. Let F be a finite group and let $\hat{Z}$ be a connected normal F-cover of Z. Let ϕ be a diffeomorphism of Z and let M be the mapping torus of ϕ. Let $\hat{\phi}$ be a lift of ϕ to $\hat{Z}$ and let $\alpha \in \operatorname{Aut}(F)$ be the automorphism defined by*

$$\hat{\phi}(zf) = \hat{\phi}(z)\, \alpha^{-1}(f) \tag{1.12}$$

for all $z \in \hat{Z}$ and $f \in F$. Put $\Gamma = F \tilde{\times}_\alpha \mathbb{Z}$, acting on $\hat{Z} \times \mathbb{R}$ on the right by

$$(z, t) \cdot (f, k) = (\hat{\phi}^k(zf), t + k). \tag{1.13}$$

For $k \in \mathbb{Z}$, define an equivalence relation $\sim_k$ on F by saying that $f \sim_k f'$ if there exists a $\gamma \in F$ such that $\gamma f \alpha^k(\gamma^{-1}) = f'$. Let $[f]_k$ be the equivalence class of $f \in F$ and let $|[f]_k|$ be its cardinality. Let $I_k(f) \in \mathbb{Z}$ be the Nielsen fixed point index of ϕ^k, evaluated at $[f]_k$. If ρ is a finite-dimensional irreducible unitary representation of $F \tilde{\times}_\alpha \mathbb{Z}$, let χ_ρ be its character. For $z \in \mathbb{C}$ with $|z|$ small, put

$$\zeta_\rho(z) = \exp\left(\sum_{\substack{f \in F \\ k>0}} \frac{z^k}{k} \chi_\rho(f, k) \frac{I_k(f)}{|[f]_k|} \right). \tag{1.14}$$

Then $\zeta_\rho(z)$ *has a meromorphic continuation to* $z \in \mathbb{C}$ *and*

$$\sum_{f,k} \chi_\rho(f,k)\, \mathcal{T}_{\langle f,k\rangle}(M) = \ln |\zeta_\rho(1)|^2. \tag{1.15}$$

Knowing (1.15) *for all* ρ *determines* $\{\mathcal{T}_{\langle f,k\rangle}(M)\}_{(f,k)\in\Gamma}$.

If the fundamental group is torsion-free then the delocalized L^2-Betti numbers vanish in all cases in which we can compute them. We do not know if this is always the case.

The calculations for the model cases in Propositions 9 and 10 are based on the work of Fried [9, 11] and Millson [23]. Anton Deitmar has also done the calculations for general nonpositively curved Hermitian symmetric spaces [8]. I thank him for telling me of his work.

The structure of the paper is as follows. In Section 2 we give the definitions of the invariants. In Sections 3–7 we prove the main results. In Section 8 we give examples to show that the results are not vacuous.

2. DEFINITIONS

Let M^d be a closed connected oriented Riemannian manifold. Let $\pi\colon \hat{M} \to M$ be a connected normal Γ-cover of M, equipped with the pullback Riemannian metric. We let $\gamma \in \Gamma$ act on $\hat{M}$ on the right by $R_\gamma \in \mathrm{Diff}(\hat{M})$. Let $\mathcal{C}$ denote the set of conjugacy classes of Γ.

DEFINITION 2.1. Let $\mathcal{A}$ be the convolution algebra of elements $a \in C^\infty(\hat{M} \times \hat{M})$ satisfying

1. $a(m\gamma, m'\gamma) = a(m, m')$ for all $\gamma \in \Gamma$ and
2. There exists an $R_a > 0$ such that if $d(m, m') \geqslant R_a$ then $a(m, m') = 0$.

The multiplication in $\mathcal{A}$ is given by

$$(ab)(m, m') = \int_{\hat{M}} a(m, m'')\, b(m'', m')\, d\,\mathrm{vol}(m''). \tag{2.1}$$

Given $a \in \mathcal{A}$ and $\langle g\rangle \in \mathcal{C}$, define $A_{\langle g\rangle} \in C^\infty(\hat{M})$ by

$$A_{\langle g\rangle}(m) = \sum_{\gamma \in \langle g\rangle} a(m\gamma, m). \tag{2.2}$$

LEMMA 1. *For all* $\gamma' \in \Gamma$,

$$A_{\langle g\rangle}(m\gamma') = A_{\langle g\rangle}(m). \tag{2.3}$$

Proof. We have

$$A_{\langle g\rangle}(m\gamma') = \sum_{\gamma\in\langle g\rangle} a(m\gamma'\gamma, m\gamma') = \sum_{\gamma\in\langle g\rangle} a(m\gamma'\gamma(\gamma')^{-1}, m) = A_{\langle g\rangle}(m). \tag{2.4}$$

Thus $A_{\langle g\rangle} = \pi^*\alpha_{\langle g\rangle}$ for a unique $\alpha_{\langle g\rangle} \in C^\infty(M)$. Q.E.D.

DEFINITION 2. Define $\mathrm{Tr}_{\langle g\rangle}: \mathscr{A} \to \mathbb{C}$ by

$$\mathrm{Tr}_{\langle g\rangle}(a) = \int_M \alpha_{\langle g\rangle} \; d\,\mathrm{vol}. \tag{2.5}$$

LEMMA 2.4. *For all $a, b \in \mathscr{A}$,*

$$\mathrm{Tr}_{\langle g\rangle}(ab) = \mathrm{Tr}_{\langle g\rangle}(ba). \tag{2.6}$$

Proof. Let $\mathscr{F}$ be a fundamental domain for the Γ-action on $\hat{M}$. Formally,

$$\begin{aligned}
\mathrm{Tr}_{\langle g\rangle}(ab) &= \int_{\mathscr{F}} \sum_{\gamma\in\langle g\rangle} (ab)(m\gamma, m)\, d\,\mathrm{vol}(m) \\
&= \int_{\mathscr{F}} \sum_{\gamma\in\langle g\rangle} \int_{\hat{M}} a(m\gamma, m')\, b(m', m)\, d\,\mathrm{vol}(m')\, d\,\mathrm{vol}(m) \\
&= \int_{\mathscr{F}} \sum_{\gamma\in\langle g\rangle} \int_{\mathscr{F}} \sum_{\gamma'\in\Gamma} a(m\gamma, m'\gamma')\, b(m'\gamma', m)\, d\,\mathrm{vol}(m')\, d\,\mathrm{vol}(m) \\
&= \sum_{\gamma\in\langle g\rangle} \sum_{\gamma'\in\Gamma} \int_{\mathscr{F}} \int_{\mathscr{F}} b(m'\gamma', m)\, a(m\gamma, m'\gamma')\, d\,\mathrm{vol}(m')\, d\,\mathrm{vol}(m) \\
&= \sum_{\gamma\in\langle g\rangle} \sum_{\gamma'\in\Gamma} \int_{\mathscr{F}} \int_{\mathscr{F}} b(m'\gamma'\gamma(\gamma')^{-1}, m\gamma(\gamma')^{-1}) \\
&\qquad \times a(m\gamma(\gamma')^{-1}, m')\, d\,\mathrm{vol}(m')\, d\,\mathrm{vol}(m) \\
&= \sum_{\gamma\in\langle g\rangle} \sum_{\gamma''\in\Gamma} \int_{\mathscr{F}} \int_{\mathscr{F}} b(m'(\gamma'')^{-1}\gamma\gamma'', m\gamma'') \\
&\qquad \times a(m\gamma'', m')\, d\,\mathrm{vol}(m')\, d\,\mathrm{vol}(m) \\
&= \sum_{\gamma\in\langle g\rangle} \sum_{\gamma''\in\Gamma} \int_{\mathscr{F}} \int_{\mathscr{F}} b(m'\gamma, m\gamma'')\, a(m\gamma'', m')\, d\,\mathrm{vol}(m)\, d\,\mathrm{vol}(m') \\
&= \sum_{\gamma\in\langle g\rangle} \int_{\mathscr{F}} \int_{\hat{M}} b(m'\gamma, m)\, a(m, m')\, d\,\mathrm{vol}(m)\, d\,\mathrm{vol}(m') \\
&= \mathrm{Tr}_{\langle g\rangle}(ba).
\end{aligned} \tag{2.7}$$

It is not hard to justify the steps in (2.7). Q.E.D.

We will need two slight extensions of the algebra $\mathscr{A}$. First, let E be a Hermitian vector bundle on M. Put $\hat{E} = \pi^* E$. For $i \in \{1, 2\}$, let π_i be projection from $\hat{M} \times \hat{M}$ onto the ith factor of $\hat{M}$. Let $\mathscr{A}$ be the convolution algebra of elements $a \in C^\infty(\hat{M} \times \hat{M}; \pi_1^* \hat{E} \otimes \pi_2^* \hat{E}^*)$ satisfying the two conditions of Definition 2.1. Equation (2.2) now has to be interpreted as

$$A_{\langle g \rangle}(m) = \sum_{\gamma \in \langle g \rangle} \operatorname{tr}((R_\gamma {}^* a)(m, m)) = \sum_{\gamma \in \langle g \rangle} \operatorname{tr}(a(m\gamma, m)). \tag{2.8}$$

Then the proof of Lemma 2.4 can be extended. Next, we can replace condition 2 of Definition 2.1 by the weaker assumption that

$$\text{for all } R > 0, \quad \sup_{d(m, m') \leqslant R} \sum_\gamma |a(m\gamma, m')| < \infty. \tag{2.9}$$

Equation (2.9) is essentially an l^1-condition on a with respect to Γ. Then we again have a convolution algebra and the proof of Lemma 2.4 still goes through.

Let $\hat{\triangle}_p$ be the p-form Laplacian on $\hat{M}$. For $t > 0$, let $e^{-t\hat{\triangle}_p}$ be the corresponding heat operator. It has a Schwartz kernel $e^{-t\hat{\triangle}_p}(m, m') \in \Lambda^p(T_m^* \hat{M}) \otimes (\Lambda^p(T_{m'}^* \hat{M}))^*$. By finite propagation speed estimates [6], $e^{-t\hat{\triangle}_p}(m, m')$ satisfies (2.9).

Definition 3. Take $g \neq e$. When the limit exists, we define the pth delocalized L^2-Betti number of M by

$$b_{p, \langle g \rangle}(M) = \lim_{t \to \infty} \operatorname{Tr}_{\langle g \rangle}(e^{-t\hat{\triangle}_p}) \tag{2.10}$$

If we were to take $g = e$ then $b_{p, \langle e \rangle}(M)$ would be the same as the pth L^2-Betti number of M [1].

Let $\{ds^2(u)\}_{u \in [-1, 1]}$ be a smooth 1-parameter family of Riemannian metrics on M. Let $\{*(u)\}_{u \in [-1, 1]}$ be the corresponding 1-parameter family of Hodge duality operators. Then

$$\begin{aligned} \frac{d}{du} \operatorname{Tr}_{\langle g \rangle}(e^{-t\hat{\triangle}_p}) &= -t \operatorname{Tr}_{\langle g \rangle}\left(e^{-t\hat{\triangle}_p} \frac{d\hat{\triangle}_p}{du}\right) \\ &= -t \operatorname{Tr}_{\langle g \rangle}\left(e^{-t\hat{\triangle}_p} \left(\left(d\left[d^*, *^{-1} \frac{d*}{du}\right] + \left[d^*, *^{-1} \frac{d*}{du}\right] d\right)\right)\right. \\ &= -2t \operatorname{Tr}_{\langle g \rangle}\left(e^{-t\hat{\triangle}_p}(dd^* - d^*d) *^{-1} \frac{d*}{du}\right). \end{aligned} \tag{2.11}$$

Let $\Pi_{\mathrm{Ker}(\hat{\triangle}_p)}$ be projection onto the L^2-kernel of $\hat{\triangle}_p$. As $\lim_{t\to\infty} e^{-t\hat{\triangle}_p} = \Pi_{\mathrm{Ker}(\hat{\triangle}_p)}$ and

$$\Pi_{\mathrm{Ker}(\hat{\triangle}_p)} dd^* = \Pi_{\mathrm{Ker}(\hat{\triangle}_p)} d^*d = 0, \tag{2.12}$$

we expect that

$$\lim_{t\to\infty} -2t\,\mathrm{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_p}(dd^* - d^*d) *^{-1}\frac{d*}{du}\right) = 0. \tag{2.13}$$

In summary, we have shown the following result.

PROPOSITION 13. *If* (2.13) *can be justified, uniformly in* u, *then* $b_{p,\langle g\rangle}(M)$ *is metric-independent and hence a* (*smooth*) *topological invariant of* M.

For the moment, let us assume that $b_{p,\langle g\rangle}(M)$ is metric-independent. For $t>0$, put

$$\mathcal{T}_{\langle g\rangle}(t) = \sum_{p=0}^{d} (-1)^p\ p\,\mathrm{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_p}). \tag{2.14}$$

Put

$$\mathcal{T}_{\langle g\rangle}(\infty) = \sum_{p=0}^{d} (-1)^p\, p b_{p,\langle g\rangle}(M). \tag{2.15}$$

DEFINITION 4. Take $g\neq e$. When the integral makes sense, we define the delocalized L^2-analytic torsion by

$$\mathcal{T}_{\langle g\rangle}(M) = -\int_0^\infty \left(\mathcal{T}_{\langle g\rangle}(t) - (1-e^{-t})\,\mathcal{T}_{\langle g\rangle}(\infty)\right)\frac{dt}{t}. \tag{2.16}$$

If we were to take $g=e$ then $T_{\langle g\rangle}(M)$ would formally be the same, up to a sign, as the L^2-analytic torsion of [17, 22]. (The latter requires a zeta-function regularization in its definition, but this is not necessary for the delocalized L^2-analytic torsion.) It follows from finite propagation speed arguments that the integrand in (2.16) is integrable for small t. Thus the question of whether the integral makes sense refers to large-t integrability.

Let $\{ds^2(u)\}_{u\in[-1,1]}$ be a smooth 1-parameter family of Riemannian metrics on M. Then for any $t>0$, as in the proof of [17, Lemma 8], we have

$$\frac{d}{du}\mathcal{T}_{\langle g\rangle}(t) = t\,\frac{d}{dt}\sum_{p=0}^{d}(-1)^p\,\mathrm{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_p} *^{-1}\frac{d*}{du}\right). \tag{2.17}$$

Proceeding formally,

$$\frac{d}{du}\mathscr{T}_{\langle g\rangle}(M) = (\lim_{t\to 0} - \lim_{t\to\infty}) \sum_{p=0}^{d} (-1)^p \operatorname{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_p} *^{-1} \frac{d*}{du}\right). \quad (2.18)$$

As we are assuming that $g \neq e$, it follows again from finite propagation speed estimates that

$$\lim_{t\to 0} \sum_{p=0}^{d} (-1)^p \operatorname{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_p} *^{-1} \frac{d*}{du}\right) = 0. \quad (2.19)$$

When it can be justified, we expect that

$$\lim_{t\to\infty} \sum_{p=0}^{d} (-1)^p \operatorname{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_p} *^{-1} \frac{d*}{du}\right) = \sum_{p=0}^{d} (-1)^p \operatorname{Tr}_{\langle g\rangle}\left(\Pi_{\operatorname{Ker}(\hat{\triangle}_p)} *^{-1} \frac{d*}{du}\right). \quad (2.20)$$

Now $\operatorname{Ker}(\hat{\triangle}_p)$ can be identified with the p-dimensional (reduced) L^2-cohomology group of M and is topological in nature [1]. In summary, we have shown the following result.

PROPOSITION 14. *If* (2.20) *can be justified, uniformly in u, and if M has vanishing L^2-cohomology then for all $g \neq e$, $\mathscr{T}_{\langle g\rangle}(M)$ is a (smooth) topological invariant of M.*

Suppose that M is a closed Hermitian manifold of complex dimension d. Let $\hat{\triangle}_{0,q}$ be the $(0,q)$-form Laplacian on $\hat{M}$, as in [32, Section 1]. For $t > 0$, let $e^{-t\hat{\triangle}_{0,q}}$ be the corresponding heat operator.

DEFINITION 5. Take $g \neq e$. When the limit exists, we define the qth delocalized holomorphic L^2-Betti number of M by

$$b^{\mathrm{hol}}_{q,\langle g\rangle}(M) = \lim_{t\to\infty} \operatorname{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_{0,q}}). \quad (2.21)$$

Let $\{ds^2(u)\}_{u\in[-1,1]}$ be a smooth 1-parameter family of Hermitian metrics on the complex manifold M. Let $\{*(u)\}_{u\in[-1,1]}$ be the corresponding 1-parameter family of duality operators. There are holomorphic analogs of equations (2.11)–(2.13) and Proposition 13.

For the moment, let us assume that $b^{\mathrm{hol}}_{q,\langle g\rangle}(M)$ is independent of the choice of Hermitian metric. For $t > 0$, put

$$\mathscr{T}^{\mathrm{hol}}_{\langle g\rangle}(t) = \sum_{q=0}^{d} (-1)^q\, q \operatorname{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_{0,q}}). \quad (2.22)$$

Put

$$\mathcal{T}^{\mathrm{hol}}_{\langle g\rangle}(\infty) = \sum_{q=0}^{d} (-1)^q \, q b^{\mathrm{hol}}_{q,\langle g\rangle}(M). \tag{2.23}$$

DEFINITION 6. Take $g \neq e$. When the integral makes sense, we define the delocalized holomorphic L^2-analytic torsion by

$$\mathcal{T}^{\mathrm{hol}}_{\langle g\rangle}(M) = -\int_0^\infty \left(\mathcal{T}^{\mathrm{hol}}_{\langle g\rangle}(t) - (1-e^{-t})\,\mathcal{T}^{\mathrm{hol}}_{\langle g\rangle}(\infty)\right) \frac{dt}{t}. \tag{2.24}$$

It follows from finite propagation speed arguments that the integrand in (2.24) is integrable for small t. Thus the question of whether the integral makes sense refers to large-t integrability.

As in equation (2.20), we expect that

$$\lim_{t\to\infty} \sum_{q=0}^{d} (-1)^q \operatorname{Tr}_{\langle g\rangle}\left(e^{-t\hat{\triangle}_{0,q}} *^{-1} \frac{d*}{du}\right) = \sum_{q=0}^{d} (-1)^q \operatorname{Tr}_{\langle g\rangle}\left(\Pi_{\operatorname{Ker}(\hat{\triangle}_{0,q})} *^{-1} \frac{d*}{du}\right). \tag{2.25}$$

Now $*(u)$, acting on $\Omega^{0,d}(\hat{M})$, is independent of u. Furthermore, $\operatorname{Ker}(\hat{\triangle}_{0,q})$ can be identified with the qth Dolbeault L^2-cohomology group of M and is independent of the choice of Hermitian metric. Then as in the discussion leading up to Proposition 14, we have

PROPOSITION 15. *If (2.25) can be justified, uniformly in u, and if M has vanishing qth Dolbeault L^2-cohomology group for $0 \leqslant q < d$, then for all $g \neq e$, $\mathcal{T}_{\langle g\rangle}(M)$ is independent of the choice of Hermitian metric on the complex manifold M.*

Now let M^d again be a smooth Riemannian manifold. Let $\hat{E}$ be a Γ-invariant Clifford module on $\hat{M}$. For simplicity, we assume that M is spin, with S denoting the spinor bundle, and that there is a Hermitian vector bundle V on M so that $\hat{E} = \pi^*(S \otimes V)$; the general case is similar. Let ∇^S be the connection on S coming from the Levi–Civita connection and let ∇^V be a Hermitian connection on V. Let D be the corresponding self-adjoint Dirac-type operator on sections of $S \otimes V$ and let $\hat{D}$ be the lifted operator on sections of $\hat{E}$. For $t > 0$, let $e^{-t\hat{D}^2}$ be the corresponding heat operator. It has a Schwartz kernel $e^{-t\hat{D}^2}(m, m') \in \hat{E}_m \otimes \hat{E}^*_{m'}$. Again, using finite propagation speed estimates it is not hard to see that $e^{-t\hat{D}^2}$ satisfies (2.9). Given a conjugacy class $\langle g\rangle$ in Γ, for $s > 0$ put

$$\eta_{\langle g\rangle}(s) = \operatorname{Tr}_{\langle g\rangle}(\hat{D} e^{-s^2\hat{D}^2}). \tag{2.26}$$

DEFINITION 7. Take $g \neq e$. When the integral makes sense, we define the delocalized L^2-eta invariant by

$$\eta_{\langle g \rangle}(M) = \frac{2}{\sqrt{\pi}} \int_0^\infty \eta_{\langle g \rangle}(s)\, ds. \tag{2.27}$$

If we were to take $g = e$ then $\eta_{\langle g \rangle}(M)$ would be the same as the L^2-eta invariant of Cheeger and Gromov [5]. In this case, it is known that the integral in (2.27) makes sense [3, 29, 30]. If $g \neq e$, then finite propagation speed arguments show that the integrand in (2.27) is integrable for small-s. Thus the question of whether the integral makes sense refers to large-s integrability. Equation (2.27) was first considered, when Γ is virtually nilpotent, in [16, Eq. (69)].

Let $\{ds^2(u)\}_{u \in [-1,1]}$, $\{h^V(u)\}_{u \in [-1,1]}$, and $\{\nabla^V(u)\}_{u \in [-1,1]}$ be smooth 1-parameter families of Riemannian metrics on M, Hermitian metrics on V, and compatible Hermitian connections on V, respectively. Then for any $s > 0$, one can check that

$$\frac{d}{du} \eta_{\langle g \rangle}(s) = \frac{d}{ds} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right). \tag{2.28}$$

Proceeding formally,

$$\frac{d}{du} \eta_{\langle g \rangle}(M) = \frac{2}{\sqrt{\pi}} \left(\lim_{s \to \infty} - \lim_{s \to 0} \right) \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right). \tag{2.29}$$

As we are assuming that $g \neq e$, it follows from finite propagation speed estimates that

$$\frac{2}{\sqrt{\pi}} \lim_{s \to 0} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 0. \tag{2.30}$$

In summary, we have shown the following result.

PROPOSITION 16. *If*

$$\frac{2}{\sqrt{\pi}} \lim_{s \to \infty} \mathrm{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 0 \tag{2.31}$$

uniformly in u then $\eta_{\langle g \rangle}(M)$ is independent of u.

Proceeding very formally,

$$\frac{2}{\sqrt{\pi}} \lim_{s \to \infty} \operatorname{Tr}_{\langle g \rangle} \left(s \frac{d\hat{D}}{du} e^{-s^2 \hat{D}^2} \right) = 2 \operatorname{Tr}_{\langle g \rangle} \left(\frac{d\hat{D}}{du} \delta(\hat{D}) \right)$$

$$= 2 \frac{d}{du} \operatorname{Tr}_{\langle g \rangle} (\operatorname{sign}(\hat{D})). \tag{2.32}$$

Thus we expect that if $\operatorname{sign}(\hat{D})$ is independent of u, as a Γ-operator, then $\eta_{\langle g \rangle}(M)$ is independent of u. In particular, we expect that this will be true in the following cases:

1. If D is the Dirac operator and for all $u \in [-1, 1]$, $(M, ds^2(u))$ has positive scalar curvature.
2. If D is the (tangential) signature operator of [2].

We now give some elementary properties of the delocalized L^2-invariants.

Proposition 17. *Suppose that Γ is finite. Let $\{\langle g_i \rangle\}$ parametrize the conjugacy classes of Γ. Let $\rho \colon \Gamma \to U(N)$ be a unitary representation of Γ. Let E_ρ be the associated flat Hermitian vector bundle on M. Let $\chi_\rho \colon \Gamma \to \mathbb{C}$ be the character of ρ.*

1. *Then*

$$b_p(M; E_\rho) = \sum_i \chi_\rho(g_i)\, b_{p, \langle g_i \rangle}(M). \tag{2.33}$$

2. *Let $\mathscr{T}(M; E_\rho) \in \mathbb{R}$ be the Ray–Singer analytic torsion* [31]. *Then*

$$\mathscr{T}(M; E_\rho) = \sum_i \chi_\rho(g_i)\, \mathscr{T}_{\langle g_i \rangle}(M). \tag{2.34}$$

3. *Let $\mathscr{T}^{\mathrm{hol}}(M; E_\rho) \in \mathbb{R}$ be the Ray–Singer holomorphic torsion* [32]. *Then*

$$\mathscr{T}^{\mathrm{hol}}(M; E_\rho) = \sum_i \chi_\rho(g_i)\, \mathscr{T}^{\mathrm{hol}}_{\langle g_i \rangle}(M). \tag{2.35}$$

4. *Let D be a Dirac-type operator on M. Let $\eta(M; E_\rho) \in \mathbb{R}$ be the Atiyah–Patodi–Singer η-invariant* [2]. *Then*

$$\eta(M; E_\rho) = \sum_i \chi_\rho(g_i)\, \eta_{\langle g_i \rangle}(M). \tag{2.36}$$

Proof. This follows from Fourier analysis on Γ, as in [20, Sect. 2]. We omit the details. ∎

Proposition 17.4 shows that when Γ is a finite group, the delocalized L^2-eta invariant has the same information as the ρ-invariant of [2].

PROPOSITION 18.

1. *We have* $b_{p,\langle g^{-1}\rangle}(M) = \overline{b_{p,\langle g\rangle}(M)}$, $\mathcal{T}_{\langle g^{-1}\rangle}(M) = \overline{\mathcal{T}_{\langle g\rangle}(M)}$, $\mathcal{T}^{\text{hol}}_{\langle g^{-1}\rangle}(M) = \overline{\mathcal{T}^{\text{hol}}_{\langle g\rangle}(M)}$, *and* $\eta_{\langle g^{-1}\rangle}(M) = \overline{\eta_{\langle g\rangle}(M)}$.
2. *Given pairs* (M_1, Γ_1) *and* (M_2, Γ_2), *we have*

$$\mathcal{T}_{\langle g_1, g_2\rangle}(M_1 \times M_2) = \delta_{g_1, e_1}\chi(M_1)\,\mathcal{T}_{\langle g_2\rangle}(M_2) + \delta_{g_2, e_2}\chi(M_2)\,\mathcal{T}_{\langle g_1\rangle}(M_1), \tag{2.37}$$

$$\mathcal{T}^{\text{hol}}_{\langle g_1, g_2\rangle}(M_1 \times M_2) = \delta_{g_1, e_1}\chi^{\text{hol}}(M_1)\,\mathcal{T}^{\text{hol}}_{\langle g_2\rangle}(M_2) + \delta_{g_2, e_2}\chi^{\text{hol}}(M_2)\,\mathcal{T}^{\text{hol}}_{\langle g_1\rangle}(M_1) \tag{2.38}$$

and

$$\eta_{\langle g_1, g_2\rangle}(M_1 \times M_2) = \delta_{g_1, e_1}\left(\int_{M_1} \hat{A}(TM_1) \cup \operatorname{ch}(E_1)\right)\eta_{\langle g_2\rangle}(M_2) + \delta_{g_2, e_2}\left(\int_{M_2} \hat{A}(TM_2) \cup \operatorname{ch}(E_2)\right)\eta_{\langle g_1\rangle}(M_1). \tag{2.39}$$

3. *If d is even then* $\mathcal{T}_{\langle g\rangle}(M) = 0$.
4. *Suppose that d is odd and D is the* (*tangential*) *signature operator* [2]. *Then* $\eta_{\langle g\rangle}(M) = 0$ *if* $d \equiv 1 \bmod 4$.

Proof. As $e^{-t\triangle_p}$ is self-adjoint and Γ-invariant,

$$\begin{aligned}\operatorname{tr}(e^{-t\hat{\triangle}_p}(m\gamma^{-1}, m)) &= \operatorname{tr}(e^{-t\hat{\triangle}_p}(m, m\gamma)) = \operatorname{tr}(e^{-t\hat{\triangle}_p}(m\gamma, m))^* \\ &= \overline{\operatorname{tr}(e^{-t\hat{\triangle}_p}(m\gamma, m))}.\end{aligned} \tag{2.40}$$

It follows that $b_{p,\langle g^{-1}\rangle}(M) = \overline{b_{p,\langle g\rangle}(M)}$. The proof of the rest of 1 is similar. The proofs of 2, 3, and 4 follow from arguments as in [2] and [17]. We omit the details. ∎

Before proceeding with the proofs of the main results, let us mention why the existence problem for the delocalized L^2-invariants is more difficult than for the ordinary L^2-invariants. The algebraic origin of the problem is as follows. Let Γ be a countable discrete group and consider the group algebra $\mathbb{C}\Gamma$ of finite sums $\sum_{\gamma\in\Gamma} c_\gamma \gamma$, with $c_\gamma \in \mathbb{C}$. Define an involution on $\mathbb{C}\Gamma$ by

$$\left(\sum_{\gamma\in\Gamma} c_\gamma \gamma\right)^* = \sum_{\gamma\in\Gamma} \overline{c_\gamma}\,\gamma^{-1}. \tag{2.41}$$

If $\langle g \rangle$ is a conjugacy class in Γ, the linear functional $\tau_{\langle g \rangle} : \mathbb{C}\Gamma \to \mathbb{C}$ given by

$$\tau_{\langle g \rangle}\left(\sum_{\gamma \in \Gamma} c_\gamma \gamma\right) = \sum_{\gamma \in \langle g \rangle} c_\gamma \tag{2.42}$$

satisfies $\tau_{\langle g \rangle}(ab) = \tau_{\langle g \rangle}(ba)$ for all $a, b \in \mathbb{C}\Gamma$. Furthermore, if $g = e$ then $\tau_{\langle e \rangle}(a^*a) \geqslant 0$ and $\tau_{\langle e \rangle}$ extends to a continuous linear functional on the group von Neumann algebra. These last two properties of $\tau_{\langle e \rangle}$, which are crucial for the usual L^2-invariants, generally fail for $\tau_{\langle g \rangle}$ if $g \neq e$.

3. PROOFS OF PROPOSITIONS 2, 3, 4 AND 5

Let $\langle g \rangle$ be a finite conjugacy class in Γ. Put

$$A = \sum_{\gamma \in \langle g \rangle} R_\gamma^* . \tag{3.1}$$

Then A is a bounded operator on $\Omega^*(\hat{M})$ which commutes with $R_{\gamma'}^*$ for all $\gamma' \in \Gamma$. Letting Tr_Γ denote the II_∞-trace [1], we have

$$\mathrm{Tr}_{\langle g \rangle}(e^{-t\hat{\triangle}_p}) = \mathrm{Tr}_\Gamma(A e^{-t\hat{\triangle}_p}). \tag{3.2}$$

Thus

$$b_{p, \langle g \rangle}(M) = \mathrm{Tr}_\Gamma(A \Pi_{\mathrm{Ker}(\hat{\triangle}_p)}). \tag{3.3}$$

Hence $b_{p, \langle g \rangle}(M)$ is well defined. It follows by standard arguments (see, for example, [5, pp. 24–27]) that it is metric-independent.

Similarly,

$$\mathscr{T}_{\langle g \rangle}(t) = \sum_{p=0} (-1)^p \, p \, \mathrm{Tr}_\Gamma(A e^{-t\hat{\triangle}_p}). \tag{3.4}$$

Then the integrand of (2.16) is

$$\frac{1}{t} \sum_{p=0} (-1)^p \, p \, \mathrm{Tr}_\Gamma[A(e^{-t\hat{\triangle}_p} - (1 - e^{-t}) \, \Pi_{\mathrm{Ker}(\hat{\triangle}_p)})]. \tag{3.5}$$

Now

$$|\mathrm{Tr}_\Gamma[A(e^{-t\hat{\triangle}_p} - \Pi_{\mathrm{Ker}(\hat{\triangle}_p)})]| \leqslant \|A\| \; \mathrm{Tr}_\Gamma[(e^{-t\hat{\triangle}_p} - \Pi_{\mathrm{Ker}(\hat{\triangle}_p)})]. \tag{3.6}$$

By assumption, there is an $\alpha_p > 0$ such that for large t,

$$\mathrm{Tr}_\Gamma[e^{-t\hat{\triangle}_p} - \Pi_{\mathrm{Ker}(\hat{\triangle}_p)}] \leqslant t^{-\alpha_p/2}. \tag{3.7}$$

It follows that the integrand in (2.16) is integrable. If M has vanishing L^2-cohomology groups then it follows as in [17] that $\mathscr{T}_{\langle g\rangle}(M)$ is metric-independent. This proves Proposition 3. The proof of Proposition 4 is similar.

Now let D be a Dirac-type operator. Then we have

$$\eta_{\langle g\rangle}(s) = \mathrm{Tr}_\Gamma(A\hat{D}e^{-s^2\hat{D}^2}). \tag{3.8}$$

Thus

$$|\eta_{\langle g\rangle}(s)| \leqslant \|A\| \ \mathrm{Tr}_\Gamma(|\hat{D}| \ e^{-s^2\hat{D}^2}). \tag{3.9}$$

It follows as in [30, Theorem 3.1.1] that the integrand in (2.27) is integrable. The rest of Proposition 5 follows as in [5].

Finally, to finish the proof of Proposition 2, suppose that $\Gamma = \mathbb{Z}^l$. Let $\hat{\Gamma}$ be the Pontryagin dual of Γ. Then an element ρ_θ of $\hat{\Gamma}$ is a representation $\rho_\theta : \Gamma \to U(1)$ of the form

$$\rho_\theta(\mathbf{k}) = e^{i\mathbf{k}\cdot\boldsymbol{\theta}}. \tag{3.10}$$

Let E_θ be the associated flat unitary line bundle on M and let $\triangle_{p,\theta}$ be the Laplacian on $\Omega^p(M; E_\theta)$. It follows as in [17, Proposition 38] that

$$\mathrm{Tr}_{\langle\mathbf{m}\rangle}(e^{-t\hat{\triangle}_p}) = \int_{\hat{\Gamma}} e^{-i\mathbf{m}\cdot\boldsymbol{\theta}} \, \mathrm{Tr}(e^{-t\triangle_{p,\theta}}) \frac{d^l\theta}{(2\pi)^l}. \tag{3.11}$$

From [33, Chap. XII], the eigenvalues $\lambda_i(\theta)$ form a sequence of non-negative algebraic functions locally on $\hat{\Gamma}$. (This corrects a claim in [17] that they are local analytic functions, which is only guaranteed if $l = 1$.) It follows that there is convergence in $L^1(\hat{\Gamma})$,

$$\lim_{t\to\infty} \mathrm{Tr}(e^{-t\triangle_{p,\theta}}) = b_p^{(2)}(M), \tag{3.12}$$

where $b_p^{(2)}(M)$ is the number of such algebraic functions which equal the zero function. Hence if $\mathbf{m} \neq 0$ then $b_{p,\langle\mathbf{m}\rangle}(M) = 0$. ∎

4. PROOFS OF PROPOSITIONS 6, 7 AND 8

To prove Proposition 6, let Λ be the reduced group C^*-algebra of Γ. We assume that there is an algebra $\mathfrak{B}$ such that

1. $\mathbb{C}\Gamma \subseteq \mathfrak{B} \subseteq \Lambda$.
2. $\mathfrak{B}$ is the projective limit of a sequence

$$\cdots \to B_{j+1} \to B_j \to \cdots \to B_0 \tag{4.1}$$

of Banach algebras $(B_j, |\cdot|_j)$ with unit, and $B_0 = \Lambda$.

3. If $i_j\colon \mathfrak{B} \to B_j$ is the induced homomorphism then i_0 is injective with dense image and $\mathfrak{B}$ is closed under the holomorphic functional calculus in Λ.

4. Given a conjugacy class $\langle g \rangle$ of Γ, define $\tau_{\langle g \rangle}\colon \mathbb{C}\Gamma \to \mathbb{C}$ as in (2.42). Then for all $\langle g \rangle \in \mathscr{C}$, $\tau_{\langle g \rangle}$ extends to a continuous linear functional on $\mathfrak{B}$, which we again denote by $\tau_{\langle g \rangle}$.

The topology on $\mathfrak{B}$ comes from the submultiplicative seminorms $\|\cdot\|_j = |i_j(\cdot)|_j$. Condition 4 is equivalent to saying that $\mathrm{HC}^0(\mathbb{C}\Gamma) = \mathrm{HC}^0(\mathfrak{B})$. If Γ is a virtually nilpotent or Gromov-hyperbolic group then conditions 1–4 are known to be satisfied by the rapid-decay algebra $\mathfrak{B}$ [14, p. 397].

If $\mathfrak{E}$ is a finitely generated right projective $\mathfrak{B}$-module then there is a continuous trace

$$\mathrm{Tr}\colon \mathrm{End}_{\mathfrak{B}}(\mathfrak{E}) \to \mathfrak{B}/\overline{[\mathfrak{B}, \mathfrak{B}]}. \tag{4.2}$$

Explicitly, suppose that $\mathfrak{E} = \mathfrak{B}^N e$ for some $N > 0$ and some projection $e \in M_N(\mathfrak{B})$. If $A \in \mathrm{End}_{\mathfrak{B}}(\mathfrak{E})$, we can think of A as an element of $M_N(\mathfrak{B})$ satisfying $A = eA = Ae$. Then

$$\mathrm{Tr}(A) = \sum_{i=1}^{N} A_{ii} \bmod \overline{[\mathfrak{B}, \mathfrak{B}]}. \tag{4.3}$$

(We quotient by the closure of $[\mathfrak{B}, \mathfrak{B}]$ to ensure that the trace takes value in a Fréchet space.)

As Λ is a C^*-algebra, there is a calculus of Λ-pseudodifferential operators on M [25]. Suppose that E^1 is a smooth Λ-vector bundle on M, meaning the fibers of E^1 are all isomorphic to a fixed finitely generated projective right Λ-module $\mathfrak{E}^1$ and the transition functions are smooth functions with value in $\mathrm{Aut}_\Lambda(\mathfrak{E}^1)$. Let E^2 be another smooth Λ-vector bundle on M. The elements of the pseudodifferential algebra $\Psi^\infty_\Lambda(M; E^1, E^2)$ map smooth sections of E^1 to smooth sections of E^2 and commute with the Λ-action.

In [19, Sect. 6.1] we extended this to a calculus of $\mathfrak{B}$-pseudodifferential operators and proved some basic properties of such operators. We only state the necessary facts, referring to [19] for details.

Let $\mathscr{E}^1$ and $\mathscr{E}^2$ be smooth $\mathfrak{B}$-vector bundles on M. By an extension of the Serre–Swan theorem, we can write $\mathscr{E}^1 = (M \times \mathfrak{B}^N)\, e^1$ for some $N > 0$ and some projection $e^1 \in C^\infty(M; M_N(\mathfrak{B}))$. Define a B_j-vector bundle by $E_j^1 = (M \times B_j^N)\, i_j(e^1)$. Then $\mathscr{E}^1$ is the projective limit of

$$\cdots \to E_{j+1}^1 \to E_j^1 \to \cdots \to E_0^1, \tag{4.4}$$

and similarly for $\mathscr{E}^2$.

For each $j \geqslant 0$, there is an algebra $\Psi_{B_j}^\infty(M; E_j^1, E_j^2)$ of B_j-pseudodifferential operators. The algebra $\Psi_{\mathfrak{B}}^\infty(M; \mathscr{E}^1, \mathscr{E}^1)$ of $\mathfrak{B}$-pseudodifferential operators is the projective limit of

$$\cdots \to \Psi_{B_j}^\infty(M; E_{j+1}^1, E_{j+1}^2) \to \Psi_{B_j}^\infty(M; E_j^1, E_j^2) \to \cdots \to \Psi_{B_0}^\infty(M; E_0^1, E_0^2). \tag{4.5}$$

Let $\mathscr{E}$ be a $\mathfrak{B}$-vector bundle on M. Given $T \in \Psi_{\mathfrak{B}}^\infty(M; \mathscr{E}, \mathscr{E})$, let $i_j(T)$ be its image in $\Psi_{B_j}^\infty(M; E_j, E_j)$.

Proposition 19 [19, Proposition 19]. *If $i_0(T)$ is invertible in $\Psi_{B_0}^\infty(M; E_0, E_0)$ then T is invertible in $\Psi_{\mathfrak{B}}^\infty(M; \mathscr{E}, \mathscr{E})$.*

Note that $\Psi_{\mathfrak{B}}^{-\infty}(M; \mathscr{E}, \mathscr{E})$ is an algebra in its own right (without unit) of smoothing operators. Given $T \in \Psi_{\mathfrak{B}}^{-\infty}(M; \mathscr{E}, \mathscr{E})$, let $\sigma_{\Psi^{-\infty}}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^{-\infty}(M; \mathscr{E}, \mathscr{E})$ and let $\sigma_{\Psi^\infty}(T)$ denote its spectrum in $\Psi_{\mathfrak{B}}^\infty(M; \mathscr{E}, \mathscr{E})$.

Lemma 3 [19, Lemma 2]. $\sigma_{\Psi^{-\infty}}(T) = \sigma_{\Psi^\infty}(T)$.

Consider the algebra $\mathfrak{A}$ of integral operators whose kernels $K(m_1, m_2) \in \mathrm{Hom}_{\mathfrak{B}}(\mathscr{E}_{m_2}, \mathscr{E}_{m_1})$ are continuous in m_1 and m_2, with multiplication

$$(KK')(m_1, m_2) = \int_Z K(m_1, m)\, K'(m, m_2)\, d\,\mathrm{vol}(m). \tag{4.6}$$

Let A_j be the analogous algebra with continuous kernels $K(m_1, m_2) \in \mathrm{Hom}_{B_j}((E_j)_{m_2}, (E_j)_{m_1})$. Give $\mathrm{Hom}_{B_j}((E_j)_{m_2}, (E_j)_{m_1})$ the Banach space norm $|\cdot|_j$ induced from $\mathrm{Hom}(B_j^N, B_j^N)$. Define a norm $|\cdot|_j$ on A_j by

$$|K|_j = (\mathrm{vol}(M))^{-1} \max_{m_1, m_2 \in M} |K(m_1, m_2)|_j. \tag{4.7}$$

Then one can check that A_j is a Banach algebra (without unit). Furthermore, $\mathfrak{A}$ is the projective limit of $\{A_j\}_{j \geqslant 0}$. Any smoothing operator $T \in \Psi_{\mathfrak{B}}^{-\infty}(M; \mathscr{E}, \mathscr{E})$ gives an element of $\mathfrak{A}$ through its Schwartz kernel. Let $\sigma_{\mathfrak{A}}(T)$ be its spectrum in $\mathfrak{A}$.

LEMMA 4 [19, Lemma 3]. $\sigma_{\mathfrak{A}}(T)=\sigma_{\Psi^{-\infty}}(T)$.

Define a continuous trace TR: $\mathscr{A}\to\mathfrak{B}/\overline{[\mathfrak{B},\mathfrak{B}]}$ by

$$\mathrm{TR}(K)=\int_M \mathrm{Tr}(K(m,m))\,d\,\mathrm{vol}(m). \tag{4.8}$$

Suppose that $0\notin\mathrm{spec}(\hat{\triangle}_p)$ or that 0 is isolated in $\mathrm{spec}(\hat{\triangle}_p)$. Let $\mathscr{D}$ be the $\mathfrak{B}$-vector bundle $\hat{M}\times_\Gamma\mathfrak{B}$ on M and put $\mathscr{E}=\Lambda^p(T^*M)\otimes\mathscr{D}$. We can lift $\triangle_p$ from M to a differential operator $\tilde{\triangle}_p\in\Psi^2_{\mathfrak{B}}(M;\mathscr{E},\mathscr{E})$. Then for all $t>0$, $e^{-t\tilde{\triangle}_p}\in\Psi^{-\infty}_{\mathfrak{B}}(M;\mathscr{E},\mathscr{E})$. As in [15, Sect. 3], one can show that

$$\mathrm{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_p})=\tau_{\langle g\rangle}(\mathrm{TR}(e^{-t\tilde{\triangle}_p})). \tag{4.9}$$

Let E be the analogous Λ-vector bundle on M whose fiber over $m\in M$ is isomorphic to $\Lambda^p(T^*_mM)\otimes\Lambda$. Recall that $i_0(\tilde{\triangle}_p)$ is the extension of $\tilde{\triangle}_p$ to an element of $\Psi^2_\Lambda(M;E,E)$. Let $N(\Gamma)$ denote the group von Neumann algebra of Γ [1]. Let $\bar{E}$ be the natural $N(\Gamma)$-vector bundle on M whose fiber over $m\in M$ is isomorphic to $\Lambda^p(T^*_mM)\otimes N(\Gamma)$. Let $\bar{\triangle}_p$ be the extension of $\tilde{\triangle}_p$ to an element of $\Psi^2_{N(\Gamma)}(M;\bar{E},\bar{E})$. As $L^2(\hat{M})$ is isomorphic to the L^2-sections of the Hilbert bundle $\hat{M}\times_\Gamma l^2(\Gamma)$, it follows that $\sigma(\hat{\triangle}_p)=\sigma(\bar{\triangle}_p)$. As the C^*-algebra Λ is a closed subalgebra of $N(\Gamma)$, it follows that $\sigma(\bar{\triangle}_p)=\sigma(i_0(\tilde{\triangle}_p))$. Using Proposition 19, it now follows that $0\notin\sigma(\tilde{\triangle}_p)$ or that 0 is isolated in $\sigma(\tilde{\triangle}_p)$. Let c be a small loop around $0\in\mathbb{C}$, oriented counterclockwise. The projection onto $\mathrm{Ker}(\tilde{\triangle}_p)$ is

$$\Pi_{\mathrm{Ker}(\tilde{\triangle}_p)}=\frac{1}{2\pi i}\int_c\frac{dz}{z-\tilde{\triangle}_p}. \tag{4.10}$$

It follows from arguments as in [25] that $\mathrm{Ker}(\tilde{\triangle}_p)$ is a finitely-generated projective right $\mathfrak{B}$-module. Let $\tilde{\triangle}'_p$ be the compression of $\tilde{\triangle}_p$ onto $\mathrm{Im}(I-\Pi_{\mathrm{Ker}(\tilde{\triangle}_p)})$. Then in terms of the trace of (4.2),

$$\mathrm{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_p})=\tau_{\langle g\rangle}(\mathrm{Tr}(I_{\mathrm{Ker}(\tilde{\triangle}_p)}))+\tau_{\langle g\rangle}(\mathrm{TR}(e^{-t\tilde{\triangle}'_p})). \tag{4.11}$$

As τ and TR are continuous, it suffices to show that there is some j such that the A_j-norm of $e^{-t\tilde{\triangle}'_p}$ is rapidly-decreasing in t.

As $\{e^{-t\tilde{\triangle}'_p}\}_{t>0}$ gives a 1-parameter semigroup in the Banach algebra A_j, it follows from [7, Theorem 1.22] that the number

$$a=\lim_{t\to\infty}t^{-1}\ln|e^{-t\tilde{\triangle}'_p}|_j \tag{4.12}$$

exists. Furthermore, for all $t>0$, the spectral radius of $e^{-t\tilde{\triangle}'_p}$ is e^{at}. Let $\lambda_0>0$ be the infimum of the spectrum of the generator $\tilde{\triangle}'_p$. Then by the spectral mapping theorem, the spectral radius of $e^{-t\tilde{\triangle}'_p}$ is $e^{-t\lambda_0}$. Thus there is a constant $C>0$ such that for $t>1$,

$$|e^{-t\tilde{\triangle}'_p}|_j \leqslant Ce^{-\lambda_0 t/2}. \tag{4.13}$$

Hence

$$b_{p,\langle g\rangle}(M) = \tau_{\langle g\rangle}(\mathrm{Tr}(I_{\mathrm{Ker}(\tilde{\triangle}_p)})) \tag{4.14}$$

is well defined. By similar arguments one can justify (2.13), showing that $b_{p,\langle g\rangle}(M)$ is metric-independent. This proves Proposition 6.

To prove Proposition 7, with its hypotheses, arguments similar to those above give

$$\mathrm{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_{0,q}}) = \tau_{\langle g\rangle}(\mathrm{Tr}(I_{\mathrm{Ker}(\tilde{\triangle}_{0,q})})) + O(e^{-c_q t}) \tag{4.15}$$

for some $c_q>0$. Then the integral in (2.24) makes sense and one can justify (2.25). This proves Proposition 7.

To prove Proposition 8, let D be a Dirac-type operator on M such that $0\notin \mathrm{spec}(\hat{D})$ or 0 is isolated in $\mathrm{spec}(\hat{D})$. Put $\mathscr{E} = S\otimes V\otimes \mathscr{D}$. We can lift D to a differential operator $\tilde{D}\in \Psi^1_{\mathfrak{B}}(M;\mathscr{E},\mathscr{E})$. Then for all $s>0$, $\tilde{D}e^{-s^2\tilde{D}^2}\in \Psi^{-\infty}_{\mathfrak{B}}(M;\mathscr{E},\mathscr{E})$. As in [15, Sect. 3], one can show that

$$\eta_{\langle g\rangle}(s) = \tau_{\langle g\rangle}(\mathrm{TR}(\tilde{D}e^{-s^2\tilde{D}^2})). \tag{4.16}$$

From finite-propagation estimates, we know that $\eta_{\langle g\rangle}(s)$ is integrable for small s. Hence we must show that $\tau_{\langle g\rangle}(\mathrm{TR}(\tilde{D}e^{-s^2\tilde{D}^2}))$ is integrable for large s. It suffices to show that there is some j such that the A_j-norm of $\tilde{D}e^{-s^2\tilde{D}^2}$ is rapidly decreasing in s.

Let E be the natural Λ-vector bundle on M whose fiber over $m\in M$ is isomorphic to $S_m\otimes V_m\otimes \Lambda$. Recall that $i_0(\tilde{D})$ is the extension of $\tilde{D}$ to an element of $\Psi^1_\Lambda(M;E,E)$. As before, $\sigma(\hat{D}) = \sigma(i_0(\tilde{D}))$. Using Proposition 4.6, it now follows that $0\notin\sigma(\tilde{D})$ or that 0 is isolated in $\sigma(\tilde{D})$. Let c be a small loop around $0\in\mathbb{C}$, oriented counterclockwise. The projection on $\mathrm{Ker}(\tilde{D})$ is

$$\Pi_{\mathrm{Ker}(\tilde{D})} = \frac{1}{2\pi i}\int_c \frac{dz}{z-\tilde{D}}. \tag{4.17}$$

Let $\tilde{D}'$ be the compression of $\tilde{D}$ onto $\mathrm{Im}(I-\Pi_{\mathrm{Ker}(\tilde{D})})$. Then

$$\tilde{D}e^{-s^2\tilde{D}^2} = 0_{\mathrm{Ker}(\tilde{D})}\oplus \tilde{D}'e^{-s^2\tilde{D}'^2}. \tag{4.18}$$

Hence we may as well assume that $0 \notin \mathrm{spec}(\tilde{D})$. Let $\lambda_0 > 0$ be the infimum of the spectrum of the generator $\tilde{D}^2$. As in (4.13), there is a constant $C > 0$ such that for $t > 1$,

$$|e^{-t\tilde{D}^2}|_j \leqslant Ce^{-\lambda_0 t/2}. \tag{4.19}$$

As

$$|\tilde{D}e^{-s^2\tilde{D}^2}|_j \leqslant |\tilde{D}e^{-\tilde{D}^2}|_j \cdot |e^{-(s^2-1)\tilde{D}^2}|_j, \tag{4.20}$$

it follows that $\eta_{\langle g \rangle}(s)$ is large-s integrable.

Suppose that $\{ds^2(u)\}_{u \in [-1,1]}$ is a smooth 1-parameter of positive-scalar-curvature metrics on M. Let $D(u)$ be the Dirac operator on M. Then for all $u \in [-1, 1]$, $\hat{D}(u)$ is invertible. Using the above methods, one sees that

$$\lim_{s \to \infty} s\tau_{\langle g \rangle}\left(\mathrm{TR}\left(\frac{d\tilde{D}}{du}\, e^{-s^2\tilde{D}^2}\right)\right) = 0. \tag{4.21}$$

From Proposition 16, $\eta_{\langle g \rangle}(M)$ is independent of u. This proves Proposition 8. ∎

Remark. If Γ is virtually nilpotent, one can also prove Propositions 6, 7, and 8 using finite propagation speed estimates on $\hat{M}$, as in [16]. This does not work when Γ is Gromov-hyperbolic, which is why we use the more indirect method of proof above.

5. PROOF OF PROPOSITIONS 9 AND 10

To prove Proposition 9, as in [9, Sect. 2], the Selberg trace formula implies that there are functions $\{G_j(t)\}_{j=0}^{d-1}$ of the form

$$G_j(t) = a_j t^{-1/2} e^{-l^2/4t} e^{-tc_j^2} \tag{5.1}$$

so that for $0 \leqslant j \leqslant d$,

$$\mathrm{Tr}_{\langle g \rangle}(e^{-t\hat{\Delta}_j}) = G_j(t) + G_{j-1}(t). \tag{5.2}$$

Here a_j and c_j are nonnegative constants whose exact values are not important for the moment. It is clear from (5.1) and (5.2) that $b_{p,\langle g \rangle}(M)$ vanishes for all p.

Now suppose that the dimension of M is $d = 2n + 1$. The Selberg trace formula gives the following result.

PROPOSITION 20 [9, Theorem 2]. *For $0 \leqslant j \leqslant 2n$, put $c_j = |n-j|$ and*

$$G_t(\sigma_j) = \frac{\mathrm{Tr}(\sigma_j(m))}{k \ \det(I - e^{-l} m^{-1})} \frac{l}{\sqrt{4\pi t}} e^{-l^2/4t} e^{-tc_j^2} e^{-nl}. \tag{5.3}$$

Then for $0 \leqslant j \leqslant d$,

$$\mathrm{Tr}_{\langle g \rangle}(e^{-t\hat{\triangle}_j}) = G_t(\sigma_j) + G_t(\sigma_{j-1}). \tag{5.4}$$

Hence from (2.14),

$$\begin{aligned} \mathscr{T}_{\langle g \rangle}(t) &= \sum_{p=0}^{d} (-1)^j \, j[G_t(\sigma_j) + G_t(\sigma_{j-1})] \\ &= \sum_{j=0}^{2n} (-1)^{j+1} \, G_t(\sigma_j). \end{aligned} \tag{5.5}$$

For $l > 0$,

$$\int_0^\infty \frac{1}{\sqrt{4\pi t}} e^{-l^2/4t} e^{-tc^2} \frac{dt}{t} = \frac{e^{-lc}}{l}. \tag{5.6}$$

Equation (1.2) now follows from (2.16).

For $r > 0$, we have

$$\begin{aligned} \mathscr{T}_{\langle g^r \rangle}(M) &= \frac{e^{-nrl}}{k \det(I - e^{-rl} m^{-r})} \sum_{j=0}^{2n} (-1)^j e^{-rl\,|n-j|} \, \mathrm{Tr}(\sigma_j(m^r)) \\ &= \frac{(-1)^n}{k} e^{-nrl} \, \mathrm{Tr}(\sigma_n(m^r)) + O(e^{-2nrl}). \end{aligned} \tag{5.7}$$

Then

$$l = \frac{1}{n} \sup \{ \alpha \in \mathbb{R} : |\mathscr{T}_{\langle g^r \rangle}(M)| = O(e^{-\alpha r}) \}. \tag{5.8}$$

Hence one recovers the marked length spectrum of M from $\{\mathscr{T}_{\langle g \rangle}(M)\}_{\langle g \rangle \in \mathscr{C}}$.

Let D be the tangential signature operator. By [23, Theorem 2.1],

$$\eta_{\langle g \rangle}(s) = (2i)^n \frac{2\pi i}{k} \frac{l^2 \ \sin(\theta_1) \cdots \sin(\theta_n)}{|\mu_1 - \mu_1^{-1}|^2 \cdots |\mu_n - \mu_n^{-1}|^2} \frac{e^{-l^2/4s^2}}{(4\pi)^{3/2} s^3}. \tag{5.9}$$

Equation (1.4) now follows from (2.27).

Equation (1.5) comes from a straightforward computation. This proves Proposition 9.

To prove Proposition 10, we first note that the complex-hyperbolic metric on Σ satisfies the hypotheses of Proposition 7. By arguments like those in [18], applied to the Dolbeault complex instead of the de Rham complex, it follows that any Hermitian metric on Σ also satisfies the hypotheses of Proposition 7. By the conclusion of Proposition 7, we can just do the computations for the complex-hyperbolic metric. In this case, we can write

$$\mathrm{Tr}_{\langle g\rangle}(e^{-t\hat{\triangle}_{0,q}}) = G_q(t) + G_{q-1}(t), \tag{5.10}$$

where $G_d(t)=0$ and for $0 \leqslant q < d$, $G_q(t)$ is given by the following proposition.

PROPOSITION 21 [11, Sect. 2].

$$G_t(\sigma_q) = \frac{\mathrm{Tr}(\sigma_q(m))}{k(1-e^{-2l})\,|\det(I-e^{-l}m^{-1})|^2}\,\frac{l}{\sqrt{4\pi t}}\,e^{-l^2/4t}e^{-t(d-q)^2}e^{-dl}. \tag{5.11}$$

Proposition 10 now follows from calculations as above. ∎

6. PROOF OF PROPOSITION 11

Let $\pi\colon M \to S^1$ be the natural projection map. For $e^{i\theta} \in U(1)$, let E_θ be the flat complex line bundle on S^1 with holonomy $e^{i\theta}$. Let $T(\theta) \in \mathbb{R}$ be the Ray–Singer analytic torsion of M, computed with the flat bundle $\pi^*(E_\theta)$. As in [17, Sect. VI], it follows from Fourier analysis that

$$\mathscr{T}_{\langle k\rangle}(M) = \int_{S^1} e^{-ik\theta}\,T(\theta)\,\frac{d\theta}{2\pi}. \tag{6.1}$$

From [24, Sect. 3] and the Cheeger–Müller theorem [4, 27],

$$T(\theta) = \sum_{p=0}^{n} (-1)^p \ln|\det(I - e^{i\theta}\phi_p^*)|^{-2}. \tag{6.2}$$

Given $\lambda \neq 0$, if $k > 0$ then

$$\int_{S^1} e^{-ik\theta} \ln|1-e^{i\theta}\lambda|^{-2}\,\frac{d\theta}{2\pi} = \frac{f(\lambda^k)}{k} \tag{6.3}$$

and if $k<0$ then

$$\int_{S^1} e^{-ik\theta} \ln |1-e^{i\theta}\lambda|^{-2} \frac{d\theta}{2\pi} = -\frac{f(\bar{\lambda}^{-k})}{k}. \tag{6.4}$$

Equation (1.9) follows from combining (6.1)–(6.4).

By standard arguments [24],

$$\zeta(z) = \prod_{p=0}^{n} \det(I - z\phi_p^*)^{(-1)^{p+1}}. \tag{6.5}$$

Equation (1.10) follows from (6.1), (6.2), and (6.5).

Now suppose that ϕ preserves D_Z. It follows that ϕ is an isometry of Z with respect to the Riemannian metric defining D_Z. In terms of the coordinates (u, z) on $\hat{M} = \mathbb{R} \times Z$, we can write

$$\hat{D} = \begin{pmatrix} -i\partial_u & D_{Z,-} \\ D_{Z,+} & i\partial_u \end{pmatrix}. \tag{6.6}$$

Then

$$\hat{D}^2 = \begin{pmatrix} -\partial_u^2 + D_{Z,-} D_{Z,+} & 0 \\ 0 & -\partial_u^2 + D_{Z,+} D_{Z,-} \end{pmatrix} \tag{6.7}$$

and

$$e^{-s^2\hat{D}^2}((u, z), (u', z'))$$

$$= \begin{pmatrix} \frac{1}{\sqrt{4\pi s^2}} e^{-(u-u')^2/4s^2} e^{-s^2 D_{Z,-} D_{Z,+}}(z, z') & 0 \\ 0 & \frac{1}{\sqrt{4\pi s^2}} e^{-(u-u')^2/4s^2} e^{-s^2 D_{Z,+} D_{Z,-}}(z, z') \end{pmatrix}. \tag{6.8}$$

It follows that

$$\operatorname{tr}(\hat{D} e^{-s^2\hat{D}^2}((u, z), (u', z'))) = i \frac{1}{\sqrt{4\pi s^2}} \frac{u-u'}{2s^2} e^{-(u-u')^2/4s^2} \operatorname{tr}_s(e^{-s^2 D_Z^2}(z, z')). \tag{6.9}$$

Hence

$$\begin{aligned}
\eta_{\langle k \rangle}(M) &= \frac{2}{\sqrt{\pi}} \int_0^\infty \int_Z i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-k^2/4s^2} \operatorname{tr}_s(e^{-s^2 D_Z^2}(\phi^k(z), z))\, d\operatorname{vol}(z)\, ds \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-k^2/4s^2} \operatorname{Tr}_s(\phi^k e^{-s^2 D_Z^2})\, ds \\
&= \frac{2}{\sqrt{\pi}} \int_0^\infty i \frac{1}{\sqrt{4\pi s^2}} \frac{k}{2s^2} e^{-k^2/4s^2} \operatorname{Tr}_s(\phi^k \mid_{\operatorname{Ker}(D_Z)})\, ds \\
&= \frac{i}{k\pi} \operatorname{Tr}_s(\phi^k \mid_{\operatorname{Ker}(D_Z)}) \int_0^\infty e^{-k^2/4s^2}\, d\left(-\frac{k^2}{4s^2}\right) \\
&= \frac{i}{k\pi} \operatorname{Tr}_s(\phi^k \mid_{\operatorname{Ker}(D_Z)}).
\end{aligned} \tag{6.10}$$

The proposition follows. ∎

7. PROOF OF PROPOSITION 12

Any irreducible unitary representation ρ of $F \tilde{\times}_\alpha \mathbb{Z}$ arises as follows [21, Sect. 10]. First, $\mathbb{Z}$ acts on the dual space $\hat{F}$. A periodic point of period j corresponds to a representation $\mu: F \to U(N)$ and a matrix $U \in U(N)$ such that $\mu(\alpha^j(f)) = U\mu(f)\, U^{-1}$. (The matrix U is determined up to multiplication by a unit complex number.) Consider the representation $v: F \tilde{\times}_\alpha j\mathbb{Z} \to U(N)$ given by

$$v(f, jr) = \mu(f)\, U^r. \tag{7.1}$$

Then ρ comes from inducing v from $F \tilde{\times}_\alpha j\mathbb{Z}$ to $F \tilde{\times}_\alpha \mathbb{Z}$. The character of ρ is

$$\chi_\rho(f, k) = \begin{cases} 0 & \text{if } \; j \nmid k, \\ \operatorname{Tr}([\mu(f) + \mu(\alpha^{-1}(f)) + \cdots + \mu(\alpha^{-(j-1)}(f))]\, U^r) & \text{if } \; k = jr. \end{cases} \tag{7.2}$$

Let $\mathscr{T}_\rho(M)$ be the analytic torsion of M, computed using the representation ρ. From Fourier analysis,

$$\mathscr{T}_\rho(M) = \sum_{f, k} \chi_\rho(f, k)\, \mathscr{T}_{\langle f, k \rangle}(M). \tag{7.3}$$

Let M' be the mapping torus of ϕ^j. Then M' is a j-fold cover of M. By [10, (VI), p. 27],

$$\mathcal{T}_\rho(M) = \mathcal{T}_\nu(M'). \tag{7.4}$$

Let E_μ be the flat $\mathbb{C}^N$-bundle on Z coming from the representation μ. Then ϕ^j acts on Z and preserves E_μ. Let $L_\mu(r)$ be the Lefschetz number of ϕ^{jr} acting on (Z, E_μ). Put

$$\zeta_\nu(z) = \exp\left(\sum_{r>0} \frac{z^r}{r} L_\mu(r)\right). \tag{7.5}$$

It follows from [24, Sect. 3] that

$$\mathcal{T}_\nu(M') = \ln |\zeta_\nu(1)|^2. \tag{7.6}$$

Take a cellular decomposition of Z. Let $\hat{Z}$ have the lifted cellular structure and let $C^*(\hat{Z})$ denote the cellular cochains on $\hat{Z}$. We let F act on $C^*(\hat{Z})$ on the right by

$$\omega \cdot f = R^*_{f^{-1}} \omega. \tag{7.7}$$

Then

$$\hat{\phi}^*(\omega \cdot f) = (\hat{\phi}^* \omega) \cdot \alpha(f). \tag{7.8}$$

We can identify $C^*(Z; E_\mu)$ with $C^*(\hat{Z}) \otimes_F \mathbb{C}^N$, with the relation $\omega \cdot f \otimes_F v = \omega \otimes_F \mu(f) v$. Then ϕ^{jr} acts on $C^*(\hat{Z}) \otimes_F \mathbb{C}^N$ by

$$\phi^{jr}(\omega \otimes_F v) = (\hat{\phi}^{jr})^* \, \omega \otimes_F U^r V. \tag{7.9}$$

Letting Tr_s denote the supertrace on $C^*(Z; E_\mu)$, we want to compute

$$L_\mu(r) = \mathrm{Tr}_s(\phi^{jr}). \tag{7.10}$$

For the moment we concentrate on $C^p(Z; E_\mu)$. Let $\{e_i\}$ be a basis of $C^p(\hat{Z})$ consisting of dual p-cells. The set of such dual p-cells has a free F-action. Write the action of $\hat{\phi}^*$ on $C^p(\hat{Z})$ as

$$\hat{\phi}^*(e_i) = \sum_l \hat{\phi}^*_{e_i \to e_l} e_l. \tag{7.11}$$

From (7.8),

$$\hat{\phi}^*_{e_i \to e_l} = \hat{\phi}^*_{e_i f \to e_l \alpha(f)}. \tag{7.12}$$

Let $\{\bar{e}_i\}$ be a set of representatives for the F-orbits of the dual p-cells. Then as a vector space,

$$C^p(\hat{Z}) \otimes_F \mathbb{C}^N = \bigoplus_i \bar{e}_i \otimes \mathbb{C}^N \tag{7.13}$$

and so

$$\begin{aligned}\phi^{jr}(\bar{e}_i \otimes v) &= \sum_{l,f} (\hat{\phi}^{jr})^*_{\bar{e}_i \to \bar{e}_l f} \bar{e}_l f \otimes_F U^r v \\ &= \sum_{l,f} (\hat{\phi}^{jr})^*_{\bar{e}_i \to \bar{e}_l f} \bar{e}_l \otimes \mu(f)\, U^r v.\end{aligned} \tag{7.14}$$

Hence

$$\mathrm{Tr}(\phi^{jr}) = \sum_{i,f} (\hat{\phi}^{jr})^*_{\bar{e}_i \to \bar{e}_i f} \mathrm{Tr}(\mu(f)\, U^r). \tag{7.15}$$

As the choice of the representatives $\{\bar{e}_i\}$ is arbitrary, we can also write

$$\mathrm{Tr}(\phi^{jr}) = \frac{1}{|F|} \sum_{i,f} (\hat{\phi}^{jr})^*_{e_i \to e_i f} \mathrm{Tr}(\mu(f)\, U^r). \tag{7.16}$$

We have

$$\begin{aligned}\mathrm{Tr}(\phi^{jr}) &= \frac{1}{|F|} \sum_{i,f} (\hat{\phi}^{jr})^*_{e_i \to e_i f} \mathrm{Tr}(\mu(f)\, U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} (\hat{\phi})^*_{e_i \to e_l} (\hat{\phi}^{jr-1})^*_{e_l \to e_i f} \mathrm{Tr}(\mu(f)\, U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} (\hat{\phi}^{jr-1})^*_{e_l \to e_i f} (\hat{\phi})^*_{e_i \to e_l} \mathrm{Tr}(\mu(f)\, U^r) \\ &= \frac{1}{|F|} \sum_{i,l,f} (\hat{\phi}^{jr-1})^*_{e_l \to e_i f} (\hat{\phi})^*_{e_i f \to e_l \alpha(f)} \mathrm{Tr}(\mu(f)\, U^r) \\ &= \frac{1}{|F|} \sum_{l,f} (\hat{\phi}^{jr})^*_{e_l \to e_l \alpha(f)} \mathrm{Tr}(\mu(f)\, U^r) \\ &= \frac{1}{|F|} \sum_{l,f} (\hat{\phi}^{jr})^*_{e_l \to e_l f} \mathrm{Tr}(\mu(\alpha^{-1}(f))\, U^r).\end{aligned} \tag{7.17}$$

Then from (7.16) and (7.17),

$$\mathrm{Tr}(\phi^{jr}) = \frac{1}{j\,|F|} \sum_{i,\,f} (\hat{\phi}^{jr})^*_{e_i \to e_i f} \,\mathrm{Tr}([\mu(f) + \mu(\alpha^{-1}(f)) + \cdots + \mu(\alpha^{-(j-1)}(f))]\, U^r). \tag{7.18}$$

Put

$$n_{p,\,jr}(f) = \frac{1}{|F|} \sum_{i,\,f} (\hat{\phi}^{jr})^*_{e_i \to e_i f}\,. \tag{7.19}$$

From (7.2), (7.10), and (7.18),

$$L_\mu(r) = \frac{1}{j} \sum_f \chi_\rho(f, jr) \sum_{p=0}^{n} (-1)^p\, n_{p,\,jr}(f). \tag{7.20}$$

Put

$$i_{p,\,jr}(f) = \sum_{f' \sim_{jr} f} \sum_i (\hat{\phi}^{jr})^*_{\bar{e}_i \to \bar{e}_i f'}\,. \tag{7.21}$$

The Nielsen fixed-point index $I_{jr}(f) \in \mathbb{Z}$ of the transformation ϕ^{jr} is defined by [12, Sect. 1]

$$I_{jr}(f) = \sum_{p=0}^{n} (-1)^p\, i_{p,\,jr}(f). \tag{7.22}$$

Put $s_{jr}(f) = |\{\gamma \in F : \gamma f \alpha^{jr}(\gamma^{-1}) = f\}|$. We have

$$\begin{aligned} i_{p,\,jr}(f) &= \frac{1}{s_{jr}(f)} \sum_{i,\,\gamma} (\hat{\phi}^{jr})^*_{\bar{e}_i \to \bar{e}_i \gamma f \alpha^{jr}(\gamma^{-1})} \\ &= \frac{1}{s_{jr}(f)} \sum_{i,\,\gamma} (\hat{\phi}^{jr})^*_{\bar{e}_i \gamma \to \bar{e}_i \gamma f} \\ &= \frac{1}{s_{jr}(f)} \sum_{i} (\hat{\phi}^{jr})^*_{e_i \to e_i f}\,. \end{aligned} \tag{7.23}$$

Then from (7.19) and (7.23),

$$n_{p,\,jr}(f) = \frac{s_{jr}(f)}{|F|}\, i_{p,\,jr}(f) = \frac{i_{p,\,jr}(f)}{|[f]_{jr}|}. \tag{7.24}$$

Hence the Lefschetz number is given in terms of the Nielsen index by

$$L_\mu(r) = \frac{1}{j} \sum_f \frac{\chi_\rho(f, jr)\, I_{jr}(f)}{|[f]_{jr}|}. \tag{7.25}$$

Substituting (7.25) into (7.6) and using (7.3)–(7.4) gives (1.15). As Γ is a type-I discrete group [21, p. 61], knowing Eq. (1.15) for all $\rho \in \hat{\Gamma}$ determines $\{\mathscr{T}_{\langle f, k \rangle}(M)\}_{(f, k) \in \Gamma}$. ∎

8. EXAMPLES

Proposition 2: It follows from Proposition 17.1 that $b_{p, \langle g \rangle}(M)$ can be nonzero if Γ is finite. For example, $b_{0, \langle g \rangle}(M) = |\langle g \rangle| / |\Gamma|$.

Proposition 3: It follows from Proposition 17.2 that $\mathscr{T}_{\langle g \rangle}(M)$ is nonzero in some examples in which M is a lens space.

Proposition 4: Applying Fourier analysis to [32, Theorem 4.1], we see that $\mathscr{T}^{\text{hol}}_{\langle g \rangle}(M)$ is nonzero if M is a 2-torus.

Proposition 5: It follows from Proposition 17.4 that $\eta_{\langle g \rangle}(M)$ is nonzero in some examples in which M is a lens space, both for the tangential signature operator and the Dirac operator.

Proposition 6: The hypotheses of the proposition are satisfied for all p if M is an even-dimensional closed real-hyperbolic manifold or a closed complex-hyperbolic manifold.

Proposition 7: Examples come from Proposition 1.

Proposition 8: Let N_1 be a closed even-dimensional spin manifold whose fundamental group is virtually nilpotent or Gromov-hyperbolic, with $\hat{A}(N_1) \neq 0$. Let N_2 be a lens space which is spin and whose Dirac operator has a nonzero ρ-invariant. Put $M = N_1 \times N_2$. Shrink N_2 so that M has positive scalar curvature. Then M satisfies the hypotheses of the proposition and $\eta_{\langle e, g \rangle}(M) = \hat{A}(N_1)\, \eta_{\langle g \rangle}(N_2)$ is nonzero for appropriate g.

Proposition 9: There are many nontrivial examples.

Proposition 10: There are many nontrivial examples.

Proposition 11: Nontrivial examples come from closed even-dimensional oriented manifolds Z with a finite-order orientation-preserving diffeomorphism ϕ such that ϕ has nonzero Lefschetz or Atiyah–Bott numbers.

Proposition 12: Any example of Proposition 11 gives an example of Proposition 12 by taking F to be the trivial group. There are also many examples with F nontrivial.

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