

NOTES ON PERELMAN'S PAPERS

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1. INTRODUCTION

These are informal notes on Perelman's papers "The Entropy Formula for the Ricci Flow and its Geometric Applications" [21], and "Ricci flow with surgery on three-manifolds", [22]. The purpose of the notes is to add detail to some of Perelman's arguments.

The notes are not meant to be complete. They are not self-contained and are meant to be read along with [21]. Selected portions of Sections 1-4 and 7-13 of [21] are covered. We include four appendices that discuss some background material and techniques that are used throughout the notes. The reader may wish to glance at these first.

In the body of the notes we will refer to Section X.Y of [21] as I.X.Y, and Section X.Y of [22] as II.X.Y.

Disclaimer : We do not guarantee the accuracy of any of the statements or proofs in these notes. It is quite possible that errors, due to the authors, have crept into the exposition. Any corrections or improvements to the notes are welcome.

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2. DERIVATION OF THE FIRST EQUATION OF SECTION 1.1

With the conventions of I, $\Delta = \text{div grad}$. From a standard formula,

$$(2.1) \quad \delta R = -\Delta v + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij}.$$

As

$$(2.2) \quad |\nabla f|^2 = g^{ij} \nabla_i f \nabla_j f,$$

we have

$$(2.3) \quad \delta |\nabla f|^2 = -v^{ij} \nabla_i f \nabla_j f + 2 \langle \nabla f, \nabla h \rangle.$$

As $dV = \sqrt{\det(g)} dx_1 \dots dx_n$, we have $\delta(dV) = \frac{v}{2} dV$, so

$$(2.4) \quad \delta(e^{-f} dV) = \left(\frac{v}{2} - h\right) e^{-f} dV.$$

Putting this together gives

$$(2.5) \quad \delta \mathcal{F} = \int_M e^{-f} \left[-\Delta v + \nabla_i \nabla_j v_{ij} - R_{ij} v_{ij} - v_{ij} \nabla_i f \nabla_j f + 2 \langle \nabla f, \nabla h \rangle + (R + |\nabla f|^2) \left(\frac{v}{2} - h\right) \right] dV.$$

As

$$(2.6) \quad \Delta e^{-f} = (|\nabla f|^2 - \Delta f) e^{-f},$$

we have

$$(2.7) \quad \int_M e^{-f} [-\Delta v] dV = - \int_M \Delta e^{-f} v dV = \int_M e^{-f} (\Delta f - |\nabla f|^2) v dV.$$

Next,

$$(2.8) \quad \begin{aligned} \int_M e^{-f} \nabla_i \nabla_j v_{ij} dV &= \int_M (\nabla_i \nabla_j e^{-f}) v_{ij} dV = - \int_M \nabla_i (e^{-f} \nabla_j f) v_{ij} dV \\ &= \int_M e^{-f} (\nabla_i f \nabla_j f - \nabla_i \nabla_j f) v_{ij} dV. \end{aligned}$$

Finally,

$$(2.9) \quad \begin{aligned} 2 \int_M e^{-f} \langle \nabla f, \nabla h \rangle dV &= -2 \int_M \langle \nabla e^{-f}, \nabla h \rangle dV = 2 \int_M (\Delta e^{-f}) h dV \\ &= 2 \int_M e^{-f} (|\nabla f|^2 - \Delta f) h dV. \end{aligned}$$

Then

$$(2.10) \quad \begin{aligned} \delta \mathcal{F} &= \int_M e^{-f} \left[\left(\frac{v}{2} - h \right) (2\Delta f - 2|\nabla f|^2) - v_{ij} (R_{ij} + \nabla_i \nabla_j f) \right. \\ &\quad \left. + \left(\frac{v}{2} - h \right) (R + |\nabla f|^2) \right] dV \\ &= \int_M e^{-f} \left[-v_{ij} (R_{ij} + \nabla_i \nabla_j f) + \left(\frac{v}{2} - h \right) (2\Delta f - |\nabla f|^2 + R) \right] dV. \end{aligned}$$

3. BASIC EXAMPLE FOR SECTION 1

Consider \mathbb{R}^n with the standard metric, constant in time. Fix $t_0 > 0$. Put $\tau = t_0 - t$ and

$$(3.1) \quad f(t, x) = \frac{|x|^2}{4\tau} + \frac{n}{2} \ln(4\pi\tau),$$

so

$$(3.2) \quad e^{-f} = (4\pi\tau)^{-n/2} e^{-\frac{|x|^2}{4\tau}}.$$

This is the standard heat kernel when considered for τ going from 0 to t_0 , i.e. for t going from t_0 to 0. One can check that f solves the second equation in I.(1.3). As

$$(3.3) \quad \int_{\mathbb{R}^n} e^{-\frac{|x|^2}{4\tau}} dV = (4\pi\tau)^{n/2},$$

f is properly normalized. Then $\nabla f = \frac{x}{2\tau}$ and $|\nabla f|^2 = \frac{|x|^2}{4\tau^2}$. Differentiating (3.3) with respect to τ gives

$$(3.4) \quad \int_{\mathbb{R}^n} \frac{|x|^2}{4\tau^2} e^{-\frac{|x|^2}{4\tau}} dV = (4\pi\tau)^{n/2} \frac{n}{2\tau},$$

so

$$(3.5) \quad \int_{\mathbb{R}^n} |\nabla f|^2 e^{-f} dV = \frac{n}{2\tau}.$$

Then $\mathcal{F}(t) = \frac{n}{2\tau} = \frac{n}{2(t_0-t)}$. In particular, this is nondecreasing as a function of $t \in [0, t_0)$.

4. I.2.2

Given a metric g , there is a unique minimizer \bar{f} of $\mathcal{F}(g, f)$ under the constraint $\int_M e^{-f} dV = 1$. To see this, write

$$(4.1) \quad \mathcal{F} = \int_M (Re^{-f} + 4|\nabla e^{-f/2}|^2) dV.$$

Putting $\Phi = e^{-f/2}$,

$$(4.2) \quad \mathcal{F} = \int_M (4|\nabla\Phi|^2 + R\Phi^2) dV = \int_M \Phi(-4\Delta\Phi + R\Phi) dV.$$

The constraint equation becomes $\int_M \Phi^2 dV = 1$. Then λ is the smallest eigenvalue of $-4\Delta + R$ and $e^{-\bar{f}/2}$ is a corresponding normalized eigenvector. As the operator is a Schrödinger operator, there is a unique normalized positive eigenvector.

If $g(t)$ is a smooth family of metrics then it follows from eigenvalue perturbation theory that $\lambda(t)$ and $\bar{f}(t)$ are smooth in t .

Now let $g(t)$ be a solution of the Ricci flow. Consider a time interval $[t_0, t_1]$, and the minimizer $\bar{f}(t_1)$. In particular, $\lambda(t_1) = \mathcal{F}(g(t_1), \bar{f}(t_1))$. Solve the backward heat equation in I.(1.3) backward on $[t_0, t_1]$, to obtain a solution $f(t)$ satisfying $f(t_1) = \bar{f}(t_1)$. By I.(1.4), $\mathcal{F}(g(t_0), f(t_0)) \leq \mathcal{F}(g(t_1), f(t_1))$. By the definition of λ , $\lambda(t_0) = \mathcal{F}(g(t_0), \bar{f}(t_0)) \leq \mathcal{F}(g(t_0), f(t_0))$. Thus $\lambda(t_0) \leq \lambda(t_1)$. (This simple argument was pointed out by Mike Anderson.)

5. BASIC EXAMPLE FOR SECTION 3.1

Consider \mathbb{R}^n with the standard metric, constant in time. Fix $t_0 > 0$. Put $\tau = t_0 - t$ and

$$(5.1) \quad f(t, x) = \frac{|x|^2}{4\tau},$$

so

$$(5.2) \quad e^{-f} = e^{-\frac{|x|^2}{4\tau}}.$$

One can check that $(g(t), f(t), \tau(t))$ satisfies I.(3.2) and I.(3.3). Now

$$(5.3) \quad \tau(|\nabla f|^2 + R) + f - n = \tau \cdot \frac{|x|^2}{4\tau^2} + \frac{|x|^2}{4\tau} - n = \frac{|x|^2}{2\tau} - n.$$

It follows from (3.3) and (3.4) that $\mathcal{W}(t) = 0$ for all $t \in [0, t_0]$.

6. DERIVATION OF I.(3.4)

Put $\sigma = \delta\tau$. Then

$$(6.1) \quad \delta((4\pi\tau)^{-n/2} e^{-f} dV) = \left(\frac{v}{2} - h - \frac{n\sigma}{2\tau}\right) (4\pi\tau)^{-n/2} e^{-f} dV.$$

Writing

$$(6.2) \quad \mathcal{W} = \int_M [\tau(R + |\nabla f|^2) + f - n] (4\pi\tau)^{-n/2} e^{-f} dV$$

we can use the fact that $\tau \in \mathbb{R}$ along with (2.10) to obtain

$$(6.3) \quad \begin{aligned} \delta\mathcal{W} = \int_M & \left[\sigma(R + |\nabla f|^2) + \tau \left(\frac{v}{2} - h \right) (2\Delta f - 2|\nabla f|^2) - \tau v_{ij} (R_{ij} + \nabla_i \nabla_j f) + \right. \\ & \left. h + [\tau(R + |\nabla f|^2) + f - n] \left(\frac{v}{2} - h - \frac{n\sigma}{2\tau} \right) \right] (4\pi\tau)^{-n/2} e^{-f} dV. \end{aligned}$$

Adding a Lie derivative, we can replace I.(3.3) by the equations

$$(6.4) \quad \begin{aligned} (g_{ij})_t &= -2(R_{ij} + \nabla_i \nabla_j f), \\ f_t &= -\Delta f - R + \frac{n}{2\tau}, \\ \tau_t &= -1. \end{aligned}$$

Thus we want to take

$$(6.5) \quad \begin{aligned} v_{ij} &= -2(R_{ij} + \nabla_i \nabla_j f), \\ h &= -\Delta f - R + \frac{n}{2\tau}, \\ \sigma &= -1. \end{aligned}$$

We see that $\frac{v}{2} - h - \frac{n\sigma}{2\tau} = 0$. Then

$$(6.6) \quad \frac{d\mathcal{W}}{dt} = \int_M \left[-(R + |\nabla f|^2) - \tau \frac{n}{2\tau} (2\Delta f - 2|\nabla f|^2) + 2\tau |R_{ij} + \nabla_i \nabla_j f|^2 - \Delta f - R + \frac{n}{2\tau} \right] (4\pi\tau)^{-n/2} e^{-f} dV.$$

Using the fact that

$$(6.7) \quad 0 = \int_M \Delta e^{-f} dV = \int_M (|\nabla f|^2 - \Delta f) e^{-f} dV,$$

we obtain

$$(6.8) \quad \begin{aligned} \frac{d\mathcal{W}}{dt} &= \int_M \left[2\tau |R_{ij} + \nabla_i \nabla_j f|^2 - 2\Delta f - 2R + \frac{n}{2\tau} \right] (4\pi\tau)^{-n/2} e^{-f} dV \\ &= \int_M 2\tau |R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}|^2 (4\pi\tau)^{-n/2} e^{-f} dV. \end{aligned}$$

The existence of a minimizer f for \mathcal{W} follows from [24] (reference indirectly from Jeff Viaclovsky).

The argument that $\nu(g(t))$ is nondecreasing along a Ricci flow is essentially the same as in Section 4 of the notes.

7. I.4.1

Remark : In the definition of noncollapsing, T could be infinite. This is why it is written that r_k^2/t_k stays bounded, while if $T < \infty$ then this is obviously the same as saying that r_k stays bounded.

In Section 5 of the notes we showed that in the case of flat \mathbb{R}^n , taking $e^{-f}(x) = e^{-\frac{|x|^2}{4\tau}}$, we get $\mathcal{W}(g, f, \tau) = 0$. So putting $\tau = r_k^2$ and $e^{-f_k}(x) = e^{-\frac{|x|^2}{4r_k^2}}$, we have $\mathcal{W}(g, f_k, r_k^2) = 0$. In the collapsing case, the idea is to use a test function f_k so that

$$(7.1) \quad e^{-f_k}(x) \sim e^{-c_k} e^{-\frac{\text{dist}_{t_k}(x, p_k)^2}{4r_k^2}},$$

where c_k is determined by the normalization condition

$$(7.2) \quad \int_M (4\pi r_k^2)^{-n/2} e^{-f_k} dV = 1.$$

The main difference between computing (7.2) in M and in \mathbb{R}^n comes from the difference in volumes, which means that $e^{-c_k} \sim \frac{1}{r_k^{-n} \text{vol}(B_k)}$. In particular, as $k \rightarrow \infty$, we have $c_k \rightarrow -\infty$.

Now that f_k is normalized correctly, the main difference between computing $\mathcal{W}(g(t_k), f_k, r_k^2)$ in M , and the analogous computation for the Gaussian in \mathbb{R}^n , comes from the f term in the integrand of \mathcal{W} . Since $f_k \sim c_k$, this will drive $\mathcal{W}(g(t_k), f_k, r_k^2)$ to $-\infty$ as $k \rightarrow \infty$, so $\mu(g(t_k), r_k^2) \rightarrow -\infty$.

To write this out precisely, let us put $\Phi = e^{-f/2}$, so that

$$(7.3) \quad \mathcal{W}(g, \Phi, \tau) = (4\pi\tau)^{-n/2} \int_M [4\tau |\nabla\Phi|^2 + (\tau R - 2 \ln \Phi - n) \Phi^2] dV.$$

For the argument it is enough to obtain small values of \mathcal{W} for positive Φ , and by approximation, it is enough to obtain small values of \mathcal{W} for nonnegative Φ . Take

$$(7.4) \quad \Phi_k(x) = e^{-c_k/2} \phi(\text{dist}_{t_k}(x, p_k)/r_k),$$

where $\phi : [0, \infty) \rightarrow [0, 1]$ is an appropriate function, to be specified later. The constant c_k is determined by

$$(7.5) \quad e^{c_k} = \int_M (4\pi r_k^2)^{-n/2} \phi^2(\text{dist}_{t_k}(x, p_k)/r_k) dV \leq (4\pi r_k^2)^{-n/2} \text{vol}(B_k).$$

Thus $c_k \rightarrow -\infty$. Next,

$$(7.6) \quad \mathcal{W}(g(t_k), \Phi_k, r_k^2) = (4\pi r_k^2)^{-n/2} \int_M [4r_k^2 |\nabla\Phi_k|^2 + (r_k^2 R - 2 \ln \Phi_k - n) \Phi_k^2] dV.$$

Let $A_k(s)$ be the mass of the distance sphere $S(p_k, r_k s)$ around p_k . Put

$$(7.7) \quad \bar{R}_k(s) = r_k^2 A_k(s)^{-1} \int_{S(p_k, r_k s)} R d \text{area}.$$

We can compute the integral in (7.6) radially to get

$$(7.8) \quad \mathcal{W}(g(t_k), \Phi_k, r_k^2) = \frac{\int_0^1 [4(\phi'(s))^2 + (\bar{R}_k(s) + c_k - 2 \ln \phi(s) - n) \phi^2(s)] A_k(s) ds}{\int_0^1 \phi^2(s) A_k(s) ds}.$$

Take ϕ to be a monotonically nonincreasing function such that $\phi(s) = 1$ if $s \in [0, 1/2]$, $\phi(s) = 0$ if $s \geq 1$ and $|\phi'(s)| \leq 10$ for $s \in [1/2, 1]$. The expression $4(\phi'(s))^2 - 2 \ln \phi(s) \phi^2(s)$ vanishes if $s \notin [1/2, 1]$, and is bounded above by $400 + e^{-1}$ if $s \in [1/2, 1]$. Then the lower bound on the Ricci curvature and the Bishop-Gromov inequality give

$$(7.9) \quad \frac{\int_0^1 [4(\phi'(s))^2 - 2 \ln \phi(s) \phi^2(s)] A_k(s) ds}{\int_0^1 \phi^2(s) A_k(s) ds} \leq 401 \frac{\text{vol}(B(p_k, r_k)) - \text{vol}(B(p_k, r_k/2))}{\text{vol}(B(p_k, r_k/2))} \\ \leq 401 \left(\frac{\int_0^1 \sinh^{n-1}(s) ds}{\int_0^{1/2} \sinh^{n-1}(s) ds} - 1 \right).$$

Next, from the upper bound on scalar curvature, $\bar{R}_k(s) \leq n(n-1)$ for $s \in [0, 1]$. Putting this together gives $\mathcal{W}(g(t_k), \Phi_k, r_k^2) \leq \text{const.} + c_k$ and so $\mathcal{W}(g(t_k), \Phi_k, r_k^2) \rightarrow -\infty$.

Thus $\mu(g(t_k), r_k^2) \rightarrow -\infty$. For any t_0 , $\mu(g(t), t_0 - t)$ is nondecreasing in t . Hence $\mu(g(0), t_k + r_k^2) \leq \mu(g(t_k), r_k^2)$, so $\mu(g(0), t_k + r_k^2) \rightarrow -\infty$. Since T is finite, t_k and r_k^2 are bounded, which gives the contradiction.

Remark : In the preceding argument we only used the upper bound on scalar curvature and the lower bound on Ricci curvature, i.e. in the definition of local collapsing one could have assumed that $R(g_{ij}(t_k)) \leq n(n-1)r_k^{-2}$ in B_k and $\text{Ric}(g_{ij}(t_k)) \geq -(n-1)r_k^{-2}$ in B_k . In fact, one can also remove the lower bound on Ricci curvature (observation of Perelman, communicated by Gang Tian). The necessary ingredients of the preceding argument were that

1. $r_k^{-n} \text{vol}(B(p_k, r_k)) \rightarrow 0$,
2. $r_k^2 R$ is uniformly bounded above on $B(p_k, r_k)$ and
3. $\frac{\text{vol}(B(p_k, r_k))}{\text{vol}(B(p_k, r_k/2))}$ is uniformly bounded above.

Suppose only that $r_k^{-n} \text{vol}(B(p_k, r_k)) \rightarrow 0$ and for all k , $r_k^2 R \leq n(n-1)$ on $B(p_k, r_k)$. If $\frac{\text{vol}(B(p_k, r_k))}{\text{vol}(B(p_k, r_k/2))} < 3^n$ for all k then we are done. If not, suppose that for a given k , $\frac{\text{vol}(B(p_k, r_k))}{\text{vol}(B(p_k, r_k/2))} \geq 3^n$. Putting $r'_k = r_k/2$, we have that $(r'_k)^{-n} \text{vol}(B(p_k, r'_k)) \leq r_k^{-n} \text{vol}(B(p_k, r_k))$ and $(r'_k)^2 R \leq n(n-1)$ on $B(p_k, r'_k)$. We replace r_k by r'_k . If now $\frac{\text{vol}(B(p_k, r_k))}{\text{vol}(B(p_k, r_k/2))} < 3^n$ then we stop. If not then we repeat the process and replace r_k by $r_k/2$. Eventually we will achieve that $\frac{\text{vol}(B(p_k, r_k))}{\text{vol}(B(p_k, r_k/2))} < 3^n$. Then we can apply the preceding argument to this new sequence of pairs $\{(p_k, r_k)\}_{k=1}^\infty$.

8. BASIC EXAMPLE FOR SECTION 7

If M is flat \mathbb{R}^n and p is the origin then

$$(8.1) \quad \gamma(\tau) = \left(\frac{\tau}{2}\right)^{1/2} \vec{q},$$

$$(8.2) \quad L(q, \bar{\tau}) = \frac{1}{2} \bar{\tau}^{-1/2} |q|^2,$$

$$(8.3) \quad l(q, \bar{\tau}) = \frac{|q|^2}{4\bar{\tau}}$$

and

$$(8.4) \quad \bar{L}(q, \bar{\tau}) = |q|^2.$$

9. REMARKS ABOUT THE REGULARITY OF \mathcal{L} -GEODESICS AND $\mathcal{L} \exp$

We are assuming that $(M, g(\cdot))$ is a Ricci flow, where the curvature operator of M is uniformly bounded on a τ -interval $[\tau_1, \tau_2]$, and each tau slice $(M, g(\tau))$ is complete for

$\tau \in [\tau_1, \tau_2]$. Thus by [15, Theorem 13.1], for every $\tau' < \tau_2$ there is a constant $D = D(\tau')$ such that

$$(9.1) \quad |\nabla R(x, \tau)| < D$$

for all $x \in M$, $\tau \leq \tau'$.

Making the change of variable $s = \sqrt{\tau}$ in the formula for \mathcal{L} -length, we get

$$(9.2) \quad \mathcal{L}(\gamma) = 2 \int_{s_1}^{s_2} \left(\frac{1}{4} \left| \frac{d\gamma}{ds} \right|^2 + s^2 R(\gamma(s)) \right) ds.$$

The Euler-Lagrange equation becomes

$$(9.3) \quad \nabla_{\hat{X}} \hat{X} - 2s^2 \nabla R + 4s \operatorname{Ric}(\hat{X}, \cdot) = 0,$$

where $\hat{X} := \frac{d\gamma}{ds} = 2sX$. Putting $s_1 = \sqrt{\tau_1}$, it follows from standard existence theory for ODE's that for each $p \in M$ and $v \in T_p M$, there is a unique solution $\gamma(s)$ to (9.3), defined on an interval $[s_1, s_1 + \epsilon)$, with $\gamma(s_1) = p$ and

$$\frac{1}{2} \gamma'(s_1) = \lim_{\tau \rightarrow \tau_1} \sqrt{\tau} \frac{d\gamma}{d\tau} = v.$$

If $\gamma(s)$ is defined for $s \in [s_1, s']$ then

$$\begin{aligned} \frac{d}{ds} |\hat{X}|^2 &= \frac{d}{ds} \langle \hat{X}, \hat{X} \rangle = 4s \operatorname{Ric}(\hat{X}, \hat{X}) + 2 \langle \nabla_{\hat{X}} \hat{X}, \hat{X} \rangle \\ &\leq C |\hat{X}|^2 \end{aligned}$$

by (9.3) and (9.1). Since the metrics $g(\tau)$ are uniformly comparable for $\tau \in [\tau_1, \tau_2]$, we conclude (by a continuity argument) that the \mathcal{L} -geodesic γ_v with $\frac{1}{2} \gamma'_v(s_1) = v$ is defined on the whole interval $[s_1, s_2)$, where $s_2 = \sqrt{\tau_2}$. In particular, for each $s = \sqrt{\tau} \in [s_1, s_2)$ and every $p \in M$, we get a globally defined and smooth \mathcal{L} -exponential map $\mathcal{L} \exp_\tau : T_p M \rightarrow M$ which takes each $v \in T_p M$ to $\gamma_v(\sqrt{\tau})$.

We now fix $p \in M$, take $\tau_1 = 0$, and let $L(q, \bar{\tau})$ be the minimizer function as in I.7.1. We can imitate the traditional Riemannian geometry proof that geodesics minimize for a short time. By the implicit function theorem, there is an $r = r(p) > 0$ (which varies continuously with p) such that for every $q \in M$ with $d(q, p) \leq r$ at $\tau = 0$, and every $\bar{\tau} \leq r^2$, there is a unique \mathcal{L} -geodesic $\gamma_{(q, \bar{\tau})} : [0, \bar{\tau}] \rightarrow M$, starting at p and ending at q , which remains within the ball $B(p, r)$ (in the $\tau = 0$ slice $(M, g(0))$), and $\gamma_{(q, \bar{\tau})}$ varies smoothly with $(q, \bar{\tau})$. Thus, the \mathcal{L} -length of $\gamma_{(q, \bar{\tau})}$ varies smoothly with $(q, \bar{\tau})$, and defines a function $\hat{L}(q, \bar{\tau})$ near $(p, 0)$. We claim that $\hat{L} = L$ near $(p, 0)$. For if $\alpha : [0, \bar{\tau}] \rightarrow M$ is a smooth curve then

$$\frac{d}{d\tau} \hat{L}(\alpha(\tau), \tau) = \langle 2\sqrt{\tau} X, \frac{d\alpha}{d\tau} \rangle + \sqrt{\tau} (R - |X|^2) \leq \sqrt{\tau} \left(R + \left| \frac{d\alpha}{d\tau} \right|^2 \right) = \frac{d}{d\tau} \left(\mathcal{L} \text{length}(\alpha|_{[0, \tau]}) \right);$$

compare [21, p. 15, b10-b8]. Thus $\gamma_{(q, \bar{\tau})}$ minimizes when $(q, \bar{\tau})$ is close to $(p, 0)$.

We can now deduce that for all $(q, \bar{\tau})$, there is an \mathcal{L} -geodesic $\gamma : [0, \bar{\tau}] \rightarrow M$ which has infimal \mathcal{L} -length among all piecewise smooth curves starting at p and ending at q (with domain $[0, \bar{\tau}]$). This can be done by imitating the usual broken geodesic argument.

Another technical issue is the justification of the change of variables from M to $T_p M$ in the proof of monotonicity of reduced volume, before I.(7.13). Fix $p \in M$ and $\tau > 0$, and let

$\mathcal{L} \exp_\tau : T_p M \rightarrow M$ be the map which takes $v \in T_p M$ to $\gamma_v(\tau)$, where $\gamma_v : [0, \tau] \rightarrow M$ is the unique \mathcal{L} -geodesic with $\sqrt{\tau} \frac{d\gamma_v}{d\tau} \rightarrow v$ as $\tau \rightarrow 0$. Let $\mathcal{B} \subset M$ be the set of points which are either endpoints of more than one minimizing \mathcal{L} -geodesic, or which are the endpoint of a minimizing geodesic $\gamma_v : [0, \tau] \rightarrow M$ where $v \in T_p M$ is a critical point of $\mathcal{L} \exp_\tau$. Let $\mathcal{G} \subset M$ be the complement of \mathcal{B} , and let $\Omega_p \subset T_p M$ be the corresponding set of initial conditions. Then Ω_p is an open set, and $\mathcal{L} \exp$ maps it diffeomorphically onto \mathcal{G} . We claim that \mathcal{B} has measure zero. By Sard's theorem, to prove this it suffices to prove that the set \mathcal{B}' of points $q \in \mathcal{B}$ which are regular values of $\mathcal{L} \exp_\tau$, has measure zero. Pick $q \in \mathcal{B}'$, and distinct points $v_1, v_2 \in T_p M$ such that $\gamma_{v_i} : [0, \tau] \rightarrow M$ are both minimizing geodesics ending at q . Then $\mathcal{L} \exp_\tau$ is a local diffeomorphism near each v_i . The first variation formula and the implicit function theorem then show that there are neighborhoods U_i of v_i , and a smooth hypersurface H passing through q , such that if we have points $w_i \in U_i$ with

$$q' := \mathcal{L} \exp_\tau(w_1) = \mathcal{L} \exp_\tau(w_2) \quad \text{and} \quad \mathcal{L} \text{ length}(\gamma_{w_1}) = \mathcal{L} \text{ length}(\gamma_{w_2}),$$

then q' lies on H . Thus \mathcal{B}' is contained in a countable union of hypersurfaces, and hence has measure zero.

Therefore one may compute the integral of any integrable function on M by pulling it back to $\Omega_p \subset T_p M$ and using the change of variables formula.

10. DERIVATION OF I.(7.7)

A remark about the notation of I.7.1 : L is a function of a point q and a time $\bar{\tau}$. The notation $L_{\bar{\tau}}$ refers to the partial derivative with respect to $\bar{\tau}$, i.e. differentiation while keeping q fixed. The notation $\frac{d}{d\tau}$ refers to differentiation along an \mathcal{L} -geodesic, i.e. simultaneously varying both the point and the time.

From the general equation for the Levi-Civita connection in terms of the metric [6, (1.29)], if $g(\tau)$ is a 1-parameter family of metrics, with $\dot{g} = \frac{dg}{d\tau}$ and $\dot{\nabla} = \frac{d\nabla}{d\tau}$, then

$$(10.1) \quad 2\langle \dot{\nabla}_X Y, Z \rangle = (\nabla_X \dot{g})(Y, Z) + (\nabla_Y \dot{g})(Z, X) - (\nabla_Z \dot{g})(X, Y).$$

In our case $\dot{g} = 2 \text{ Ric}$ and so

$$(10.2) \quad \begin{aligned} \frac{d}{d\tau} \langle \nabla_Y Y, X \rangle &= \langle \nabla_X \nabla_Y Y, X \rangle + \langle \nabla_Y Y, \nabla_X X \rangle + 2 \text{ Ric}(\nabla_Y Y, X) + \langle \dot{\nabla}_Y Y, X \rangle \\ &= \langle \nabla_X \nabla_Y Y, X \rangle + \langle \nabla_Y Y, \nabla_X X \rangle + \\ &\quad 2 \text{ Ric}(\nabla_Y Y, X) + 2(\nabla_Y \text{ Ric})(Y, X) - (\nabla_X \text{ Ric})(Y, Y). \end{aligned}$$

On the other hand,

$$(10.3) \quad \begin{aligned} 2Y \cdot \text{ Ric}(Y, X) - X \cdot \text{ Ric}(Y, Y) &= 2(\nabla_Y \text{ Ric})(Y, X) + 2 \text{ Ric}(\nabla_Y Y, X) + 2 \text{ Ric}(Y, \nabla_Y X) \\ &\quad - (\nabla_X \text{ Ric})(Y, Y) - 2 \text{ Ric}(\nabla_X Y, Y) \\ &= 2 \text{ Ric}(\nabla_Y Y, X) + 2(\nabla_Y \text{ Ric})(Y, X) \\ &\quad - (\nabla_X \text{ Ric})(Y, Y) - 2 \text{ Ric}([X, Y], Y). \end{aligned}$$

We are assuming that the variation field Y satisfies $[X, Y] = 0$ (this was used in I.(7.1)). Hence one obtains the formula

$$(10.4) \quad \frac{d}{d\tau} \langle \nabla_Y Y, X \rangle = \langle \nabla_X \nabla_Y Y, X \rangle + \langle \nabla_Y Y, \nabla_X X \rangle + 2Y \cdot \text{Ric}(Y, X) - X \cdot \text{Ric}(Y, Y).$$

Next, using (10.2) and I.(7.2),

(10.5)

$$\begin{aligned} 2\bar{\tau}^{1/2} \langle \nabla_Y Y, X \rangle &= 2 \int_0^{\bar{\tau}} \frac{d}{d\tau} (\tau^{1/2} \langle \nabla_Y Y, X \rangle) d\tau \\ &= \int_0^{\bar{\tau}} \tau^{1/2} \left[\frac{1}{\tau} \langle \nabla_Y Y, X \rangle + 2 \frac{d}{d\tau} \langle \nabla_Y Y, X \rangle \right] d\tau \\ &= \int_0^{\bar{\tau}} \tau^{1/2} \left[\frac{1}{\tau} \langle \nabla_Y Y, X \rangle + 2 \langle \nabla_X \nabla_Y Y, X \rangle + 2 \langle \nabla_Y Y, \nabla_X X \rangle + \right. \\ &\quad \left. 4 \text{Ric}(\nabla_Y Y, X) + 4(\nabla_Y \text{Ric})(Y, X) - 2(\nabla_X \text{Ric})(Y, Y) \right] d\tau \\ &= \int_0^{\bar{\tau}} \tau^{1/2} [2 \langle \nabla_X \nabla_Y Y, X \rangle + (\nabla_Y Y)R + \\ &\quad 4(\nabla_Y \text{Ric})(Y, X) - 2(\nabla_X \text{Ric})(Y, Y)] d\tau. \end{aligned}$$

It follows that

$$\begin{aligned} (10.6) \quad \delta_Y^2 \mathcal{L} - 2\bar{\tau}^{1/2} \langle \nabla_Y Y, X \rangle &= \int_0^{\bar{\tau}} \tau^{1/2} [Y \cdot Y \cdot R - (\nabla_Y Y)R + 2 \langle R(Y, X)Y, X \rangle + \\ &\quad 2 |\nabla_X Y|^2 - 4(\nabla_Y \text{Ric})(Y, X) + 2(\nabla_X \text{Ric})(Y, Y)] d\tau \\ &= \int_0^{\bar{\tau}} \tau^{1/2} [\text{Hess}(Y, Y)R + 2 \langle R(Y, X)Y, X \rangle + 2 |\nabla_X Y|^2 \\ &\quad - 4(\nabla_Y \text{Ric})(Y, X) + 2(\nabla_X \text{Ric})(Y, Y)] d\tau. \end{aligned}$$

11. DERIVATION OF I.(7.10)

If $\{Y_i\}$ is an orthonormal basis at $\gamma(\bar{\tau})$, let $Y_i(\tau)$ be the corresponding vector field as in I.(7.8). Then as in the computation below I.(7.8), it follows that $\langle Y_i(\tau), Y_j(\tau) \rangle = \frac{\tau}{\bar{\tau}} \delta_{ij}$. Putting $Y_i(\tau) = \left(\frac{\tau}{\bar{\tau}}\right)^{1/2} e_i(\tau)$, substituting into I.(7.9) and summing over i gives

$$(11.1) \quad \Delta L \leq \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} \tau^{3/2} \sum_i H(X, e_i) d\tau,$$

where

(11.2)

$$\sum_i H(X, e_i) = -\Delta R + 2 \text{Ric}(X, X) -$$

$$4(\langle \nabla R, X \rangle - \sum_i \nabla_{e_i} \text{Ric}(e_i, X)) - 2 \sum_i \text{Ric}_\tau(e_i, e_i) + 2|\text{Ric}|^2 - \frac{1}{\tau} R.$$

Tracing the second Bianchi identity gives

$$(11.3) \quad \sum_i \nabla_{e_i} \text{Ric}(e_i, X) = \frac{1}{2} \langle \nabla R, X \rangle.$$

From a known identity,

$$(11.4) \quad R_\tau = -\Delta R - 2|\text{Ric}|^2.$$

On the other hand,

$$(11.5) \quad R_\tau = (g^{ij} R_{ij})_\tau = -2|\text{Ric}|^2 + \sum_i \text{Ric}_\tau(e_i, e_i),$$

so

$$(11.6) \quad \sum_i \text{Ric}_\tau(e_i, e_i) = -\Delta R.$$

Putting all this together gives $\sum_i H(X, e_i) = H(X)$, where $H(X)$ is defined in I.(7.3). Thus

$$(11.7) \quad \begin{aligned} \Delta L &\leq \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}} \int_0^{\bar{\tau}} \tau^{3/2} H(X) d\tau \\ &= \frac{n}{\sqrt{\bar{\tau}}} - 2\sqrt{\bar{\tau}}R - \frac{1}{\bar{\tau}} K. \end{aligned}$$

12. DERIVATION OF I.(7.11)

The Hessian of \mathcal{L} is given by

$$(12.1) \quad \text{Hess}_L(Y, Y) = \delta_Y^2 \mathcal{L} - \delta_{\nabla_Y Y} \mathcal{L} = \delta_Y^2 \mathcal{L} - 2\sqrt{\bar{\tau}} \langle X, \nabla_Y Y \rangle,$$

which is given by the right-hand-side of (10.6). As with ordinary geodesics, the equation for an \mathcal{L} -Jacobi field will differ from the integrand in (10.6) by a total derivative, namely

$$(12.2) \quad \frac{d}{d\tau} (2\tau^{1/2} \langle Y, \nabla_X Y \rangle).$$

Then if Y is an \mathcal{L} -Jacobi field,

$$(12.3) \quad \text{Hess}_L(Y, Y) = 2\bar{\tau}^{1/2} \langle Y, \nabla_X Y \rangle = 2\bar{\tau}^{1/2} \langle Y, \nabla_Y X \rangle.$$

13. MONOTONICITY OF \tilde{V}

Fixing $p \in M$, we can write

$$(13.1) \quad \tilde{V}(\tau) = \int_{T_p M} \tau^{-n/2} e^{-l(X, \tau)} J(X, \tau) dX,$$

where the integration is up to the cut locus, $l(X, \tau)$ is computed along the \mathcal{L} -exponential of X and similarly for $J(X, \tau)$. The monotonicity of \tilde{V} follows from that of $\tau^{-n/2} e^{-l(X, \tau)} J(X, \tau)$.

14. LAST STATEMENT IN I.7.1

We have

$$(14.1) \quad R_\tau + \frac{2}{n} R^2 \leq -\Delta R,$$

so

$$(14.2) \quad R^{-2} R_\tau + \frac{2}{n} \leq -R^{-2} \Delta R.$$

If $r(\tau) = \min_x R(x, \tau)$ then

$$(14.3) \quad -\frac{dr^{-1}}{d\tau} + \frac{2}{n} \leq 0,$$

so $r^{-1}(\tau) - \frac{2\tau}{n}$ is monotonically nondecreasing.

Suppose that for some $\tau \in [0, \tau_0]$, we have $r(\tau) < -\frac{n}{2(\tau_0 - \tau)}$. Then

$$(14.4) \quad r^{-1}(\tau) - \frac{2\tau}{n} > -\frac{2\tau_0}{n}.$$

As

$$(14.5) \quad r^{-1}(\tau) - \frac{2\tau}{n} \leq r^{-1}(\tau_0) - \frac{2\tau_0}{n},$$

it follows that $r^{-1}(\tau_0) > 0$. As $r^{-1}(\tau) < 0$, r^{-1} must vanish in $[\tau, \tau_0]$, which is a contradiction.

15. I.7.2

Put

$$(15.1) \quad \begin{aligned} P_{abc} &= \nabla_a R_{bc} - \nabla_b R_{ac}, \\ M_{ab} &= \Delta R_{ab} - \frac{1}{2} \nabla_a \nabla_b R + 2 R_{acbc} R_{cd} - R_{ac} R_{bc} + \frac{R_{ab}}{2t}. \end{aligned}$$

Given a 2-form U and a 1-form W , put

$$(15.2) \quad Z(U, W) = M_{ab} W_a W_b + 2 P_{abc} U_{ab} W_c + R_{abcd} U_{ab} U_{cd}.$$

Hamilton's Harnack inequality says that with a nonnegative curvature operator, if $Z(t_0) \geq 0$ then $Z(t) \geq 0$ for all $t \geq t_0$ [15, Theorem 14.1]. Taking $W_a = Y_a$ and $U_{ab} = (Y_a X_b - X_a Y_b)/2$, and using the fact that

$$(15.3) \quad \text{Ric}_\tau(Y, Y) = -(\Delta R_{ab}) Y^a Y^b - 2 R_{acbd} R_{cd} Y^a Y^b + 2 R_{ac} R_{bc} Y^a Y^b,$$

one gets that $H(X, Y) = 2Z(U, W)$.

The choice of time parameter τ is arbitrary up to a constant. Defining M by $H(X, Y) = M(\tau) - \frac{\text{Ric}(Y, Y)}{\tau}$, it follows that for all a , we can equally well apply the Harnack inequality to $M(\tau) - \frac{\text{Ric}(Y, Y)}{\tau+a}$. Now suppose that for some $\tau \in (0, \tau_0)$, we have

$$(15.4) \quad H(X, Y) < -\text{Ric}(Y, Y) \left(\frac{1}{\tau} + \frac{1}{\tau_0 - \tau} \right).$$

Then $M(\tau) - \frac{\text{Ric}(Y,Y)}{\tau-\tau_0} < 0$. We would get a contradiction with the Harnack inequality unless for all $\tau' \in (\tau, \tau_0)$, we have $M(\tau') - \frac{\text{Ric}(Y,Y)}{\tau'-\tau_0} < 0$. However, as $\text{Ric}(Y,Y) \geq 0$, if $\text{Ric}(Y,Y) \neq 0$ then this implies that $M(\tau')$ is unbounded below as $\tau' \rightarrow \tau_0$, which is a contradiction. We are interested in generic Y and if $\text{Ric} = 0$ then g is flat, in which case the first inequality at the bottom of p. 18 of I is trivially true.

The inequality at the top of p. 19 of I should read

$$(15.5) \quad H(X) \geq -nR \left(\frac{1}{\tau} + \frac{1}{\tau_0 - \tau} \right),$$

but this will not affect the argument. From I.(7.6),

$$(15.6) \quad \begin{aligned} 4\tau|\nabla l|^2 &= -4\tau R + 4l - \frac{4}{\sqrt{\tau}} \int_0^\tau \tilde{\tau}^{3/2} H(X) d\tilde{\tau} \\ &\leq -4\tau R + 4l + \frac{4n}{\sqrt{\tau}} \int_0^\tau \tilde{\tau}^{3/2} R \left(\frac{1}{\tilde{\tau}} + \frac{1}{\tau_0 - \tilde{\tau}} \right) d\tilde{\tau} \\ &= -4\tau R + 4l + \frac{4n}{\sqrt{\tau}} \int_0^\tau \tilde{\tau}^{1/2} R \frac{\tau_0}{\tau_0 - \tilde{\tau}} d\tilde{\tau} \\ &\leq -4\tau R + 4l + \frac{4n}{c\sqrt{\tau}} \int_0^\tau \tilde{\tau}^{1/2} R d\tilde{\tau} \\ &\leq -4\tau R + 4l + \frac{8nl}{c}, \end{aligned}$$

where the last line uses the first equation in I.7.1. Equation I.(7.16) follows, and the proof of I.(7.17) is similar.

16. I.7.3

The proof is an argument by contradiction, assuming initially that $\epsilon_k \rightarrow 0$.

Given an \mathcal{L} -geodesic $\gamma(\tau)$ with velocity vector $X(\tau) = \frac{d\gamma}{d\tau}$, its initial vector is $X_0 = \lim_{\tau \rightarrow 0} \sqrt{\tau} X(\tau)$. If $|X_0|$ is small compared to $\epsilon_k^{-1/2}$, we want to show that γ does not escape from B_k in time $\epsilon_k r_k^2$. (We will not worry about constants.)

We have

$$(16.1) \quad \begin{aligned} \frac{d}{d\tau} \langle X(\tau), X(\tau) \rangle &= 2 \text{Ric}(X, X) + 2 \langle X, \nabla_X X \rangle \\ &= 2 \text{Ric}(X, X) + \langle X, \nabla R - \frac{1}{\tau} X - 4 \text{Ric}(X, \cdot) \rangle \\ &= -\frac{|X|^2}{\tau} - 2 \text{Ric}(X, X) + \langle X, \nabla R \rangle, \end{aligned}$$

so

$$(16.2) \quad \frac{d}{d\tau} (\tau |X|^2) = -2\tau \text{Ric}(X, X) + \tau \langle X, \nabla R \rangle.$$

Letting C denote a generic n -dependent constant, for $x \in B(p, r_k/2)$ and $t \in [t_k - r_k^2/2, t_k]$, the fact that g satisfies the Ricci flow gives an estimate $|\nabla R|(x, t) \leq Cr_k^{-3}$ [15, Theorem

13.1]. Then

$$(16.3) \quad \left| \frac{d}{d(\tau/r_k^2)} (\tau|X|^2) \right| \leq C \tau |X|^2 + C (\tau/r_k^2)^{1/2} (\tau|X|^2)^{1/2}.$$

Equivalently,

$$(16.4) \quad \left| \frac{d}{d(\tau/r_k^2)} (\sqrt{\tau} |X|) \right| \leq C \sqrt{\tau} |X| + C (\tau/r_k^2)^{1/2},$$

from which

$$(16.5) \quad \sqrt{\tau} |X| \leq e^{C\tau/r_k^2} |X_0| + C \int_0^{\tau/r_k^2} \sqrt{s} e^{C(\tau/r_k^2 - s)} ds.$$

Thus

$$(16.6) \quad r_k^{-1} \int_0^{\bar{\tau}} |X(\tau)| d\tau \leq |X_0| \int_0^{\bar{\tau}/r_k^2} u^{-1/2} e^{Cu} du + C \int_0^{\bar{\tau}/r_k^2} u^{-1/2} \int_0^u \sqrt{s} e^{C(u-s)} ds du.$$

Taking $\bar{\tau} = \epsilon_k r_k^2$ and $|X_0| \leq c \epsilon_k^{-1/2}$, with c small and ϵ_k very small, it follows that

$$(16.7) \quad \int_0^{\epsilon_k r_k^2} |X(\tau)| d\tau \leq 2 c r_k.$$

From the Ricci flow equation $g_\tau = 2 \text{ Ric}$, it follows that the metrics $g(\tau)$ between $\tau = 0$ and $\bar{\tau} = \epsilon_k r_k^2$ are $e^{C\epsilon_k}$ -close to each other. Then for ϵ_k small, the length of γ , as measured with the metric at time t_k , will be at most $3 c r_k$.

As $\tau \rightarrow 0$, $\tau^{-n/2} e^{-l(X_0, \tau)} J(X_0, \tau) \rightarrow e^{-|X_0|^2/4}$. Then from monotonicity,

$$(16.8) \quad \int_{T_p M - B(0, c \epsilon_k^{-1/2})} \tau^{-n/2} e^{-l(X_0, \tau)} J(X_0, \tau) d^n X_0 \leq \int_{T_p M - B(0, c \epsilon_k^{-1/2})} e^{-|X_0|^2/4} d^n X_0 \leq C e^{-\frac{c^2}{4\epsilon_k}}$$

for k large.

17. I.8.3(B)

If γ is a minimal geodesic from x_0 to x_1 then for any vector field V along γ that vanishes at the endpoints,

$$(17.1) \quad \int_0^{d(x_0, x_1)} \left(|\nabla_X V|^2 + \langle R(V, X)V, X \rangle \right) ds \geq 0.$$

Take V to be linear from $s = 0$ to $s = r_0$, parallel from $s = r_0$ to $s = d(x_0, x_1) - r_0$ and linear from $s = d(x_0, x_1) - r_0$ to $s = d(x_0, x_1)$. Adding over an orthonormal frame of parallel vectors gives

$$(17.2) \quad - \int_\gamma \text{Ric}(X, X) \geq -2(n-1) \left(\frac{2}{3} K r_0 + r_0^{-1} \right).$$

Corollary 17.3. [15, Theorem 17.2] *If $\text{Ric} \leq K$ globally, with $K > 0$, then for all $x_0, x_1 \in M$,*

$$(17.4) \quad \frac{d}{dt} \text{dist}_t(x_0, x_1) \geq -\text{const.}(n) K^{1/2}.$$

Proof. Put $r_0 = K^{-1/2}$. If $\text{dist}_t(x_0, x_1) > 2r_0$ then the corollary follows from Lemma I.8.3(b). If $\text{dist}_t(x_0, x_1) \leq 2r_0$ then we have the estimate

$$(17.5) \quad \int_{\gamma} -\text{Ric}(X, X) \geq -K \text{dist}_t(x_0, x_1) \geq -2K^{1/2}.$$

□

18. I.8.2

We want a lower bound on $\bar{L}(x, \frac{1}{2})$ (as defined using \mathcal{L} -geodesics from x_0) for x satisfying $\text{dist}_{\frac{1}{2}}(x, x_0) \leq \frac{1}{10}$. Applying the maximum principle to I.(7.15) gave a global lower bound on \bar{L} . The idea is to spatially localize this estimate near x_0 , by means of a radial function ϕ .

To get I.(8.1), it suffices to take $\phi(t) = \frac{1}{e^{(2A+100n)(\frac{1}{10}-t)} - 1}$ for t near $\frac{1}{10}$.

To see that $\bar{L} + 2n + 1 \geq 1$ for $t \geq \frac{1}{2}$: From the last remark at the end of I.7.1,

$$(18.1) \quad R(\cdot, \tau) \geq -\frac{n}{2(1-\tau)}.$$

Then for $\bar{\tau} \in [0, \frac{1}{2}]$,

$$(18.2) \quad L(q, \bar{\tau}) \geq -\int_0^{\bar{\tau}} \sqrt{\tau} \frac{n}{2(1-\tau)} d\tau \geq -n \int_0^{\bar{\tau}} \sqrt{\tau} d\tau = -\frac{2n}{3} \bar{\tau}^{3/2}.$$

Hence

$$(18.3) \quad \bar{L}(q, \bar{\tau}) = 2\bar{\tau}^{1/2} L(q, \bar{\tau}) \geq -\frac{4n}{3} \bar{\tau}^2 \geq -\frac{n}{3}.$$

19. I.(9.1)

The right-hand-side of I.(9.1) should be multiplied by u .

To derive it, we first claim that

$$(19.1) \quad \frac{d\Delta}{dt} = 2 R_{ij} \nabla_i \nabla_j.$$

To see this, for $f_1, f_2 \in C_c^\infty(M)$, we have

$$(19.2) \quad \int_M f_1 \Delta f_2 dV = - \int_M \langle df_1, df_2 \rangle dV,$$

so

$$(19.3) \quad \int_M f_1 \frac{d\Delta}{dt} f_2 dV - \int_M f_1 \Delta f_2 R dV = -2 \int_M \text{Ric}(df_1, df_2) dV + \int_M \langle df_1, df_2 \rangle R dV,$$

so

$$(19.4) \quad \frac{d\Delta}{dt} f_2 - R \Delta f_2 = 2 \nabla_i (R_{ij} \nabla_j f_2) - \nabla_i (R \nabla_i f_2).$$

Then (19.1) follows from the traced second Bianchi identity.

Next, one can check that $\square^* u = 0$ is equivalent to

$$(19.5) \quad (\partial_t + \Delta) f = \frac{n}{2} \frac{1}{T-t} + |\nabla f|^2 - R.$$

Then one obtains

$$(19.6) \quad \begin{aligned} u^{-1} \square^* v &= - (\partial_t + \Delta) [(T-t)(2\Delta f - |\nabla f|^2 + R) + f] - \\ &\quad 2 \langle \nabla [(T-t)(2\Delta f - |\nabla f|^2 + R) + f], u^{-1} \nabla u \rangle \\ &= 2\Delta f - |\nabla f|^2 + R - (T-t)(\partial_t + \Delta)(2\Delta f - |\nabla f|^2 + R) \\ &\quad - (\partial_t + \Delta) f + 2(T-t) \langle \nabla(2\Delta f - |\nabla f|^2 + R), \nabla f \rangle + 2|\nabla f|^2. \end{aligned}$$

Now

$$(19.7) \quad \begin{aligned} (\partial_t + \Delta)(2\Delta f - |\nabla f|^2 + R) &= 2(\partial_t \Delta) f + 2\Delta(\partial_t + \Delta) f \\ &\quad - (\partial_t + \Delta) |\nabla f|^2 + (\partial_t + \Delta) R \\ &= 4 R_{ij} \nabla_i \nabla_j f + 2\Delta(|\nabla f|^2 - R) - 2 \operatorname{Ric}(df, df) \\ &\quad - 2 \langle \nabla f_t, \nabla f \rangle - \Delta |\nabla f|^2 + \Delta R + 2|\operatorname{Ric}|^2 + \Delta R \\ &= 4 R_{ij} \nabla_i \nabla_j f + 2\Delta|\nabla f|^2 - 2 \operatorname{Ric}(df, df) \\ &\quad - 2 \langle \nabla(-\Delta f + |\nabla f|^2 - R), \nabla f \rangle - \Delta |\nabla f|^2 + 2|\operatorname{Ric}|^2. \end{aligned}$$

Hence the term in $u^{-1} \square^* v$ proportionate to $(T-t)^{-1}$ is

$$(19.8) \quad -\frac{n}{2} \frac{1}{T-t}.$$

The term proportionate to $(T-t)^0$ is

$$(19.9) \quad 2\Delta f - |\nabla f|^2 + R - |\nabla f|^2 + R + 2|\nabla f|^2 = 2(\Delta f + R).$$

The term proportionate to $(T-t)$ is $(T-t)$ times

$$(19.10) \quad \begin{aligned} &- 4 R_{ij} \nabla_i \nabla_j f - 2\Delta|\nabla f|^2 + 2 \operatorname{Ric}(df, df) + \\ &2 \langle \nabla(-\Delta f + |\nabla f|^2 - R), \nabla f \rangle + \Delta|\nabla f|^2 - 2|\operatorname{Ric}|^2 + \\ &2 \langle \nabla(2\Delta f - |\nabla f|^2 + R), \nabla f \rangle = \\ &- 4 R_{ij} \nabla_i \nabla_j f - \Delta|\nabla f|^2 + 2 \operatorname{Ric}(df, df) + 2 \langle \nabla \Delta f, \nabla f \rangle - 2|\operatorname{Ric}|^2 = \\ &- 4 R_{ij} \nabla_i \nabla_j f - 2|\operatorname{Hess}(f)|^2 - 2|\operatorname{Ric}|^2. \end{aligned}$$

Putting this together gives

$$(19.11) \quad \square^* v = -2(T-t) \left| R_{ij} + \nabla_i \nabla_j f - \frac{1}{2(T-t)} g_{ij} \right|^2 u.$$

20. I.9.2

The statement of the corollary should have $\max v/u$ instead of $\min v/u$.

To prove it, we have

$$(20.1) \quad (\partial_t + \Delta) \frac{v}{u} = \frac{v \square^* u - u \square^* v}{u^2} - \frac{2}{u^3} \langle \nabla u, (u \nabla v - v \nabla u) \rangle.$$

Then using the fact that $\square^* v \leq 0$, at a maximum point p of v/u we have

$$(20.2) \quad \left(\partial_t \frac{v}{u} \right) (p) \geq 0.$$

21. I.9.3

We have

$$(21.1) \quad \frac{d}{dt} \int_M h v dV = \int_M ((\partial_t - \Delta) h v + h(\partial_t + \Delta - R)v) dV = - \int_M h \square^* v dV \geq 0.$$

As $t \rightarrow T$, the computation of $\int_M h v$ approaches the flat-space calculation, which one finds to be zero.

22. I.9.5

The statement at the top of page 23 of I should be $\square^* ((4\pi\tau)^{-n/2} e^{-l}) \leq 0$.

From this and the fact that $\square^* ((4\pi\tau)^{-n/2} e^{-f}) = 0$, the argument of the proof of Corollary I.9.2 gives that $\max e^{f-l}$ is nondecreasing in t , so $\max(f-l)$ is nondecreasing in t . As $t \rightarrow T$ one obtains the flat-space result, namely that $f-l$ vanishes. Thus $f(t) \leq l(T-t)$ for all $t \in [0, T)$.

To give an alternative proof, putting $\tau = T - t$, Corollary I.9.4 says that

$$(22.1) \quad \frac{d}{d\tau} f(\gamma(\tau), \tau) \leq \frac{1}{2} \left(R(\gamma(\tau), \tau) + |\dot{\gamma}(t)|^2 \right) - \frac{1}{2\tau} f(\gamma(\tau), \tau),$$

or

$$(22.2) \quad \frac{d}{d\tau} (\tau^{1/2} f(\gamma(\tau), \tau)) \leq \frac{1}{2} \tau^{1/2} \left(R(\gamma(\tau), \tau) + |\dot{\gamma}(t)|^2 \right).$$

For small τ ,

$$(22.3) \quad f(\gamma(\tau), \tau) \sim d(p, \gamma(\tau))^2/4\tau = O(\tau^0).$$

Then integration gives $\tau^{1/2} f \leq \frac{1}{2} L$, or $f \leq l$.

23. IDEA OF PROOF OF I.10.1

As described after the statement of Theorem I.10.1, the proof of I.10.1 is an argument by contradiction. One assumes that one has a Ricci flow solution, and picks a point (\bar{x}, \bar{t}) where the desired curvature bound does not hold. One can assume, roughly speaking, that (\bar{x}, \bar{t}) is the first point in the given solution where the bound does not hold; see I.(10.4). One now considers the solution u to the conjugate heat equation, starting as a δ -function at (\bar{x}, \bar{t}) , and

the corresponding function v . We know that $v \leq 0$ and the goal is to get a negative upper bound for its integral at time $t = 0$. Using monotonicity of $\int v$, it suffices to establish such an upper bound for the integral of v over an appropriate ball B at a time \tilde{t} near \bar{t} (Claim 3 of I.10.1). The argument to get such a bound is by contradiction. If there were not such a bound then one could consider a rescaled sequence of counterexamples with $\int_B v \rightarrow 0$, and try to take a limit. If the limit exists then one obtains a solution with $\int_B v = 0$, which implies that the solution is a gradient shrinking soliton, which violates the assumptions. If the limit doesn't exist then one can do a further rescaling to see that in fact $\int_B v \rightarrow -\infty$ for some subsequence, which is a contradiction.

The argument now uses the fact that $\int v$ is the expression that appears in the logarithmic Sobolev inequality. If the isoperimetric constant is sufficiently close to the Euclidean value c_n , one concludes that $\int v$ must be bounded below by a constant close to zero, which contradicts the negative upper bound on $\int v$. In order to make the argument precise, before invoking the logarithmic Sobolev inequality, one multiplies u by a bump function h .

24. ARGUMENT FOR CLAIM 2 OF I.10.1

From Claim 1 of I.10.1, we know that I.(10.1) is satisfied under the condition I.(10.2). The spacetime region described in I.(10.2) is not a product region, due to the fact that $d(x, t)$ is time-dependent. The goal is to obtain the estimate I.(10.1) on a product region in spacetime. To do so, one needs to estimate how fast distances are changing with respect to t .

We claim first that if (x, t) satisfies $\bar{t} - \frac{1}{2}\alpha Q^{-1} \leq t \leq \bar{t}$ and $d(x, t) \leq d(\bar{x}, \bar{t}) + AQ^{-1/2}$ then $|Rm|(x, t) \leq 4Q$. To see this, if $(x, t) \in M_\alpha$ then it is true by Claim 1. If $(x, t) \notin M_\alpha$ then $|Rm|(x, t) < \alpha t^{-1}$. As $(\bar{x}, \bar{t}) \in M_\alpha$, we know that $Q \geq \alpha \bar{t}^{-1}$. Then $t \geq \frac{1}{2}\bar{t}$ and so $|Rm|(x, t) < 2\alpha \bar{t}^{-1} \leq 2Q$.

Thus we have a uniform curvature bound on the time- t distance ball $B(x_0, d(\bar{x}, \bar{t}) + AQ^{-1/2})$, provided that $\bar{t} - \frac{1}{2}\alpha Q^{-1} \leq t \leq \bar{t}$. Suppose that (x, t) satisfies I.(10.3) but does not satisfy I.(10.1). As stated in I, x lies in the time- \bar{t} distance ball $B(x_0, d(\bar{x}, \bar{t}) + \frac{1}{10}AQ^{-1/2})$. Applying Lemma I.8.3(b) with $r_0 = \frac{1}{2}AQ^{-1/2}$ and the above curvature bound, we obtain

$$(24.1) \quad \text{dist}_t(x, \bar{x}) - \text{dist}_{\bar{t}}(x, \bar{x}) \leq \frac{1}{2}\alpha Q^{-1} \cdot 2(n-1) \left(\frac{2}{3} \cdot 4Q \left(\frac{1}{2}AQ^{-1/2} \right) + 2A^{-1}Q^{1/2} \right).$$

Assuming that $A \geq 1$ (we'll take $A \rightarrow \infty$ later) and using the fact that $\alpha < \frac{1}{100n}$, it follows that $d(\bar{x}, t) \leq d(\bar{x}, \bar{t}) + \frac{1}{2}AQ^{-1/2}$. This shows that it is self-consistent to use the curvature bound in the above application of Lemma I.8.3(b).

25. ARGUMENT FOR CLAIM 3 OF I.10.1

If the injectivity radii of the scaled metrics are bounded away from zero : From Corollary I.9.3, $v \leq 0$. If $\int_B v = 0$ at time \tilde{t} then v vanishes on B at time \tilde{t} . Let h be a solution to the heat equation on $M \times [\tilde{t}, \bar{t}]$ with $h(x, \tilde{t})$ a nonnegative nonzero function supported in B . As in the proof of Corollary I.9.3, $\int_M hv \, dV$ is nondecreasing in t and vanishes for $t = \tilde{t}$

and $t \rightarrow \bar{t}$. Thus $\int_M hv \, dV$ vanishes for all $t \in [\tilde{t}, \bar{t})$. However, for $t \in (\tilde{t}, \bar{t})$, h is strictly positive and v is nonpositive. Thus v vanishes on M for all $t \in (\tilde{t}, \bar{t})$, and so

$$(25.1) \quad R_{ij} + \nabla_i \nabla_j f - \frac{1}{2(\bar{t} - t)} g_{ij} = 0.$$

on this interval. From the evolution equation,

$$(25.2) \quad \frac{dg_{ij}}{dt} = -2R_{ij} = 2\nabla_i \nabla_j f - \frac{1}{\bar{t} - t} g_{ij}.$$

It follows that the sectional curvatures go like $(\bar{t} - t)^{-1}$ (after performing diffeomorphisms), which contradicts the fact that $|Rm|(\bar{x}, \bar{t}) = 1$.

If the injectivity radii of the scaled metrics tend to zero : The flat limit space L can be described as the total space of a flat orthogonal \mathbb{R}^m -bundle over a flat compact manifold C . The heat kernel u on L will be Gaussian in the fiber directions and will decay exponentially fast to a constant in the base directions, i.e. $u(x, \tau) \sim (4\pi\tau)^{-m/2} e^{-\frac{|x|^2}{4\tau}} \frac{1}{\text{vol}(C)}$, where $|x|$ is the fiber norm. With this for u , one finds that $v = (m - n) \left(1 + \frac{1}{2} \ln(4\pi\tau)\right) u$. With B the ball around a basepoint of radius $\sqrt{\tau}$, the integral of u over B has a positive limit as $\tau \rightarrow \infty$, and so $\lim_{\tau \rightarrow \infty} \int_B v = -\infty$. When we do the rescaling the allowed range for \tilde{t} goes to $-\infty$.

26. ARGUMENT FOR THEOREM I.10.1

Recalling that $t \in [0, \epsilon^2]$, if $\phi' \neq 0$ then if A is sufficiently large, we have $9A\epsilon \leq d(y, t) \leq 21A\epsilon$. We will apply Lemma I.8.3(a) with the parameter r_0 of Lemma I.8.3(a) equal to \sqrt{t} . As $r_0 \leq \epsilon$, we have $y \notin B(x_0, r_0)$. From I.(10.4), on $B(x_0, r_0)$ we have $|Rm|(\cdot, t) \leq \alpha t^{-1} + 2\epsilon^{-2}$. Then from Lemma I.8.3(a), at (y, t) we have

$$(26.1) \quad \begin{aligned} d_t - \Delta d &\geq -(n-1) \left(\frac{2}{3}(\alpha t^{-1} + 2\epsilon^{-2})t^{1/2} + t^{-1/2} \right). \\ &= -(n-1) \left(1 + \frac{2}{3}\alpha + \frac{4}{3}\epsilon^{-2}t \right) t^{-1/2}. \end{aligned}$$

It follows that $d_t - \Delta d + \frac{100n}{\sqrt{t}} \geq 0$.

To construct ϕ , take the function which is $1 - 2(x-1)^2$ on $[1, 3/2]$ and $2(x-2)^2$ on $[3/2, 2]$, and smooth it slightly.

We have that u is positive for $t \in [0, \bar{t})$ and $\int_M u$ is constant in t . Then

$$(26.2) \quad \left(\int_M hu \right)_t = \int_M (\square h)u \leq -\frac{1}{(10A\epsilon)^2} \int_M \phi'' u \leq \frac{10}{(10A\epsilon)^2} \int_M u = \frac{10}{(10A\epsilon)^2}.$$

Similarly, using the result of Corollary I.9.3 that $v \leq 0$,

$$(26.3) \quad \left(- \int_M hv \right)_t \leq \int_M -(\square h)v \leq \frac{1}{(10A\epsilon)^2} \int_M \phi'' v \leq \frac{10}{(10A\epsilon)^2} \int_M -\phi v = \frac{10}{(10A\epsilon)^2} \int_M -hv.$$

In Claim 3 of I.10.1, $\tilde{t} \in [\bar{t}/2, \bar{t}]$. Then $\sqrt{\bar{t} - \tilde{t}} \leq 2^{-1/2} \epsilon$ and so for large A , h will be one on the ball B at time \tilde{t} of radius $\sqrt{\bar{t} - \tilde{t}}$ centered at \bar{x} . Then at time \tilde{t} ,

$$(26.4) \quad \int_M -hv \geq \int_B -v \geq \beta.$$

Thus

$$(26.5) \quad \int_M -hv \Big|_{t=0} \geq \beta e^{-\frac{\tilde{t}}{(A\epsilon)^2}} \geq \beta e^{-\frac{\bar{t}}{(A\epsilon)^2}} \geq \beta \left(1 - \frac{\bar{t}}{(A\epsilon)^2}\right) \geq \beta(1 - A^{-2}).$$

To verify the second equality in the next equation, it is enough to check that

$$(26.6) \quad \int_M (-2\Delta f + |\nabla f|^2) h e^{-f} dV = \int_M \left(-|\nabla \tilde{f}|^2 + \frac{|\nabla h|^2}{h^2}\right) h e^{-f} dV.$$

This follows from

$$(26.7) \quad \begin{aligned} \int_M (-2\Delta f + |\nabla f|^2) h e^{-f} dV &= \int_M (2\langle \nabla f, \nabla(h e^{-f}) \rangle + |\nabla f|^2 h e^{-f}) dV \\ &= \int_M \left(2\langle \nabla f, \frac{\nabla h}{h} - \nabla f \rangle + |\nabla f|^2\right) h e^{-f} dV \\ &= \int_M \langle \nabla f, 2\frac{\nabla h}{h} - \nabla f \rangle h e^{-f} dV \\ &= \int_M \langle \nabla \tilde{f} + \frac{\nabla h}{h}, \frac{\nabla h}{h} - \nabla \tilde{f} \rangle h e^{-f} dV \\ &= \int_M \left(-|\nabla \tilde{f}|^2 + \frac{|\nabla h|^2}{h^2}\right) h e^{-f} dV. \end{aligned}$$

To derive the last inequality of the equation (up to unimportant constants), we have $\frac{|\nabla h|^2}{h} \leq \frac{10}{(10A\epsilon)^2}$ and $-Rh \leq 1$. Then

$$(26.8) \quad \int_M \tilde{t} \left(\frac{|\nabla h|^2}{h} - Rh\right) u \leq \epsilon^2 \left(\frac{10}{(10A\epsilon)^2} + 1\right) \leq A^{-2} + \epsilon^2.$$

Also,

$$(26.9) \quad \int_M -uh \log h \leq \int_{M-B(x_0, 10A\epsilon)} u = 1 - \int_{B(x_0, 10A\epsilon)} u \leq 1 - (1 - cA^{-2}) = cA^{-2}$$

for an appropriate constant c .

The paper now renames $\tilde{u} \rightarrow u$ and $\tilde{f} \rightarrow f$. Then there is a sequence of functions f with support inside an open ball such that $\int u \rightarrow 1$ and $\int \left(-\frac{1}{2} |\nabla f|^2 - f + n\right) u$ is bounded away from zero by a positive constant. On the other hand, the Euclidean isoperimetric inequality implies the logarithmic Sobolev inequality

$$(26.10) \quad \int \left(-\frac{1}{2} |\nabla f|^2 - f + n\right) u \leq 0,$$

provided that $u = (2\pi)^{-n/2} e^{-f}$ satisfies $\int u = 1$. A proof, just assuming the Euclidean isoperimetric inequality, is given in [3], where an equivalent form of the inequality is stated [3, I.(8)].

27. PROOF OF COROLLARY I.10.2

If the corollary is not true, we can center ourselves around the collapsing balls $B(x, \sqrt{t})$ to obtain functions f as before. As in the proof of Theorem I.4.1, the volume condition along with the fact that $\int (2\pi)^{-n/2} e^{-f} \rightarrow 1$ means that $f \rightarrow -\infty$, which implies that $\int (-\frac{1}{2} |\nabla f|^2 - f + n) u \rightarrow -\infty$.

28. ARGUMENT FOR COROLLARY I.10.4

Using Theorem I.10.1 and Corollary I.10.2, we can apply Theorem I.8.2 starting at time $A^{-1}(\epsilon r_0)^2$.

29. ARGUMENT FOR I.10.5

One must assume an upper diameter bound. The idea is to apply the Ricci flow to get noncollapsing manifolds with bounded curvature and diameter, for which the finiteness of diffeomorphism types is known. One has to know that the Ricci flow exists for a uniform time interval, depending on r_0 and δ . If not, suppose that there is a sequence of solutions $\{g_i\}_{i=1}^\infty$ so that (after a slight rescaling) g_i only exists up to time $i^{-2}r_0^2$, and at time 0, $R \geq -r_0^{-2}$ and $\text{vol}(\partial\Omega)^n \geq (1 - \delta_i) c_n \text{vol}(\Omega)^{n-1}$ for Ω in an r_0 -ball, with $\delta_i \rightarrow 0$. From [15, Theorem 8.1], a sectional curvature of g_i must blow up as $t \rightarrow i^{-2}r_0^2$. Rescaling in space by i/r_0 and time by i^2/r_0^2 , we get solutions $\{\hat{g}_i\}_{i=1}^\infty$ that exist for $t \in [0, 1)$, have $R(x) \geq -i^{-2}$ at time 0 and satisfy $\text{vol}(\partial\Omega)^n \geq (1 - \delta_i) c_n \text{vol}(\Omega)^{n-1}$ for Ω in an i -ball. Given $\alpha > 0$, for i large, we can apply Theorem I.10.1 with $\epsilon = i^{-1}$ and the r_0 of the theorem equal to i , to get $|Rm(\hat{g}_i)|(x, t) \leq \alpha t^{-1} + 1$ whenever $t \in (0, 1)$. This contradicts the assumption that a sectional curvature of \hat{g}_i blows up as $t \rightarrow 1$.

30. EXAMPLES FOR I.11.2

We will use the terminology of II.1 that a κ -solution is an ancient solution of the type described in I.11.1.

There is an ancient solution on the cylinder $\mathbb{R} \times S^{n-1}(r)$, where the radius satisfies $r^2(t) = r_0^2 - 2(n-2)t$, that is κ -noncollapsed on all scales. On the other hand, the quotient solution on $S^1 \times S^{n-1}(r)$ is κ -collapsed for all κ .

Bryant's gradient steady soliton is given by $g(t) = \phi_t^* g_0$, where $g_0 = dr^2 + \mu(r) d\Theta^2$ is a certain rotationally symmetric metric on \mathbb{R}^3 . It has sectional curvatures that go like r^{-1} , and $\mu(r) \sim r$. The gradient function f satisfies $R_{ij} + \nabla_i \nabla_j f = 0$, with $f(r) \sim -2r$. Then for r and $r - 2t$ large, $\phi_t(r, \Theta) \sim (r - 2t, \Theta)$. In particular, if $R_0 \in C^\infty(\mathbb{R}^3)$ is the scalar curvature function of g_0 then $R(t, r, \Theta) \sim R_0(r - 2t, \Theta)$.

To check the conclusion of I.11.8 in this case, given a point $(r_0, \Theta) \in \mathbb{R}^3$ at time 0, the scalar curvature goes like r_0^{-1} . Multiplying the soliton metric by r_0^{-1} and sending $t \rightarrow r_0 t$ gives the asymptotic metric

$$(30.1) \quad d(r/\sqrt{r_0})^2 + \frac{r - 2r_0 t}{r_0} d\Theta^2.$$

Putting $u = (r - r_0)/\sqrt{r_0}$, the rescaled metric is approximately

$$(30.2) \quad du^2 + \left(1 + \frac{u}{\sqrt{r_0}} - 2t\right) d\Theta^2.$$

Given $\epsilon > 0$, this will be ϵ -biLipschitz close to the evolving cylinder $du^2 + (1 - 2t) d\Theta^2$ provided that $|u| \leq \epsilon\sqrt{r_0}$, i.e. $|r - r_0| \leq \epsilon r_0$. To have an ϵ -neck, we want this to hold whenever $|r - r_0|^2 \leq (\epsilon r_0^{-1})^{-1}$. This will be the case if $r_0 \geq \epsilon^{-3}$. Thus M_ϵ is approximately

$$(30.3) \quad \{(r, \Theta) \in \mathbb{R}^3 : r \leq \epsilon^{-3}\}$$

and $Q = R(x_0, 0) \sim \epsilon^3$. Then $\text{diam}(M_\epsilon) \sim \epsilon^{-3}$ and at the origin $0 \in M_\epsilon$, $R(0, 0) \sim \epsilon^0$. It follows that for the value of κ corresponding to this solution, $C(\epsilon, \kappa)$ must grow at least as fast as ϵ^{-3} as $\epsilon \rightarrow 0$.

31. I.(11.1)

Given a Ricci flow solution $g(t')$ for $t' \in [0, \Omega)$, put $H(X) = R_{t'} + \frac{R}{t'} + 2\langle X, \nabla R \rangle + 2 \text{Ric}(X, X)$. The trace Harnack inequality says that if the curvature operator is nonnegative then $H(X) \geq 0$ for all $t' > 0$. In particular, we can apply this to the Ricci flow solution corresponding to the piece of the ancient solution starting at an arbitrary time $t_0 \in (-\infty, 0)$, with $t' = t - t_0$. Then $R_t + \frac{R}{t-t_0} + 2\langle X, \nabla R \rangle + 2 \text{Ric}(X, X) \geq 0$. Taking $t_0 \rightarrow -\infty$ gives $R_t + 2\langle X, \nabla R \rangle + 2 \text{Ric}(X, X) \geq 0$.

In particular, $R_t \geq 0$. This implies the essential equivalence of the noncollapsing definitions I.4.2 and I.8.1 for ancient solutions. Namely, if a solution is κ -collapsed in the sense of I.8.1 then it is automatically κ -collapsed in the sense of I.4.2. Conversely, if a time- t_0 slice of an ancient solution is collapsed in the sense of I.4.2 then the fact that $R_t \geq 0$, together with bounds on distance distortion, implies that it is collapsed in the sense of I.8.1 (possibly for a different value of κ).

32. I.11.2

For a fixed τ , I.(7.16) implies that $|\nabla l^{1/2}|^2 \leq \frac{C}{4\tau}$, and so

$$(32.1) \quad |l^{1/2}(q) - l^{1/2}(q(\tau))| \leq \sqrt{\frac{C}{4\tau}} \text{dist}_{t_0-\tau}(q, q(\tau)).$$

Also from I.(7.16) we have $R \leq \frac{Cl}{\tau}$, and we can plug the previous bound on l into the right-hand-side to get a bound on R .

Instead of the interval $[1/2, 1]$ considered in I.11.2, we will use the interval $[1, 2]$. Given any $\bar{\tau}$, we apply the above bounds initially at time $\tau = \bar{\tau}$. From the noncollapsing and the fact that R is nonincreasing in τ , we obtain bounds on the geometry for $\tau \in [\bar{\tau}, 2\bar{\tau}]$, at least

on a smaller ball. Then as in the proof in I.7.3, we obtain bounds on l for $\tau \in [\bar{\tau}, 2\bar{\tau}]$ and on a possibly smaller ball (but with a uniform bound on how much smaller it is). Then we also get bounds on R in this region.

As the rescaled solutions are uniformly noncollapsing and have uniform curvature bounds on balls, Appendix D implies that we can take a subsequence that converges to a solution $\bar{g}_{ij}(\tau)$, $1 \leq \tau < \infty$. We may assume that we have locally Lipschitz convergence of l .

We define the reduced volume $\bar{V}(\tau)$ for the limit solution using the limit function \bar{l} . We claim that for any τ , the number $\bar{V}(\tau)$ for the limit solution is the limit of numbers $\tilde{V}(\tau_i)$ for the original solution, with $\tau_i \rightarrow \infty$. To see the convergence, note that the monotonicity of I.7.1 implies that in general, the integrand for \tilde{V} satisfies

$$(32.2) \quad \tau^{-n/2} e^{-l(X,\tau)} J(X,\tau) \leq \text{const. } e^{-|X|^2/4}.$$

Thus we can apply dominated convergence, along with the pointwise convergence of the integrands of $\tilde{V}(\tau_i)$ to the integrand of $\bar{V}(\tau)$.

As there is a uniform upper bound on l on an appropriate ball around $q(\tau_i)$, and a lower volume bound on the ball, it follows that as $i \rightarrow \infty$, $\tilde{V}(\tau_i)$ is uniformly bounded away from zero. From this argument and the monotonicity of \tilde{V} , $\bar{V}(\tau)$ is a positive constant as a function of τ .

We write I.(7.13) as

$$(32.3) \quad \partial_\tau (\tau^{-n/2} e^{-l} dV) - \Delta (\tau^{-n/2} e^{-l}) dV \leq 0.$$

As this inequality holds for each rescaled solution, it follows that it holds distributionally for \bar{l} . In particular, the nonpositivity implies that the left-hand-side of (32.3), when computed for the limit solution, is actually a nonpositive measure. If the left-hand-side of (32.3) (for the limit solution) were not strictly zero then after multiplying by appropriate cutoff functions and integrating, we would conclude that $\frac{d\bar{V}}{d\tau}$ is somewhere negative, which is a contradiction. Thus we must have equality in (32.3), or equivalently, in I.(7.13). This implies equality in I.(7.10), which implies equality in I.(7.14). Writing I.(7.14) as

$$(32.4) \quad (4\Delta - R) e^{-\frac{l}{2}} = \frac{l-n}{\tau} e^{-\frac{l}{2}},$$

elliptic theory (presumably) gives smoothness of \bar{l} .

One then wishes to say that equality in I.(7.14) implies equality in I.(7.12), which implies that one has a gradient shrinking soliton. There is an apparent problem with this argument, as the use of I.(7.12) implicitly assumes that the solution is defined for all $\tau \geq 0$, which we do not know. However, one can instead use Proposition I.(9.1), with $f = \bar{l}$. Equality in (7.14) implies that $v = 0$, so I.(9.1) directly gives the gradient shrinking soliton equation. (The problem with the argument using I.(7.12), and its resolution using I.(9.1), were pointed out by the UCSB group.)

If the gradient shrinking soliton $\bar{g}_{ij}(\tau)$ is flat then $\partial_i \partial_j \bar{l} = \frac{\bar{g}_{ij}}{2\tau}$ and $\Delta \bar{l} = \frac{n}{2\tau}$. Putting this into the equality I.(7.14) gives $|\nabla \bar{l}|^2 = \frac{\bar{l}}{\tau}$. It follows that the level sets of \bar{l} are distance spheres of radius $\sqrt{4\tau}$. From the smoothness of \bar{l} , M must be \mathbb{R}^n and I.(7.13) and I.(7.14) imply that with an appropriate choice of origin, $\bar{l} = \frac{|x|^2}{4\tau}$. This gives the contradiction.

We remark that the gradient soliton constructed here does not, a priori, have bounded curvature on time slices, i.e. it may not be a κ -solution. In the 2 and 3-dimensional cases one can prove this using additional reasoning. See section 36 where it is shown that 2-dimensional κ -solutions are round spheres, and 39 where the 3-dimensional case is discussed.

33. I.11.3

The paper Perelman cites only deals with compact solitons. One can employ a somewhat different line of reasoning to prove Corollary 11.3, see section 36.

34. I.11.4

We will give a proof of I.11.4 that is logically independent of I.11.3, i.e. also covers the case $n = 2$. We refer to Appendix A, and also recall that Hamilton [13] has proven a Harnack inequality for Ricci flows with nonnegative curvature operator. For ancient solutions it says that for $t_1 < t_2$, $x_1, x_2 \in M$, we have

$$(34.1) \quad R(x_2, t_2) \geq \exp\left(-\frac{d_{t_1}^2(x_1, x_2)}{2(t_2 - t_1)}\right) R(x_1, t_1).$$

For later use, we note that if $R(x, t) = 0$ for some (x, t) then the solution is flat.

Now suppose $(M, g(t))$ is a κ -solution on an n -manifold M , where $n \geq 2$, pick $p \in M$, and consider the time- t_0 slice $(M, g(t_0))$. Let \mathcal{R} be the asymptotic scalar curvature ratio. We will be considering a sequence of pointed rescaled Ricci flows $(M, x_k, g_k(t))$, where $x_k \in M$ is a sequence with $d_{t_0}(x_k, p) \rightarrow \infty$, $g_k(t) := R(x_k, t_0)g(t_0 + \frac{t}{R(x_k, t_0)})$, and $t \in (-\infty, 0]$.

Case 1: $\mathcal{R} = \infty$. In this case we may assume there are sequences $x_k \in M$, $r_k > 0$ such that $d(x_k, p) \rightarrow \infty$, $\frac{r_k}{d(x_k, p)} \rightarrow 0$, $R(x_k)r_k^2 \rightarrow \infty$, and $R(x) \leq 2R(x_k)$ for all $x \in B(x_k, r_k)$, where distances and scalar curvatures are taken at time t_0 [15, Lemma 22.2]. Consider the sequence of pointed, rescaled Ricci flows $(M, x_k, g_k(t))$ as above. By (34.1) we have $R_k(x, t) \leq 2$ whenever $t \leq 0$ and $d_k(x, x_k) \leq R(x_k)^{1/2}r_k$, where d_k is the distance function for $g_k(0)$ and $R_k(\cdot, \cdot)$ is the scalar curvature of $g_k(\cdot)$. The κ -noncollapsed assumption gives a uniform positive lower bound on the injectivity radius of $g_k(0)$ at x_k , and so by Appendix D we may extract a pointed limit solution $(M_\infty, x_\infty, g_\infty(t))$, $t \in (-\infty, 0]$, of the sequence of pointed Ricci flows. By relative volume comparison, $(M_\infty, x_\infty, g_\infty(0))$ has positive asymptotic volume growth. By Appendix A, the Riemannian manifold $(M_\infty, x_\infty, g_\infty(0))$ is isometric to an Alexandrov space which splits off a line, which means that it is a Riemannian product $\mathbb{R} \times N$. This implies a product structure for earlier times [11, Section 8]. Now when $n = 2$, we have a contradiction, since $R(x_\infty, 0) = 1$ but $(M_\infty, g_\infty(0))$ is a product surface, and must therefore be flat. When $n > 2$ we obtain a κ -solution on an $(n - 1)$ -manifold with positive asymptotic volume growth at time zero, and by induction this is impossible.

Case 2: $0 < \mathcal{R} < \infty$. Here we choose a sequence $x_k \in M$ such that $d(x_k, p) \rightarrow \infty$ and $R(x_k, t_0)d^2(x_k, p) \rightarrow \mathcal{R}$. Consider the rescaled pointed solution $(M, x_k, g_k(t))$ for $t \in (-\infty, 0]$ as above. We have $R_k(x_k, 0) = 1$, and for all $b > 0$, for sufficiently large k , we have $R_k(x, t) \leq R_k(x, 0) \leq \frac{2\mathcal{R}}{d_k^2(x, p)}$ for all x such that $d_k(x, p) > b$. Fixing $b < \sqrt{\mathcal{R}} < B$, and applying the derivative estimates and κ -noncollapsing assumptions as in case 1, we

may extract a pointed limit flow $(M_\infty, x_\infty, g_\infty(t))$ from the sequence $(M_k, x_k, g_k(t))$ where $M_k := \{x \in M \mid b < d_k(x, p) < B\}$. Note that $(M_\infty, g_\infty(0))$ is isometric to an annular portion of a nonflat metric cone, since $(M_k, p, g_k(0))$ Gromov-Hausdorff converges to the Tits cone $C_T(M, g(t_0))$. When $n = 2$ this contradicts the fact that $R_\infty(x_\infty, 0) = 1$. When $n \geq 3$, we will derive a contradiction from Hamilton's curvature evolution equation

$$(34.2) \quad \text{Rm}_t = \Delta \text{Rm} + Q(\text{Rm}).$$

Choose an orthonormal frame e_1, \dots, e_n in the tangent space of $(M_\infty, g_\infty(0))$ at x_∞ such that e_1 points radially outward (with respect to the cone structure), and e_2, e_3 span a 2-plane with strictly positive curvature. Let P denote the 2-plane spanned by e_1 and e_2 . In terms of the curvature operator, the fact that $\text{Rm}_\infty(e_1, e_2, e_2, e_1) = 0$ is equivalent to $\langle P, \text{Rm}_\infty P \rangle = 0$. As the curvature operator is nonnegative, it follows that $\text{Rm}_\infty P = 0$. (In fact, this is true for any metric cone.) Then

$$(34.3) \quad \langle P, (\nabla_{e_i} \text{Rm}_\infty) P \rangle = e_i \langle P, \text{Rm}_\infty P \rangle - \langle \nabla_{e_i} P, \text{Rm}_\infty P \rangle - \langle P, \text{Rm}_\infty \nabla_{e_i} P \rangle = 0.$$

Similarly,

$$(34.4) \quad \langle P, (\Delta \text{Rm}_\infty) P \rangle = \sum_i \langle P, (\nabla_{e_i} \nabla_{e_i} - \nabla_{\nabla_{e_i} e_i}) \text{Rm}_\infty P \rangle = \sum_i \langle \nabla_{e_i} P, \text{Rm}_\infty \nabla_{e_i} P \rangle.$$

However, $\nabla_{e_3} P$ has a nonradial component $\frac{1}{r} e_2 \wedge e_3$. Thus $(\Delta \text{Rm}_\infty)(e_1, e_2, e_2, e_1) > 0$. The zeroth order quadratic term $Q(\text{Rm})$ appearing in (34.2) is nonnegative when Rm is nonnegative, so we conclude that $\partial_t \text{Rm}_\infty(e_1, e_2, e_2, e_1) > 0$ at $t = 0$. This means that $\text{Rm}_\infty(-\epsilon)(e_1, e_2, e_2, e_1) < 0$ for $\epsilon > 0$ sufficiently small, which is impossible.

Case 3: $\mathcal{R} = 0$. From [23], the universal cover of any noncompact nonnegatively curved complete manifold M with $\mathcal{R} = 0$ splits isometrically as a Euclidean space times a surface. With the additional assumption that $\mathcal{V} > 0$, if $n \geq 3$ then it follows that M is flat.

To give a direct argument in our case, let us take any sequence $x_k \in M$ with $d(x_k, p) \rightarrow \infty$. Set $r_k := d_{t_0}(x_k, p)$, put

$$g_k(t) := r_k^{-2} g(t_0 + r_k^2 t)$$

for $t \in (-\infty, 0]$, and let d_k be the distance function associated with $g_k(0)$. For any $0 < b < B$, put

$$M_k(b, B) := \{x \in M \mid 0 < b < d_k(x, p) < B\}.$$

Since $\mathcal{R} = 0$, we get that $\sup_{x \in M_k(b, B)} |\text{Rm}_k(x, 0)| \rightarrow 0$ as $k \rightarrow \infty$. Invoking the κ -noncollapsed assumption as in the previous cases, we may assume that $(M, g_k(0), p)$ Gromov-Hausdorff converges to a metric cone $(M_\infty, g_\infty, p_\infty)$ (the Tits cone $C_T(M, g(t_0))$) which is flat and smooth away from the vertex p_∞ , and the convergence is smooth away from p_∞ .

The ‘‘unit sphere’’ in $C_T(M, g(t_0))$ defines a compact smooth hypersurface S_∞ in $(M_\infty \setminus \{p_\infty\}, g_\infty(0))$ whose principal curvatures are identically 1. If $n \geq 3$ then S_∞ must be a quotient of the standard $(n - 1)$ -sphere by the free action of a finite group of isometries. We have a sequence $S_k \subset M_k$ of approximating smooth hypersurfaces whose principal curvatures (with respect to $g_k(0)$) go to 1 as $k \rightarrow \infty$. In view of the convergence to $(M_\infty, g_\infty, p_\infty)$, for sufficiently large k , the inward principal curvatures of S_k with respect to $g_k(0)$ are close to 1. As M has nonnegative curvature, S_k is diffeomorphic to a sphere [10, Theorem A]. Thus S_∞ is isometric to the standard $(n - 1)$ -sphere, and so $C_T(M, g(t_0))$ is isometric to

n -dimensional Euclidean space. Then $(M, g(t_0))$ is isometric to \mathbb{R}^n , which contradicts the definition of a κ -solution.

To also handle the case $n = 2$, we use the fact that $g_k(t)$ is a Ricci flow solution, to extract a limiting smooth incomplete time-independent Ricci flow solution $(M_\infty \setminus p_\infty, g_\infty(t))$ for $t \in [-1, 0]$. Note that this solution is unpointed. In view of the convergence to the limiting solution, for sufficiently large k , the inward principal curvatures of S_k with respect to $g_k(t)$ are close to 1 for all $t \in [-1, 0]$. This implies that S_k bounds a domain $B_k \subset M$ whose diameter with respect to $g_k(t)$ is uniformly bounded above, say by 10 (see Appendix A).

Applying the Harnack inequality with $y_k \in S_k$ (at time 0) and $x \in B_k$ (at time 1), we see that $\sup_{x \in B_k} |\text{Rm}_k(x, -1)| \rightarrow 0$ as $k \rightarrow \infty$. Thus $(B_k, g_k(-1), p)$ Gromov-Hausdorff converges to a flat manifold $(B_\infty, g_\infty(-1), \bar{p}_\infty)$ with convex boundary. As all of the principal curvatures of ∂B_∞ are 1, B_∞ must be isometric to a Euclidean unit ball. This implies that S_∞ is isometric to the standard $(n - 1)$ -sphere, and we obtain a contradiction as before.

Corollary 34.5. *The $(M, g(\cdot))$ be a noncompact κ -solution. Then the asymptotic scalar curvature ratio \mathcal{R} is infinite.*

Proof. This is a consequence of the proof of I.11.4, since the cases $0 < \mathcal{R} < \infty$ and $\mathcal{R} = 0$ led to contradictions. \square

35. COROLLARIES OF I.11.4

Corollary 35.1. *1. If $B(x_0, r_0)$ is a ball in a time slice of a κ -solution, then the normalized volume $r_0^{-n} \text{vol}(B(x_0, r_0))$ is controlled (i.e. bounded away from zero) \iff the normalized scalar curvature $r_0^2 R(x_0)$ is controlled (i.e. bounded above).*

2. If $B(x_0, r_0)$ is a ball in a time slice of a κ -solution, then the normalized volume $r_0^{-n} \text{vol}(B(x_0, r_0))$ is almost maximal \iff the normalized scalar curvature $r_0^2 R(x_0)$ is almost zero.

3. (Precompactness) If $(M_k, g_k(\cdot), (x_k, t_k))$ is a sequence of pointed κ -solutions (without the assumption that $R(x_k, t_k) = 1$) and for some $r > 0$, the r -balls $B(x_k, r) \subset (M_k, g_k(t_k))$ have controlled normalized volume, then a subsequence converges to an ancient solution $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$ which has nonnegative curvature operator, and is κ -noncollapsed (though a priori the curvature may be unbounded on a given time slice).

4. There is a constant $\eta = \eta(\kappa)$ such that for every κ -solution $(M, g(\cdot))$, and all $x \in M$, we have $|\nabla R|(x, t) \leq \eta R^{\frac{3}{2}}$, $|R_t| \leq \eta R^2$. More generally, there are scale invariant bounds on all derivatives on curvature tensor which depend only on κ .

5. There is a function $\alpha : [0, \infty) \rightarrow [0, \infty)$ depending only on κ such that $\lim_{s \rightarrow \infty} \alpha(s) = \infty$, and for every κ -solution $(M, g(\cdot))$ and $x, y \in M$, we have $R(y)d^2(x, y) \geq \alpha(R(x)d^2(x, y))$.

Proof. Assertion 1, \implies . Suppose we have a sequence of κ -solutions $(M_k, g_k(\cdot))$, and sequences $t_k \in (-\infty, 0]$, $x_k \in M_k$, $r_k > 0$, such that at time t_k , the normalized volume of $B(x_k, r_k)$ is $\geq c > 0$, and $R(x_k, t_k)r_k^2 \rightarrow \infty$. By Appendix B, for each k , we can find

$y_k \in B(x_k, 5r_k)$, $\bar{r}_k \leq r_k$, such that $R(y_k, t_k)\bar{r}_k^2 \geq R(x_k, t_k)r_k^2$, and $R(z, t_k) \leq 2R(y_k, t_k)$ for all $z \in B(y_k, \bar{r}_k)$. Note that by relative volume comparison,

$$(35.2) \quad \frac{\text{vol}(B(y_k, \bar{r}_k))}{\bar{r}_k^n} \geq \frac{\text{vol}(B(y_k, 10r_k))}{(10r_k)^n} \geq \frac{\text{vol}(B(x_k, r_k))}{(10r_k)^n} \geq \frac{c}{10^n}.$$

Rescaling the sequence of pointed solutions $(M_k, g_k(\cdot), (y_k, t_k))$ by $R(y_k, t_k)$, we get a sequence satisfying the hypotheses of Appendix D (we use here the fact that $R_t \geq 0$ for an ancient solution), so it accumulates on a limit flow $(M_\infty, g_\infty(\cdot), (y_\infty, 0))$ which is a κ -solution. By (35.2), the asymptotic volume ratio of $(M, g_\infty(0))$ is $\geq \frac{c}{10^n} > 0$. This contradicts I.11.4.

Assertion 3. By relative volume comparison, it follows that every r -ball in $(M_k, g_k(t_k))$ has normalized volume bounded below by a (k -independent) function of its distance to x_k . By 1, this implies that the curvature of $(M_k, g_k(t_k))$ is bounded by a k -independent function of the distance to x_k , and hence we can apply Appendix D to extract a smoothly converging subsequence.

Assertion 1, \Leftarrow . Suppose we have a sequence $(M_k, g_k(\cdot))$ of κ -solutions, and sequences $x_k \in M_k$, $r_k > 0$, such that $R(x_k, t_k)r_k^2 < c$ for all k , but $r_k^{-n} \text{vol}(B(x_k, r_k)) \rightarrow 0$. For large k , we can choose $\bar{r}_k \in (0, r_k)$ such that $\bar{r}_k^{-n} \text{vol}(B(x_k, \bar{r}_k)) = \frac{1}{2}c_n$ where c_n is the volume of the unit Euclidean n -ball. By relative volume comparison, $\frac{\bar{r}_k}{r_k} \rightarrow 0$. Applying 3, we see that the pointed sequence $(M_k, g_k(\cdot), (x_k, t_k))$, rescaled by the factor \bar{r}_k^{-2} , accumulates on a pointed ancient solution $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$, such that the ball $B(x_\infty, 1) \subset (M_\infty, g_\infty)$ has normalized volume $\frac{1}{2}c_n$ at $t = 0$.

Suppose the ball $B(x_\infty, 1) \subset (M_\infty, g_\infty(0))$ were flat. Then by Hamilton's Harnack inequality (applied to the approximators) we would have $R_\infty(x, t) = 0$ for all $x \in M_\infty$, $t \leq 0$, i.e. $(M_\infty, g_\infty(t))$ would be a time independent flat manifold. But flat manifolds other than Euclidean space have zero asymptotic volume, which contradicts the assumption that the sequence $(M_k, g_k(\cdot))$ is κ -noncollapsed. Thus $B(x_\infty, 1) \subset (M_\infty, g_\infty(0))$ is not flat, which means, by the Harnack inequality, that the scalar curvature of $g_\infty(0)$ is strictly positive everywhere. Therefore, with respect to g_k we have

$$\liminf_{k \rightarrow \infty} R(x_k, t_k)r_k^2 = \liminf_{k \rightarrow \infty} (R(x_k, t_k)\bar{r}_k^2) \left(\frac{r_k}{\bar{r}_k} \right)^2 \geq \text{const.} \liminf_{k \rightarrow \infty} \left(\frac{r_k}{\bar{r}_k} \right)^2 = \infty,$$

which is a contradiction.

Assertion 2, \Rightarrow . Apply 1, the precompactness criterion, and the fact that a nonnegatively-curved manifold whose balls have normalized volume c_n must be flat.

Assertion 2, \Leftarrow . Apply 1, the precompactness criterion, and Hamilton's Harnack inequality (to the approximators).

Assertion 4. This follows by rescaling g so that $R(x, t) = 1$, and applying 1 and 3.

Assertion 5. The quantity $R(z)d^2(u, v)$ is scale invariant. If the assertion failed, we would have sequences $(M_k, g_k(\cdot))$, $x_k, y_k \in M_k$, such that $R(y_k) = 1$, $d(x_k, y_k)$ remains bounded, but the curvature at x_k blows up. This contradicts 3.

36. AN ALTERNATE PROOF OF I.11.3 USING I.11.4 AND COROLLARY 35.1

To clarify the chain of logical dependence, we remark that this section is concerned with 2-dimensional κ -solutions, and does not use anything from I.11.2 or I.11.3. It does use I.11.4; however, we avoid circularity here because the proof of I.11.4 given in section 34, unlike Perelman's proof, does not use I.11.3 either.

Lemma 36.1. *There is a constant $v = v(\kappa) > 0$ such that if $(M, g(\cdot))$ is a 2-dimensional κ -solution (a priori either compact or noncompact), $x, y \in M$ and $r = d(x, y)$ then*

$$(36.2) \quad \text{vol}(B_t(x, r)) \geq vr^2.$$

Proof. If the lemma were not true then there would be a sequence $(M_k, g_k(\cdot))$ of 2-dimensional κ -solutions, and sequences $x_k, y_k \in M_k$, $t_k \in \mathbb{R}$ such that $r_k^{-2} \text{vol}(B_{t_k}(x_k, r_k)) \rightarrow 0$, where $r_k = d(x_k, y_k)$. Let z_k be the midpoint of a shortest segment from x_k to y_k in the t_k -time slice $(M_k, g_k(t_k))$. For large k , choose $\bar{r}_k \in (0, r_k/2)$ such that

$$(36.3) \quad \bar{r}_k^{-2} \text{vol}(B_{t_k}(z_k, \bar{r}_k)) = \frac{\pi}{2},$$

i.e. half the area of the unit disk in \mathbb{R}^2 . As

$$(36.4) \quad \frac{\pi}{2} = \bar{r}_k^{-2} \text{vol}(B_{t_k}(z_k, \bar{r}_k)) \leq \bar{r}_k^{-2} \text{vol}(B_{t_k}(x_k, r_k)) = (\bar{r}_k/r_k)^{-2} r_k^{-2} \text{vol}(B_{t_k}(x_k, r_k)),$$

it follows that $\lim_{k \rightarrow \infty} \frac{\bar{r}_k}{r_k} = 0$. Then by part 3 of Corollary 35.1, the sequence of pointed Ricci flows $(M_k, g_k(\cdot), (z_k, t_k))$, when rescaled by \bar{r}_k^{-2} , accumulates on a Ricci flow $(M_\infty, g_\infty(\cdot), (z_\infty, 0))$. The segments from z_k to x_k and y_k accumulate on a line in $(M_\infty, g_\infty(0))$, and hence $(M_\infty, g_\infty(0))$ splits off a line. By (36.3), $(M_\infty, g_\infty(0))$ cannot be isometric to \mathbb{R}^2 , and hence must be a cylinder. Considering the approximating Ricci flows, we get a contradiction to the κ -noncollapsed assumption. \square

Lemma 36.1 implies that the asymptotic volume of any noncompact 2-dimensional κ -solution is at least $v > 0$. By I.11.4 (including the 2-dimensional case in section 34) we therefore conclude that every 2-dimensional κ -solution is compact. (This was implicitly assumed in the proof of Corollary I.11.3 in [21], as its reference [H10] is about compact surfaces.)

Consider the family \mathcal{F} of 2-dimensional κ -solutions $(M, g(\cdot), (x, 0))$ with $\text{diam}(M, g(0)) = 1$. By Lemma 36.1, there is uniform lower bound on the volume of the $t = 0$ time slices of κ -solutions in \mathcal{F} . Thus \mathcal{F} is compact in the smooth topology by part 3 of Corollary 35.1 (the precompactness leads to compactness in view of the diameter bound). This implies (recall that $R > 0$) that there is a constant $K \geq 1$ such that every time slice of every 2-dimensional κ -solution has K -pinched curvature.

Hamilton has shown that volume-normalized Ricci flow on compact surfaces with positively pinched initial data converges exponentially fast to a constant curvature metric [12].

His argument shows that there is a small $\epsilon > 0$, depending continuously on the initial data, so that when the volume of the (unnormalized) solution has been reduced by a factor of at least ϵ , the pinching is at most the square root of the initial pinching. By the compactness of the family \mathcal{F} , this ϵ can be chosen uniformly when we take the initial data to be the $t = 0$ time slice of a κ -solution in \mathcal{F} .

Now let K_0 be the worst pinching of a 2-dimensional κ -solution, and let $(M, g(\cdot))$ be a κ -solution where the curvature pinching of $(M, g(0))$ is K_0 . Choosing $t < 0$ such that $\epsilon \text{vol}(M, g(t)) = \text{vol}(M, g(0))$, the previous paragraph implies the curvature pinching of $(M, g(t))$ is at least K_0^2 . This would contradict the fact that K_0 is the upper bound on the pinching for all κ -solutions, unless $K_0 = 1$.

37. I.11.5

“Compactly contained” means that $B(x_k, r_k)$ has compact closure. If the corollary is not true, there are an $\epsilon > 0$ and sequences of solutions with $\text{vol}(B(x_k, A_k/\sqrt{Q_k})) > \epsilon(A_k/\sqrt{Q_k})^n$ for k large, where $A_k \rightarrow \infty$. The blowup limit has $\mathcal{V}(0) > 0$. Suppose that for each $\kappa > 0$, there are a point (x_κ, t_κ) and a radius r_κ so that $|\text{Rm}(x_\kappa, t_\kappa)| \leq r_\kappa^{-2}$ on the time- t_κ ball $B(x_\kappa, r_\kappa)$, and $\text{vol}(B(x_\kappa, r_\kappa)) < \kappa r_\kappa^n$. From the Bishop-Gromov inequality, $\mathcal{V}(t_\kappa) < \kappa$. As is said in I, letting R be the supremum of the scalar curvature and taking $\kappa \rightarrow 0$ gives $\mathcal{V}(0) = 0$, which is a contradiction.

38. I.11.6

For part (a), to argue as in the proof of Claim 1 of I.10.1, put $r_0 = 1$ and suppose that $g_{ij}(t)$ satisfies the Ricci flow equation for $t \in [t_0, 0]$, with $|R(x, t)| > C(1 + (t - t_0)^{-1})$ for some (x, t) satisfying $d_t(x, x_0) < \frac{1}{4}$. Following the notation of the proof of I.10.1, put $A = \lambda C^{1/2}$ and $\alpha = \lambda^2 C^{1/2}$, where we will take λ to be a sufficiently small number that only depends on n . Put

$$(38.1) \quad M_\alpha = \{(x, t) : R(x, t) \geq \alpha(t - t_0)^{-1}\}.$$

As in Claim 1 of I.10.1, with λ small enough, one can find $(\bar{x}, \bar{t}) \in M_\alpha$ with $d_{\bar{t}}(\bar{x}, x_0) < \frac{1}{3}$ such that

$$(38.2) \quad R(x, t) \leq 2R(\bar{x}, \bar{t})$$

whenever

$$(38.3) \quad (x, t) \in M_\alpha, \quad t \in [t_0, \bar{t}], \quad d_t(x, x_0) \leq d_{\bar{t}}(\bar{x}, x_0) + AR^{-1/2}(\bar{x}, \bar{t}).$$

We want to show that (38.2) holds whenever

$$(38.4) \quad \bar{t} - \frac{1}{2}\alpha Q^{-1} \leq t \leq \bar{t}, \quad \text{dist}_{\bar{t}}(x, \bar{x}) \leq \frac{1}{10}AQ^{-1/2},$$

where $Q = R(\bar{x}, \bar{t})$. As in the proof of Claim 2 of I.10.1, we have a uniform bound $R(x, t) \leq 2Q$ on the time- t distance ball $B(x_0, \text{dist}_{\bar{t}}(\bar{x}, x_0) + AQ^{-1/2})$, provided that $\bar{t} - \frac{1}{2}\alpha Q^{-1} \leq t \leq \bar{t}$. We want to show that if (x, t) satisfies (38.4) then it satisfies (38.2) or (38.3). If (x, t) satisfies (38.4) and does not satisfy (38.2) then it belongs to M_α and x lies in

the time- \bar{t} ball $B(x_0, \text{dist}_{\bar{t}}(\bar{x}, x_0) + \frac{1}{10}AQ^{-1/2})$. Applying Lemma I.8.3(b) with $r_0 = \frac{1}{2}Q^{-1/2}$ gives

$$(38.5) \quad \text{dist}_t(x, \bar{x}) - \text{dist}_{\bar{t}}(x, \bar{x}) \leq \text{const.}(n) \alpha Q^{-1/2} \leq \lambda \text{const.}(n) AQ^{-1/2}.$$

Taking λ small enough, we have

$$(38.6) \quad \text{dist}_t(\bar{x}, x_0) \leq \text{dist}_{\bar{t}}(\bar{x}, x_0) + \frac{1}{2}AQ^{-1/2},$$

and so it is consistent to use the above curvature bound in the application of Lemma I.8.3(b).

Hence (38.2) holds whenever $\bar{t} - \frac{1}{2}\alpha Q^{-1} \leq t \leq \bar{t}$ and $\text{dist}_{\bar{t}}(x, \bar{x}) \leq \frac{1}{10}AQ^{-1/2}$. Renaming A gives that (38.2) holds whenever $\bar{t} - AQ^{-1} \leq t \leq \bar{t}$ and $\text{dist}_{\bar{t}}(x, \bar{x}) \leq AQ^{-1/2}$, where A tends to infinity with C .

Applying Corollary I.11.5 at (\bar{x}, \bar{t}) gives that for any $\epsilon > 0$,

$$(38.7) \quad \text{vol}(B(\bar{x}, A/\sqrt{Q_k})) \leq \epsilon(A/\sqrt{Q_k})^n,$$

where the A of Corollary I.11.5 only depends on ϵ . By the Bishop-Gromov inequality, $\text{vol}(B(\bar{x}, 1)) \leq \epsilon$ whenever $A/\sqrt{Q_k} \leq 1$. Taking $Q_k \rightarrow \infty$ and $\epsilon \rightarrow 0$, we get a contradiction to the fact that $\text{vol}(B(\bar{x}, 1)) \geq \text{vol}(B(x_0, \frac{2}{3})) \geq (\frac{2}{3})^n \text{vol}(B(x_0, 1)) \geq (\frac{2}{3})^n w$.

For part (b), the idea is that if τ_0 is sufficiently small then we will still have the estimate $\text{vol}(B(x_0, r_0)) \geq 5^{-n}wr_0^n$ for the time- t ball $B(x_0, r_0)$ when $t \in [-\tau r_0^2, 0]$, and so we can apply part (a) with w replaced by $\frac{w}{5}$. More precisely, putting $r_0 = 1$, let τ be the largest number in $[0, -t_0]$ so that the time- t ball $B(x_0, 1)$ satisfies $\text{vol}(B(x_0, 1)) \geq 5^{-n}w$ whenever $t \in [-\tau, 0]$. It suffices to get a uniform lower bound on τ . If $\tau < t_0$ then $\text{vol}(B(x_0, 1)) = 5^{-n}w$. The conclusion of part (a) holds in the sense that

$$(38.8) \quad R(x, t) \leq C(5^{-n}w) + B(5^{-n}w)(t + \tau)^{-1}$$

whenever $t \in [-\tau, 0]$ and $\text{dist}_t(x, x_0) \leq \frac{1}{4}$. Lemma 8.3(b), along with (38.8), implies that the time- $(-\tau)$ ball $B(x_0, \frac{1}{4})$ contains the time-0 ball $B(x_0, \frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}))$. From the nonnegative curvature, the time- $(-\tau)$ volume of the first ball is at least as large as the time-0 volume of the second ball. Then

$$(38.9) \quad \begin{aligned} 5^{-n}w &= \text{vol}(B(x_0, 1)) \geq \text{vol}(B(x_0, \frac{1}{4})) \\ &\geq \text{vol}(B(x_0, \frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}))) \\ &\geq (\frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}))^n \text{vol}(B(x_0, 1)) \\ &\geq (\frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}))^n w, \end{aligned}$$

where the balls on the top line of (38.9) are at time- $(-\tau)$, and the other balls are at time-0. Thus $\frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}) \leq \frac{1}{5}$, and so $\tau \geq \tau_0$ where $\frac{1}{4} - 10(n-1)(\tau_0\sqrt{C} + 2\sqrt{B\tau_0}) = \frac{1}{5}$.

If we instead assume that the curvature operator is bounded below by $-r_0^{-2}$ in the time-dependent ball of radius r_0 , centered at x_0 , then the only difference is that the volume of the time- $(-\tau)$ ball $B(x_0, \frac{1}{4})$ will be at least $e^{-\text{const.}\tau r_0^{-2}}$ times the volume of the time-0 ball $B(x_0, \frac{1}{4} - 10(n-1)(\tau\sqrt{C} + 2\sqrt{B\tau}))$.

39. I.11.7

We remark that in the first paragraph of I.11.7, the correct statement is that a solution on a closed manifold, whose asymptotic gradient shrinking soliton is also closed, is a quotient of the round S^3 or $S^2 \times \mathbb{R}$. There are solutions on compact manifolds with noncompact asymptotic gradient shrinking soliton; see II.1.4.

For the proof of the theorem of I.11.7, the first claim is that the scalar curvature is bounded as a function of the distance to the basepoint. This is proved in I.11.7; alternatively, it follows from part 5 of Corollary 35.1.

The statement about splitting off a line, in the rescaling at $(y_i, 0)$, follows as in the case $\mathcal{R} = \infty$ of I.11.4.

For the last sentence in I.11.7, consider M equipped with the time zero metric, and let D_i denote the sequence of neck regions, with their cross-sections tending to zero as $i \rightarrow \infty$. Note that M has to be 1-ended. Otherwise, it would contain a line, and would therefore have to split off a line isometrically [7, Theorem 8.17]. But then M , the product of a line and a surface, could not have neck regions with cross-sections tending to zero.

From the theory of nonnegatively curved manifolds [7, Chapter 8.5], there is an exhaustion $M = \bigcup_{t \geq 0} C_t$ by totally convex compact sets C_t so that $(t_1 \leq t_2) \Rightarrow (C_{t_1} \subset C_{t_2})$, and

$$(39.1) \quad C_{t_1} = \{q \in C_{t_2} : \text{dist}(q, \partial C_{t_2}) \geq t_2 - t_1\}.$$

It follows from (39.1) that a neck region which is sufficiently close to a cylinder will intersect some ∂C_t in an approximate 2-sphere cross-section. Fix such a neck region D_0 and let C_{t_0} be the corresponding convex set. As M has one end, ∂C_{t_0} has only one connected component, namely the approximate 2-sphere cross-section.

For all $t > t_0$, there is a distance-nonincreasing retraction $r : C_t \rightarrow C_{t_0}$ which maps $C_t - C_{t_0}$ onto ∂C_{t_0} [25]. Let D be a neck region with a very small cross-section and let C_t be a convex set so that ∂C_t intersects D in an approximate 2-sphere cross-section. Then ∂C_t consists entirely of this approximate cross-section. The restriction of r to ∂C_t is distance-nonincreasing, but will map the 2-sphere ∂C_t onto the 2-sphere ∂C_{t_0} . This is a contradiction.

Remark 39.2. One may wonder where we have used the fact that we have a Ricci flow solution, i.e. whether the curvature is bounded for any κ -noncollapsed Riemannian 3-manifold with nonnegative sectional curvature. Following the above argument, we could again split off a line in a rescaling around high-curvature points. However, we would not necessarily know that the ensuing nonnegatively-curved surface is compact. (A priori, it could be a smoothed-out cone, for example.) In the case of a Ricci flow, the compactness comes from I.11.3.

Corollary 39.3. *Let $(M, g(\cdot))$ be a 3-dimensional κ -solution. Then any asymptotic soliton constructed as in I.11.2 is also a κ -solution.*

Note that the proof of I.11.7 requires I.11.4 and its consequences. We avoid logical circularity here by using section 36 and the proof of I.11.4 which avoids I.11.3.

40. ARGUMENT FOR A WEAK VERSION OF COROLLARY I.11.8

In this section we give a direct argument to prove the claims of Corollary I.11.8, except for the diameter bound. In the next section we give another argument which also proves the diameter bound

We claim first that M_ϵ is compact. Suppose not. Then there is a sequence of points $x_k \in M_\epsilon$ going to infinity. Fix a basepoint $x_0 \in M$. Then $R(x_0) \operatorname{dist}_0^2(x_0, x_k) \rightarrow \infty$. By part 5 of Corollary 35.1 above, $R(x_k) \operatorname{dist}_0^2(x_0, x_k) \rightarrow \infty$. Rescaling around $(x_k, 0)$ to make its scalar curvature one, we can use Theorem I.11.7 to extract a convergent subsequence (M_∞, x_∞) . As in the case $\mathcal{R} = \infty$ of I.11.4, we can say that (M_∞, x_∞) splits off a line. Hence for large k , x_k is the center of an ϵ -neck, which is a contradiction.

Next we claim that for any ϵ , there exists $C = C(\epsilon, \kappa) > 0$ such that if $g_{ij}(t)$ satisfies the assumptions in I.11.1 then for any point $x \in M_\epsilon$, there is a point $x_0 \in \partial M_\epsilon$ such that $\operatorname{dist}_0(x, x_0) \leq CQ^{-1/2}$ and $C^{-1}Q \leq R(x, 0) \leq CQ$, where $Q = R(x_0, 0)$.

If not then there is a sequence $\{M_i\}_{i=1}^\infty$ of ancient solutions satisfying the assumptions in I.11.1 along with points $x_i \in M_{i,\epsilon}$ such that for each $y_i \in \partial M_{i,\epsilon}$, we have

1. $\operatorname{dist}_0^2(x_i, y_i) R(y_i, 0) \geq i$ or
2. $R(y_i, 0) \geq i R(x_i, 0)$ or
3. $R(x_i, 0) \geq i R(y_i, 0)$.

Rescale the metric on M_i so that $R(x_i, 0) = 1$. From Theorem I.11.7, a subsequence of the pointed spaces (M_i, x_i) will converge smoothly to an ancient solution (M_∞, x_∞) satisfying the assumptions of I.11.1. Also, $x_\infty \in M_{\infty,\epsilon}$.

Taking a subsequence, we can assume that 1. occurs for each i , or 2. occurs for each i , or 3. occurs for each i . If $M_\infty \neq M_{\infty,\epsilon}$, choose $y_\infty \in \partial M_{\infty,\epsilon}$. Then y_∞ is the limit of a subsequence of points $y_i \in \partial M_{i,\epsilon}$.

If 1. occurs for each i then $\operatorname{dist}_0^2(x_\infty, y_\infty) R(y_\infty, 0) = \infty$, which is impossible. If 2. occurs for each i then $R(y_\infty, 0) = \infty$, which is impossible. If 3. occurs for each i then $R(y_\infty, 0) = 0$. It follows from (34.1) that M_∞ is flat, which is impossible, as $R(x_\infty, 0) = 1$.

Hence $M_\infty = M_{\infty,\epsilon}$, i.e. no point in the noncompact ancient solution M_∞ is the center of an ϵ -neck. This contradicts the previous conclusion that $M_{\infty,\epsilon}$ is compact.

41. ARGUMENT FOR A STRONG VERSION OF COROLLARY I.11.8

The following is an application of the compactness result Theorem I.11.7 :

Corollary 41.1. *For all $\kappa > 0$, there exists an $\epsilon_0 > 0$ such that for all $0 < \epsilon < \epsilon_0$ there exists an $\alpha = \alpha(\epsilon, \kappa)$ with the property that for any κ -solution $(M, g(\cdot))$, and at any time t , precisely one of the following holds (M_ϵ denotes the set of points which are not centers of ϵ -necks at time t):*

A. $(M, g(\cdot))$ is round cylindrical flow, and so every point at every time is the center of an ϵ -neck for all $\epsilon > 0$.

B. M is noncompact, $M_\epsilon \neq \emptyset$, and for all $x, y \in M_\epsilon$, we have $R(x)d^2(x, y) < \alpha$.

C. M is compact, and there is a pair of points $x, y \in M_\epsilon$ such that $R(x)d^2(x, y) > \alpha$,

$$M_\epsilon \subset B(x, \alpha R(x)^{-\frac{1}{2}}) \cup B(y, \alpha R(y)^{-\frac{1}{2}}),$$

and every $z \in M \setminus M_\epsilon$ satisfies $R(z)d^2(z, \overline{xy}) < \alpha$.

D. M is compact and there exists a point $x \in M_\epsilon$ such that $R(x)d^2(x, z) < \alpha$ for all $z \in M$.

Lemma 41.2. *For all $\epsilon > 0$, $\kappa > 0$, there exists $\alpha = \alpha(\epsilon, \kappa)$ with the following property. Suppose $(M, g(\cdot))$ is any κ -solution, $x, y, z \in M$, and at time t we have $x, y \in M_\epsilon$ and $R(x)d^2(x, y) > \alpha$. Then at time t either $R(x)d^2(z, x) < \alpha$ or $R(y)d^2(z, y) < \alpha$ or $R(z)d^2(z, \overline{xy}) < \alpha$ and $z \notin M_\epsilon$.*

Proof. Pick $\epsilon > 0$, $\kappa > 0$, and suppose no such α exists. Then there is a sequence $\alpha_k \rightarrow \infty$, a sequence of κ -solutions $(M_k, g_k(\cdot))$, and sequences $x_k, y_k, z_k \in M_k$, $t_k \in \mathbb{R}$ violating the α_k -version of the statement for all k . Thus we must have

$$R(x_k, t_k)d_{t_k}^2(x_k, y_k) \rightarrow \infty, \quad R(x_k, t_k)d_{t_k}^2(z_k, x_k) \rightarrow \infty, \quad \text{and} \quad R(y_k, t_k)d_{t_k}^2(z_k, y_k) \rightarrow \infty.$$

Let $z'_k \in \overline{x_k y_k}$ be a point in $\overline{x_k y_k}$ nearest z_k in $(M_k, g_k(t_k))$.

We first show that $R(x_k, t_k)d_{t_k}^2(z'_k, x_k) \rightarrow \infty$. If not, we may pass to a subsequence on which $R(x_k, t_k)d_{t_k}^2(z'_k, x_k)$ remains bounded. Applying Theorem I.11.7, we may pass to a subsequence and rescale by $R(x_k, t_k)$, to make the sequence $(M_k, g_k(\cdot), (x_k, t_k))$ converge to a κ -solution $(M_\infty, g_\infty(\cdot), (x_\infty, 0))$, the segments $\overline{x_k y_k} \subset (M_k, g_k(t_k))$ converge to a ray $\overline{x_\infty \xi} \subset (M_\infty, g_\infty(0))$, and the segments $\overline{z'_k z_k}$ converge to a ray $\overline{z'_\infty \eta}$. Recall that the comparison angle $\tilde{\angle}_{z'_\infty}(u, v)$ tends to the Tits angle $\partial_T(\xi, \eta)$ as $u \in \overline{z'_\infty \xi}$, $v \in \overline{z'_\infty \eta}$ tend to infinity. Since $d(z_k, z'_k) = d(z_k, \overline{x_k y_k})$ we must have $\partial_T(\xi, \eta) \geq \frac{\pi}{2}$. Now consider a sequence $u_k \in \overline{z'_\infty \xi}$ tending to infinity. By Theorem I.11.7, part 5 of Corollary 35.1, and the remarks about Alexandrov spaces in Appendix A, if we rescale $(M_\infty, g_\infty(\cdot), (u_k, 0))$ by $R(u_k, 0)$, we get round cylindrical flow as a limit. When k is sufficiently large, we may find an almost product region $D \subset (M_\infty, g_\infty(\cdot))$ containing u_k which is disjoint from $\overline{z'_\infty \eta}$, and whose cross-section $\Sigma \times \{0\} \subset \Sigma \times (-1, 1) \simeq D$ intersects the ray $\overline{z'_\infty \xi}$ transversely at a single point. This implies that $\Sigma \times \{0\}$ separates the two ends of $\overline{z'_\infty \xi} \cup \overline{z'_\infty \eta}$ from each other; hence M_∞ is two-ended, and $(M_\infty, g_\infty(\cdot))$ is round cylindrical flow. This contradicts the assumption that x_k is not the center of an ϵ -neck. Hence $R(x_k, t_k)d_{t_k}^2(z'_k, x_k) \rightarrow \infty$, and similar reasoning shows that $R(y_k, t_k)d_{t_k}^2(z'_k, y_k) \rightarrow \infty$.

By part 5 of Corollary 35.1, we therefore have $R(z'_k, t_k)d_{t_k}^2(z'_k, x_k) \rightarrow \infty$ and $R(z'_k, t_k)d_{t_k}^2(z'_k, y_k) \rightarrow \infty$. Rescaling the sequence $(M_k, g_k(\cdot), (z'_k, t_k))$ by $R(z'_k, t_k)$, we get convergence to round cylindrical flow (since any limit flow contains a line), and $\overline{z'_k z_k}$ subconverges to a segment orthogonal to the \mathbb{R} -factor, which implies that $R(z'_k, t_k)d_{t_k}^2(z_k, z'_k)$ is bounded and z_k is the center of an ϵ -neck for large k . This contradicts our assumption that the α_k -version of the lemma is violated for each k . \square

Proof of Corollary 41.1. Let $(M, g(\cdot))$ be a κ -solution, and $\epsilon > 0$.

Case 1: Every $x \in (M, g(t))$ is the center of an ϵ -neck. In this case, if $\epsilon > 0$ is sufficiently small, M fibers over a 1-manifold with fiber S^2 . If the 1-manifold is homeomorphic to \mathbb{R} , then M has two ends, which implies that the flow $(M, g(\cdot))$ is an evolving round cylinder. If the base of the fibration were a circle, then the universal cover $(\tilde{M}, \tilde{g}(t))$ would split off a line, which would imply that the universal covering flow would be a round cylindrical flow; but this would violate the κ -noncollapsed assumption at very negative times. Thus A holds in this case.

Case 2: There exist $x, y \in M_\epsilon$ such that $R(x)d^2(x, y) > \alpha$. By Lemma 41.2 and Corollary 35.1 part 5, for all $z \in M - \left(B(x, \alpha R(x)^{-\frac{1}{2}}) \cup B(y, \alpha R(y)^{-\frac{1}{2}}) \right)$, we have $R(z)d^2(z, \overline{xy}) < \alpha$ and $z \notin M_\epsilon$. This implies (again by Corollary 35.1 part 5) that there exists a $\gamma = \gamma(\epsilon, \kappa)$ such that for every $z \in M$ there is a $z' \in \overline{xy}$ for which $R(z')d^2(z', z) < \gamma$, which means that M must be compact, and C holds.

Case 3: $M_\epsilon \neq \emptyset$, and for all $x, y \in M_\epsilon$, we have $R(x)d^2(x, y) < \alpha$. If M is noncompact then we are in case B and are done, so assume that M is compact. Pick $x \in M_\epsilon$, and suppose $z \in M$ maximizes $R(x)d^2(\cdot, x)$. If $R(x)d^2(z, x) \geq \alpha$, then z is the center of an ϵ -neck, and we may look at the cross-section Σ of the neck region. If Σ separates M , then when $\epsilon > 0$ is sufficiently small, we get a contradiction to the assumption that z maximizes $R(x)d^2(x, \cdot)$. Hence Σ cannot separate M , and there is a loop passing through x which intersects Σ transversely at one point. It follows that the universal covering flow $(\tilde{M}, \tilde{g}(\cdot))$ is cylindrical flow, a contradiction. Hence $R(x)d^2(x, z) < \alpha$ for all $z \in M$, so D holds. \square

42. MORE PROPERTIES OF κ -SOLUTIONS

Proposition 42.1. *For all $\kappa > 0$, $\alpha > 0$, $\theta > 0$, there exists a $\beta(\kappa, \alpha, \theta) < \infty$ such that if $(M, g(t))$ is a time slice of a κ -solution, $x, y_1, y_2 \in M$, $R(x)d^2(x, y_i) > \beta$ for $i = 1, 2$, and $\angle_x(y_1, y_2) \geq \theta$, then (a) x is the center of an α -neck, and (b) $\angle_x(y_1, y_2) \geq \pi - \alpha$.*

Proof. The proof of this is similar to the first part of the proof of Lemma 41.2. Note that when α is small, then after enlarging β if necessary, the neck region around x will separate y_1 from y_2 ; this implies (b). \square

Corollary 42.2. *For all $\kappa > 0$, $\epsilon > 0$, there exists a $\rho = \rho(\kappa, \epsilon)$ such that if $(M, g(t))$ is a time slice of a κ -solution, $\eta \subset (M, g(t))$ is a minimizing geodesic segment with endpoints y_1, y_2 , $z \in M$, $z' \in \eta$ is a point in η nearest z , and $R(z')d^2(z', y_i) > \rho$ for $i = 1, 2$, then z, z' are centers of ϵ -necks, and $\max(R(z)d^2(z, z'), R(z')d^2(z, z')) < 4\pi^2$.*

Proof. Pick $\epsilon' > 0$. Under the assumptions, if

$$\min(R(z')d^2(z', y_1), R(z')d^2(z', y_2))$$

is sufficiently large, we can apply the preceding proposition to the triple z', y_1, y_2 , to conclude that z' is the center of an ϵ' -neck. Since the shortest segment from z to z' is orthogonal to η , when ϵ' is small enough the segment $\overline{zz'}$ will lie close to an S^2 cross-section in the approximating round cylinder, which gives $R(z')d^2(z, z') \lesssim 2\pi^2$. \square

43. I.11.9

We show that there exists $\kappa_0 > 0$ such that if an ancient solution on a noncompact three-manifold satisfies the assumptions in I.11.1 with some $\kappa > 0$, then it would satisfy these assumptions with $\kappa = \kappa_0$.

Suppose that for a given κ , M is κ -collapsed at some scale. After rescaling, we can assume that there is a point $(x_0, 0)$ so that $|\text{Rm}(x, t)| \leq 1$ for all (x, t) satisfying $\text{dist}_0(x, x_0) < 1$ and $t \in [-1, 0]$, with $\text{vol}(B(x_0, 1)) < \kappa$. Let $\tilde{V}(t)$ denote the reduced volume as a function of $t \in (-\infty, 0]$, as defined using curves from $(x_0, 0)$. It is nondecreasing in t . As in I.7.3, there is an estimate $\tilde{V}(-\kappa) \leq 3\kappa^{n/2}$. Take a sequence of times $t_i \rightarrow -\infty$. For each t_i , choose $q_i \in M$ so that $l(q_i, t_i) \leq \frac{n}{2}$. From the proof of I.11.2, for all $\epsilon > 0$, there is a $\delta > 0$ such that $l(q, t)$ does not exceed δ^{-1} whenever $t \in [2t_i, t_i]$ and $\text{dist}_{2t_i}^2(q, q_i) \leq \epsilon^{-1}t_i$. Given the monotonicity of \tilde{V} and the upper bound on $l(q, t)$, we obtain an upper bound on the volume of the time- $2t_i$ ball $B(q_i, \sqrt{t_i/\epsilon})$ of the form $3(2t_i)^{n/2} e^{\delta^{-1}} \kappa^{n/2}$. On the other hand, from I.11.2, a subsequence of the rescalings of the ancient solution around (q_i, t_i) converges to a nonflat gradient shrinking soliton. From Lemma II.1.2, the gradient shrinking soliton must be an evolving cylinder or its \mathbb{Z}_2 -quotient. Fixing ϵ , this gives a lower bound on κ .

Remark : John Morgan pointed out that the hypotheses of Lemma II.1.2 assume a global upper bound on the sectional curvature of any time slice, which is not a priori the case for the rescaled solution in I.11.2. However, in our 3-dimensional case, the argument of I.11.7 shows that there is such an upper bound.

44. II.1.2

The basic example of a gradient shrinking soliton is the metric on $\mathbb{R} \times S^2$ which gives the 2-sphere a radius of $\sqrt{-2t}$ at time $t \in (-\infty, 0)$. With coordinates (s, θ) on $\mathbb{R} \times S^2$, the function f is given by $f(t, s, \theta) = -\frac{s^2}{4t}$.

Applying ∇_i to II.(1.1) gives

$$(44.1) \quad \Delta \nabla_j f + \nabla_i R_{ij} = 0.$$

As $\nabla_i R_{ij} = \frac{1}{2} \nabla_j R$ and $\Delta \nabla_j f = \nabla_j \Delta f + R_{jk} \nabla_k f = \nabla_j (-R - \frac{n}{2t}) + R_{jk} \nabla_k f$, equation II.(1.2) follows.

The fact that $\int_0^{\bar{s}} \text{Ric}(X, X) ds \leq \text{const.}$ follows as in the proof of Lemma I.8.3(b). If $\{Y_i\}_{i=1}^3$ are orthonormal vector fields along γ then

$$(44.2) \quad \left(\int_0^{\bar{s}} |\text{Ric}(X, Y_1)| ds \right)^2 \leq \bar{s} \int_0^{\bar{s}} |\text{Ric}(X, Y_1)|^2 ds \leq \bar{s} \sum_{i=1}^3 \int_0^{\bar{s}} |\text{Ric}(X, Y_i)|^2 ds.$$

Thinking of Ric as a self-adjoint linear operator on TM , $\sum_{i=1}^3 |\text{Ric}(X, Y_i)|^2 = \langle X, \text{Ric}^2 X \rangle$. In terms of a pointwise orthonormal frame $\{e_i\}$ of eigenvectors of Ric, with eigenvalues λ_i ,

write $X = \sum_{i=1}^3 X_i e_i$. Then

$$(44.3) \quad \langle X, \text{Ric}^2 X \rangle = \sum_{i=1}^3 \lambda_i^2 X_i^2 \leq \left(\sum_{i=1}^3 \lambda_i \right) \left(\sum_{i=1}^3 \lambda_i X_i^2 \right) = R \cdot \text{Ric}(X, X).$$

Hence

$$(44.4) \quad \left(\int_0^{\bar{s}} |\text{Ric}(X, Y_1)| ds \right)^2 \leq (\sup_M R) \bar{s} \int_0^{\bar{s}} \text{Ric}(X, X) ds \leq \text{const. } \bar{s}.$$

Multiplying II.(1.1) by $X^i X^j$ and summing gives $\frac{d^2 f(\gamma(s))}{ds^2} + \text{Ric}(X, X) - \frac{1}{2} = 0$. Then

$$(44.5) \quad \left. \frac{df(\gamma(s))}{ds} \right|_{s=\bar{s}} = \left. \frac{df(\gamma(s))}{ds} \right|_{s=0} + \frac{1}{2} \bar{s} - \int_0^{\bar{s}} \text{Ric}(X, X) ds \geq \frac{1}{2} \bar{s} - \text{const.}$$

Multiplying II.(1.1) by $X^i Y^j$ and summing gives $\frac{d}{ds}(Y \cdot f)(\gamma(s)) + \text{Ric}(X, Y) = 0$. Then

$$(44.6) \quad (Y \cdot f)(\gamma(\bar{s})) = (Y \cdot f)(\gamma(0)) - \int_0^{\bar{s}} \text{Ric}(X, Y) ds$$

and $|(Y \cdot f)(\gamma(\bar{s}))| \leq \text{const. } \sqrt{\bar{s}}$.

Take points x_α tending toward infinity, with $R(x_\alpha) \rightarrow \bar{R}$. Putting $r_\alpha = \sqrt{\text{dist}_{-1}(x_0, x_\alpha)}$, we have $\frac{r_\alpha}{\text{dist}_{-1}(x_0, x_\alpha)} \rightarrow 0$ and $R(x_\alpha) r_\alpha^2 \rightarrow \infty$. Then the argument of the case $\mathcal{R} = \infty$ of the proof of I.11.4 shows that any convergent subsequence of the rescalings around $(x_\alpha, -1)$ splits off a line.

Let N denote a level surface of f . At a point of N , choose an orthonormal frame $\{e_1, e_2, e_3\}$ with $e_3 = X$ normal to N . From the Gauss-Codazzi equation,

$$(44.7) \quad R^N = 2 K^N(e_1, e_2) = 2(K^M(e_1, e_2) + \det(S)),$$

where S is the shape operator. As $R = 2(K^M(e_1, e_2) + K^M(e_1, e_3) + K^M(e_2, e_3))$ and $\text{Ric}(X, X) = K^M(e_1, e_3) + K^M(e_2, e_3)$, we obtain

$$(44.8) \quad R^N = R - 2 \text{Ric}(X, X) + 2 \det(S).$$

The shape operator is given by $S = \frac{\text{Hess} f|_{TN}}{|\nabla f|}$. From II.(1.1), $\text{Hess} f = \frac{1}{2} - \text{Ric}$. We can

diagonalize $\text{Ric}|_{TN}$ to write $\text{Ric} = \begin{pmatrix} r_1 & 0 & c_1 \\ 0 & r_2 & c_2 \\ c_1 & c_2 & r_3 \end{pmatrix}$, where $r_3 = \text{Ric}(X, X)$. Then

$$(44.9) \quad \det(\text{Hess} f|_{TN}) = \left(\frac{1}{2} - r_1 \right) \left(\frac{1}{2} - r_2 \right) = \frac{1}{4} \left((1 - r_1 - r_2)^2 - (r_1 - r_2)^2 \right) \\ \leq \frac{1}{4} (1 - r_1 - r_2)^2 = \frac{1}{4} (1 - R + \text{Ric}(X, X))^2.$$

This shows that the scalar curvature of N is bounded above by

$$(44.10) \quad R - 2 \text{Ric}(X, X) + \frac{(1 - R + \text{Ric}(X, X))^2}{2|\nabla f|^2}.$$

If $|\nabla f|$ is large then $1 - R + \text{Ric}(X, X) < 2|\nabla f|^2$. As $1 - R + \text{Ric}(X, X)$ is positive when the distance from x to x_0 is large enough,

$$(44.11) \quad (1 - R + \text{Ric}(X, X))^2 < 2(1 - R + \text{Ric}(X, X))|\nabla f|^2 \\ \leq 2(1 - R + \text{Ric}(X, X))|\nabla f|^2 + 2|\nabla f|^2 \text{Ric}(X, X)$$

and so

$$(44.12) \quad \frac{(1 - R + \text{Ric}(X, X))^2}{2|\nabla f|^2} < 1 - R + 2 \text{Ric}(X, X).$$

Hence

$$(44.13) \quad R - 2 \text{Ric}(X, X) + \frac{(1 - R + \text{Ric}(X, X))^2}{2|\nabla f|^2} < 1.$$

45. I.12.1

We give a proof which differs in some points from the proof in I.12.1, but which has the same ingredients.

Suppose that the statement were false. Then for some $\epsilon, \kappa > 0$, we would have a sequence of ϕ -nonnegatively curved 3-dimensional Ricci flows $(M_k, g_k(\cdot))$ defined on intervals $[0, T_k]$, and sequences $r_k \rightarrow 0$, $\hat{x}_k \in M_k$, $\hat{t}_k \geq 1$ such that M_k is κ -noncollapsed on scales $< r_k$ and $Q_k = R(\hat{x}_k, \hat{t}_k) \geq r_k^{-2}$, but the Q_k -rescaled solution in $B_{(\epsilon Q_k)^{-1/2}}(\hat{x}_k) \times [\hat{t}_k - (\epsilon Q_k)^{-1}, \hat{t}_k]$ is not ϵ -close to the corresponding subset of an ancient solution satisfying the assumptions of I.11.1.

We note that if the statement were false for ϵ then it would also be false for any smaller ϵ . Because of this, somewhat paradoxically, we will begin the argument with a given ϵ but will allow ourselves to make ϵ small enough later so that the argument works.

The goal is to get a contradiction based on the ‘‘bad’’ points (\hat{x}_k, \hat{t}_k) . In a sense, the method of proof of Theorem I.12.1 is an induction on the curvature scale. It consists of four steps. Step 1 consists of replacing the sequence (\hat{x}_k, \hat{t}_k) by another sequence of ‘‘bad’’ points (x_k, t_k) which have the property that points near (x_k, t_k) with distinctly higher scalar curvature are ‘‘good’’ points. It then suffices to get a contradiction based on the existence of the sequence (x_k, t_k) . In steps 2-4 one uses the points (x_k, t_k) to build up a κ -solution, whose existence then contradicts the ‘‘badness’’ of the points (x_k, t_k) .

More precisely, let $(M_k, \bar{g}_k(\cdot), (x_k, t_k))$ be the result of rescaling $g_k(\cdot)$ by $R(x_k, t_k)$. We will show that the sequence of pointed flows $(M_k, \bar{g}_k(\cdot), (x_k, t_k))$ accumulates on a κ -solution $(M_\infty, \bar{g}_\infty(\cdot), (x_\infty, t_0))$, thereby obtaining a contradiction.

In step 2 one takes a pointed limit of the manifolds $(M_k, \bar{g}_k(t_k), x_k)$ in order to construct what will become the final time slice of the κ -solution, $(M_\infty, \bar{g}_\infty(t_0), x_\infty)$. In order to take this limit, it is necessary to show that the manifolds $(M_k, \bar{g}_k(t_k), x_k)$ have uniformly bounded curvature on distance balls of a fixed radius. If this were not true then for some radius, a subsequence of the manifolds $(M_k, \bar{g}_k(t_k), x_k)$ would have curvatures that asymptotically blowup on the ball of that radius. One shows that geometrically, the curvature blowup is due to the asymptotic formation of a cone-like point at the blowup radius. Doing a further rescaling at this cone-like point, one obtains a Ricci flow solution that ends on a part of a

nonflat metric cone. This gives a contradiction as in the case $0 < \mathcal{R} < \infty$ of Proposition I.11.4

Thus one can construct the pointed limit $(M_\infty, \bar{g}_\infty(t_0), x_\infty)$. The goal now is to show that $(M_\infty, \bar{g}_\infty(t_0), x_\infty)$ is the final time slice of a κ -solution $(M_\infty, \bar{g}_\infty(\cdot), (x_\infty, t_0))$. In step 3 one shows that $(M_\infty, \bar{g}_\infty(t_0), x_\infty)$ extends backward to a Ricci flow solution on some time interval $[t_0 - \Delta, t_0]$, and that the time slices have bounded nonnegative curvature. In step 4 one shows that the Ricci flow solution can be extended all the way to time $(-\infty, t_0]$, thereby constructing a κ -solution

Step 1: Adjusting the choice of basepoints.

The phrase “nearly the smallest curvature Q ” in I.12.1 should read “nearly the largest curvature Q ”. This is clear from the sentence in parentheses that follows.

Lemma 45.1. *We may find $H_k \rightarrow \infty$, $x_k \in M_k$, and $t_k \geq \frac{1}{2}$ such that $Q_k := R(x_k, t_k) \rightarrow \infty$, and for all k the conclusion of Theorem I.12.1 fails at (x_k, t_k) , but holds for any $(y, t) \in (M_k \times [t_k - H_k Q_k^{-1}, t_k], g_k(\cdot))$ for which $R(y, t) \geq 2R(x_k, t_k)$.*

Proof. Choose $H_k \rightarrow \infty$ such that $H_k(R(x_k, t_k))^{-1} \leq \frac{1}{10}$ for all k . Set $(x_k, t_k) := (\hat{x}_k, \hat{t}_k)$. For each k , look for a point in $M_k \times [t_k - H_k Q_k^{-1}, t_k]$ at which Theorem I.12.1 fails, and the scalar curvature is at least $2R(x_k, t_k)$. If such a point exists, replace (x_k, t_k) by this point; otherwise do nothing. Repeat this until the second alternative occurs. This process must terminate with a new choice of (x_k, t_k) satisfying the lemma. \square

Hereafter we use this modified sequence (x_k, t_k) . Let $(M_k, \bar{g}_k(\cdot), (x_k, t_k))$ be the result of rescaling $g_k(\cdot)$ by $Q_k = R(x_k, t_k)$; we use \bar{R}_k to denote its scalar curvature. Note that the rescaled time interval of Lemma 45.1 has duration $H_k \rightarrow \infty$; this is what we want in order to try to extract an ancient solution.

Step 2: For every $\rho < \infty$, the scalar curvature \bar{R}_k is uniformly bounded on the ρ -balls $B(x_k, \rho) \subset (M_k, \bar{g}_k(t_k))$ (the argument for this is essentially equivalent to the proof of Claim 2 in I.12.1). Before proceeding, we need some bounds which come from our choice of basepoints, and the derivative bounds inherited (by approximation) from κ -solutions.

Lemma 45.2. *There is a constant $C = C(\kappa)$ so that for any (x, t) in a Ricci flow solution, if $R(x, t) > 0$ and the solution in $B_t(x, (\epsilon R(x, t))^{-1/2}) \times [t - (\epsilon R(x, t))^{-1}, t]$ is ϵ -close to a corresponding subset of an ancient solution satisfying the assumptions in I.11.1 then $|\nabla R^{-1/2}|(x, t) \leq C$ and $|\partial_t R^{-1}| \leq C$.*

Proof. This follows from the compactness in Theorem I.11.7. \square

Note that the same value of C in Lemma 45.2 also works for smaller ϵ .

Lemma 45.3 (Claim 1 of I.12.1). *For each (\bar{x}, \bar{t}) with $t_k - \frac{1}{2} H_k Q_k^{-1} \leq \bar{t} \leq t_k$, we have $R_k(x, t) \leq 4\bar{Q}_k$ whenever $\bar{t} - c\bar{Q}_k^{-1} \leq t \leq \bar{t}$ and $\text{dist}_{\bar{t}}(x, \bar{x}) \leq c\bar{Q}_k^{-1/2}$, where $\bar{Q}_k = Q_k + R_k(\bar{x}, \bar{t})$ and $c = c(\kappa) > 0$ is a small constant.*

Proof. If $R_k(x, t) \leq 2Q_k$ then there is nothing to show. If $R_k(x, t) > 2Q_k$, consider a spacetime curve γ that goes linearly from (x, t) to (x, \bar{t}) , and then goes from (x, \bar{t}) to (\bar{x}, \bar{t}) along a minimizing geodesic. If there is a point on γ with curvature $2Q_k$, let p be the nearest such point to (x, t) . If not, put $p = (\bar{x}, \bar{t})$. We can apply Lemma 45.2 along γ from (x, t) to p . The claim follows from integrating the ensuing derivative bounds along γ . \square

Lemma 45.4. *In terms of the rescaled solution $\bar{g}_k(t)$, for each (\bar{x}, \bar{t}) with $t_k - \frac{1}{2}H_k \leq \bar{t} \leq t_k$, we have $\bar{R}_k(x, t) \leq 4\tilde{Q}_k$ whenever $\bar{t} - c\tilde{Q}_k^{-1} \leq t \leq \bar{t}$ and $\text{dist}_{\bar{t}}(x, \bar{x}) \leq c\tilde{Q}_k^{-1/2}$, where $\tilde{Q}_k = 1 + \bar{R}_k(\bar{x}, \bar{t})$.*

Proof. This is just the rescaled version of Lemma 45.3. \square

For all $\rho \geq 0$, put

$$D(\rho) := \sup\{\bar{R}_k(x, t_k) \mid k \geq 1, x \in B(x_k, \rho) \subset (M_k, \bar{g}_k(t_k))\},$$

and let ρ_0 be the supremum of the ρ 's for which $D(\rho) < \infty$. Note that $\rho_0 > 0$, in view of Lemma 45.4 (taking $(\bar{x}, \bar{t}) = (x_k, t_k)$). Suppose that $\rho_0 < \infty$. After passing to a subsequence if necessary, we may find a sequence $y_k \in M_k$ with $\text{dist}_{t_k}(x_k, y_k) \rightarrow \rho_0$ and $\bar{R}(y_k, t_k) \rightarrow \infty$. Let $\eta_k \subset (M_k, \bar{g}_k(t_k))$ be a minimizing geodesic segment from x_k to y_k . Let $z_k \in \eta_k$ be the point on η_k closest to y_k at which $\bar{R}(z_k, t_k) = 2$, and let γ_k be the subsegment of η_k running from y_k to z_k . By Lemma 45.4 the length of γ_k is bounded away from zero independent of k . Due to the ϕ -pinching (see (C.3) and (C.5)), for all $\rho < \rho_0$, we have a uniform bound on $|\text{Rm}|$ on the balls $B(x_k, \rho) \subset (M_k, \bar{g}_k(t_k))$. The injectivity radius is also controlled in $B(x_k, \rho)$, in view of the curvature bounds and the κ -noncollapsing. Therefore after passing to a subsequence, we can assume that the pointed sequence $(B(x_k, \rho_0), \bar{g}_k(t_k), x_k)$ converges in the pointed Gromov-Hausdorff topology (i.e. for all $\rho < \rho_0$ we have the usual Gromov-Hausdorff convergence) to a pointed C^1 -Riemannian manifold $(Z, \bar{g}_\infty, x_\infty)$, the segments η_k converge to a segment (missing an endpoint) $\eta_\infty \subset Z$ emanating from x_∞ , and γ_k converges to $\gamma_\infty \subset \eta_\infty$. Let \bar{Z} denote the completion of (Z, \bar{g}_∞) , and $y_\infty \in \bar{Z}$ the limit point of η_∞ . Note that by part 4 of Corollary 35.1, the Riemannian structure near γ_∞ may be chosen to be many times differentiable; in particular the scalar curvature \bar{R}_∞ is defined, differentiable, and satisfies the bound in Lemma 45.2 near γ_∞ .

Lemma 45.5. *1. There is a function $c : (0, \infty) \rightarrow \mathbb{R}$ depending only on κ , with $\lim_{t \rightarrow 0} c(t) = \infty$, such that if $w \in \gamma_\infty$ then $\bar{R}_\infty(w)(\rho_0 - d(w, x_\infty))^2 > c(\epsilon)$.*

2. There is a function $\epsilon' : (0, \infty) \rightarrow \mathbb{R} \cup \infty$ depending only on κ , with $\lim_{t \rightarrow 0} \epsilon'(t) = 0$, such that if $w \in \gamma_\infty$ and $d(y_\infty, w)$ is sufficiently small then the pointed manifold $(Z, \bar{R}_\infty(w)\bar{g}_\infty, w)$ is $2\epsilon'(\epsilon)$ -close to a round cylinder in the C^2 topology.

Proof. It follows from Lemma 45.1 that for all $w \in \gamma_\infty$, the pointed Riemannian manifold $(Z, \bar{R}_\infty(w)\bar{g}_\infty, w)$ is 2ϵ -close to (a time slice of) a pointed κ -solution. From the definition of (2ϵ) -closeness, there is an embedded region around w , large on the scale defined by $\bar{R}_\infty(w)$, which is close to the corresponding subset of a pointed κ -solution. This gives a lower bound on the distance $\rho_0 - d(w, x_\infty)$ to the point of curvature blowup, thereby proving part 1 of the Lemma.

We know that $(Z, \bar{R}_\infty(w)\bar{g}_\infty, w)$ is 2ϵ -close to a pointed κ -solution $(N, h(t), \star)$ in the pointed C^2 -topology. By Lemma 45.2, we know $\bar{R}_\infty(w)$ tends to ∞ as $d(w, x_\infty) \rightarrow \rho_0$, for $w \in \gamma_\infty$. In particular, $\bar{R}_\infty(w)d^2(w, x_\infty) \rightarrow \infty$. From part 1 above, we can choose ϵ small enough in order to make $\bar{R}_\infty(w)d^2(w, y_\infty)$ large enough to apply Proposition 42.1. Hence the pointed manifold $(N, R(\star)h(t), \star)$ is $\epsilon'(\epsilon)$ -close to a round cylinder, where $\epsilon' : (0, \infty) \rightarrow \mathbb{R}$ is a function with $\lim_{t \rightarrow 0} \epsilon'(t) = 0$. \square

It follows from part 2 of Lemma 45.5 that when ϵ is small, if we form the union of the balls $B(w, 4\pi(R_\infty(w))^{\frac{1}{2}})$ as w ranges over the points in γ_∞ , we obtain a Riemannian manifold W , and by adding in the point y_∞ , we get a metric space \bar{W} which is locally complete, and geodesic near y_∞ . As the original manifolds M_k had ϕ -almost nonnegative curvature, it follows that W is nonnegatively curved. Furthermore, y_∞ cannot be an interior point of any geodesic segment in \bar{W} , since such a geodesic would have to pass through a cylindrical region near y_∞ twice. The usual proof of the Toponogov triangle comparison inequality now applies near y_∞ since minimizers remain in the smooth nonnegatively curved part of \bar{W} .

This implies that blow-ups of (\bar{W}, y_∞) converge to the tangent cone $C_{y_\infty}\bar{W}$. As \bar{W} is three-dimensional, so is $C_{y_\infty}\bar{W}$ [5, Corollary 7.11]. It will be C^2 -smooth away from the vertex and nowhere flat, by part 2 of Lemma 45.5. Pick $z \in C_{y_\infty}\bar{W}$ such that $d(z, y_\infty) = 1$. Then the ball $B(z, \frac{1}{2}) \subset C_{y_\infty}\bar{W}$ is the Gromov-Hausdorff limit of a sequence of rescaled balls $B(z_k, r_k) \subset (M_k, \bar{g}_k(t_k))$ where $r_k \rightarrow 0$, whose center points (z_k, t_k) satisfy the conclusions of I.12.1. Applying Lemma 45.4 and Appendix C, we get the curvature bounds needed to extract a limiting Ricci flow solution whose time zero slice is isometric to $B(z, \frac{1}{2})$. Now we can apply the reasoning from the $0 < \mathcal{R} < \infty$ case of Proposition I.11.4 to get a contradiction. This completes step 2.

Step 3: The sequence of pointed flows $(M_k, \bar{g}_k(\cdot), (x_k, t_k))$ accumulates on a pointed Ricci flow $(M_\infty, \bar{g}_\infty(\cdot), (x_\infty, t_0))$ which is defined on a time interval $[t', t_0]$ with $t' < t_0$. By step 2, we know that the scalar curvature of $(M_k, \bar{g}_k(t_k))$ at $y \in M_k$ is bounded by a function of the distance from y to x_k . Lemma 45.4 extends this curvature control to a backward parabolic neighborhood centered at y whose radius depends only on $d(y, x_k)$. Thus we can conclude that all derivatives of the curvature $(M_k, \bar{g}_k(t_k))$ are controlled as a function of the distance from x_k , which means that the sequence of pointed manifolds $(M_k, \bar{g}_k(t_k), x_k)$ accumulates to a smooth manifold $(M_\infty, \bar{g}_\infty)$.

We claim that M_∞ has nonnegative sectional curvature. This is a general fact about blowup solutions. To see it, for $m \in M_\infty$, let $m_k \in (M_k, \bar{g}_k(t_k))$ be a sequence of approximants to m . Then $\lim_{k \rightarrow \infty} \overline{\text{Rm}}(m_k) = \overline{\text{Rm}}_\infty(m)$, where $\overline{\text{Rm}}(m_k) = Q_k^{-1} \text{Rm}(m_k)$. Also, as $t_k R(m_k) \geq -\frac{3}{2}$, the numbers $R(m_k)$ are uniformly bounded below. There are two possibilities: either the numbers $R(m_k)$ are uniformly bounded above or they are not. If they are uniformly bounded above then $\text{Rm}(m_k)$ is uniformly bounded below (from the comment after (C.3)) and so $\overline{\text{Rm}}_\infty(m) = 0$. Suppose on the other hand that a subsequence of the numbers $R(m_k)$ tends to infinity. Applying (C.4) to this subsequence gives that $\overline{\text{Rm}}_\infty(m)$,

the limit of

$$(45.6) \quad Q_k^{-1} \operatorname{Rm}(m_k) = \bar{R}(m_k) \frac{\operatorname{Rm}(m_k)}{R(m_k)},$$

is nonnegative.

Furthermore, M_∞ must have bounded curvature; otherwise we could use Lemma 45.1 and the argument from Section 39 to get a contradiction. By Lemma 45.4 again, we now get curvature control on $(M_k, \bar{g}_k(\cdot))$ for a time interval $[t_k - \Delta, t_k]$ for some $\Delta > 0$, and hence we can extract a subsequence which converges to a pointed Ricci flow $(M_\infty, \bar{g}_\infty(\cdot), (x_\infty, t_0))$ defined for $t \in [t_0 - \Delta, t_0]$, which has nonnegative curvature and bounded curvature on each time slice.

Step 4: Getting an ancient solution. Let $(t', t_0]$ be the maximal time interval on which we can extract a limiting solution $(M_\infty, \bar{g}_\infty(\cdot))$. Suppose that $t' > -\infty$. By Lemma 45.4 the maximum of the scalar curvature on the time slice $(M_\infty, \bar{g}_\infty(t))$ must tend to infinity as $t \rightarrow t'$. From the trace Harnack inequality, $R_t + \frac{R}{t-t'} \geq 0$, and so

$$(45.7) \quad \bar{R}_\infty(x, t) \leq Q \frac{t_0 - t'}{t - t'},$$

where Q is the maximum of the scalar curvature on $(M_\infty, \bar{g}_\infty(t_0))$. Combining this with Corollary 17.3, we get

$$\frac{d}{dt} d_t(x, y) \geq \operatorname{const.} \sqrt{Q \frac{t_0 - t'}{t - t'}}.$$

Since the right hand side is integrable on $(t', t_0]$, and using the fact that distances are nonincreasing in time (since $\operatorname{Rm} \geq 0$), it follows that there is a constant C such that

$$(45.8) \quad |d_t(x, y) - d_{t_0}(x, y)| < C$$

for all $x, y \in M_\infty$, $t \in (t', t_0]$.

If M_∞ is compact then by (45.8) the diameter of $(M_\infty, \bar{g}_\infty(t))$ is bounded independent of $t \in (t', t_0]$. Since the minimum of the scalar curvature is increasing in time, it is also bounded independent of t . Now the argument in step 2 shows that the curvature is bounded everywhere independent of t .

We may therefore assume M_∞ is noncompact. To be consistent with the notation of I.12.1, we now relabel the basepoint (x_∞, t_0) as (x_0, t_0) . Since nonnegatively curved manifolds are asymptotically conical (see Appendix A), there is a constant D such that if $y \in M_\infty$, and $d_{t_0}(y, x_0) > D$, then there is a point $x \in M_\infty$ such that

$$(45.9) \quad d_{t_0}(x, y) = d_{t_0}(y, x_0) \quad \text{and} \quad d_{t_0}(x, x_0) \geq \frac{3}{2} d_{t_0}(y, x_0);$$

by (45.8) the same conditions hold at all times $t \in (t', t_0]$, up to error C . If for some such y , and some $t \in (t', t_0]$ the scalar curvature were large, then $(M_\infty, \bar{R}_\infty(y, t) \bar{g}_\infty(t), (y, t))$ would be 2ϵ -close to a κ -solution $(N, h(\cdot), (z, t_0))$. When ϵ is small we could use Proposition 42.1 to see that y lies in a neck region U in $(M_\infty, \bar{g}_\infty(t))$ of diameter $\approx R(y, t)^{-\frac{1}{2}} \ll 1$ which separates x_0 from x . Since \bar{g}_∞ has nonnegative curvature, we have $\operatorname{diam}_{t_0}(U) \leq \operatorname{diam}_t(U) \ll 1$. Hence at time t_0 the set U separates x_0 from x , where x_0, y , and x satisfy (45.9). Since

$(M_\infty, \bar{g}_\infty(t_0))$ has bounded geometry, there must be a uniform upper bound on $R(y, t)$. Thus the curvature of $(M_\infty, \bar{g}_\infty)$ is uniformly bounded outside a set of uniformly bounded diameter. Repeating the reasoning from step 2, we get uniform bounds everywhere. This contradicts our assumption that the curvature blows up as $t \rightarrow t'$.

46. I.12.2

We first write out the corrected version of the theorem (see II.6.2).

Theorem 46.1. *For any $A < \infty$, there exists $K = K(A) < \infty$ and $\rho = \rho(A) > 0$ with the following property. Suppose in dimension three we have a solution to the Ricci flow with ϕ -almost nonnegative curvature. Suppose that for $x_0 \in M$ and $r_0 > 0$, the solution is defined for $0 \leq t \leq r_0^2$ and has $|\text{Rm}|(x, t) \leq r_0^{-2}$ for all (x, t) satisfying $\text{dist}_0(x, x_0) < r_0$. Suppose in addition that the volume of the metric ball $B(x_0, r_0)$ at time zero is at least $A^{-1}r_0^3$, and $\phi(r_0^{-2}) < \rho$. Then $R(x, r_0^2) \leq Kr_0^{-2}$ whenever $\text{dist}_{r_0^2}(x, x_0) < Ar_0$.*

The added restriction $\phi(r_0^{-2}) < \rho$ is necessary, as otherwise one could conclude that neck pinches do not occur.

There is an apparent gap in the proof of I.12.2, in the sentence (There is a little subtlety...). We instead follow the proof of Proposition II.6.3(b,c), which proves the same statement in the presence of surgeries.

The volume assumption in the theorem is used to guarantee noncollapsing, by means of Theorem I.8.2.

The first claim is that for any $\epsilon > 0$ there exists $K = K(A, \epsilon) < \infty$ so that for any r_0 , whenever we have a solution as in the statement of the theorem and $\text{dist}_{r_0^2}(x, x_0) < Ar_0$, then $R(x, r_0^2) < Kr_0^{-2}$ or a neighborhood of (x, r_0^2) is ϵ -close to the corresponding subset of a κ -solution. One can think of this claim as a localized analog of Theorem 12.1, where ‘‘localized’’ refers to the fact that both the hypotheses and the conclusion involve the point x_0 .

To prove the claim, suppose that there is a sequence of such pointed solutions $(M_k, g_k(t), x_{0,k})$, along with points $\hat{x}_k \in M_k$, so that $\text{dist}_{r_0^2}(\hat{x}_k, x_{0,k}) < Ar_0$ and $r_0^2 R(\hat{x}_k, r_0^2) \rightarrow \infty$, but each (\hat{x}_k, r_0^2) does not have a neighborhood that is ϵ -close to the corresponding subset of a κ -solution. As in the proof of Theorem I.12.1, we will allow ourselves to make ϵ smaller during the course of the proof.

We first show that there is a sequence $D_k \rightarrow \infty$ and modified points (\bar{x}_k, \bar{t}_k) with $\frac{3}{4}r_0^2 \leq \bar{t}_k \leq r_0^2$, $\text{dist}_{\bar{t}_k}(\bar{x}_k, x_{0,k}) < \frac{3}{2}Ar_0$ and $Q_k = R(\bar{x}_k, \bar{t}_k) \rightarrow \infty$, so that any point (x'_k, t'_k) with $R(x'_k, t'_k) > 2Q_k$, $\bar{t}_k - D_k^2 Q_k^{-1} \leq t'_k \leq \bar{t}_k$ and $\text{dist}_{t'_k}(x'_k, x_{0,k}) < \text{dist}_{\bar{t}_k}(\bar{x}_k, x_{0,k}) + D_k Q_k^{-1/2}$ has a neighborhood that is ϵ -close to the corresponding subset of a κ -solution, but (\bar{x}_k, \bar{t}_k) does not have a neighborhood that is ϵ -close to ...

To find (\bar{x}_k, \bar{t}_k) , put $D_k = \frac{r_0 R(\hat{x}_k, r_0^2)^{1/2}}{10}$. Start with $(x_k, t_k) = (\hat{x}_k, r_0^2)$ and look if there is a point (x'_k, t'_k) with $R(x'_k, t'_k) > 2R(x_k, t_k)$, $\bar{t}_k - D_k^2 R(x_k, t_k)^{-1} \leq t'_k \leq \bar{t}_k$ and $\text{dist}_{t'_k}(x'_k, x_{0,k}) < \text{dist}_{t_k}(x_k, x_{0,k}) + D_k R(x_k, t_k)^{-1/2}$, but which does not have a neighborhood that is ϵ -close to the corresponding subset of a κ -solution. If there is such a point, we replace (x_k, t_k)

by (x'_k, t'_k) and repeat the process. The process must terminate after a finite number of steps to give a point (\bar{x}_k, \bar{t}_k) with the property that any point (x'_k, t'_k) with $R(x'_k, t'_k) > 2Q_k$, $\bar{t}_k - D_k^2 Q_k^{-1} \leq t'_k \leq \bar{t}_k$ and $\text{dist}_{t'_k}(x'_k, x_{0,k}) < \text{dist}_{\bar{t}_k}(\bar{x}_k, x_{0,k}) + D_k Q_k^{-1/2}$ is ϵ -close to ..., but (\bar{x}_k, \bar{t}_k) is not ϵ -close to ...

(Note that the condition $\text{dist}_{t'_k}(x'_k, x_{0,k}) < \text{dist}_{\bar{t}_k}(\bar{x}_k, x_{0,k}) + D_k Q_k^{-1/2}$ involves the metric at time t'_k . In order to construct an ancient solution, one of the issues will be to replace this by a condition that only involves the metric at time \bar{t}_k , i.e. that involves a parabolic neighborhood around (\bar{x}_k, \bar{t}_k) .)

Let $\bar{g}_k(t)$ denote the rescaling of the solution $g_k(t)$ by Q_k . We normalize the time interval of the rescaled solution by fixing a number t_∞ and saying that for all k , the time- \bar{t}_k slice of (M_k, g_k) corresponds to the time- t_∞ slice of (M_k, \bar{g}_k) . Then the scalar curvature \bar{R}_k of \bar{g}_k satisfies $\bar{R}_k(\bar{x}_k, t_\infty) = 1$.

By the argument of Step 2 of the proof of I.12.1, a subsequence of the pointed spaces $(M_k, \bar{g}_k(t_\infty), \bar{x}_k)$ will smoothly converge to a nonnegatively-curved pointed space $(M_\infty, \bar{g}_\infty, \bar{x}_\infty)$. By the pointpicking, if $m \in M_\infty$ has $\bar{R}(m) \geq 3$ then a neighborhood of m is ϵ -close to the corresponding region in a κ -solution. It follows, as in Step 3 of the proof of Theorem I.12.1, that the sectional curvature of M_∞ will be bounded above by some $C < \infty$. Using Lemma 45.4, the metric on M_∞ is the time- t_∞ slice of a nonnegatively-curved Ricci flow solution defined on some time interval $[t_\infty - c, t_\infty]$, with $c > 0$, and one has convergence of a subsequence $\bar{g}_k(t) \rightarrow \bar{g}_\infty(t)$ for $t \in [t_\infty - c, t_\infty]$. As $\bar{R}_t \geq 0$, the scalar curvature on this time interval will be uniformly bounded above by $6C$, and so from Appendix C, the sectional curvature will be uniformly bounded above on the time interval. Hence we can apply Lemma I.8.3(b) to get a uniform additive bound on the length distortion between times $t_\infty - c$ and t_∞ (see Step 4 of the proof of I.12.1). More precisely, in applying Lemma I.8.3(b), we use the curvature bound coming from the hypotheses of the theorem near x_0 , and the just-derived upper curvature bound near \bar{x}_k .

It follows that for a given $A > 0$, for large k , if $t'_k \in [\bar{t}_k - cQ_k^{-1}/2, \bar{t}_k]$ and $\text{dist}_{\bar{t}_k}(x'_k, \bar{x}_k) < AQ_k^{-1/2}$ then $\text{dist}_{t'_k}(x'_k, x_{0,k}) < \text{dist}_{\bar{t}_k}(\bar{x}_k, x_{0,k}) + D_k Q_k^{-1/2}$. In particular, if a point (x'_k, t'_k) lies in the parabolic neighborhood given by $t'_k \in [\bar{t}_k - cQ_k^{-1}/2, \bar{t}_k]$ and $\text{dist}_{\bar{t}_k}(x'_k, \bar{x}_k) < AQ_k^{-1/2}$, and has $R(x'_k, t'_k) > 2Q_k$, then it has a neighborhood that is ϵ -close to the corresponding subset of a κ -solution.

As in Step 4 of the proof of Theorem I.12.1, we now extend $(M_\infty, \bar{g}_\infty, \bar{x}_\infty)$ backward to a κ -solution $\bar{g}_\infty(t)$, defined for $t \in (-\infty, t_\infty]$. To do so, we use the fact that if the solution is defined backward to a time- t slice then the length distortion bound, along with the pointpicking, implies that a point m in the time- t slice with $\bar{R}_\infty(m) > 3$ has a neighborhood that is ϵ -close to the corresponding subset of a κ -solution. Then we obtain smooth convergence of parabolic regions of the points (\bar{x}_k, \bar{t}_k) to the κ -solution, which is a contradiction to the choice of the (\bar{x}_k, \bar{t}_k) 's.

Now we know that regions of high scalar curvature modeled by corresponding regions in κ -solutions. To continue with the proof of the theorem, fix A large. Suppose that the theorem is not true. Then there are

1. Numbers $\rho_k \rightarrow 0$,

2. Numbers $r_{0,k}$ with $\phi(r_{0,k}^{-2}) \leq \rho_k$,
3. Solutions $(M_k, g_k(t))$ defined for $0 \leq t \leq r_{0,k}^2$,
4. Points $x_{0,k} \in M_k$ and
5. Points $x_k \in M_k$

so that

- a. $|\text{Rm}|(x, t) \leq r_{0,k}^{-2}$ for all $(x, t) \in M_k \times [0, r_{0,k}^2]$ satisfying $\text{dist}_0(x, x_{0,k}) < r_{0,k}$,
- b. The volume of the metric ball $B(x_{0,k}, r_{0,k})$ at time zero is at least $A^{-1}r_{0,k}^3$ and
- c. $\text{dist}_{r_{0,k}^2}(x_k, x_{0,k}) < Ar_{0,k}$, but
- d. $r_{0,k}^2 R(x_k, r_{0,k}^2) \rightarrow \infty$.

We now apply step 2 of the proof of Theorem I.12.1 to obtain a contradiction. The only difference is that in Theorem I.12.1, the nonnegative curvature of W came from the ϕ -almost nonnegative curvature assumption on the original manifolds M_k along with the fact (with the notation of I.12.1) that the numbers $Q_k = R(x_k, t_k)$, which we used to rescale, go to infinity. In the present case we do not know that the scalar curvatures at the basepoints $(x_{0,k}, r_{0,k}^2)$, when multiplied by $r_{0,k}^2$, go to infinity. However, we do know that if $r_{0,k}^2 R(x_k, r_{0,k}^2)$ is large then $\phi(R(x_k, r_{0,k}^2)) = \phi((r_{0,k}^2 R(x_k, r_{0,k}^2)) r_{0,k}^{-2}) \leq \phi(r_{0,k}^{-2})$. This is enough to give the nonnegative curvature of W and carry out the argument.

47. I.12.3

We pick a point (x_0, t_0) and a radius $r_0 > 0$ such that the assumptions of the theorem do hold whereas the conclusion does not. We claim that we may assume that any other point (x', t') and radius $r' > 0$ with that property has either $t' > t_0$ or $t' < t_0 - 2\tau r_0^2$, or $2r' > r_0$. To see this, suppose that it is not true with the given x_0, t_0, r_0 . Then there are (x'_0, t'_0) and r'_0 with $t'_0 \in [t_0 - 2\tau r_0^2, t_0]$ and $r'_0 \leq \frac{1}{2}r_0$, for which the assumptions of the theorem hold but the conclusion does not. In particular, there is a point (x, t) with $t \in [t'_0 - \tau(r'_0)^2, t'_0]$, such that $R(x, t) \geq K(r'_0)^{-2} \geq 4Kr_0^{-2}$. If the triple (x'_0, t'_0, r'_0) satisfies the claim then we stop, and otherwise we iterate the procedure. The iteration must terminate, which provides the desired triple $(\hat{x}_0, \hat{t}_0, \hat{r}_0)$. Note that $\hat{t}_0 > t_0 - 4\tau r_0^2 \geq 0$.

To make a slight correction to the statement of the corollary, we want to assume that the ball $B(x_0, r_0)$ has sectional curvatures strictly greater than $-r_0^{-2}$ at each point, in order to ensure that $\tau' > 0$. From estimates on the length and volume distortion under the Ricci flow, we know that there is a $w' = w'(w) > 0$ so that the time- t' ball $B(x_0, r_0)$ has volume at least $w'r_0^n$.

The Alexandrov statement is that if $\text{Rm} \geq -r_0^{-2}$ on $B(x_0, r_0)$ and $\text{vol}(B(x_0, r_0)) \geq w'r_0^n$ then there is a ball $B(x', r') \subset B(x_0, r_0)$ with $r' \geq c(w')r_0$ such that $\text{vol}(B(x', r')) \geq \frac{1}{2}\omega_n(r')^n$. To see this, suppose that it is not true. Rescale so that $r_0 = 1$. Then there is a sequence of Riemannian manifolds $\{M_i\}_{i=1}^\infty$ with balls $B(x_i, 1) \subset M_i$ so that $\text{Rm} \Big|_{B(x_i, 1)} \geq -1$ and $\text{vol}(B(x_i, 1)) \geq w'$, but with the property that all balls $B(x'_i, r') \subset B(x_i, 1)$ with $r' \geq i^{-1}$ satisfy $\text{vol}(B(x'_i, r')) < \frac{1}{2}\omega_n(r')^n$. After taking a subsequence, we can assume that $\lim_{i \rightarrow \infty} (B(x_i, 1), x_i) = (X, x_\infty)$ in the pointed Gromov-Hausdorff topology. From [5, Theorem 10.8], the Riemannian volume forms $d\text{vol}_{M_i}$ converge weakly to the three-dimensional Hausdorff measure μ of X . From [5, Corollary 6.7 and Section 9], for any

$\epsilon > 0$, there are balls $B(x'_\infty, r') \subset X$ with $\mu(B(x'_\infty, r')) \geq (1 - \epsilon) \omega_n(r')^n$. This gives a contradiction.

To finish the proof, we have the estimate $R \leq K(r')^{-2}$ whenever $t \in [t' - \tau(r')^2, t']$ and $\text{dist}_t(x, x') \leq \frac{1}{4}r'$. From (C.5), we have a bound $|\text{Rm}| \leq \text{const} \cdot K(r')^{-2} \phi(K(r')^{-2})$ at such a point. Taking the r_0 of Theorem I.12.2 to be $\frac{1}{4}r'$, we obtain a bound of the form $R \leq K'(r'/4)^{-2}$ whenever $t \in [t' - \frac{1}{2}\tau(r')^2, t']$ and $\text{dist}_t(x, x') \leq 10r_0$, where K' is the constant of Theorem I.12.2. Then at such a point, $\text{Rm} \geq -K'(r'/4)^{-2} \phi(K'(r'/4)^{-2})$. If r_0 is small then this is greater than $-r_0^{-2}$, because of the ϕ -factor, which is a contradiction.

We note that in the application of Theorem I.12.2 at the end of the proof, we must take into account the extra hypothesis in Theorem I.12.2 that $\phi(r_0^{-2}) < \rho$ (see II.6.2). This does not affect the argument; in our case it becomes the assumption that $\phi((r'/4)^{-2}) < \rho$. This will be satisfied if r_0 is small enough, which is what we are assuming.

48. I.12.4

Suppose that the corollary is not true, i.e. for any $\rho > 0$, there is a solution as in the statement of the corollary with $r_0 < \rho$, $\min \text{Rm}(x, t_0) \Big|_{B(x_0, r_0)} = -\frac{1}{r_0^2}$ and $\text{vol}(B(x_0, r_0)) > \omega r_0^n$. The idea is to go backwards in time using Corollary I.12.3 to get a curvature bound on a smaller ball, and then to go forwards in time using Corollary I.12.2 to get a curvature bound on the original ball.

From Corollary I.12.3, $R(x, t) < Kr_0^{-2}$ whenever $t_0 \geq 4\tau r_0^2$, $t \in [t_0 - \tau r_0^2, t_0]$ and $\text{dist}_t(x, x_0) \leq \frac{1}{4}r_0$, provided that $\phi(r_0^{-2})$ is less than the ρ of Corollary I.12.3. If $\phi(Kr_0^{-2}) < 1$ then the ϕ -almost nonnegative curvature condition implies that $|\text{Rm}(x, t)| \leq Kr_0^{-2}$.

Choose the largest $\widehat{r}_0 > 0$ so that $\widehat{r}_0^2 \leq \tau r_0^2$ and $Kr_0^{-2} \leq \widehat{r}_0^{-2}$. From the Bishop-Gromov inequality, and the bounds on length and volume distortion under Ricci flow, there is an $A > 0$ (depending on K) so that the volume of $B(x_0, \widehat{r}_0)$ at time $t_0 - \widehat{r}_0^2$ is bounded below by $A^{-1}\widehat{r}_0^n$. We may also assume that A is such that $\frac{r_0}{\widehat{r}_0} \leq A$. Then we can apply Theorem I.12.2 to the ball $B(x_0, \widehat{r}_0)$ and the time interval $[t_0 - \widehat{r}_0^2, t_0]$ to conclude that at time t_0 , $R(x, t_0) \Big|_{B(x_0, r_0)} \leq K(A) \widehat{r}_0^{-2}$, where $K(A)$ is as in the statement of Theorem I.12.2. From the ϕ -almost nonnegative curvature condition,

$$(48.1) \quad \text{Rm} \Big|_{B(x_0, r_0)} \geq -K(A) \widehat{r}_0^{-2} \phi(K(A) \widehat{r}_0^{-2}).$$

For small r_0 , because of the ϕ -factor, this contradicts the fact that $\min \text{Rm}(x, t_0) \Big|_{B(x_0, r_0)} = -\frac{1}{r_0^2}$.

49. I.13.1

Choose $s > 0$ small enough so that $s < \rho(w)$, where $\rho(w)$ is as in Corollary I.12.4, and $\phi(s^{-2}) < \rho(w)$ where $\rho(w)$ is now as in Corollary I.12.3.

We rescale the solution so that at time zero, the eigenvalues of the curvature operator are bounded below by -1 . Take $t_0 = 1$. Then for $t \geq 1$, we have a bound of the form (C.4). As (C.4) is explicitly scale-invariant, it is enough to give the arguments in the case $t = 1$.

Given $x \in M$, put

$$(49.1) \quad \hat{r} = \sup\{r : \text{Rm} \Big|_{B(x,r)} \geq -\frac{1}{r^2}\}.$$

Put $\hat{\rho} = \min(s, \sup_{y \in M} \text{dist}_1(x, y))$. There are the following possibilities :

I. $\hat{r} < \hat{\rho}$.

As $\hat{r} < \sup_{y \in M} \text{dist}_1(x, y)$, we have $\min \text{Rm} \Big|_{B(x,\hat{r})} = -\frac{1}{\hat{r}^2}$. As $\hat{r} < s$, Corollary I.12.4 implies that $\text{vol}(B(x, \hat{r})) \leq w\hat{r}^n$. Thus $x \in M_{thin}$.

II. $\hat{r} \geq \hat{\rho}$.

A. $\text{vol}(B(x, r)) \leq wr^n$ for some $r \leq \hat{\rho}$.

As $r \leq \hat{r}$, we have $\text{Rm} \Big|_{B(x,r)} \geq -\frac{1}{\hat{r}^2} \geq -\frac{1}{r^2}$. Then $x \in M_{thin}$.

B. $\text{vol}(B(x, r)) > wr^n$ for all $r \leq \hat{\rho}$.

As $\text{Rm} \Big|_{B(x,\hat{\rho})} \geq -\frac{1}{\hat{r}^2} \geq -\frac{1}{\hat{\rho}^2}$ and $\phi(\hat{\rho}^{-2}) \leq \phi(s^{-2})$, Corollary I.12.3 implies that $R \Big|_{B(x,\hat{\rho}/4)} < K\hat{\rho}^{-2}$. Hence $|\text{Rm}| \leq \text{const.} (K+1)\hat{\rho}^{-2}$ on $B(x, \hat{\rho}/4)$. As we also have $\text{vol}(B(x, \hat{\rho}/4)) > w(\hat{\rho}/4)^n$, we will consider x to be in M_{thick} . (Our definition of M_{thick} differs slightly from that of I.13.1.)

50. II. NOTATION AND TERMINOLOGY

Definition 50.1. *An open set U in a Riemannian 3-manifold M is an ϵ -neck if modulo rescaling, it has distance less than ϵ , in the $C^{[1/\epsilon]+1}$ -topology, to the product of the round 2-sphere of scalar curvature 1 (and therefore Gaussian curvature $\frac{1}{2}$) with an interval I of length at least $2\epsilon^{-1}$. If a point $x \in M$ and a neighborhood U of x are specified then we will understand that “distance refers to the pointed topology, where the basepoint in $S^2 \times I$ projects to the center of I . A subset of the form $U \times [a, b] \subset M \times [a, b]$, where $U \subset M$ is open, sitting in the spacetime of a Ricci flow is a strong ϵ -neck if after parabolic rescaling and time shifting, it has distance less than ϵ to the product Ricci flow defined on the time interval $[-1, 0]$ which, at its final time, is isometric to the product of a round 2-sphere of scalar curvature 1 with an interval of length greater than $2\epsilon^{-1}$. (Evidently, the time-0 slice of the product has 3-dimensional scalar curvature equal to 1.)*

Our definition of an ϵ -neck differs in an insubstantial way from that on p. 1 of II. In the definition of II, a ball $B(x, t, \epsilon^{-1}r)$ is called an ϵ -neck if, after rescaling the metric with a factor r^{-2} , it is ϵ -close to the corresponding subset of the standard neck $S^2 \times \mathbb{I} \dots$ (italicized words added by us). Clearly after a slight change of the constants, an ϵ -neck in our sense is contained in an ϵ -neck in the sense of II, and vice versa. An important fact is that the

notion of (x, t) being contained in an ϵ -neck is an open condition with respect to the pointed $C^{1/\epsilon+1}$ -topology on Ricci flow solutions.

With an ϵ -approximation $f : S^2 \times \mathbb{I} \rightarrow U$ being understood, a cross-sectional sphere in U will mean the image of $S^2 \times \{\lambda\}$ under f , for some $\lambda \in (-\epsilon, \epsilon)$. Any curve γ in U that intersects both $f(S^2 \times \{\epsilon\})$ and $f(S^2 \times \{-\epsilon\})$ must intersect each cross-sectional sphere. If γ is a minimizing geodesic and ϵ is small enough then γ will intersect each cross-sectional sphere exactly once.

\mathbb{I} is an open interval and B^3 is an open ball.

There is a typo in the definition of a strong ϵ -neck: the parabolic neighborhood should be $P(x, t, \epsilon^{-1}r, -r^2)$, i.e. it should go backward in time rather than forward.

An ϵ -cap is the result of capping off an ϵ -neck by a 3-ball with an arbitrary metric. A capped ϵ -horn is the result of capping off an ϵ -horn by a 3-ball with an arbitrary metric. Let U be an ϵ -neck in an ϵ -tube (or horn) and let S be a cross-sectional sphere in U . We claim that S separates the tube (or horn). Suppose not. Then there is a curve γ joining the two ends of the tube (or horn) that doesn't intersect S . Minimizing the length among such curves, we can assume that γ is a geodesic. Let x be a point on γ that is closest to S . Since x itself lies in an ϵ -neck, we obtain a contradiction.

For an example of an ϵ -horn, consider the metric $g = dr^2 + r^{2\alpha}d\theta^2$ on $(0, R) \times S^2$, with $\alpha > 1$. Rescaling around r_0 to have unit radius, we put $s = r_0^{-\alpha}(r - r_0)$, with $s \in (-r_0^{1-\alpha}, r_0^{-\alpha}(R - r_0))$, and write $r_0^{-2\alpha}g = ds^2 + (1 + r_0^{\alpha-1}s)^{2\alpha}d\theta^2$. The region with $s \in (-\frac{1}{\epsilon}, \frac{1}{\epsilon})$ will be ϵ -biLipschitz close to the standard cylinder provided that $\alpha r_0^{\alpha-1}$ is roughly smaller than ϵ . Note that as $r_0 \rightarrow 0$, the constant ϵ improves; this is related to Lemma II.4.3.

Throughout II, ϵ denotes a small positive constant that is meant to be universal. The precise value of ϵ is unspecified. When going through the proofs in II, one is allowed to make ϵ small enough so that the arguments work, but one is only allowed to make a finite number of such reductions. If the statement of a lemma or theorem invokes ϵ then the statement is meant to be true uniformly with respect to the other variables.

51. II.1.2

See section 44.

52. II.1.5

As the $t = 0$ slice of a noncompact κ -solution is nonflat and has nonnegative sectional curvature, we know immediately that it is diffeomorphic to $\mathbb{R} \times S^2$, $\mathbb{R} \times_{\mathbb{Z}_2} S^2 = \mathbb{R}P^3 - \overline{B^3}$, \mathbb{R}^3 , or $S^1 \times \mathbb{R}^2$ [8, Theorem 8.1]. A solution diffeomorphic to $\mathbb{R} \times S^2$ is an isometric product, and a solution diffeomorphic to $\mathbb{R} \times_{\mathbb{Z}_2} S^2$ is covered by an isometric product. Now Corollary I.11.8 rules out the case of $S^1 \times \mathbb{R}^2$, which does not have the right geometry at infinity. In the remaining case of \mathbb{R}^3 , Corollary I.11.8 shows that for small $\epsilon > 0$, the solution has a compact nonneck part M_ϵ diffeomorphic to $\overline{B^3}$.

A compact κ -solution M is necessarily diffeomorphic to a quotient of S^3 [17]. It has an asymptotic soliton (Proposition I.11.2) that is either compact or noncompact. If the asymptotic soliton is compact then it must be a shrinking quotient of the round S^3 [17], so the same is true of M . If the asymptotic soliton is noncompact then for $t \ll 0$, $M_\epsilon(t)$ has two connected components, and $M(t) - M_\epsilon(t)$ is a long tube joining them; see Corollary 41.1 in section 41. Using the compactness result of Theorem I.11.7, it follows that each component of M_ϵ is diffeomorphic to the nonneck part of a noncompact ancient solution, and so from the preceding paragraph it is diffeomorphic to $\overline{B^3}$ or $\mathbb{R}P^3 - B^3$. As M has finite fundamental group, no more than one of the two components can be diffeomorphic to $\mathbb{R}P^3 - B^3$, and so M is diffeomorphic to S^3 or $\mathbb{R}P^3$.

The “important conclusion” is that every solution which is a κ -solution *for some* κ is either a κ_0 -solution or a metric quotient of the round sphere.

Regarding the end of II.1.5, we claim that for every sufficiently small $\epsilon > 0$ one can find $C_1 = C_1(\epsilon)$ such that for each point (x, t) in every ancient κ -solution there is a radius $r \in [C_1^{-1}R(x, t)^{-1/2}, C_1R(x, t)^{-1/2}]$ and a neighborhood B , $B(x, t, r) \subset B \subset B(x, t, 2r)$, which falls into one of the four categories :

- (a) B is a strong ϵ -neck (more precisely, the slice of a strong ϵ -neck at its maximal time), or
- (b) B is an ϵ -cap, or
- (c) B is a closed manifold, diffeomorphic to S^3 or $\mathbb{R}P^3$, or
- (d) B is a closed manifold of constant positive sectional curvature.

Note that the statement is slightly stronger than that in II, in that we have $C_1^{-1}R(x, t)^{-1/2} \leq r \leq C_1R(x, t)^{-1/2}$ as opposed to $r > 0$.

To prove the claim, we may assume that we are talking about a κ_0 -solution, as if M is a metric quotient of a round sphere then it falls into category (d) for any $r > \pi(R(x_i, t_i)/6)^{-1/2}$ (since then $M = B(x_i, t_i, r) = B(x_i, t_i, 2r)$).

Fix a small ϵ and suppose that the claim is not true. Then there is a sequence of κ_0 -solutions M_i that together provide a counterexample. That is, there is a sequence $C_i \rightarrow \infty$ and a sequence of points $(x_i, t_i) \in M_i \times (-\infty, 0]$ so that for any $r \in [C_i^{-1}R(x_i, t_i)^{-1/2}, C_iR(x_i, t_i)^{-1/2}]$ one cannot find a B between $B(x_i, t_i, r)$ and $B(x_i, t_i, 2r)$ falling into one of the four categories. Take a convergent subsequence of (M_i, x_i, t_i) to obtain a limit κ_0 -solution $(M_\infty, x_\infty, t_\infty)$. Then for any $r > 0$, one cannot find a B between $B(x_\infty, t_\infty, r)$ and $B(x_\infty, t_\infty, 2r)$ falling into one of the four categories. However, if M_∞ is compact then for any r greater than the diameter of the time- t_∞ slice of M_∞ , $B(x_\infty, t_\infty, r) = M_\infty = B(x_\infty, t_\infty, 2r)$ falls into category (c) or (d), which is a contradiction. If M_∞ is noncompact then Corollary I.11.8 implies that for some $r > 0$, there will be a B between $B(x_\infty, t_\infty, r)$ and $B(x_\infty, t_\infty, 2r)$ falling into category (a) or (b).

The fact that the scalar curvature in B at time t is between $C_2^{-1}R(x, t)$ and $C_2^{-1}R(x, t)$ comes from the fact that the subset B in M_∞ has positive bounded scalar curvature. Similarly, the lower bound on the volume of B in cases (a), (b) and (c) comes from the positive volume of B in M_∞ , and the lower bound on the sectional curvature of B in case (c) comes from the fact that a compact κ_0 -solution has positive sectional curvature [14].

PUT IN THE CONDITION $C_1 > 30\epsilon^{-1}$. SEE PROLONGATION OF SURGERY FLOW.

DISCUSS OPENNESS OF THE CONDITIONS.

53. CLAIM 2 OF II.2

Suppose that we have a Ricci flow solution $g(t)$, $t \in [0, T]$, with $g(0) = g_0$. Suppose that there is a sequence of points x_i going to infinity for which the pointed solutions $(M, g, (x_i, 0))$ do not converge in the C^∞ -topology to the shrinking cylinder. As in the proof of [14], a subsequence of the solutions will converge to a solution $(M_\infty, g_\infty, x_\infty)$ with $(M_\infty, g_\infty(0))$ isometric to $\mathbb{R} \times S^2$. Using the uniqueness of Ricci flow solutions on a compact manifold, $(M_\infty, g_\infty, x_\infty)$ must be a shrinking cylinder, which is a contradiction.

In particular, $T \leq 1$.

54. CLAIM 1 OF II.2

Since the time-zero metric g_0 is rotationally symmetric, it is clear, by separation of variables, that there is a rotationally symmetric solution for some time interval $[0, T]$. The issue is to show that any Ricci flow solution $g(t)$, $t \in [0, T]$, $T \leq 1$, with $g(0) = g_0$, is rotationally symmetric. (This is needed, for example, in II.4.5.) If one had uniqueness of solutions then the result would be automatic. One does have uniqueness for Ricci flow solutions on a compact manifold [15, Section 6], but it is not automatic for solutions on a noncompact manifold.

Following II, we first construct Killing vector fields. Suppose that

$$(54.1) \quad u_t^m = u^m_{;k}{}^k + R^m_i u^i.$$

Then

$$(54.2) \quad \begin{aligned} \partial_t(u^m_{;i}) &= u^m_{t;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= (u^m_{;k}{}^k + R^m_k u^k)_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= (u^m_{;k}{}^k + R^m_k u^k)_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= u^m_{;ki}{}^k + R^m_{k;i} u^k + R^m_k u^k_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= u^m_{;ik}{}^k - R^m_{lki} u^l{}^k - R^k_{lki} u^m{}^l + R^m_{k;i} u^k + R^m_k u^k_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= u^m_{;ki}{}^k - R^m_{lki} u^l{}^k - R_{li} u^m{}^l + R^m_{k;i} u^k + R^m_k u^k_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= u^m_{;ik}{}^k - (R^m_{lki} u^l)_{;k} - R^m_{lki} u^l{}^k - R_{ki} u^m{}^k + R^m_{k;i} u^k + R^m_k u^k_{;i} + (\partial_t \Gamma^m_{ki}) u^k \\ &= u^m_{;ik}{}^k - R^m_{lki}{}^k u^l - 2R^m_{lki} u^l{}^k - R_{ik} u^m{}^k + R^m_{k;i} u^k + R^m_k u^k_{;i} + (\partial_t \Gamma^m_{ki}) u^k. \end{aligned}$$

Contracting the second Bianchi identity gives

$$(54.3) \quad R_{mlki}{}^k = R_{il;m} - R_{im;l}.$$

Also,

$$(54.4) \quad \begin{aligned} \partial_t \Gamma^m_{ki} &= \partial_t (g^{ml} \Gamma_{lki}) = 2R^{ml} \Gamma_{lki} - g^{ml} (R_{lk,i} + R_{li,k} - R_{ik,l}) \\ &= -R^m_{k;i} - R^m_{i;k} + R_{ik}{}^m. \end{aligned}$$

Substituting (54.3) and (54.4) in (54.2) gives

$$(54.5) \quad \partial_t(u_{;i}^m) = u_{;ik}^m{}^k - 2R_{lki}^m u^l{}_{;k} - R_{ik} u^m{}_{;k} + R_{;k}^m u^k{}_{;i}.$$

Then

$$(54.6) \quad \begin{aligned} \partial_t(u_{j;i}) &= \partial_t(g_{jm}u_{;i}^m) = -2R_{jm}u_{;i}^m + g_{jm}\partial_t(u_{;i}^m) \\ &= u_{j;ik}^k - 2R_{jlk}u^l{}_{;k} - R_{ik}u_j{}^k - R_{jk}u_{;i}^k \\ &= u_{j;ik}^k + 2R_{ikjl}u^l{}_{;k} - R_{ik}u_j{}^k - R_{kj}u_{;i}^k. \end{aligned}$$

Equivalently, writing $v_{ij} = u_{j;i}$ gives

$$\partial_tv_{ij} = v_{ij;k}^k + 2R_{i j}{}^{kl}v_{kl} - R_i{}^k v_{kj} - R_j{}^k v_{ik}.$$

Then putting $L_{ij} = v_{ij} + v_{ji}$ gives

$$\partial_tL_{ij} = L_{ij;k}^k + 2R_{i j}{}^{kl}L_{kl} - R_i{}^k L_{kj} - R_j{}^k L_{ik}.$$

For any $\lambda \in \mathbb{R}$, we have

$$(54.7) \quad \partial_t(e^{2\lambda t}L_{ij}L^{ij}) = 2\lambda(e^{2\lambda t}L_{ij}L^{ij}) + (e^{2\lambda t}L_{ij}L^{ij})_{;k}{}^k - 2e^{2\lambda t}L_{ij;k}L^{ij;k} + Q(\text{Rm}, e^{\lambda t}L),$$

where $Q(R, L)$ is an algebraic expression that is linear in the curvature tensor Rm and quadratic in L . Putting $M_{ij} = e^{\lambda t}L$ gives

$$(54.8) \quad \partial_t(M_{ij}M^{ij}) = 2\lambda M_{ij}M^{ij} + (M_{ij}M^{ij})_{;k}{}^k - 2M_{ij;k}M^{ij;k} + Q(\text{Rm}, M).$$

Suppose that we have a Ricci flow solution $g(t)$, $t \in [0, T]$, with $g(0) = g_0$. Let $u(0)$ be a rotational Killing vector field for g_0 . Let $u_\infty(0)$ be its restriction to (any) S^2 , which we will think of as the 2-sphere at spatial infinity. Solve (54.1) for $t \in [0, T]$; due to the asymptotics coming from Claim 2, there is no problem in doing so. Arguing as in the proof of Claim 2, one can show that for any $t \in [0, T]$, at spatial infinity $u(t)$ converges to $u_\infty(0)$. Construct $M_{ij}(t)$ from $u(t)$. As $u(0)$ is a Killing vector field, $M_{ij}(0) = 0$. For any $t \in [0, T]$, at spatial infinity the tensor $M_{ij}(t)$ converges smoothly to zero. Suppose that λ is sufficiently negative, relative to the L^∞ -norm of the sectional curvature on the time interval $[0, T]$. We can apply the maximum principle to (54.8) to conclude that $M_{ij}(t) = 0$ for all $t \in [0, T]$. Thus $u(t)$ is a Killing vector field for all $t \in [0, T]$.

Let $\{u_\infty^{(a)}\}_{a=1}^3$ be a basis for the Killing vector fields on S^2 . Starting with initial conditions given by the rotational vector fields for g_0 , we obtain linearly independent solutions $\{u^{(a)}(t)\}_{a=1}^3$ of (54.1) with spatial asymptotics given by $u^{(a)}(t) \rightarrow u_\infty^{(a)}$.

For any $t \in [0, T]$, there is a unique Busemann function $B(t)$ (up to constants) for the metric $g(t)$. For $t > 0$, the strong maximum principle implies that $g(t)$ has positive sectional curvature. Then Cheeger-Gromoll theory [7] implies that $B(t)$ has a unique maximum point $p(t)$. We normalize $B(t)$ so that $B(t)(p(t)) = 0$. Putting $r(t) = -B(t)$, we have a Riemannian submersion $r(t) : \mathbb{R}^3 - \{p(t)\} \rightarrow (0, \infty)$. The level sets of $r(t)$ are 2-spheres. As $B(t)$ is preserved by isometries, the flows of $\{u^{(a)}(t)\}_{a=1}^3$ preserve each level set. Let $\partial_{r(t)}$ be the dual vector field to the 1-form $dr(t)$. The flowlines of $\partial_{r(t)}$ are geodesics emanating from $p(t)$. As the Lie derivative $\mathcal{L}_{u^{(a)}(t)}dr(t) = d(\mathcal{L}_{u^{(a)}(t)}r(t))$ vanishes, we also have $\mathcal{L}_{\partial_{r(t)}}u^{(a)}(t) = [\partial_{r(t)}, u^{(a)}(t)] = -\mathcal{L}_{u^{(a)}(t)}\partial_{r(t)} = 0$. Thus the flow of $\partial_{r(t)}$ allows us to identify the actions of $u^{(a)}(t)$ on different level sets. In view of the spatial asymptotics of

$u^{(a)}(t)$, it follows that on $\mathbb{R}^3 - \{p(t)\}$, the metric $g(t)$ can be written in the form $g(t) = dr(t)^2 + f^2(t, r(t)) g_{S^2}$, where g_{S^2} is the standard metric on S^2 and $f(t, r)$ is some positive function. That is, $g(t)$ is rotationally symmetric.

55. CLAIM 3 OF II.2

Consider Ricci flow starting from an initial rotationally symmetric metric $g(0) = dr^2 + F^2(r, 0) g_{S^2}$. The solution is of the form $g(t) = H^2(r, t) dr^2 + F^2(r, t) g_{S^2}$.

We let $'$ denote differentiation with respect to r . A computation gives

$$(55.1) \quad \begin{aligned} \text{Ric}(\partial_r, \partial_r) &= -2 \frac{H}{F} \left(\frac{F'}{H} \right)', \\ \text{Ric}(X, Y) &= \left(1 - \frac{F}{H} \left(\frac{F'}{H} \right)' - \left(\frac{F'}{H} \right)^2 \right) g_{S^2}(X, Y), \end{aligned}$$

where X and Y are tangent to S^2 . Then the Ricci flow equation becomes

$$(55.2) \quad \begin{aligned} \frac{\partial H^2}{\partial t} &= -2 \left(-2 \frac{H}{F} \left(\frac{F'}{H} \right)' \right) \\ \frac{\partial F^2}{\partial t} &= -2 \left(1 - \frac{F}{H} \left(\frac{F'}{H} \right)' - \left(\frac{F'}{H} \right)^2 \right). \end{aligned}$$

The boundary conditions are $H(0, t) = 1$, $H'(0, t) = 0$, $F(0, t) = 0$, $F'(0, t) = 1$. One finds that

$$(55.3) \quad \frac{\partial(F^2 H^{-1})}{\partial t} = -\frac{2}{H} \left(1 - \left(\frac{F'}{H} \right)^2 \right),$$

where the right-hand-side has no second derivatives. This suggests a change of variable to F and $FH^{-1/2}$. TO BE CONTINUED

Remark 55.4. An alternative approach to proving Claims 1 and 3 would be to show uniqueness for solutions on \mathbb{R}^3 that asymptotically approach the standard solution on the cylinder, in the sense of Claim 2. Following the lines of [15, Section 6], the issue comes down to uniqueness of solutions of a certain parabolic equation whose spatial asymptotics come from Claim 2. If \widehat{g}_1 and \widehat{g}_2 are two solutions with the same initial conditions, the uniqueness should come from showing that the equation implies that $e^{-Ct} \int_M |\widehat{g}_1 - \widehat{g}_2|^2$ is nonincreasing in t , for sufficiently large $C > 0$.

56. ARGUMENT FOR CLAIM 4 OF II.2

We sketch a proof of Claim 4 that does not use Theorem I.10.1. Let us write the rotationally symmetric metric $g(t)$ as $dr^2 + f^2(r, t) d\theta^2$, with $d\theta^2$ the standard metric on S^2 . We can assume that the initial metric $g(0)$ has nonnegative curvature operator. Then the same will be true for $g(t)$ when $t > 0$. Nonnegative sectional curvature implies that f is concave in r . By Claim 2 of II.2, $\lim_{r \rightarrow \infty} f^2(r, t) = 2(1 - t)$.

If $T < 1$ is the time of the first singularity then either a unique singularity develops at the midpoint p of the cap or there is a sequence $t_i \rightarrow T$ such that $f(\cdot, t_i)$ converges to zero uniformly on compact r -intervals. In the second case, for large i , we can find points $x_i \in M$ so that $f^2(x_i, t_i) = 1 - T$. The concavity of f implies that $\lim_{i \rightarrow \infty} d_{t_i}(p, x_i) = \infty$. As the Ricci flow decreases distances, the points $(x_i, 0)$ go off to infinity on the time-zero slice. Consider the pointed solutions $(M, g(t), (x_i, 0))$. We can take a convergent subsequence to get a Ricci flow solution $(M_\infty, g_\infty(t), (x_\infty, 0))$ defined for $t \in [0, T]$. Since $(x_i, 0)$ goes off to infinity, $(M_\infty, g(0))$ is the round cylinder. However, the time- T slice of M_∞ has $f^2(x_\infty, T) = 1 - T$, which contradicts the uniqueness in Claim 3 of II.2, as the shrinking cylinder has $f^2(\cdot, T) = 2(1 - T)$.

Thus the singularity forms at p , and we obtain a contradiction by looking at the corresponding conical point of the limit time- T slice, as in the proof of I.12.1.

57. CLAIM 5 OF II.2

The statement about having an ϵ -neck, but not a strong ϵ -neck, refers to the fact that there is no backward extension of the solution to $t < 0$. Hence if $B(x, t, \epsilon^{-1}r)$ is an ϵ -neck, with $t < r^2$, we can only consider its backward spacetime neighborhood $P(x, t, \epsilon^{-1}r, -t)$, instead of the parabolic neighborhood $P(x, t, \epsilon^{-1}r, -r^2)$.

To see that $R_{min}(t) \geq \text{const.}(1-t)^{-1}$, suppose that there are sequences $t_i \rightarrow 1$ and $c_i \rightarrow 0$ so that $R_{min}(t_i) \leq c_i(1-t_i)^{-1}$. For i large, choose $x_i \in M$ so that $R(x_i, t_i) \leq c_i(1-t_i)^{-1}$. Put $\bar{g}_i(t) = (1-t_i)^{-1} g_i(1 - \frac{1-t}{1-t_i})$. The pointed solutions $(M, \bar{g}_i, (x_i, 0))$ are defined for $t \in [1 - \frac{1}{1-t_i}, 1)$ and have a warped product metric $dr^2 + f_i^2(r, t)d\theta^2$, with $\lim_{r \rightarrow \infty} f_i^2(r, 0) = 2$. Using the techniques of I.11 and I.12, we can take a subsequence that converges to an ancient solution $(M_\infty, \bar{g}_\infty, (x_\infty, 0))$, defined for $t \in (-\infty, 1)$. As $R(x_\infty, 0) = 0$, the Harnack inequality implies that the ancient solution must be flat. From the noncollapsing, it must be \mathbb{R}^3 . Then for large i , $(M, \bar{g}_i(0))$ has a large almost Euclidean region. However, this contradicts the concavity of $f_i(\cdot, 0)$.

58. II.3

The fact that a closed positively curved manifold becomes extinct at a finite time T comes from II.(7.1). It implies that

$$(58.1) \quad \frac{d}{dt} R_{min} \geq \frac{2}{3} R_{min}^2,$$

which implies that

$$(58.2) \quad R_{min}^{-1}(t) \leq R_{min}^{-1}(0) - \frac{2}{3} t.$$

We also use the fact that the solution is smooth until $R_{min} = \infty$, which is true since the manifold has positive sectional curvature [17]. We note that the upper bound on $R_{min}^{-1}(t)$ holds more generally for smooth flows on manifolds with positive scalar curvature. In the case of a Ricci flow with surgeries, (58.1) will hold between surgeries. The surgery procedure is such that R_{min} is unchanged under surgeries, since the surgery is only done in a high-scalar-curvature region. The conclusion is that (58.2) is valid for the Ricci flow starting

from any closed manifold with positive scalar curvature. In particular, such a flow has $\lim_{t \rightarrow T} R_{min}(t) = \infty$ for some finite T . This is used in II.6.1.

By its definition, Ω_ρ is closed in Ω . It is compact because it lies in the complement of some open neighborhood of $M - \Omega$, and so is a closed subset of the compact manifold M .

We remark that a priori, the structure of $M - \Omega$ can be quite complicated. For example, it is not ruled out that an accumulating sequence of 2-spheres simultaneously shrinks. That is, $M - \Omega$ could have a subset of the form $(\{0\} \cup \{\frac{1}{i}\}_{i=1}^\infty) \times S^2 \subset (-1, 1) \times S^2$, the picture being that Ω contains a sequence of smaller and smaller adjacent double horns. One could even imagine a Cantor set's worth of 2-spheres simultaneously shrinking, although there may be additional arguments to rule out both of these cases.

Given this, one may worry that a slight perturbation of this situation could cause a case in which one singularity occurs shortly after the previous one, giving rise to an accumulation of singularity times. This could indeed be an issue if one tries to define a smooth flow through singularities. However, in the example of $M - \Omega$ containing an accumulating sequence of 2-spheres, the surgery of II.4.4 is done in such a way that at the first singularity time, an entire region $(-\epsilon, \epsilon) \times S^2$ is removed from $(-1, 1) \times S^2$, thereby removing all but a finite number of the shrinking 2-spheres.

We will not assume that the diameter stays bounded as one approaches the singular time, although this is probably the case.

59. RICCI FLOW WITH SURGERY

GIVE MORE EXAMPLES

SHOULD REMARK SOMEWHERE THAT HAMILTON'S COMPACTNESS APPLIES TO RICCI FLOWS WITH SURGERY

The principal purpose of sections II.4 and II.5 is to show that one can prescribe the surgery procedure in such way that Ricci flow with surgery is well-defined for all time. This involves showing that

- One can give a sufficiently precise description of the formation of singularities so that one can envisage defining a geometric surgery. In the case of the formation of the first singularity, such a description was given in II.3.
- The sequence of surgery times cannot accumulate.

The argument in II.3 strongly uses both the κ -noncollapsing of I.7 and the characterization of the geometry in a spacetime region around a point (x_0, t_0) with large scalar curvature, as given in Theorem I.12.1. The proofs of both of these results use the smoothness of the solution at times before t_0 . If surgeries occur before t_0 then one must have strong control on the scales at which the surgeries occur, in order to extend the arguments of I.7 and I.12.1. This forces one to consider time-dependent scales. See the overview [20] for additional nontechnical discussion of this argument and its place in the larger scheme.

Section II.4 introduces Ricci flow with surgery, in varying degrees of generality. Our treatment of this material follows Perelman's. We have added some terminology to help

formalize the surgery process. There is some arbitrariness in this formalization, but the version given below seems adequate.

For later use, we now summarize the relevant notation that we introduce. More precise definitions will be given below. We will avoid using new notation as much as possible.

- \mathcal{M} is a Ricci flow with surgery.
- \mathcal{M}_t is the time- t slice of \mathcal{M} .
- \mathcal{M}_{reg} is the set of regular points of \mathcal{M} .
- If T is a singular time then M_T^- is the limit of time slices \mathcal{M}_t as $t \rightarrow T^-$ (called Ω in II.4.1) and M_T^+ is the outgoing time slice (for example, the result of performing surgery on Ω). If T is a nonsingular time then $\mathcal{M}_T^- = \mathcal{M}_T^+ = \mathcal{M}_T$.

The basic notion of a Ricci flow with surgery is simply a sequence of Ricci flows which “fit together” in the sense that the final (possibly singular) time slice of each flow is isometric, modulo surgery, to the initial time slice of the next one.

Definition 59.1. *A Ricci flow with surgery is given by*

- A collection of Ricci flows $\{(M_k \times [t_k^-, t_k^+), g_k(\cdot))\}_{1 \leq k \leq N}$, where $N \leq \infty$, M_k is a compact¹ manifold, $t_k^+ = t_{k+1}^-$ for all $1 \leq k < N$, and g_k goes singular at t_k^+ for each $k < N$. We allow t_N^+ to be ∞ .
- A collection of limits $\{(\Omega_k, \bar{g}_k)\}_{1 \leq k \leq N}$ in the sense of II.3, at the respective final times t_k^+ that are singular if $k < N$. (Recall that Ω_k is an open subset of M_k .)
- A collection of isometric embeddings $\{\psi_k : X_k^+ \rightarrow X_{k+1}^-\}_{1 \leq k < N}$ where $X_k^+ \subset \Omega_k$ for $1 \leq k < N$ and $X_k^- \subset M_k$ for $1 < k \leq N$ are compact 3-dimensional submanifolds with boundary. The X_k^\pm 's are the subsets which survive the transition from one flow to the next, and the ψ_k 's give the identifications between them.

A Ricci flow with surgery does not necessarily have to have any real surgeries, i.e. it could be a smooth nonsingular flow. Our definition allows Ricci flows with surgery that are more general than those appearing in the argument for geometrization, where the transitions/surgeries have a very special form. Before turning to these more special flows in section 63, we first discuss some basic features of Ricci flow with surgery.

It will be convenient to associate a (non-manifold) space-time \mathcal{M} to the Ricci flow with surgery. This is constructed by taking the disjoint union of the smooth manifolds with boundary

$$(M_k \times [t_k^-, t_k^+)) \cup (\Omega_k \times \{t_k^+\}) \subset M_k \times [t_k^-, t_k^+]$$

for $1 \leq k \leq N$ and making identifications using the ψ_k 's as gluing maps. We denote the quotient map by π . We will sometimes use \mathcal{M} to refer to the whole Ricci-flow-with-surgery structure rather than just the associated spacetime. The *time- t slice* \mathcal{M}_t of \mathcal{M} is the image of the time- t slices of the constituent Ricci flows under the quotient map.

Let us summarize the structure of \mathcal{M} . The subset $\Omega_k \subset M_k$ survives to time t_k^+ . When we compare the time slices at times $t_k^+ - s$ and $t_{k+1}^- + s$, for small $s > 0$, the topological

¹We will permit M_k to be empty.

change is that we remove $M_k - X_k^+$ from M_k and add $M_{k+1} - X_{k+1}^-$. At time $t_k^+ = t_{k+1}^-$, Ω_k and M_{k+1} coexist in the time slice.

If $t = t_k^+ = t_{k+1}^-$ is a surgery time then we put $\mathcal{M}_t^- = \pi(\Omega_k \times \{t_k^+\})$ and $\mathcal{M}_t^+ = \pi(M_{k+1} \times \{t_{k+1}^-\})$, and $\mathcal{M}_t^+ = \mathcal{M}_t^- = \mathcal{M}_t$ otherwise; we refer to \mathcal{M}_t^+ and \mathcal{M}_t^- as the forward and backward time slices, respectively. We let $\mathcal{M}_{(t,t')}$ denote the *time slab* between t and t' , i.e. the union of the times slices between t and t' , $\mathcal{M}_{(t,t')} := \cup_{\bar{t} \in (t,t')} \mathcal{M}_{\bar{t}}$; the closed time slab $\mathcal{M}_{[t,t']}$ is defined to be the closure of $\mathcal{M}_{(t,t')}$ in \mathcal{M} , so $\mathcal{M}_{[t,t']} = \mathcal{M}_t^+ \cup \mathcal{M}_{(t,t')} \cup \mathcal{M}_{t'}^-$. We (ab)use the notation (x, t) to denote a point $x \in \mathcal{M}$ lying in the time t slice \mathcal{M}_t , even though \mathcal{M} is no longer a product, in general.

The space-time \mathcal{M} has 3 types of points: the 4-manifold points, which include all points at non-surgery times and all points in $\pi(\text{Interior}(X_k^+) \times \{t_k^+\})$ (or $\pi(\text{Interior}(X_k^-) \times \{t_k^-\})$); the boundary points of \mathcal{M} , which are the images in \mathcal{M} of $M_1 \times \{t_1^-\}$, $(\Omega_k \setminus X_k^+) \times \{t_k^+\}$ for $1 \leq k < N$, and $(M_k \setminus X_k^-) \times \{t_k^-\}$ for $1 < k \leq N$; and the ‘‘splitting’’ points, which are the images in \mathcal{M} of $\partial X_k^+ \times \{t_k^+\}$ for $1 \leq k < N$. (Here the classification of points is according to the smooth structure, not the topology; in fact \mathcal{M} is a topological manifold with boundary.) We say that (x, t) is *regular* if it is either a 4-manifold point, or it lies in the initial or final time slices $\mathcal{M}_{t_1^-}$ and $\mathcal{M}_{t_N^+}$. Let \mathcal{M}_{reg} denote the set of regular points; this has a natural C^∞ structure since the gluing maps ψ_k , being isometries between smooth Riemannian manifolds, are smooth maps.

Note that the Ricci flows on the M_k 's define a Riemannian metric g on the ‘‘horizontal’’ subbundle of the tangent bundle of \mathcal{M}_{reg} . It follows from the definition of the Ricci flow that g is actually smooth on \mathcal{M}_{reg} (though it won't be real analytic at a typical surgery time).

We metrize each time slice \mathcal{M}_t , and the forward/backward time slices, by infimizing the path length of piecewise smooth paths; we are allowing our distance functions to be infinite, since the infimum will be infinite when points lie in different components. If $(x, t) \in \mathcal{M}_t$, $r > 0$, we let $B(x, t, r)$ denote the corresponding metric ball; similarly $B^\pm(x, t, r)$ denotes the ball in \mathcal{M}_t^\pm centered at $(x, t) \in \mathcal{M}_t^\pm$. A ball $B(x, t, r) \subset \mathcal{M}_t$ is *proper* if the distance function $d_{(x,t)} : B(x, t, r) \rightarrow [0, r]$ is a proper function; a proper ball ‘‘avoids singularities’’, except possibly at its frontier. Proper balls $B^\pm(x, t, r) \subset \mathcal{M}_t^\pm$ are defined likewise.

An *admissible curve* in \mathcal{M} is a path $\gamma : [c, d] \rightarrow \mathcal{M}$ such that $\gamma(t) \in \mathcal{M}_t$ for all $t \in [c, d]$, such that for each k , the part of γ landing in $\mathcal{M}_{[t_k^-, t_k^+]}$ lifts to a smooth map into $M_k \times [t_k^-, t_k^+] \cup \Omega_k \times \{t_k^+\}$. We will use $\dot{\gamma}$ to denote the ‘‘horizontal’’ part of the velocity of an admissible curve γ . A point $(x, t) \in \mathcal{M}$ is *accessible from* $(x_0, t_0) \in \mathcal{M}$ if there is an admissible curve running from (x, t) to (x_0, t_0) . An admissible curve $\gamma : [c, d] \rightarrow \mathcal{M}$ is *static* if its lifts to the product spaces have constant first component; the points in the image of a static curve are ‘‘the same’’, modulo the passage of time and identifications taking place at surgery times. A *barely admissible curve* is an admissible curve whose image is not contained in \mathcal{M}_{reg} . If $\gamma : [c, d] \rightarrow \mathcal{M}$ is barely admissible, either one of its endpoints is not a regular point, or there is a surgery time $t = t_k^+ = t_{k+1}^- \in (c, d)$ such that $\gamma(t)$ lies in

$$\pi(\partial X_k^+ \times \{t_k^+\}) = \pi(\partial X_{k+1}^- \times \{t_{k+1}^-\}).$$

If $(x, t) \in \mathcal{M}_t^+$, $r > 0$, and $\Delta t > 0$, we define the forward *parabolic ball* $P(x, t, r, \Delta t)$ to be the union of (the images of) the static admissible curves $\gamma : [t, t + \Delta t] \rightarrow \mathcal{M}$ starting in $B^+(x, t, r)$; when $\Delta t < 0$, the parabolic ball $P(x, t, r, \Delta t)$ is defined similarly using static admissible curves ending in $B^-(x, t, r)$.

If $Y \subset \mathcal{M}_t$, and $t \in [c, d]$, we say that Y is *unscathed in* $[c, d]$ if every point $(x, t) \in Y$ lies on a static curve defined on the time interval $[c, d]$. If, for instance, $d = t$, then this will force $Y \subset \mathcal{M}_t^-$. The term “unscathed” is intended to capture the idea that the set is unaffected by singularities and surgery. (Sometimes Perelman uses the phrase “the solution is defined in $P(x, t, r, \Delta t)$ ” as synonymous with “the solution is unscathed in $P(x, t, r, \Delta t)$ ”, for example in the definition of canonical neighborhood in II.4.1.) We may use the notation $Y \times [c, d]$ for the set of points lying on static curves $\gamma : [c, d] \rightarrow \mathcal{M}$ which pass through Y , when Y is unscathed on $[c, d]$. Note that if Y is open and unscathed on $[c, d]$ then we can think of the Ricci flow on $Y \times [c, d]$ as an ordinary (i.e. surgery-free) Ricci flow.

The definitions of ϵ -neck, ϵ -cap, ϵ -tube, (capped/double) ϵ -horn, do not require modification for a Ricci flow with surgery, since they are just special types of Riemannian manifolds; they will turn up as subsets of forward or backward time slices of a Ricci flow with surgery. A *strong ϵ -neck* is a subset of the form $U \times [c, d] \subset \mathcal{M}$, where $U \subset \mathcal{M}_d^-$ is an open set unscathed on the interval $[c, d]$, which is a strong ϵ -neck in the sense of section 50.

60. II.4.1

The following definition captures the geometric structure that emerges by combining I.12.1 (and the version for Ricci flows with surgery in II.5) with the geometric description of κ -solutions. The idea is that blow-ups either yield κ -solutions, whose structure is well-understood by I.11, or there are surgeries nearby in the recent past, in which case the local geometry resembles that of the standard solution. Both alternatives produce canonical neighborhoods.

Definition 60.1. (*Canonical neighborhoods*) Let $\epsilon > 0$ be admissible in II.1.5 and Claim 5 of II.2, let $C_1 = C_1(\epsilon) > 30\epsilon^{-1}$, $C_2 = C_2(\epsilon)$ be as in II.1.5, and let $r : [a, b] \rightarrow (0, \infty)$ be a positive nonincreasing function. A Ricci flow with surgery \mathcal{M} defined on the time interval $[a, b]$ satisfies the r -canonical neighborhood assumption if every $(x, t) \in \mathcal{M}_t^\pm$ with scalar curvature $R(x, t) \geq r(t)^{-2}$ has a canonical neighborhood in the corresponding (forward/backward) time slice, as in II.1.5. More precisely, there is an $\hat{r} \in (C_1^{-1}R(x, t)^{-\frac{1}{2}}, C_1R(x, t)^{-\frac{1}{2}})$ and an open set $U \subset \mathcal{M}_t^\pm$ such that $\overline{B^\pm(x, t, \hat{r})} \subset U \subset B^\pm(x, t, 2\hat{r})$ falls into one of the following categories²:

(a) $U \times [t - \Delta t, t] \subset \mathcal{M}$ is a strong ϵ -neck for some $\Delta t > 0$. Note that after parabolic rescaling the scalar curvature at (x, t) becomes 1, so the scale factor must be $\approx R(x, t)$, which implies that $\Delta t \approx R(x, t)^{-1}$.

(b) U is an ϵ -cap.

(c) U is a closed manifold diffeomorphic to S^3 or RP^3 .

(d) U is a closed manifold of constant positive sectional curvature.

²Note that the smaller of the two balls is closed, to make it easier to check the openness of the condition

Moreover, the scalar curvature in U lies in a compact subinterval of $(C_2^{-1}R(x, t), C_2R(x, t))$, and its volume in cases (a), (b), and (c) is greater than $C_2^{-1}R(x, t)^{-\frac{3}{2}}$, and in case (c) the minimum of the sectional curvature is greater $C_2^{-1}R(x, t)$.

Finally, we require that

$$(60.2) \quad |\nabla R(x, t)| < \eta R(x, t)^{\frac{3}{2}}, \quad \left| \frac{\partial R}{\partial t}(x, t) \right| < \eta R(x, t)^3,$$

where η is the constant from II.(1.3). Here the time derivative $\frac{\partial R}{\partial t}(x, t)$ should be interpreted as a one-sided derivative when the point (x, t) is added or removed during surgery at time t .

Remark 60.3. We remark that when surgery is performed according to the recipe of II.4.4, if (p, t) lies in $\mathcal{M}_t^+ \setminus \mathcal{M}_t^-$, i.e. it is “added” by surgery, then it will sit in an ϵ -cap because \mathcal{M}_t^+ will resemble the standard solution from II.2 near (p, t) ; points lying somewhat further out on the capped neck will belong to a strong ϵ -neck which extends backward in time prior to the surgery.

Remark 60.4. There is a constant $C_3 = C_3(C_1, C_2)$ such that if $(x, t) \in \mathcal{M}_t^\pm$, and $\gamma \subset \mathcal{M}_t^\pm$ is a minimizing geodesic with

$$d((x, t), \gamma)R(x, t)^{\frac{1}{2}} \leq 10 \quad \text{and} \quad d((x, t), \partial\gamma)R(x, t)^{\frac{1}{2}} \geq C_3,$$

then the canonical neighborhood for (x, t) must be of type (a) – a strong ϵ -neck. To see this, observe that type (b) is ruled out because the segment would have to “double back” through an ϵ -neck; types (c) and (d) are excluded because the length of γ would exceed the diameter of the components.

EXPLAIN SOMEWHERE THAT EVERY END WILL LIE IN AN ϵ -HORN WHICH MEETS Ω_ρ (??) OR A DOUBLE/CAPPED ϵ -HORN.

The next condition, which will ultimately be guaranteed by the Hamilton-Ivey curvature pinching result and careful surgery, is also essential in blow-up arguments à la section I.12.1.

Definition 60.5. (*ϕ -pinching*) Let $\phi : [1, \infty) \rightarrow (0, \infty)$ be a decreasing function with $\lim_{R \rightarrow \infty} \phi(R) = 0$. The Ricci flow with surgery \mathcal{M} satisfies the ϕ -pinching assumption if the inequality $\text{Rm} \geq -\phi(R)R$ holds at every point $(x, t) \in \mathcal{M}$ with $R(x, t) \geq 1$.

Definition 60.6. A Ricci flow with surgery satisfies the a priori assumptions if it satisfies the ϕ -pinching and r -canonical neighborhood assumptions on the time interval of the flow. Note that the a priori assumptions depend on ϵ , the function $r(t)$ of Definition 60.1 and the function ϕ of Definition 60.5.

PERHAPS ONE COULD DISCUSS THE GEOMETRY OF ENDS (AT SINGULAR TIMES) HERE.

61. II.4.2

Let \mathcal{M} be a Ricci flow with surgery that satisfies the a priori assumptions. Subsection II.4.2 has the surgery flow analogs of Claim 1 and Claim 2 from I.12.1.

Lemma 61.1. (cf. Claim 1 of II.4.2) Suppose $(x_0, t_0) \in \mathcal{M}$, and set $Q := R(x_0, t_0) + r(t_0)^{-2}$. Then $R(x, t) \leq 8Q$ for all $(x, t) \in P(x_0, t_0, \frac{1}{2}\eta^{-1}Q^{-\frac{1}{2}}, -\frac{1}{8}\eta^{-1}Q^{-1})$, where η is the constant from (60.2).

The proof follows from the estimates (60.2) and a continuity argument, running along paths in $B(x_0, t_0, \frac{1}{2}\eta^{-1}Q^{-\frac{1}{2}})$, and then backward in time along static paths.

Lemma 61.2. (cf. Claim 2 of II.4.2) For any $A < \infty$ there exist $Q = Q(A, \epsilon, r, \phi)$ and $\xi = \xi(A, \epsilon, r, \phi)$ with the following property. Suppose \mathcal{M} is a Ricci flow with surgery that satisfies the a priori assumptions, and $\gamma \subset \mathcal{M}_{t_0}^\pm$ is a minimizing geodesic with endpoints (x_0, t_0) and (x, t_0) , such that $R(y, t_0) > r(t_0)^{-2}$ for all $(y, t_0) \in \gamma$, and $Q_0 := R(x_0, t_0)$ is large enough that $\phi(Q_0) < \xi$. Finally, let $(z, t_0) \in \gamma$ be a point satisfying $R(z, t_0) > 10C_2R(x_0, t_0)$, where C_2 is as in II.1.5. Then the distance $d((x_0, t_0), (z, t_0))$ in $\mathcal{M}_{t_0}^\pm$ is at least $AQ_0^{-\frac{1}{2}}$ provided $R(x, t_0) > QQ_0$. DO WE REQUIRE ϵ TO BE SUFFICIENTLY SMALL HERE?

The proof follows the same lines as the proof of Claim 2 of I.12.1, except that the canonical neighborhood assumption replaces the point-picking argument. WHAT ELSE SHOULD WE SAY HERE?

FORMULATE special case of Claim 2 that is used most often: that the curvature is controlled as a function of distance, when the curvature is large.

62. II.4.3

In the statement of the next lemma we will write Ω synonymously with the \mathcal{M}_T^- of Section 59.

Lemma 62.1. (cf. Lemma II.4.3) Let \mathcal{M} be a Ricci flow with surgery that satisfies the a priori assumptions, defined on an interval ending at $T < \infty$, which goes singular at time T . Let Ω denote the time- T limit, in the sense of Section II.3. Pick $0 < \delta < 1$, put $\rho := \delta r(T)$, and

$$\Omega_\rho := \{(x, t) \in \Omega \mid R(x, T) \leq \rho^{-2}\}.$$

Then there is a radius $0 < h < \delta\rho$, depending only on δ, ϵ, r and ϕ with the following property. Suppose that (x, T) is in an ϵ -horn $\mathcal{H} \subset \Omega$ whose boundary is contained in Ω_ρ . Suppose that $R(x, T) \geq h^{-2}$. Then the parabolic ball $P(x, T, \delta^{-1}R(x, T)^{-\frac{1}{2}}, -R(x, T)^{-1})$ is contained in a strong δ -neck. MAY WE ASSUME THAT h IS MONOTONE IN δ ? (As usual, ϵ is a fixed constant that is small enough so that the result holds uniformly with respect to the other variables.)

Proof. Fix $\delta > 0$. Suppose that the claim is not true. Then there is a sequence of Ricci flows with surgery \mathcal{M}^α and points $(x^\alpha, T^\alpha) \in \mathcal{M}^\alpha$ such that

1. \mathcal{M}^α satisfies the ϕ -pinching and r -canonical neighborhood assumptions,
2. \mathcal{M}^α goes singular at time T^α ,
3. (x^α, T^α) belongs to an ϵ -horn $\mathcal{H}^\alpha \subset \Omega^\alpha$ whose boundary is contained in Ω_ρ^α , and
4. $R(x^\alpha, T^\alpha) \rightarrow \infty$, but
5. For each α , $P(x^\alpha, T^\alpha, \delta^{-1}R(x^\alpha, T^\alpha)^{-\frac{1}{2}}, -R(x^\alpha, T^\alpha)^{-1})$ is not contained in a strong δ -neck.

Recall that when ϵ is small enough, any cross-sectional 2-sphere sitting in an ϵ -neck $V \subset \mathcal{H}^\alpha$ is separating; see section 50. We may find a properly embedded minimizing geodesic $\gamma^\alpha \subset \mathcal{H}^\alpha$ which joins the two ends of \mathcal{H}^α . As γ^α must intersect a cross-sectional 2-sphere containing (x^α, T^α) , it must pass within distance $\leq 10R(x^\alpha, T^\alpha)^{-\frac{1}{2}}$ of (x^α, T^α) , when ϵ is small. For large α , both ends of γ^α will exit the ball $B(x^\alpha, T^\alpha, C_3R(x, t)^{-\frac{1}{2}}) \subset \Omega$, where C_3 is the constant from Remark 60.4; this follows from Lemma 61.2, since the scalar curvature tends to infinity on one end of γ^α , and to ρ^{-2} on the other, while $R(x^\alpha, T^\alpha)\rho^{-2} \rightarrow \infty$. Thus by Remark 60.4, the canonical neighborhood of (x^α, T^α) is of type (a), i.e. a strong ϵ -neck.

By Lemma 61.2, after rescaling the metric on the time- T^α slice by $R(x^\alpha, T^\alpha)$ we have uniform curvature bounds on distance balls. We also have a uniform lower bound on the injectivity radius at (x^α, T^α) of the rescaled solution. Hence after passing to a subsequence, we may take a pointed limit (M^∞, x^∞) of the time- T^α slices. By the ϕ -pinching assumption, M^∞ will have nonnegative curvature. Furthermore, it will contain a complete minimizing geodesic (a limit of the γ^α 's). Thus by Toponogov's theorem, M^∞ splits off an \mathbb{R} -factor. As the canonical neighborhood of (x^α, T^α) is an ϵ -neck, it follows that $M^\infty = \mathbb{R} \times S^2$ for some nonnegatively curved metric on S^2 . In particular, M^∞ has scalar curvature uniformly bounded above.

Any point $\hat{x} \in M^\infty$ is a limit of points $(\hat{x}^\alpha, T^\alpha) \in \mathcal{M}^\alpha$. As $R(\hat{x}) > 0$ and $R(x^\alpha, T^\alpha) \rightarrow \infty$, it follows that $R(\hat{x}^\alpha, T^\alpha) \rightarrow \infty$. Then for large α , $(\hat{x}^\alpha, T^\alpha)$ is in a canonical neighborhood which, in view of the \mathbb{R} -factor in M^∞ , must be a strong ϵ -neck. From the upper bound on the scalar curvature of M^∞ , along with the time interval involved in the definition of a strong ϵ -neck, it follows that we can parabolically rescale the pointed flows $(\mathcal{M}^\alpha, x^\alpha, T^\alpha)$ by $R(x^\alpha, T^\alpha)$, shift time and extract a smooth pointed limiting Ricci flow $(\mathcal{M}^\infty, x^\infty, 0)$ which is defined on a time interval $(\xi, 0]$, for some $\xi < 0$. The time-0 slice \mathcal{M}_0^∞ splits off an \mathbb{R} -factor, which means that the same will be true of all time slices.

Let ξ be the minimal negative number so that after parabolically rescaling the pointed flows $(\mathcal{M}^\alpha, x^\alpha, T^\alpha)$ by $R(x^\alpha, T^\alpha)$, we can extract a limit Ricci flow \mathcal{M}^∞ which is the product of \mathbb{R} with a Ricci flow on S^2 , and is defined on the time interval $(\xi, 0]$. We claim that $\xi = -\infty$. Suppose not, i.e. $\xi > -\infty$. Given $(x, t) \in \mathcal{M}^\infty$, as $R(x, t) > 0$ and $R(x^\alpha, T^\alpha) \rightarrow \infty$, it follows that (x, t) is a limit of points $(x^\alpha, t^\alpha) \in \mathcal{M}^\alpha$ that lie in canonical neighborhoods. In view of the \mathbb{R} -factor in \mathcal{M}^∞ , for large α these canonical neighborhoods must be strong ϵ -necks. This implies in particular that $(R^{-2} \frac{\partial R}{\partial t})(x, t) > 0$, so $\frac{\partial R}{\partial t}(x, t) > 0$. Then there is a uniform upper bound Q for the scalar curvature on \mathcal{M}^∞ . Extending backward from a time- $(\xi + \frac{1}{100Q})$ slice, we can construct a limit Ricci flow that exists on some time interval $(\xi', 0]$ with $\xi' < \xi$, which is a contradiction.

Thus we obtain an ancient solution \mathcal{M}^∞ with the property that each point (x, t) lies in a strong ϵ -neck. Removing the \mathbb{R} -factor gives an ancient solution on S^2 . In view of the fact that each time slice is ϵ -close to the round S^2 , up to rescaling, it follows that the ancient solution on S^2 must be the standard shrinking solution (see Section 36). Then \mathcal{M}^∞ is the standard shrinking solution on $S^2 \times \mathbb{R}$. Hence for an infinite number of α , $P(x^\alpha, T^\alpha, \delta^{-1}R(x^\alpha, T^\alpha)^{-\frac{1}{2}}, -R(x^\alpha, T^\alpha)^{-1})$ is in fact in a strong δ -neck, which is a contradiction. \square

Remark 62.2. If a given h makes Lemma 62.1 work for a given function r then one can check that logically, h also works for any r' with $r' \geq r$. Because of this, we may assume that h only depends on $\min r = r(T)$ and is monotonically nondecreasing as a function of $r(T)$.

63. II.4.4

ADD MORE DISCUSSION AND DETAIL TO THE SURGERY PROCESS.

The surgery is done in a δ -neck, which will generically have some negative sectional curvatures. The standard cylinder has nonnegative sectional curvature, and the goal is to meld the two metrics together so as to not make any negative sectional curvature on the original piece more negative. Following [18, Section 4], let $z \in [0, c]$ be a lengthwise coordinate for the result of the surgery, $c \in (3\lambda, 4\lambda)$, for an appropriate parameter λ . The metric is specified differently on $[0, \lambda] \times S^2$, $[\lambda, 2\lambda] \times S^2$, $[2\lambda, 3\lambda] \times S^2$ and $[3\lambda, c] \times S^2$. On $[0, \lambda] \times S^2$, the new metric is the same as the original metric g . On $[\lambda, 2\lambda] \times S^2$, it equals $e^{-2f}g$, where $f = f(z)$ will be specified. On $[2\lambda, 3\lambda] \times S^2$ it is of the form $\phi e^{-2f}g + (1-\phi)r_0^2\bar{g}$, where ϕ is a bump function that is one near 2λ and zero near 3λ , r_0 is a parameter to be specified and \bar{g} is the metric on a part of the standard solution of II.2. On $[3\lambda, c] \times S^2$, it equals $r_0^2\bar{g}$. The idea is to take f on $[\lambda, c]$ to be $f(z) = c_0 e^{-\frac{z}{\lambda}}$, where the numbers c_0 and p are to be specified, so that the squeezing of the metric by the factor e^{-2f} gives it positive sectional curvature.

The surgery addition is done starting at a 2-sphere near the center of a δ -neck, which corresponds to $\{\lambda\} \times S^2$. From the statement of Lemma II.5.3, it appears that ϵ , the small but fixed constant in the paper, is taken so that the distance from the midpoint p of the added cap to $\{\lambda\} \times S^2$ is roughly $\epsilon^{-1}h$. To be consistent with the rest of the paper, the surgery will be done on a 2-sphere $\{\lambda\} \times S^2$ where the scalar curvature is roughly h^{-2} .

Let X_0 denote the Riemannian product of a round 2-sphere of scalar curvature 1 with $[0, \infty)$, and let $x_0 \in X_0$ be a point lying on the boundary 2-sphere. The next lemma says that it's possible to perform surgery without disrupting the Hamilton-Ivey pinching condition (see Appendix C), so that the end result looks locally like the initial condition of the standard solution \mathcal{S} .

Lemma 63.1. *There exists $\delta' = \delta'(\delta)$ with $\lim_{\delta \rightarrow 0} \delta'(\delta) = 0$ with the following property. Suppose (X, x) is a pointed Riemannian 3-manifold with boundary which is δ -close to the pointed manifold (X_0, x_0) , and assume that X satisfies the Hamilton-Ivey pinching condition. Then provided δ is sufficiently small, there is a pointed Riemannian manifold (X', x') with the following properties:*

- (1) X' satisfies the Hamilton-Ivey pinching condition.
- (2) (X', x') is $\delta'(\delta)$ -close to $(\mathcal{S}_0, (c, 0))$ – the initial condition of the standard solution with basepoint at the center of the cap. In particular, when $\delta > 0$ is sufficiently small, the scalar curvature within the ball $B(x', \delta'^{-1}) \subset X'$ will fall in the interval $(\frac{1}{2}, 4)$.
- (3) There are subsets $K \subset B(x, 20) \subset X$ and $K' \subset B(x', 20) \subset X'$ such that $X \setminus K$ is isometric to $X' \setminus K'$.

Definition 63.2. (*Ricci flow with cutoff*) Let \mathcal{M} be a Ricci flow with surgery defined on $[a, b]$ that satisfies the a priori assumptions, where the pinching condition is with respect to the function ϕ_0 associated with Hamilton-Ivey pinching. Let $\delta : [a, b] \rightarrow (0, \infty)$ be a nonincreasing function. Then \mathcal{M} is a Ricci flow with (r, δ) -cutoff if :

- (1) \mathcal{M} satisfies the Hamilton-Ivey pinching condition.
- (2) At each singular time t_k , the forward time slice $\mathcal{M}_{t_k}^+$ is obtained from the backward time slice $\Omega = \mathcal{M}_{t_k}^-$ by applying the following procedure:
 - A. Discard each component of Ω which is disjoint from

$$\Omega_\rho := \{(x, t_k) \in \Omega \mid R(x, t_k) \leq \rho^{-2}\},$$

where $\rho := \delta(t_k)r(t_k)$.

B. In each ϵ -horn \mathcal{H}_{ij} of each of the remaining components Ω_i , find a point (x_{ij}, t_k) such that $R(x_{ij}, t_k) = h_k^{-2}$, where $h_k = h(t_k)$ is as in Lemma 62.1.

C. Find a strong δ -neck $U_{ij} \times [t_k - h_k^2, t_k]$ containing $P(x_{ij}, t_k, \delta^{-1}R(x_{ij}, t_k)^{-\frac{1}{2}}, R(x_{ij}, t_k)^{-1})$; this is guaranteed to exist by Lemma 62.1.

D. For each ij , let $S_{ij} \subset U_{ij}$ be a cross-sectional 2-sphere containing (x_{ij}, t_k) , and cut $\cup_i \Omega_i$ open along the S_{ij} 's, to obtain a manifold X which has a pair of spherical boundary components for each ij .

E. Apply Lemma 63.1 to glue spherical caps onto the resulting manifold with boundary.

Note that the definition of Ricci flow with (r, δ) -cutoff also depends on the function $r(t)$ through the a priori assumption.

Remark 63.3. We recall the reasoning, from II.3, why it is legitimate to discard the components in step A. above. Let \mathcal{C} be a component of Ω so that $R(x, t_k) > \rho^{-2}$ for all $x \in \mathcal{C}$. Each point of \mathcal{C} is in a canonical neighborhood. If \mathcal{C} is a closed positively curved 3-manifold then we know its topological type from [17]. Otherwise, every point in \mathcal{C} has a neighborhood that is an ϵ -horn or an ϵ -cap. If \mathcal{C} is compact then it must be diffeomorphic to S^3 or $S^1 \times S^2$, and so can be safely discarded. THIS POSSIBILITY SEEMS TO BE NEGLECTED IN II.3 AND II.4.4. If \mathcal{C} is noncompact then it must be a capped ϵ -horn or a double ϵ -horn. In either case, the topological type of M_t for t slightly less than t_k can be reconstructed from that of $M_{t_k}^+$ by adding 1-handles, i.e. taking connected sums of components or taking connected sums with $S^1 \times S^2$.

Remark 63.4. When $\delta > 0$ is sufficiently small, we will have $\text{vol}(\mathcal{M}_t^+) < \text{vol}(\mathcal{M}_t^-) - h(t)^3$ for each surgery time $t \in (a, b)$. This is because each component that is discarded in step E contains "half" of the δ -neck U_{ij} , which has volume $> \text{const.} \delta^{-1}h(t)^3$, while the cap added has volume $< \text{const.} h(t)^3$.

POINT OUT THAT THE FLOW MAY BECOME EXTINCT.

DISCUSS COMPACTNESS CRITERIA SOMEWHERE: Assume that some open ball has compact closure, is unscathed on a certain time interval, has curvature bounds, and injectivity radius bounds. Then get local limit.

Lemma 63.5. (*Prolongation of Ricci flows with cutoff*) Suppose that r and δ are nonincreasing positive functions defined on $[a, b]$. Let \mathcal{M} be a Ricci flow with (r, δ) -cutoff defined on an interval $[a, c] \subset [a, b]$. Provided $\sup \delta$ is sufficiently small, either

- (1) \mathcal{M} can be prolonged to a Ricci flow with (r, δ) -cutoff defined on $[a, b]$, or
- (2) There is an extension of \mathcal{M} to a Ricci flow with surgery defined on an interval $[a, T]$ with $T \in (c, b]$, where
 - a. The restriction of the flow to any subinterval $[a, T']$, $T' < T$, is a Ricci flow with (r, δ) -cutoff, but
 - b. The r -canonical neighborhood assumption fails at some point $(x, T) \in \mathcal{M}_T^-$.

In particular, the only obstacle to prolongation of Ricci flows with (r, δ) -cutoff is the potential breakdown of the r -canonical neighborhood assumption.

Proof. Consider the time slice of \mathcal{M} at time c , which we call \mathcal{M}_c^- . If it is singular then we perform steps A-E of Definition 63.2 to produce \mathcal{M}_c^+ ; otherwise we set $\mathcal{M}_c^+ = \mathcal{M}_c^-$. Since the surgery is done using Lemma 63.1, provided $\delta > 0$ is sufficiently small, the forward time slice \mathcal{M}_c^+ will satisfy the Hamilton-Ivey pinching assumption.

We claim that the r -canonical neighborhood assumption holds in \mathcal{M}_c^+ . To see this, observe that provided δ is sufficiently small, every point $(x, c) \in \mathcal{M}_c^-$ lying in one of the strong δ -necks $U_{ij} \times [c - h^2, c]$ whose distance to ∂U_{ij} is $> 2\epsilon^{-1}h$ will lie in a canonical neighborhood of type (a) whose distance to (x_{ij}, c) is at least $d((x, c), (x_{ij}, c)) - 2\epsilon^{-1}h$. Therefore, if $(x, c) \in \mathcal{M}_c^+ \setminus \mathcal{M}_c^-$, and $d((x, c), \mathcal{M}_c^+ \setminus \mathcal{M}_c^-) \in [4\epsilon^{-1}h, \frac{1}{10}\delta^{-1}h]$, then (x, c) will lie in a neighborhood of type (a) in \mathcal{M} . Using the time c slices of these strong ϵ -necks, we can find ϵ -caps C_{ij} which provide canonical neighborhoods for each point $(x, c) \in \mathcal{M}_c^+$ lying in $N_{10\epsilon^{-1}h}(\mathcal{M}_c^+ \setminus \mathcal{M}_c^-)$, assuming $C_1 > 30\epsilon^{-1}$. Finally, it is not hard to see that when δ is sufficiently small and $(x, c) \in \mathcal{M}_c^+ \setminus N_{\frac{1}{20}\delta^{-1}h}(\mathcal{M}_c^+ \setminus \mathcal{M}_c^-)$, then the canonical neighborhood that existed prior to surgery will be unaffected by the surgery procedure.

We now prolong \mathcal{M} by Ricci flow with initial condition \mathcal{M}_c^+ . If the flow extends smoothly up to time b then we are done. Otherwise, there is some time $t_{\text{sing}} \leq b$ at which it goes singular. We add the singular limit Ω at time t_{sing} to obtain a Ricci flow with surgery defined on $[a, t_{\text{sing}}]$. From the Hamilton-Ivey estimates [16], \mathcal{M} satisfies the Hamilton-Ivey pinching condition on $[a, t_{\text{sing}}]$. It follows from Definition 60.1 that the set of times $t \in (c, t_{\text{sing}}]$ for which the r -canonical neighborhood assumption holds is relatively open. **DOUBLE-CHECK THIS.** If the set of such times is (c, T) for some $T \leq t_{\text{sing}}$ then we are done. Otherwise, the r -canonical neighborhood assumption holds at t_{sing} . In this case we repeat the construction with c replaced by t_{sing} , and iterate if necessary. Either we will reach time b after a finite number of iterations, or we will reach a time T satisfying (2), or we will perform an infinite number of surgeries before time b . However, the last possibility cannot occur. Each surgery removes a volume of at least h^3 , but the lower bound on the scalar curvature during the flow, coming from the maximum principle, gives a finite upper bound on the total volume growth during the complement of the surgery times. \square

64. II.4.5

Let \mathcal{M} be a Ricci flow with (r, δ) -cutoff. The lemma says that provided δ is small, after a surgery at scale h there is a ball B of radius $Ah \gg h$ centered in the surgery cap, such that the ball's evolution is close to that of the standard solution for an elapsed time close to h^2 , unless another surgery occurs during which the entire ball is thrown away. Note that the elapsed time h^2 corresponds, modulo parabolic rescaling, to the duration of the standard solution.

Lemma 64.1. (*cf. II.4.5*)

For any $A < \infty$, $\theta \in (0, 1)$ and $\hat{r} > 0$, one can find $\hat{\delta} = \hat{\delta}(A, \theta, \hat{r}) > 0$ with the following property. Suppose that we have a Ricci flow with (r, δ) -cutoff defined on a time interval $[a, b]$ with $\min r = r(b) \geq \hat{r}$. Suppose that there is a surgery time $T_0 \in (a, b)$, with $\delta(T_0) \leq \hat{\delta}$. Let $(p, T_0) \in \mathcal{M}_{T_0}$ be the center of the surgery cap. Let $\hat{h} := h(\delta(T_0), \epsilon, r(T_0), \phi_0)$ be the surgery scale given by Lemma 62.1 and put $T_1 := \min(b, T_0 + \theta \hat{h}^2)$. Then one of the two following possibilities occurs :

(1) The solution is unscathed on $P(p, T_0, A\hat{h}, T_1 - T_0)$. The pointed solution there (with respect to the basepoint (p, T_0)) is, modulo parabolic rescaling, A^{-1} -close to the pointed flow on $U_0 \times [0, (T_1 - T_0)\hat{h}^{-2}]$, where U_0 is an open subset of the initial time slice \mathcal{S}_0 of the standard solution \mathcal{S} and the basepoint is the center c of the cap in \mathcal{S}_0 .

(2) Assertion (1) holds with T_1 replaced by some $t^+ \in [T_0, T_1)$, where t^+ is a surgery time. Moreover, the entire ball $B(p, T_0, A\hat{h})$ becomes extinct at time t^+ , i.e. $P(p, T_0, A\hat{h}, t^+ - T_0) \cap \mathcal{M}_{t^+} \subset \mathcal{M}_{t^+}^- \setminus \mathcal{M}_{t^+}^+$.

Proof. We give a proof with the same ingredients as the proof in II, but which is slightly rearranged. We first show the following result, which is almost the same as Lemma 64.1.

Lemma 64.2. For any $A < \infty$, $\theta \in (0, 1)$ and $\hat{r} > 0$, one can find $\hat{\delta} = \hat{\delta}(A, \theta, \hat{r}) > 0$ with the following property. Suppose that we have a Ricci flow with (r, δ) -cutoff defined on a time interval $[a, b]$ with $\min r = r(b) \geq \hat{r}$. Suppose that there is a surgery time $T_0 \in (a, b)$, with $\delta(T_0) \leq \hat{\delta}$. Let $(p, T_0) \in \mathcal{M}_{T_0}$ be the center of the surgery cap. Let $\hat{h} := h(\delta(T_0), \epsilon, r(T_0), \phi_0)$ be the surgery scale given by Lemma 62.1 and put $T_1 := \min(b, T_0 + \theta \hat{h}^2)$. Suppose that the solution is unscathed on $P(p, T_0, A\hat{h}, T_1 - T_0)$. Then the pointed solution there (with respect to the basepoint (p, T_0)) is, modulo parabolic rescaling, A^{-1} -close to the pointed flow on $U_0 \times [0, (T_1 - T_0)\hat{h}^{-2}]$, where U_0 is an open subset of the initial time slice \mathcal{S}_0 of the standard solution \mathcal{S} and the basepoint is the center c of the cap in \mathcal{S}_0 .

Proof. Fix θ and \hat{r} . Suppose that the lemma is not true. Then for some $A > 0$, there is a sequence $\{\mathcal{M}^\alpha, (p^\alpha, T_0^\alpha)\}_{\alpha=1}^\infty$ of pointed Ricci flows with $(r^\alpha, \delta^\alpha)$ -cutoff that together provide a counterexample. In particular,

1. $\lim_{\alpha \rightarrow \infty} \delta^\alpha(T_0^\alpha) = 0$.
2. \mathcal{M}^α is unscathed on $P(p^\alpha, T_0^\alpha, A\hat{h}^\alpha, T_1^\alpha - T_0^\alpha)$.
3. If $(\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0))$ is the pointed Ricci flow arising from $(\mathcal{M}^\alpha, (p^\alpha, T_0^\alpha))$ by a time shift of T_0^α and a parabolic rescaling by \hat{h}^α then $P(\hat{p}^\alpha, 0, A, (T_1^\alpha - T_0^\alpha)(\hat{h}^\alpha)^{-2})$ is not A^{-1} -close to a pointed subset of the standard solution.

Put $T_2 = \inf_\alpha (T_1^\alpha - T_0^\alpha)(\hat{h}^\alpha)^{-2}$. (We do not exclude that $T_2 = 0$.) Clearly $T_2 \leq \theta$. Let \mathcal{T} be the set of times $\tau \in [0, T_2]$ with the property that we can apply Theorem D.1 of Appendix D, if we want, to take a convergent subsequence of the pointed solutions $(\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0))$ on the time interval $[0, \tau]$. That is, $\tau \in \mathcal{T}$ if and only if for each $\lambda > 0$, eventually (in α) there are unscathed sets $P(\hat{p}^\alpha, 0, \lambda, \tau)$ whose curvatures are uniformly bounded in α .

Sublemma 64.3. $\mathcal{T} = [0, T_2]$.

Proof. Suppose not. The interval \mathcal{T} can be written as either $[0, T_3]$, with $T_3 < T_2$, or $[0, T_3)$, with $T_3 \leq T_2$. Suppose first that $T_3 < T_2$. Given $\sigma \in (0, T_2 - T_3]$, as $T_3 + \sigma \notin \mathcal{T}$, it follows that for some $\lambda > 0$ we can find an infinite subsequence of $\{\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0)\}_{\alpha=1}^\infty$, which we relabel by α , so that either

1. For each α the set $B(\hat{p}^\alpha, 0, \lambda)$ becomes scathed on $[0, T_3 + \sigma]$, or
2. For each α the set $P(\hat{p}^\alpha, 0, \lambda, T_3 + \sigma)$ is unscathed but $\lim_{\alpha \rightarrow \infty} \sup_{P(\hat{p}^\alpha, 0, \lambda, T_3 + \sigma)} |\text{Rm}| = \infty$.

By Theorem D.1 of Appendix D, after passing to a subsequence, there is a complete limit solution $(\widehat{\mathcal{M}}^\infty, (\hat{p}^\infty, 0))$ defined on the time interval \mathcal{T} . Relabel the subsequence by α . By the uniqueness result of II.2, $(\widehat{\mathcal{M}}^\infty, (\hat{p}^\infty, 0))$ must be the same as the restriction of the standard solution to \mathcal{T} . From the bounded curvature of the standard solution on \mathcal{T} , the canonical neighborhood assumption and equation (60.2), we can choose $\sigma \in (0, T_2 - T_3]$ so that we have uniform curvature bounds on balls in the pointed solutions $(\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0))$, up to time $T_3 + \sigma$. Then we can take a further subsequence that converges, again necessarily to the standard solution, on the time interval $[0, T_3 + \sigma]$. However, since $\lim_{\alpha \rightarrow \infty} \delta^\alpha(T_0^\alpha) = 0$, and surgeries only occur near the centers of δ -necks, the balls $B(\hat{p}^\alpha, 0, \lambda)$ cannot all be scathed on $[0, T_3 + \sigma]$. And we do have uniform curvature bounds on the sets $P(\hat{p}^\alpha, 0, \lambda, T_3 + \sigma)$. This is a contradiction.

The case $\mathcal{T} = [0, T_2)$ is similar. □

Returning to the original sequence $\{\mathcal{M}^\alpha, (p^\alpha, T_0^\alpha)\}_{\alpha=1}^\infty$ and its rescaling $\{\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0)\}_{\alpha=1}^\infty$, we can now take a subsequence that converges on the time interval $[0, T_2]$, again necessarily to the standard solution. Then there will be an infinite subsequence $\{\widehat{\mathcal{M}}^{\alpha_\beta}, (\hat{p}^{\alpha_\beta}, 0)\}_{\beta=1}^\infty$ of $\{\widehat{\mathcal{M}}^\alpha, (\hat{p}^\alpha, 0)\}_{\alpha=1}^\infty$, with $\lim_{\beta \rightarrow \infty} (T_1^{\alpha_\beta} - T_0^{\alpha_\beta})(\hat{h}^{\alpha_\beta})^{-2} = T_2$, so that $P(\hat{p}^{\alpha_\beta}, 0, A, (T_1^{\alpha_\beta} - T_0^{\alpha_\beta})(\hat{h}^{\alpha_\beta})^{-2})$ is A^{-1} -close to a pointed subset of the standard solution. This is a contradiction. □

We now finish the proof of Lemma 64.1. If the solution is unscathed on $P(p, T_0, A\hat{h}, T_1 - T_0)$ then we can apply Lemma 64.2 to see that we are in case (1) of the conclusion of Lemma 64.1. Suppose, on the other hand, that the solution is scathed on $P(p, T_0, A\hat{h}, T_1 - T_0)$. Let t^+ be the largest t so that the solution is unscathed on $P(p, T_0, A\hat{h}, t - T_0)$. We can apply Lemma 64.2 to see that conclusion (1) of Lemma 64.1 holds with T_1 replaced by t^+ . As surgery is always performed near the middle of a δ -neck, if $\hat{\delta} \ll A^{-1}$ then the final time slice in the parabolic neighborhood $P(p, T_0, A\hat{h}, t^+ - T_0)$ cannot intersect a 2-sphere where a surgery is going to be performed. The only other possibility is that the entire ball $B(p, T_0, A\hat{h})$ becomes extinct at time t^+ . □

65. II.4.6

Let \mathcal{M} be a Ricci flow with (r, δ) -cutoff. Corollary II.4.6 says that when δ is sufficiently small, an admissible curve γ which comes close to a surgery cap at a surgery time will have a large value of $\int_{\gamma} (R(\gamma(t)) + |\dot{\gamma}(t)|^2) dt$. Note that the latter quantity is not quite the same as $\mathcal{L}(\gamma)$, and is invariant under parabolic rescaling.

Corollary II.4.6 is used in the extension of Section I.7 to Ricci flows with surgery. The idea is that if δ is small and $L(x, t)$ isn't too large then any \mathcal{L} -minimizing sequence of admissible curves joining the basepoint (x_0, t_0) to (x, t) must avoid surgery regions, and will therefore accumulate on a minimizing \mathcal{L} -geodesic.

Corollary 65.1. (cf. Corollary II.4.6) For any $l < \infty$ and $\hat{r} > 0$, we can find $A = A(l, \hat{r}) < \infty$ and $\theta = \theta(l, \hat{r})$ with the following property. Suppose that we are in the situation of Lemma 64.1, with $\delta < \hat{\delta}(A, \theta, \hat{r})$. As usual, \hat{h} will be the surgery scale coming from Lemma 62.1. Let $\gamma : [T_0, T_\gamma] \rightarrow \mathcal{M}$ be an admissible curve, with $T_\gamma \in (T_0, T_1]$. Suppose that $\gamma(T_0) \in B(p, T_0, \frac{A\hat{h}}{2})$, $\gamma([T_0, T_\gamma]) \subset P(p, T_0, A\hat{h}, T_\gamma - T_0)$, and either

$$a. T_\gamma = T_1 = T_0 + \theta(\hat{h})^2,$$

or

$$b. \gamma(T_\gamma) \in \partial B(p, T_0, A\hat{h}) \times [T_0, T_\gamma].$$

Then

$$\int_{T_0}^{T_\gamma} (R(\gamma(t)) + |\dot{\gamma}(t)|^2) dt > l.$$

Proof. For the moment, fix $A < \infty$ and $\theta \in (0, 1)$. Choose $\hat{\delta} = \hat{\delta}(A, \theta, \hat{r})$ so as to satisfy Lemma 64.1. Let \mathcal{M} , (p, T_0) , etc., be as in the hypotheses of Lemma 64.1. Let $\gamma : [T_0, T_\gamma] \rightarrow \mathcal{M}$ be a curve as in the hypotheses of the Corollary. From Lemma 64.1, we know that the parabolic region $P(p, T_0, A\hat{h}, T_\gamma - T_0) \subset \mathcal{M}$, with basepoint (p, T_0) , is (after parabolic rescaling by \hat{h}^{-2}) A^{-1} -close to a pointed flow $U_0 \times [0, \hat{T}_\gamma] \subset \mathcal{S}$, the latter having basepoint $(c, 0)$. Here $U_0 \subset \mathcal{S}_0$ and $\hat{T}_\gamma := (T_\gamma - T_0)\hat{h}^{-2}$. Then the image of γ , under the diffeomorphism implicit in the definition of A^{-1} -closeness, gives rise to a smooth curve $\gamma_0 : [0, \hat{T}_\gamma] \rightarrow U_0 \times [0, \hat{T}_\gamma]$ so that (if A is sufficiently large) :

$$(65.2) \quad \gamma_0(0) \in B(c, 0, \frac{3}{5}A),$$

$$(65.3) \quad \int_{T_0}^{T_\gamma} |\dot{\gamma}|^2 dt \geq \frac{1}{2} \int_0^{\hat{T}_\gamma} |\dot{\gamma}_0|^2 dt,$$

$$(65.4) \quad \int_{T_0}^{T_\gamma} R(\dot{\gamma}) dt \geq \frac{1}{2} \int_0^{\hat{T}_\gamma} R(\dot{\gamma}_0) dt,$$

and

$$(a) \hat{T}_\gamma = \theta,$$

or

$$(b) \gamma_0(\hat{T}_\gamma) \notin P(c, 0, \frac{4}{5}A, \hat{T}_\gamma).$$

In case (a) we have, by Claim 5 of II.2,

$$\begin{aligned} \int_{T_0}^{T_\gamma} R(\gamma(t), t) dt &\geq \frac{1}{2} \int_0^\theta R(\gamma_0(t)) dt \\ &\geq \frac{1}{2} \int_0^\theta \text{const.}(1-t)^{-1} dt = \text{const.} \log(1-\theta). \end{aligned}$$

If we choose θ is sufficiently close to 1 then in this case, we can ensure that

$$(65.5) \quad \int_{T_0}^{T_\gamma} (R(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt \geq \int_{T_0}^{T_\gamma} R(\gamma(t), t) dt > l.$$

In case (b), we may use the fact that the Ricci curvature of the standard solution is everywhere positive, and hence the metric tensor is nonincreasing with time. So if $\pi : \mathcal{S} = \mathcal{S}_0 \times [0, 1) \rightarrow \mathcal{S}_\theta$ is projection to the time- θ slice and we put $\eta := \pi \circ \gamma_0$ then

$$(65.6) \quad \int_{T_0}^{T_\gamma} |\dot{\gamma}(t)|^2 dt \geq \frac{1}{2} \int_0^{\hat{T}_\gamma} |\dot{\gamma}_0(t)|^2 dt \geq \frac{1}{2} \int_0^{\hat{T}_\gamma} |\dot{\eta}(t)|^2 dt \geq \frac{1}{2} \left(d(\eta(0), \eta(\hat{T}_\gamma)) \right)^2.$$

With our given value of θ , in view of (b), if we take A large enough then we can ensure that $\frac{1}{2} \left(d(\eta(0), \eta(\hat{T}_\gamma)) \right)^2 > l$. This proves the lemma. \square

66. II.4.7

Corollary 66.1. (cf. Corollary II.4.7) For any $Q < \infty$ $\hat{r} > 0$, there is a $\theta = \theta(Q, \hat{r}) \in (0, 1)$ with the following property. Suppose that we are in the situation of Lemma 64.1, with $\delta < \hat{\delta}(A, \theta, \hat{r})$ and $A > \epsilon^{-1}$. If $\gamma : [T_0, T_x] \rightarrow \mathcal{M}$ is a static curve starting in $B(p, T_0, A\hat{h})$, and

$$(66.2) \quad Q^{-1}R(\gamma(t)) \leq R(\gamma(T_x)) \leq Q(T_x - T_0)^{-1}$$

for all $t \in [T_0, T_x]$, then $T_x \leq T_0 + \theta\hat{h}^2$.

Proof. If $T_x > T_0 + \theta\hat{h}^2$ then by Lemma 64.1 and Claim 5 of II.2,

$$R(\gamma(T_0 + \theta\hat{h}^2)) \geq \text{const.}(1-\theta)^{-1}\hat{h}^{-2}.$$

Thus by (66.2) we get

$$Q^{-1} \text{const.}(1-\theta)^{-1}\hat{h}^{-2} \leq R(\gamma(T_x)) \leq Q(T_x - T_0)^{-1},$$

or

$$T_x - T_0 \leq \text{const.} Q^2(1-\theta)\hat{h}^2,$$

If θ is close enough to 1 then $\text{const.} Q^2(1-\theta)\hat{h}^2$ is less than $\theta\hat{h}^2$, which is a contradiction. \square

67. THE L -FUNCTION OF I.7 AND RICCI FLOWS WITH SURGERY

In this section we examine several points which arise when one adapts the noncollapsing argument of I.7.3 to Ricci flows with surgery. This material is implicit background for Lemma II.5.2 and Proposition II.6.3. We will use notation and terminology introduced in our discussion of II.4.

Let \mathcal{M} be a Ricci flow with surgery, and fix a point $(x_0, t_0) \in \mathcal{M}$. One may define the \mathcal{L} -length of an admissible curve γ from (x_0, t_0) to some (x, t) , for $t < t_0$, using the formula from I.7:

$$\mathcal{L}(\gamma) = \int_t^{t_0} \sqrt{t_0 - \bar{t}} (R + |\dot{\gamma}|^2) d\bar{t},$$

where $\dot{\gamma}$ denotes the spatial part of the velocity of γ . One defines the L -function on $\mathcal{M}_{(-\infty, t_0)}$ by setting $L(x, t)$ to be the infimal \mathcal{L} -length of the admissible curves from (x_0, t_0) to (x, t) if such an admissible curve exists, and infinity otherwise.

If γ is an admissible curve landing in \mathcal{M}_{reg} , then the first variation formula and the geodesic equation apply (when appropriate). If γ is a stable \mathcal{L} -geodesic in \mathcal{M}_{reg} , then the proof of the monotonicity along γ of the weighted Jacobian $\tau^{-\frac{3}{2}} \exp(-l(\tau))J(\tau)$ remains valid. Similarly, if $U \subset \mathcal{M}_{(-\infty, t_0)}$ is an open set such that every $(x, t) \in U$ is accessible from (x_0, t_0) by a minimizing \mathcal{L} -geodesic (i.e. an \mathcal{L} -geodesic of \mathcal{L} -length $L(x, t)$) contained in \mathcal{M}_{reg} , then the arguments of I.7.1 imply that the differential inequality I.(7.15),

$$(67.1) \quad \bar{L}_\tau + \Delta \bar{L} \leq 2n$$

holds in U , in the barrier sense.

Lemma 67.2. *Let \mathcal{M} be a Ricci flow with surgery defined on $[a, b]$. Suppose that we are given a time $t_0 \in [a, b]$, and a point $x_0 \in \mathcal{M}_{t_0} \cap \mathcal{M}_{\text{reg}}$. Let $t \in [a, t_0)$ be another time and let $(x, t) \in \mathcal{M}$ be a point so that $L(x, t) < \infty$. If there is an $\alpha > 0$ such that any barely admissible curve from (x, t) to (x_0, t_0) has \mathcal{L} -length at least $L(x, t) + \alpha$, then there is a minimizing \mathcal{L} -geodesic from (x, t) to (x_0, t_0) which lies in \mathcal{M}_{reg} .*

Proof. Let $\{\gamma_j : [t, t_0] \rightarrow \mathcal{M}\}_{j=1}^\infty$ be a sequence of admissible curves from (x, t) to (x_0, t_0) such that $L(x, t) = \lim_{j \rightarrow \infty} \mathcal{L}(\gamma_j)$. By restricting the sequence, we may assume that $\sup_j \mathcal{L}(\gamma_j) < 2L(x, t)$. We claim that there is a subsequence of the γ_j 's that

(a) converges uniformly to some $\gamma_\infty : [t, t_0] \rightarrow \mathcal{M}$,

and

(b) converges weakly to γ_∞ in $W^{1,2}$ on any subinterval $[t, t'] \subset [t, t_0)$ such that (t, t') is free of surgeries.

To see this, note that on any time interval $[c, d] \subset [t, t_0)$ which is free of surgery times, one may apply the Schwartz inequality to the \mathcal{L} -length, to conclude that the γ_j 's are uniformly Hölder on $[c, d]$. We know that $\gamma_j(t')$ lies in $\mathcal{M}_{t'}^- \cap \mathcal{M}_{t'}^+$ for each surgery time $t' \in (t, t_0)$, and so one can use similar reasoning to get Hölder control on a short time interval of the form $[t'', t']$. Using a change of variable as in (9.2), one obtains uniform Hölder control near t_0 after reparametrizing with s . Thus the γ_j 's are equicontinuous and (a) follows. To show

(b), let $t < t_1 < t_2 < \dots < t_k < t'$ be the surgery times lying in (t, t') , and apply weak compactness to each of the sequences

$$\{\gamma_j|_{[t, t_1]}\}, \dots, \{\gamma_j|_{[t_k, t']}\};$$

this is justified by the fact that by (a), the γ_j 's remain in a part of \mathcal{M} with bounded curvature, and hence bounded geometry.

The weak lower semicontinuity of \mathcal{L} -length now guarantees that γ_∞ is a $W^{1,2}$ path from (x, t) to (x_0, t_0) with $\mathcal{L}(\gamma_\infty) \leq \liminf_{j \rightarrow \infty} \mathcal{L}(\gamma_j) = L(x, t)$. Hence γ_∞ is actually minimizing, and must therefore be piecewise smooth, which means that it is admissible. But then by our assumption, it follows that $\gamma_\infty \subset \mathcal{M}_{\text{reg}}$. \square

The fact that I.(7.15) can hold locally allows one to appeal – under appropriate conditions – to the maximum principle as in I.7.1 to prove that $\min l \leq \frac{3}{2}$ on every time slice:

Lemma 67.3. *Suppose \mathcal{M} is a Ricci flow with surgery defined on $[a, b]$, $t_0 \in [a, b]$, and $(x_0, t_0) \in \mathcal{M}_{t_0} \cap \mathcal{M}_{\text{reg}}$. Assume that there is an $\alpha > 0$ such that every barely admissible curve starting at a regular point and ending at (x_0, t_0) has reduced-length greater than $\frac{3}{2} + \alpha$. Then for all $t \in [a, t_0)$ there is a point $x \in \mathcal{M}_t$ where $l(x, t) \leq \frac{3}{2}$.*

Proof. We use the maximum principle as in I.7.1. For $\tau > 0$ let $\beta(\tau)$ be the minimum of $\bar{L} - 6\tau = 4\tau(l - \frac{3}{2})$ in the τ -slice (i.e. $\mathcal{M}_{t_0 - \tau}$). Then β is continuous and negative for small $\tau > 0$. Moreover, as long as $\beta(\tau) < 4\tau\alpha$, the minimum of β will be attained at a point (x, t) where $l < \frac{3}{2} + \alpha$; therefore any (\bar{x}, \bar{t}) close to (x, t) will be joined to (x_0, t_0) by a minimizing \mathcal{L} -geodesic by Lemma 67.2, and hence the differential inequality $\bar{L}_\tau + \Delta L \leq 6$ will hold, in the sense of barriers, at (x, t) . This implies that $D^+\beta(\tau) \leq 0$, where $D^+\beta$ denotes the upper right derivate. By a continuity argument, one then concludes that $\beta(\tau) \leq 0$ for all τ . \square

The notion of local collapsing can be adapted to Ricci flows with surgery, as follows.

Definition 67.4. *Suppose \mathcal{M} is a Ricci flow with surgery defined on $[a, b]$, $(x_0, t_0) \in \mathcal{M}$, $r > 0$, $t_0 - r^2 \geq a$, $B(x_0, t_0, r) \subset \mathcal{M}_{t_0}^-$ is a proper ball, and the parabolic ball $P(x_0, t_0, r, -r^2)$ is unscathed. Then \mathcal{M} is κ -collapsed at (x_0, t_0) at scale r if $|\text{Rm}| \leq r^{-2}$ on $P(x_0, t_0, r, -r^2)$ and $\text{vol}(B(x_0, t_0, r)) < \kappa r^3$; otherwise it is κ -noncollapsed.*

We make use of the following variant of the non-collapsing argument from I.7.3:

Lemma 67.5. *(Local version of reduced volume comparison) There is a function $\kappa' : \mathbb{R}_+ \rightarrow \mathbb{R}_+$, satisfying $\lim_{\kappa \rightarrow 0} \kappa'(\kappa) = 0$, with the following property. Let \mathcal{M} be a Ricci flow with surgery defined on $[a, b]$, $t_0 \in (a, b]$, $(x_0, t_0) \in \mathcal{M}_{t_0} \cap \mathcal{M}_{\text{reg}}$, $t \in [a, t_0)$, $0 < r^2 < t_0 - t$. Let Y be the set of points $(x, t) \in \mathcal{M}_t$ which are accessible from (x_0, t_0) by minimizing \mathcal{L} -geodesics which remain in \mathcal{M}_{reg} . Assume in addition that \mathcal{M} is κ -collapsed at (x_0, t_0) at scale r : $P(x_0, t_0, r, -r^2) \subset \mathcal{M}_{\text{reg}}$, $|\text{Rm}| \leq r^{-2}$ on $P(x_0, t_0, r, -r^2)$, and $\text{vol}(B(x_0, t_0, r)) < \kappa r^3$. Then the reduced volume of Y is at most $\kappa'(\kappa)$.*

Proof. Let $\hat{Y} \subset T_{x_0}\mathcal{M}_{t_0}$ be the set of vectors $v \in T_{x_0}\mathcal{M}_{t_0}$ such that there is a minimizing \mathcal{L} -geodesic $\gamma : [t, t_0] \rightarrow \mathcal{M}_{\text{reg}}$ running from (x_0, t_0) to some point in Y , with

$$\lim_{\bar{t} \rightarrow t_0} \sqrt{t_0 - \bar{t}} \dot{\gamma}(\bar{t}) = -v.$$

The calculations from I.7.1 apply to \mathcal{L} -geodesics sitting in \mathcal{M}_{reg} , and in particular the monotonicity of the weighted Jacobian $\tau^{-\frac{n}{2}} \exp(-l(\tau))J(\tau)$ holds. Now one repeats the proof of I.7.3, working with the set \hat{Y} instead of the set of initial velocities of *all* minimizing \mathcal{L} -geodesics. \square

68. II.5

Our presentation of this material follows Perelman's, except for some shuffling of the material. We will be using some terminology introduced in section 59.

A compact Riemannian 3-manifold is *normalized* if $|\text{Rm}| \leq 1$ everywhere, and the volume of every unit ball is at least half the volume of the Euclidean unit ball.

The main result of this section is Proposition 68.1 (cf. II.5.1), which implies that one can choose positive nonincreasing functions $r : \mathbb{R}_+ \rightarrow (0, \infty)$, $\delta : \mathbb{R}_+ \rightarrow (0, \infty)$ such that the Ricci flow with (r, δ) -surgery flow starting with any normalized initial condition will be defined for all time.

The actual statement is structured to facilitate a proof by induction:

Proposition 68.1. *(cf. Proposition II.5.1) There exist decreasing sequences $0 < r_j < \epsilon^2$, $\kappa_j > 0$, $0 < \bar{\delta}_j < \epsilon^2$ for $1 \leq j < \infty$, such that for any normalized initial data and any function $\delta : [0, \infty) \rightarrow (0, \infty)$ such that $\delta < \bar{\delta}_j$ on $[2^{j-1}\epsilon, 2^j\epsilon]$, the Ricci flow with (r, δ) -cutoff is defined for all time and is κ -noncollapsed at scales below ϵ .*

Proof. Here, and in the rest of this section, r and κ will always denote functions defined on an interval $[0, T] \subseteq [0, \infty)$ with the property that $r(t) = r_j$ and $\kappa(t) = \kappa_j$ for all $t \in [0, T] \cap [2^{j-1}\epsilon, 2^j\epsilon]$. Recall that ϵ is a "global" parameter which is assumed to be small, i.e. all statements involving ϵ (explicitly or otherwise) are true provided ϵ is sufficiently small.

We first give a sketch of the proof. SHOULD THIS GO IN AN OVERVIEW? We warn the reader that the sketch presents the steps of the argument in a somewhat different order than they appear in the detailed proof.

Given positive nonincreasing functions r and δ , if one has a normalized initial condition $(M, g(0))$, then there will be a maximal time interval on which the Ricci flow with (r, δ) -cutoff is defined. It is not difficult to see that this interval can be finite only if it is of the form $[0, T)$ for some $T < \infty$, and the Ricci flow with (r, δ) on $[0, T)$ extends to a Ricci flow with surgery on $[0, T]$ for which the r -canonical neighborhood assumption fails at time T , see Lemma 63.5; the main point here is that the r -canonical neighborhood assumption allows one to run the flow forward up to the singular time, and then perform surgery, while volume considerations rule out an accumulation of surgery times. Thus the crux of the proof is showing that the functions r and δ can be chosen so that the r -canonical neighborhood

assumption will continue to hold, and the Ricci flow with surgery satisfies a noncollapsing condition.

The strategy is to argue by induction on i that r_i , $\bar{\delta}_i$, and κ_i can be chosen (and $\bar{\delta}_{i-1}$ can be adjusted) so that the statement of the proposition holds on the finite time interval $[0, 2^i\epsilon]$. In the induction step, one establishes the canonical neighborhood assumption using an argument by contradiction similar to the proof of I.12.1 (we recommend that the reader review this before proceeding). The main difference between the proof of I.12.1 and that of Proposition II.5.1 is that the non-collapsing assumption, the key ingredient that allows one to implement the blow-up argument, is no longer available as a direct consequence of I.7.3, due to the presence of surgeries.

We now discuss the augmentations to the non-collapsing argument of I.7.3. Recall that it involves a collapsed parabolic ball $P(x_0, t_0, r_0, -r_0^2)$, the L -function with basepoint (x_0, t_0) , and the \mathcal{L} -exponential map based at (x_0, t_0) . One compares two estimates for the reduced volume of a suitably chosen time slice \mathcal{M}_t : the lower bound comes from the selection of a point where the reduced distance is at most $\frac{3}{2}$, which comes from an application of the maximum principle to the L -function; the upper bound comes from the monotonicity of the weighted Jacobian of the \mathcal{L} -exponential map. In fact, the second estimate works without significant modification provided one considers only those minimizing \mathcal{L} -geodesics which can be prolonged to the time t slice; this yields an upper bound for the reduced volume of their endpoints in the time t slice. To salvage the lower estimate, Perelman shows that by making the surgery parameter δ small, one can force the \mathcal{L} -length of any curve passing close to the surgery locus to be large (Lemma II.5.3). This implies that if (x, t) is a point where L isn't too large, then (x, t) will be the endpoint of a minimizing \mathcal{L} -geodesic from (x_0, t_0) , and therefore the calculations from I.7 will be valid near (x, t) ; the maximum principle can then be applied as before to show that the minimum of the reduced length is $\leq \frac{3}{2}$ on each time slice. To pass from this to a quantitative lower bound on the reduced volume, one uses a time t -slice for some t lying in the preceding time interval $[2^{i-1}\epsilon, 2^i\epsilon]$, where inductively one has some control on the geometry.

We remark that in II.5, ϵ plays the double role of both being the small universal constant of the paper II, and being a small time parameter. That is, it acquires units of length squared. This can be traced to the normalization that the solution at the initial time has sectional curvatures bounded above by one. It may be more natural to introduce a length space D , take the initial normalization to be that sectional curvatures are bounded above by D^{-2} , take the small time parameter to be ϵD^2 , etc. However, we will refrain from doing so.

We now begin the proof of Proposition 68.1 by establishing a noncollapsing condition.

Lemma 68.2. *(cf. Lemma II.5.2) Suppose positive sequences $\bar{\delta}_j$, κ_j , r_j have been defined for $1 \leq j \leq i$ and satisfy the proposition on the interval $[0, 2^i\epsilon]$. Then there exist $\hat{\kappa} > 0$ and $\hat{\delta} = \hat{\delta}(\hat{r}) > 0$, depending on the sequences above (and ϵ), with the following property. Given $T \in [2^i\epsilon, 2^{i+1}\epsilon]$ and $0 < \hat{r} < \epsilon^2$, suppose that $\delta, r : [0, T] \rightarrow (0, \infty)$ are functions satisfying*

1. $\delta \leq \bar{\delta}$ on $[0, 2^{i-1}\epsilon]$ and $\delta \leq \hat{\delta}$ on $[2^{i-1}\epsilon, T]$ (thus we are likely forcing δ to be still smaller on the preceding time interval $[2^{i-1}\epsilon, 2^i\epsilon]$),

2. $r \equiv r_j$ on $[2^{j-1}\epsilon, 2^j\epsilon]$ for all $1 \leq j \leq i$, and $r \equiv \hat{r}$ on $[2^i, T]$.

Then a Ricci flow with (r, δ) -cutoff flow on $[0, T]$ and a normalized initial condition is $\hat{\kappa}$ -noncollapsed on scales $< \epsilon$ on the interval $[2^i\epsilon, T]$. (Recall that the definition of noncollapsing involves unscathed parabolic neighborhoods; see Definition 67.4.)

Proof. Let δ , r , and \hat{r} , be as in the statement of the lemma, and \mathcal{M} be a Ricci flow with (r, δ) -cutoff with normalized initial condition, defined on $[0, T]$. Given $t_0 \in [2^i\epsilon, T]$, $(x_0, t_0) \in \mathcal{M}$ and $0 < r_0 < \epsilon$, suppose that $B(x_0, t_0, r_0)$ is a proper ball that is unscathed on the time interval $[t_0 - r_0^2, t_0]$, with $|\text{Rm}| \leq r_0^{-2}$ on $P(x_0, t_0, r_0, -r_0^2)$. We may assume that $r_0 \geq \hat{r}$, since otherwise the lower bound on $\text{vol}(B(x_0, t_0, r_0))$ follows from the canonical neighborhood assumption.

The first step is to show that by making the surgery parameter $\bar{\delta}$ small, one can make the \mathcal{L} -length of curves passing close to the surgery large. Let \mathcal{L}_+ denote the functional on the collection of admissible curves obtained from \mathcal{L} by replacing the scalar curvature R with the quantity $R_+ := \max(R, 0)$, and define the function L_+ on $\mathcal{M}_{[0, t_0]}$ by infimizing the \mathcal{L}_+ -lengths of admissible curves.

Sublemma 68.3. (cf. Lemma II.5.3) For any $\Lambda < \infty$ one can find $\delta' = \delta'(\Lambda, r_0)$ with the following property. WE NEED EITHER CONTINUOUS OR MONOTONE DEPENDENCE OF δ' ON r_0 . Suppose that $\hat{\delta} \leq \delta'(\Lambda, r_0)$ and $T_0 \in [2^{i-1}\epsilon, t_0]$ is a surgery time. Let $\gamma : [T_0, t_0] \rightarrow \mathcal{M}$ be an admissible curve from $B(p, T_0, \epsilon^{-1}h)$ to (x_0, t_0) , where p is the center of the cap inserted during surgery, and h is the radius of the δ -neck. Then

$$\mathcal{L}_+(\gamma) = \int_{T_0}^{t_0} \sqrt{t_0 - t} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt \geq \Lambda$$

where $R_+ := \max(R, 0)$.

Proof. Put $\Delta t := \epsilon r_0^4 \Lambda^{-2}$. We may assume that $\Delta t \leq r_0^2$, which will be the case when $\epsilon < 1$ and $\Lambda > r_0$.

Consider the behavior of γ near (x_0, t_0) , and set

$$\tilde{t} := \inf\{t \in [t_0 - \Delta t, t_0] \mid \gamma([t, t_0]) \subset P(x_0, t_0, r_0, \Delta t)\}.$$

If $\tilde{t} > t_0 - \Delta t$, then $\gamma(\tilde{t}) \in \partial P(x_0, t_0, r_0, \Delta t)$, so by the Schwarz inequality

$$\int_{\tilde{t}}^{t_0} \sqrt{t_0 - t} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt$$

is bounded below by

$$(68.4) \quad \int_{\tilde{t}}^{t_0} \sqrt{t_0 - t} |\dot{\gamma}(t)|^2 dt \geq \left(\int_{\tilde{t}}^{t_0} |\dot{\gamma}(t)| dt \right)^2 \left(\int_{\tilde{t}}^{t_0} (t_0 - t)^{-1/2} dt \right)^{-1} \\ \geq \text{const. } r_0^2 (\Delta t)^{-1/2} = \text{const. } \epsilon^{-\frac{1}{2}} \Lambda,$$

where the const. comes from the length distortion estimate arising from the fact that $|\text{Rm}| \leq r_0^{-2}$ on $P(x_0, t_0, r_0, -r_0^2)$. If $\epsilon \ll 1$ then this will be bounded below by Λ . Therefore we may assume that $\gamma([t_0 - \Delta t, t_0]) \subset P(x_0, t_0, r_0, -\Delta t)$.

We now apply Corollary II.4.6 with $l = (\Delta t)^{-\frac{1}{2}}\Lambda$ to obtain A and θ , and enlarge A if necessary to ensure that $\gamma(T_0) \in B(p, T_0, \epsilon^{-1}h) \subset B(p, T_0, \frac{Ah}{2})$. We then assume (as in the corollary) that $\hat{\delta} < \bar{\delta}(A, \theta)$, where $\bar{\delta}(A, \theta)$ denotes the quantity from Lemma II.4.5. Put

$$T_\gamma := \sup\{t \in [T_0, T_0 + \theta h^2] \mid \gamma([T_0, t]) \subset P(p, T_0, Ah, \theta h^2)\}.$$

One of the two conditions in the statement of Corollary II.4.6 must hold, and we conclude that

$$(68.5) \quad \int_{T_0}^{T_\gamma} (R_+ + |\dot{\gamma}|^2) dt > (\Delta t)^{-\frac{1}{2}}\Lambda.$$

When $\hat{\delta}$ is sufficiently small, by Lemma II.4.5 we know that the curvature on $P(p, T_0, Ah, \theta h^2)$ is $> \frac{h^{-2}}{10}$, while the curvature on $P(x_0, t_0, r_0, -r_0^2)$ is $\leq r_0^{-2} \leq \hat{r}^{-2} \ll h^{-2}$. Thus γ must arrive in $P(x_0, t_0, r_0, -r_0^2)$ after leaving $P(p, T_0, Ah, \theta h^2)$, i.e. $T_\gamma < t_0 - \Delta t$. Therefore

Hence

$$(68.6) \quad \begin{aligned} \int_{T_0}^{t_0} \sqrt{t_0 - t} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt &\geq \int_{T_0}^{T_\gamma} \sqrt{t_0 - t} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt \geq \\ (\Delta t)^{1/2} \int_{T_0}^{T_\gamma} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt &\geq (\Delta t)^{1/2} l = \Lambda. \end{aligned}$$

□

We continue with the proof of Lemma 68.2.

By the maximum principle and the fact that initial condition is normalized, we have $R \geq -6$; this implies that $\mathcal{L}_+ \leq \mathcal{L} + 4T^{\frac{3}{2}}$. Assume that

$$(68.7) \quad \hat{\delta} \leq \min\{\delta'(2\epsilon^{-2}T^{\frac{3}{2}}, \bar{r}) \mid \bar{r} \in [\hat{r}, \epsilon^2]\},$$

where $\delta'(\cdot, \cdot)$ is the function in Sublemma 68.3. If $t \in [2^{i-1}\epsilon, t_0)$, and $\gamma : [t, t_0] \rightarrow \mathcal{M}$ is a barely admissible curve starting at a regular point (x, t) , then there is a surgery time $T_0 \in (t, t_0)$ such that $\gamma(T_0)$ lies on the boundary of a cap inserted by surgery. Therefore by Sublemma 68.3, we have

$$(68.8) \quad \mathcal{L}(\gamma) \geq \mathcal{L}_+(\gamma) - 4T^{\frac{3}{2}} \geq \int_{T_0}^{t_0} \sqrt{t_0 - t} (R_+(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt - 4T^{\frac{3}{2}} \geq 2\epsilon^{-2}T^{\frac{3}{2}} - 4T^{\frac{3}{2}},$$

so its reduced length is

$$(68.9) \quad \frac{\mathcal{L}(\gamma)}{2\sqrt{t_0 - t}} \geq (\epsilon^{-2} - 4)T \geq (\epsilon^{-2} - 4)2^i\epsilon.$$

When ϵ is sufficiently small this is $\geq 2 = \frac{3}{2} + \frac{1}{2}$, so we may apply Lemmas 67.2 and 67.3 with $a = 2^{i-1}\epsilon$ and $\alpha = \frac{1}{2}$ to conclude that there is a point $(x', 2^{i-1}\epsilon) \in \mathcal{M}_{\text{reg}}$ where $l(x', 2^{i-1}\epsilon) \leq \frac{3}{2}$, and a minimizing \mathcal{L} -geodesic $\gamma_0 : [2^{i-1}\epsilon, t_0] \rightarrow \mathcal{M}_{\text{reg}}$ with $\gamma_0(2^{i-1}\epsilon) = (x', 2^{i-1}\epsilon)$.

Claim: There exists $\bar{t} \in [2^{i-1}\epsilon, \frac{3}{2}2^{i-1}\epsilon]$ such that $R(\gamma_0(\bar{t})) \leq r_i^{-2}$. Otherwise, we would have $R(\gamma_0(t)) > r_i^{-2}$ for all $t \in [2^{i-1}\epsilon, \frac{3}{2}2^{i-1}\epsilon]$, and then we would get

(68.10)

$$\begin{aligned} \mathcal{L}_+(\gamma_0) &\geq \int_{2^{i-1}\epsilon}^{\frac{3}{2}2^{i-1}\epsilon} \sqrt{2^i\epsilon - t} r_i^{-2} dt = r_i^{-2} \int_{2^{i-2}\epsilon}^{2^{i-1}\epsilon} \sqrt{\sigma} d\sigma = \frac{2}{3} (1 - 2^{-3/2}) (2^{i-1}\epsilon)^{\frac{3}{2}} r_i^{-2} \\ &\geq \frac{2}{3} (1 - 2^{-3/2}) (2^{i-1}\epsilon)^{\frac{3}{2}} \epsilon^{-4}, \end{aligned}$$

while on the other hand,

$$(68.11) \quad \begin{aligned} \mathcal{L}_+(\gamma_0) &\leq \mathcal{L}(\gamma_0) + 4T^{\frac{3}{2}} = l(\gamma_0(2^{i-1}\epsilon), 2^{i-1}\epsilon) (2\sqrt{t_0 - 2^{i-1}\epsilon}) + 4T^{\frac{3}{2}} \\ &\leq 3 \left((2^{i+1}\epsilon)^{\frac{1}{2}} + 4(2^{i+1}\epsilon)^{\frac{3}{2}} \right). \end{aligned}$$

When ϵ is sufficiently small this gives a contradiction, proving the claim.

Put $\bar{x} := \gamma_0(\bar{t})$. We now estimate $l(\bar{x}, \bar{t})$. First,

$$(68.12) \quad \begin{aligned} \mathcal{L}(\gamma_0|_{[\bar{t}, t_0]}) &= \mathcal{L}(\gamma_0|_{[2^{i-1}\epsilon, t_0]}) - \int_{2^{i-1}\epsilon}^{\bar{t}} \sqrt{t_0 - t} (R(\gamma(t), t) + |\dot{\gamma}(t)|^2) dt \\ &\leq 3 (2^{i+1}\epsilon)^{1/2} + 4(2^{i+1}\epsilon)^{3/2}. \end{aligned}$$

Then

$$(68.13) \quad l(\bar{x}, \bar{t}) = \frac{L(\bar{x}, \bar{t})}{2\sqrt{t_0 - \bar{t}}} \leq \frac{3 (2^{i+1}\epsilon)^{1/2} + 4(2^{i+1}\epsilon)^{3/2}}{2 (2^{i-2}\epsilon)^{1/2}}.$$

Applying Claim 1 of II.4.2 at (\bar{x}, \bar{t}) we have $Q \leq 2r_i^{-2}$, and hence the scalar curvature is at most $16r_i^{-2}$ on the parabolic ball $\hat{P} := P(\bar{x}, \bar{t}, \alpha r_i, -\alpha r_i^2)$, where $0 < \alpha < 1$ is universal. This means that $\hat{P} \subset \mathcal{M}_{\text{reg}}$, since a singular point would have curvature comparable to $h^{-2} \gg r_i^{-2}$. Let $Y := \hat{P} \cap \mathcal{M}_{\bar{t} - \alpha r_i^2}$. By the noncollapsing assumption and curvature bounds on \hat{P} , we therefore have $\text{vol}(Y) \geq v_i$, where v_i depends on κ_i , ϵ , and r_i . Every point $(x, t) \in Y$ can be joined to (\bar{x}, \bar{t}) by an admissible curve Γ along which the \mathcal{L} -length (computed with the basepoint at (x_0, t_0)) is at most $c_1 = c_1(i)$. Concatenating this with $\gamma_0|_{[\bar{t}, t_0]}$, we can get an admissible curve from any $(x, t) \in Y$ to (x_0, t_0) with \mathcal{L} -length at most $c_2 = c_2(i)$. By making $\hat{\delta}$ sufficiently small, we can use Sublemma 68.3 to guarantee that any barely admissible curve from any point in Y to (x_0, t_0) has \mathcal{L} -length at least $c_2 + 1$. Lemma 67.2 implies that every $(x, t) \in Y$ can be joined to (x_0, t_0) by a minimizing \mathcal{L} -geodesic. Finally, Lemma 67.5 gives the desired noncollapsing estimate at (x_0, t_0) at scale r_0 . This completes the proof of Lemma 68.2 \square

Remark 68.14. The argument in II to prove Lemma II.5.2 constructs a point (\bar{x}, \bar{t}) as above, but with $\bar{t} \in [2^{i-1}\epsilon, 2^i\epsilon]$. This seems to be inadequate to estimate $l(\bar{x}, \bar{t})$.

Proof of Proposition 68.1. The proof is by induction on i . To start the induction process, the initial normalization $|\text{Rm}| \leq 1$ at $t = 0$, along with the maximum principle, implies that a smooth solution exists for some definite time. From I.7.3, the solution is κ -noncollapsed at

some scale, within this time interval. From I.12.1, the ρ -canonical neighborhood assumption holds in this time interval, for some $\rho > 0$.

Now assume inductively that r_j , κ_j , and $\bar{\delta}_j$ have been selected for $1 \leq j \leq i$, thereby defining the functions r , κ , and $\bar{\delta}$ on $[0, 2^i\epsilon]$, such that for any function δ on $[0, 2^i\epsilon]$ satisfying $0 < \delta(t) \leq \bar{\delta}(t)$, if one has normalized initial data then the Ricci flow with (r, δ) -cutoff is defined on $[0, 2^i\epsilon]$ and is $\kappa(t)$ -noncollapsed at scales $< \epsilon$.

Suppose it is not possible to choose r_{i+1} , κ_{i+1} , and $\bar{\delta}_{i+1}$ (and, if necessary, make $\bar{\delta}_i$ smaller), such that the statement above holds with i replaced by $i + 1$. Then there are sequences $r^\alpha \rightarrow 0$ and $\bar{\delta}^{\alpha\beta} \rightarrow 0$ so that for each (α, β) there is a counterexample to the $(r^\alpha, \bar{\delta}^{\alpha\beta})$ -statement. The relevance of having $\bar{\delta}^{\alpha\beta} \rightarrow 0$ is that for any $A > 0$ and $\theta \in (0, 1)$, for large β we will have $\bar{\delta}^{\alpha\beta} \leq \hat{\delta}(A, \theta, r^\alpha)$, where $\hat{\delta}$ is the quantity coming from Lemma 68.2. So we may fix a sequence $A^\alpha \rightarrow \infty$ and just assume that we have sequences $r^\alpha \rightarrow 0$ and $\bar{\delta}^\alpha \rightarrow 0$ so that $\bar{\delta}^\alpha \leq \hat{\delta}(A^\alpha, 1 - (A^\alpha)^{-1}, r^\alpha)$ and for each α , there is a counterexample $(M^\alpha, g^\alpha(0))$ to the $(r^\alpha, \bar{\delta}^\alpha)$ -statement. By Lemma 68.2 and Lemma 63.5, each counterexample must prolong to a Ricci flow with surgery \mathcal{M}^α defined on a time interval $[0, T^\alpha]$ with $T^\alpha \in [2^i\epsilon, 2^{i+1}\epsilon]$, which restricts to a Ricci flow with $(r^\alpha, \bar{\delta}^\alpha)$ -cutoff on any proper subinterval $[0, \tau]$ of $[0, T^\alpha]$, but for which the r^α -canonical neighborhood assumption fails at some point $(\bar{x}^\alpha, T^\alpha)$ lying in the backward time slice $\mathcal{M}_{T^\alpha}^-$.

Let $(\widehat{\mathcal{M}}^\alpha, (\bar{x}^\alpha, 0))$ be the pointed Ricci flow with surgery obtained from $(\mathcal{M}^\alpha, (\bar{x}^\alpha, T^\alpha))$ by shifting time by T^α and parabolically rescaling by $R(\bar{x}^\alpha, T^\alpha)$. In brief, the rest of the proof goes as follows. If surgeries occur further and further away from $(\bar{x}^\alpha, 0)$ in spacetime as $\alpha \rightarrow \infty$, then the reasoning of I.12.1 applies and we obtain a κ -solution as a limit. This would contradict the fact that $(\bar{x}^\alpha, T^\alpha)$ does not have a canonical neighborhood. Thus there must be surgeries in a parabolic ball of a fixed size centered at $(\bar{x}^\alpha, 0)$, for arbitrarily large α . Then one argues using II.4.5 that the solution will be close to the (suitably rescaled and time-shifted) standard solution, which again leads to a canonical neighborhood and a contradiction.

Let $\mathcal{T} \subset (-\infty, 0]$ be the set of numbers τ' such that for all $\lambda < \infty$, the ball $B(\bar{x}^\alpha, 0, \lambda) \subset \widehat{\mathcal{M}}_0^\alpha$ is proper and unscathed on $[\tau', 0]$ for sufficiently large α . Then the arguments of I.12.1 apply in the time interval \mathcal{T} , so that after passing to a subsequence if necessary, the pointed flows $(\widehat{\mathcal{M}}^\alpha, (\bar{x}^\alpha, 0))$ converge on the time interval \mathcal{T} to a Ricci flow (without surgery) $(\mathcal{M}^\infty, (\bar{x}^\infty, 0))$. It will necessarily have nonnegative curvature and scalar curvature globally bounded by some number $Q < \infty$. If we can show that $\mathcal{T} = (-\infty, 0]$ then $(\mathcal{M}^\infty, (\bar{x}^\infty, 0))$ will be a κ -solution, which will contradict the assumption that $(\bar{x}^\alpha, T^\alpha)$ does not admit a canonical neighborhood.

Suppose that $\mathcal{T} \neq (-\infty, 0]$. We know that for all $\tau' \in \mathcal{T}$ and $\lambda < \infty$, the scalar curvature in $P(\bar{x}^\alpha, 0, \lambda, \tau')$ is bounded by $Q + 1$ when α is sufficiently large. By Claim 1 of II.4.2, there exists $\sigma < \inf \mathcal{T}$ such that for all $\lambda < \infty$, if (for large α) the solution $\widehat{\mathcal{M}}^\alpha$ is unscathed on $P(\bar{x}^\alpha, 0, \lambda, t^\alpha)$ for some $t^\alpha > \sigma$ then

$$(68.15) \quad R(x, t) < 2(Q + 1) \quad \text{for all } (x, t) \in P(\bar{x}^\alpha, 0, \lambda, t^\alpha).$$

By the definition of \mathcal{T} , and after passing to a subsequence if necessary, there exist $\lambda < \infty$ and a sequence $\gamma^\alpha : [\sigma^\alpha, 0] \rightarrow \widehat{\mathcal{M}}^\alpha$ of static curves so that

1. $\gamma^\alpha(0) \in B(\bar{x}^\alpha, 0, \lambda)$ and
2. The point $\gamma^\alpha(\sigma^\alpha)$ is inserted during surgery at time $\sigma^\alpha > \sigma$.

For each α , we may assume that σ^α is the largest number having this property. Put $\xi^\alpha := \sigma^\alpha + (h^\alpha(\sigma^\alpha))^2$. (In the notation of II, $(h^\alpha(\sigma^\alpha))^2$ would be written as $R(\bar{x}, \bar{t}) h^2(T_0)$. Note that we have no a priori control on $h^\alpha(\sigma^\alpha)$.) Then ξ^α is the blow-up time of the rescaled and shifted standard solution that Lemma II.4.5 compares with $(\widehat{\mathcal{M}}^\alpha, \gamma^\alpha(\sigma^\alpha))$. We claim that $\liminf_{\alpha \rightarrow \infty} \xi^\alpha > 0$. Otherwise, Lemma II.4.5 would imply that after passing to a subsequence, there are regions of $\widehat{\mathcal{M}}^\alpha$, starting from time σ^α , that are better and better approximated by rescaled and shifted standard solutions whose blowup times ξ^α have a limit that is nonpositive, thereby contradicting (68.15). Lemma II.4.5, along with the fact that $R(\bar{x}^\alpha, 0) = 1$, also gives a uniform upper bound on ξ^α .

Now Lemma II.4.5 implies that for large α , the restriction of $\widehat{\mathcal{M}}^\alpha$ to the time interval $[\sigma^\alpha, 0]$ is well approximated by the restriction to $[\sigma^\alpha, 0]$ of a rescaled and shifted standard solution. Then Claim 5 of II.2 implies that $(\bar{x}^\alpha, T^\alpha)$ has a canonical neighborhood, which is a contradiction. \square

69. II.6.3

For II.6.3(a) : After rescaling to $r_0 = 1$, the proof of I.8.2 used the maximum principle to argue that $h(y, t) = \phi(d(y, t) - A(2t - 1))(\bar{L}(y, 1 - t) + 2n + 1)$ has a minimum that satisfies $(\min h)(t) \leq (2n + 1) e^{(2n+C(A))(1-t)}$. Here $\bar{L}(q, \tau) = 2\sqrt{\tau} L(q, \tau)$ and $\phi \geq 1$ is a certain function. If one knew that the possible contribution of a barely admissible curve to $h(y, t)$ was greater than $(2n + 1) e^{(2n+C(A))(1-t)} + \epsilon$ then one could still apply the maximum principle. For this, it suffices to know that the possible contribution of a barely admissible curve to $\bar{L}(q, \tau)$ can be bounded below by a sufficiently large number. However, Lemma II.5.3 only says that we can make the contribution of a barely admissible curve to L large. Because of the factor $2\sqrt{\tau}$ in the definition of $\bar{L}(q, \tau)$, we cannot necessarily say that its contribution to $\bar{L}(q, \tau)$ is large.

After rescaling so that $r_0 = 1$, the time interval $[t_0 - r_0^2, t_0]$ is shifted to $[0, 1]$. By assumption, the sectional curvatures are bounded below by -1 at $t = 0$. The maximum principle implies that $R \geq -3$ for $t \in [\frac{1}{2}, 1]$; see II.(7.2). From this, if $0 < \tau \leq \frac{1}{2}$ then $\bar{L}(y, \tau) \geq -6\sqrt{\tau} \int_0^\tau \sqrt{v} dv = -4\tau^2$, so $\widehat{L}(y, \tau) > 0$.

Using the fact that $\frac{d}{dt}\sqrt{\tau} = -\frac{d}{d\tau}\sqrt{\tau} = -\frac{1}{2\sqrt{\tau}}$, the computations of I.8.2 give

$$\begin{aligned}
 (69.1) \quad \square h &\geq -(\bar{L} + 2\sqrt{\tau}) C(A) \phi - 6\phi - \frac{1}{\sqrt{\tau}} \phi \\
 &= -C(A)h - \left(6 + \frac{1}{\sqrt{\tau}}\right) \phi.
 \end{aligned}$$

Then

$$\begin{aligned}
 (69.2) \quad \frac{d}{d\tau} \left(\log \left(\frac{h_0(\tau)}{\sqrt{\tau}} \right) \right) &= h_0^{-1} \frac{dh_0}{d\tau} - \frac{1}{2\tau} \leq C(A) + \left(6 + \frac{1}{\sqrt{\tau}} \right) \frac{\phi}{h_0} - \frac{1}{2\tau} \\
 &= C(A) + \left(6 + \frac{1}{\sqrt{\tau}} \right) \frac{1}{\bar{L} + 2\sqrt{\tau}} - \frac{1}{2\tau} \\
 &= C(A) + \frac{6\sqrt{\tau} + 1}{\sqrt{\tau}\bar{L} + 2\tau} - \frac{1}{2\tau}.
 \end{aligned}$$

As $\bar{L} \geq -4\tau^2$,

$$(69.3) \quad \frac{d}{d\tau} \left(\log \left(\frac{h_0(\tau)}{\sqrt{\tau}} \right) \right) \leq C(A) + \frac{6\sqrt{\tau} + 1}{2\tau - 4\tau^2\sqrt{\tau}} - \frac{1}{2\tau} \leq C(A) + \frac{50}{\sqrt{\tau}}.$$

As $\tau \rightarrow 0$, the Euclidean space computation gives $\bar{L}(q, \tau) \sim |q|^2$, so $\lim_{\tau \rightarrow 0} \frac{h_0(\tau)}{\sqrt{\tau}} = 2$. Then

$$(69.4) \quad h_0(\tau) \leq 2\sqrt{\tau} \exp(C(A)\tau + 100\sqrt{\tau}).$$

It now suffices to show that for a barely admissible curve γ ,

$$(69.5) \quad \int_0^\tau \sqrt{v} (R(\gamma(1-v), v) + |\dot{\gamma}(v)|^2) dv \geq \exp(C(A)\tau + 100\sqrt{\tau}) + \epsilon,$$

where $0 < \tau \leq \frac{1}{2}$ and $\gamma(1-\tau) \in B(p, 1-\tau, \epsilon^{-1}h)$. This follows from Lemma II.5.3. Then we can apply the maximum principle and follow the proof of I.8.2.

The proof of II.6.3(b) is similar to the proof of I.12.2 in section 46, and the proof of II.5.4.

To prove II.6.3(c), we can rescale t_0 to 1 and then apply Claim 2 of II.4.2; see the proof of Theorem I.12.2 in section 46. We note that II.6.3(c) is an analog of Theorem I.12.2, but the hypotheses are slightly different. In II.6.3 one assumes a lower bound on the volume of the time- t_0 ball $B(x_0, t_0, r_0)$, while in I.12.2 one assumes a lower bound on the volume of the time- $(t_0 - r_0^2)$ ball $B(x_0, t_0 - r_0^2, r_0)$. In view of the curvature assumption on $P(x_0, t_0, r_0, -r_0^2)$, the hypotheses are equivalent.

70. II.6.5

At the end of I.11.6, a statement similar to Lemma II.6.5 is proven with the hypotheses that the sectional curvature is at least -1 on the family of time-dependent balls $\{(x, t) : t \in [-\tau, 0] \text{ and } \text{dist}_t(x_0, x) \leq 1\}$, and the volume of $B(x_0, 0, 1)$ is bounded below in terms of the volume of the corresponding Euclidean ball. The conclusion is then that $R(x, t) \leq K_0\tau^{-1}$ if $t \in [-\tau/2, 0]$ and $\text{dist}_t(x_0, x) \leq \frac{1}{2}$. The fact that the volume of the ball $B(x_0, \frac{1}{4}, -\tau)$ is bounded below in terms of the volume of the corresponding Euclidean ball, follows from the proof of I.11.6(b).

The hypothesis of Lemma II.6.5 instead has a lower sectional curvature bound on the parabolic neighborhood $P(x_0, 0, 1, -\tau)$. If one had nonnegative sectional curvature on this neighborhood then one would immediately get the same statement on the family

of time-dependent balls. With a lower sectional curvature bound of -1 , if τ_0 is sufficiently small then one obtains the same lower bound on a family of slightly smaller time-dependent balls. Applying Theorem I.11.6(b), one obtains an upper curvature bound on $\{(x, t) : t \in [-\tau/2, 0] \text{ and } \text{dist}_t(x_0, x) \leq \frac{1}{2}\}$. Then if τ_0 is small enough, one can apply the length distortion estimate of Lemma I.8.3(b) to get an upper scalar curvature bound on $P(x_0, 0, \frac{1}{4}, -\tau/2)$ as stated in Lemma II.6.5(a). The conclusion of Lemma II.6.5(b) again follows from the proof of Theorem I.11.6(b).

71. II.6.6

The proof is similar to what appears in the proof of Corollary I.12.3; see section 47. Suppose that the lemma is not true. Then there is a sequence of Riemannian manifolds $\{M_i\}_{i=1}^\infty$ with balls $B(x_i, 1) \subset \overline{B(x_i, 1)} \subset M_i$ so that $\text{Rm}|_{B(x_i, 1)} \geq -1$ and $\text{vol}(B(x_i, 1)) \geq w$, along with a sequence $r'_i \rightarrow 0$ so that each subball $B(x'_i, r'_i) \subset B(x_i, 1)$ has a subball $B(x''_i, r''_i) \subset B(x'_i, r'_i)$ with $\text{vol}(B(x''_i, r''_i)) < (1 - \epsilon) \omega_n (r''_i)^n$. After taking a subsequence, we can assume that $\lim_{i \rightarrow \infty} (B(x_i, 1), x_i) = (X, x_\infty)$ in the pointed Gromov-Hausdorff topology, where (X, x_∞) is a pointed Alexandrov space with curvature bounded below by -1 . From [5, Theorem 10.8], the Riemannian volume forms $d \text{vol}_{M_i}$ converge weakly to the three-dimensional Hausdorff measure μ of X . From [5, Corollary 6.7 and Section 9], for every small $\delta > 0$, there is a ball $B(x'_\infty, \delta) \subset X$ with $\mu(B(x'_\infty, \delta)) \geq (1 - \frac{\epsilon}{10}) \omega_n \delta^n$. Then for large i , there is a ball $B(y_i, \delta_i) \subset M_i$ with $\frac{\delta}{2} \leq \delta_i \leq 2\delta$ and $\text{vol}(B(y_i, \delta_i)) \geq (1 - \frac{\epsilon}{5}) \omega_n \delta_i^n$. Given $\sigma > 0$, for large i , take a ball $B(x''_i, r''_i) \subset B(y_i, \delta_i)$ with $d_{M_i}(x''_i, y_i) < \sigma$, $r''_i < \sigma$ and $\text{vol}(B(x''_i, r''_i)) < (1 - \epsilon) \omega_n (r''_i)^n$. Clearly $\text{vol}(B(y_i, \delta_i)) \leq \text{vol}(B(x''_i, \delta_i + \sigma))$. From the Bishop-Gromov inequality,

$$(71.1) \quad \frac{\text{vol}(B(x''_i, \delta_i + \sigma))}{\text{vol}(B(x''_i, r''_i))} \leq \frac{\int_0^{\delta_i + \sigma} \sinh^2(s) ds}{\int_0^{r''_i} \sinh^2(s) ds}.$$

Taking $\sigma \rightarrow 0$ gives

$$(71.2) \quad \text{vol}(B(x''_i, \delta_i)) \leq (1 - \epsilon) \omega_n \int_0^{\delta_i} \sinh^2(s) ds.$$

Taking $\delta \rightarrow 0$ then gives a subsequence of the (y_i, δ_i) 's so that $\text{vol}(B(y_i, \delta_i)) \leq (1 - \frac{\epsilon}{2}) \omega_n \delta_i^n$, which is a contradiction.

72. II.6.4

Proposition II.6.4 (along with Corollary II.6.8) is an analog of Theorem I.12.3. However, the proof of II.6.4 is more complicated, due to the need to deal with possible surgeries.

- The main goal is to obtain an upper scalar curvature bound. The idea of the proof is to
1. Restrict to a subball on which the proposition is true.
 2. Go backwards in time using the proposition on the subball.
 3. Use Proposition II.6.3(c) to get a (large) upper scalar curvature bound on a parabolic neighborhood that includes the original ball.
 4. Use the ϕ -almost nonnegative curvature to get a standard lower sectional curvature

bound on the parabolic neighborhood.

5. Go to the beginning time slice of the parabolic neighborhood and repeat the process.

One does this until one goes far enough back in time to apply Lemma II.6.5(a) which, under the assumption of a lower sectional curvature bound, gives the desired upper scalar curvature bound.

The numbers C_1 , K and τ are fixed, but the requirements on them will be specified during the proof. In the conclusion of the theorem, the statement “the solution is defined in $P(x_0, t_0, r_0/4, -\tau r_0^2)$ ” means that the solution is smooth in that parabolic region, i.e. no surgeries occur.

If $r_0 \leq r(t_0)$ and $R(x, t_0) > r_0^{-2}$ for some $x \in B(x_0, t_0, \frac{r_0}{2})$ then $R(x, t_0) > r(t_0)^{-2}$. This implies that $R(x, t_0)$ is in a canonical neighborhood, which contradicts the almost-Euclidean-volume assumption on subballs of $B(x_0, t_0, r_0)$. Thus $R(x, t_0) \leq r_0^{-2}$. The bound II.(1.3) on a canonical neighborhood implies that for some small $\epsilon > 0$, $R \leq 2 r_0^{-2}$ on $P(x_0, t_0, \frac{1}{4}r_0, -\epsilon\eta^{-1}r_0^2)$. Furthermore, if

(*.1) $C_1 \geq 100$

then $R \leq \frac{1}{2h^2}$ on $P(x_0, t_0, r_0/4, -\tau r_0^2)$. As surgeries only occur when $R \geq h^{-2}$, there cannot be any surgeries in the region. Hence if we have

(*.2) $K \geq 200$ and

(*.3) $\tau \leq \epsilon\eta^{-1}$

then there are no counterexamples to the proposition with $r_0 \leq r(t_0)$.

Continuing with II.6.7, we have a sequence of solutions M^α satisfying the assumptions of the proposition, with $\bar{r}^\alpha \rightarrow 0$, so that the conclusion of the proposition is violated for M^α . Let t_0^α be the first time when the conclusion is violated for M^α and let $B(x_0^\alpha, t_0^\alpha, r_0^\alpha)$ be a time- t_0^α ball of smallest radius which provides a counterexample. That is, either there is a surgery in $P(x_0^\alpha, t_0^\alpha, r_0^\alpha/4, -\tau(r_0^\alpha)^2)$ or $R \geq K(r_0^\alpha)^{-2}$ somewhere on $P(x_0^\alpha, t_0^\alpha, r_0^\alpha/4, -\tau(r_0^\alpha)^2)$. From the previous paragraph, $r_0^\alpha \geq r(t_0^\alpha)$. Also, $(r_0^\alpha)^2(t_0^\alpha)^{-1} = (\bar{r}^\alpha)^2 \rightarrow 0$.

Removing the α superscripts, put $\hat{t} = t_0$.

The iterative procedure is now the following. Find a ball $B(x_1, \hat{t}, r) \subset B(x_0, \hat{t}, r_0)$ so that $B(x_1, \hat{t}, r)$ satisfies the assumptions of the proposition, where $r = \theta_0(\frac{1}{10})\frac{r_0}{4} < r_0$. Here θ_0 is the function of Lemma II.6.6. More precisely, we are assuming that $B(x_1, \hat{t}, r)$ has sectional curvatures at least $-r^{-2}$ at each point, and the volume of any subball $B(x', \hat{t}, r') \subset B(x_1, \hat{t}, r)$ has volume at least $(1 - \epsilon)$ times the volume of the Euclidean ball of the same radius. (For the first iteration, we can take any radius- r subball of $B(x_0, \hat{t}, r_0)$.)

The conclusion of the proposition holds for $B(x_1, \hat{t}, r)$. Thus there is a smooth solution in $P(x_1, \hat{t}, \frac{r}{4}, -\tau r^2)$ with the curvature bound $R < Kr^{-2}$.

In the proof, we will only need to consider values of \hat{t} in $[t_0 - \tau_0 r_0^2, t_0]$, where τ_0 is the constant from Lemma II.6.5. Then

$$(72.1) \quad r_0^2 \hat{t}^{-1} \leq \frac{r_0^2}{t_0 - \tau_0 r_0^2} = \frac{1}{(r_0^2 t_0^{-1})^{-1} - \tau_0}.$$

Taking α large enough, $r_0^2 t_0^{-1}$ can be made sufficiently small so that $r_0^2 \hat{t}^{-1}$ is small enough to apply Proposition II.6.3(c).

We denote by \tilde{r}_0 and \tilde{t}_0 the parameters of Proposition II.6.3(c). Take $\tilde{t}_0 \in [\hat{t} - \tau r^2/2, \hat{t}]$ and $\tilde{r}_0 = K^{-1/2} r$. From the above statement about $P(x_1, \hat{t}, \frac{r}{4}, -\tau r^2)$, we have $|\text{Rm}| \leq \tilde{r}_0^{-2}$ on $P(x_1, \tilde{t}_0, \tilde{r}_0, -\tilde{r}_0^2)$ if

$$(*.4) \quad K^{-1} = \frac{1}{2}\tau.$$

Applying length and volume distortion estimates on $P(x_1, \hat{t}, \frac{r}{4}, -\tau r^2)$, we will also have $\text{vol}(B(x_1, \tilde{t}_0, \tilde{r}_0)) \leq A_0^{-1} \tilde{r}_0^3$ for some universal constant $A_0 > 0$. Then Proposition II.6.3(c) implies that $R(x, t) \leq K_2(A) \tilde{r}_0^{-2} = K_2(A) K r^{-2} \equiv K' r^{-2}$ for (x, t) satisfying $t \in [\hat{t} - \tau r^2/2, \hat{t}]$ and $\text{dist}_t(x, x_1) \leq A r_0$. Take $A = 100 r_0 r^{-1}$.

In order to apply Lemma II.6.5(b), we will want to know that the solution in $\{(x, t) : t \in [\hat{t} - \tau r^2/2, \hat{t}] \text{ and } \text{dist}_t(x, x_0) \leq r_0\}$ is smooth. For this, it suffices to have $K'(A) r^{-2} \leq \frac{1}{4} h^{-2}$, as surgeries only occur where $R = h^{-2}$. This is equivalent to $r_0 \geq 2 (K'(A))^{\frac{1}{2}} \frac{r_0}{r} h$, so we want

$$(*.5) \quad C_1 \geq (K'(A))^{\frac{1}{2}} \frac{r_0}{r}.$$

The pinching estimate implies that if $t \in [\hat{t} - \tau r^2/2, \hat{t}]$ and $\text{dist}_t(x, x_1) \leq r_0$ then

$$(72.2) \quad \text{Rm} \geq -K'(A) r^{-2} \phi(K'(A) r^{-2} (1 + \hat{t})) = -K'(A) (r/r_0)^{-2} r_0^{-2} \phi(K'(A) (r/r_0)^{-2} r_0^{-2} (1 + \hat{t})).$$

As $\frac{r}{r_0}$ is fixed, if $r_0^2 \hat{t}^{-1}$ is small enough then $\text{Rm} \geq -r_0^{-2}$ on this region. By Lemma II.6.5(b), the volume of $B(x_0, \hat{t} - \tau r^2/2, \frac{r_0}{4})$ is at least $\frac{1}{10}$ of the volume of the Euclidean ball of the same radius.

By Lemma II.6.6, there is a subball $B(x_2, \hat{t} - \tau r^2/2, r) \subset B(x_0, \hat{t} - \tau r^2/2, r_0)$ so that the assumptions of the proposition are satisfied on $B(x_2, \hat{t} - \tau r^2/2, r)$. This means that we have the setup to repeat the iterative procedure, replacing \hat{t} by $\hat{t} - \tau r^2/2$. We can repeat the iteration down to time $t_0 - \tau_0 r_0^2$, at which point Lemma II.6.5(b) stops working. We know that $\text{Rm} \geq -r_0^{-2}$ whenever $t \in [t_0 - \tau_0 r_0^2, t_0]$ and $x \in B_t(x_0, r_0)$. We can now apply Lemma II.6.5(a), in the version stated in Section 70 for a family of time-dependent balls, to obtain $R \leq K_0 \tau_0^{-1} r_0^{-2}$ on $B_{t_0}(x_0, r_0)$. This is a contradiction if

$$(*.6) \quad K > K_0 \tau_0^{-1}.$$

Looking at the conditions (*), K and τ are related by (*.4). We can choose K large enough so that (*.2), (*.3) and (*.6) are satisfied. Then C_1 can be taken large enough so that (*.1) and (*.5) are satisfied.

73. II.7.1

For the statement on the bottom of page 17 of II, suppose that $\bar{V} > 0$. From II.(7.2), $\bar{R} \bar{V}^{-2/3} \geq -\frac{3}{2}$. Suppose that $\bar{R} \bar{V}^{-2/3} = c > -\frac{3}{2}$. Then for large t , $R_{\min}(t) \sim \frac{c}{t}$. This would imply from II.(7.3) that for any $\epsilon > 0$, $V(t) \leq \text{const. } t^{\epsilon-c}$ whenever t is sufficiently large. Then $V(t) t^{-3/2} \leq \text{const. } t^{-(c+3/2-\epsilon)}$. If $\epsilon = \frac{1}{2}(c + \frac{3}{2})$, this contradicts the assumption that $\bar{V} \geq 0$.

For the statement on the top of page 18 of II, suppose that the limit solution on $P(\bar{x}, 1, r, -r^2)$ has $\int_{B(\bar{x}, \tilde{t}, r)} (R'_{\min}(\tilde{t}) - R(x, \tilde{t})) dV \leq c < 0$ whenever $\tilde{t} \in (t_0 - \epsilon, t_0 + \epsilon) \subset [1 - r^2, 1]$, where

$R'_{min}(\tilde{t})$ is the minimum of R over $B(\bar{x}, \tilde{t}, r)$. Then for large α ,

$$(73.1) \quad \int_{B(x^\alpha, \tilde{t}t^\alpha, r\sqrt{t^\alpha})} (R'_{min}(\tilde{t}t^\alpha) - R(x, \tilde{t}t^\alpha)) dV < \frac{c}{2} \sqrt{t^\alpha},$$

where R'_{min} is now the minimum of R over $B(x^\alpha, \tilde{t}t^\alpha, r\sqrt{t^\alpha})$. Then

$$(73.2) \quad \int_{B(x^\alpha, \tilde{t}t^\alpha, r\sqrt{t^\alpha})} (R_{min}(\tilde{t}t^\alpha) - R(x, \tilde{t}t^\alpha)) dV < \frac{c}{2} \sqrt{t^\alpha}.$$

This will give positive contributions on the order of $\frac{1}{t}$ to the right-hand-side of II.(7.4), which contradicts the fact that \hat{R} has a limit. Thus R is spatially constant on $P(\bar{x}, 1, r, -r^2)$ and so must equal $-\frac{3}{2t}$. Then from II.(7.1), the sectional curvature on $P(\bar{x}, 1, r, -r^2)$ is $-\frac{1}{4t}$.

74. II.7.2

We note that in the presence of surgeries, away from a surgery time we still have $\frac{d}{dt}R_{min} \geq \frac{2}{3}R_{min}^2$. A surgery cannot change R_{min} , so II.(7.2) is still valid. Similarly, if $R_{min} \leq 0$ then away from a surgery time we still have $\frac{d}{dt}\hat{R}(t) \geq 0$. A surgery will decrease the volume, so $\hat{R}(t)$ will still be nondecreasing.

In the hypotheses of Lemma II.7.2, the lower bound on the volume of the ball should be $wr^3t_0^{-3/2}$. (Note that r is dimensionless.) The conclusion is that the rescaled curvature is close to constant, i.e.

$$(74.1) \quad (2tR_{ij}(x_0, t_0) + g_{ij})(2tR^{ij}(x_0, t_0) + g^{ij}) < \xi.$$

To prove part (a), suppose that there is a sequence of points (x_0, t_0) that provide a counterexample. If $r > \bar{r}(w)$ then the hypotheses will still be satisfied upon replacing r by $\bar{r}(w)$, after possibly changing w . Rescaling by t_0 , taking a limit (using Corollary II.6.8) and applying II.7.1 gives a contradiction.

For part (b), we apply Proposition II.6.3 to the parabolic neighborhood $P(x_0, t_0, r'_0, -(r'_0)^2)$ where $Kr_0^{-2} = (r'_0)^{-2}$. By II.6.3(b), each point in $B(x_0, t_0, Ar\sqrt{t_0})$ with scalar curvature at least $Q = K'(A)r_0^{-2}$ has a canonical neighborhood. If there were such a point, let $x'_0 \in B(x_0, t_0, Ar\sqrt{t_0})$ be a point with scalar curvature Q . It also has a canonical neighborhood. We can apply part (a) to a ball around x'_0 with a radius on the order of $(K'(A))^{-1/2}r_0$, and with a value of w coming from the canonical neighborhood hypothesis, to get a contradiction. Thus $R \leq K'(A)r_0^{-2}$ on $B(x_0, t_0, Ar\sqrt{t_0})$. If T is large enough then the ϕ -almost nonnegative curvature implies that $|\text{Rm}| \leq K'(A)r_0^{-2}$. Then the noncollapsing in Proposition II.6.3(a) gives a lower local volume bound. Hence we can apply part (a) to appropriate-sized balls in $B(x_0, t_0, \frac{A}{2}r\sqrt{t_0})$. As A is arbitrary, this proves part (b).

For part (c), if the statement is not true then consider a sequence that gives a counterexample. That is, there is a sequence of times t_0 going to infinity and some $\xi > 0$ so that there are points in $P(x_0, t_0, Ar\sqrt{t_0}, Ar^2t_0)$ violating the conclusion. Given t_0 , let $t' \in [t_0, t_0 + Ar^2t_0]$ be the first time at which the conclusion is violated. Then $|2t'R_{ij}(x', t') + g_{ij}|^2 > \xi$ for some x' , with $(x', t') \in P(x_0, t_0, Ar\sqrt{t_0}, Ar^2t_0)$. From length-distortion estimates on the time interval $[t_0, t']$, one obtains an estimate $\text{dist}_{t'}(x_0, x') \leq \text{const. } Ar\sqrt{t_0}$, where const. is

uniform with respect to t_0 . Then one can apply part (b) to the ball $B(x_0, t_0, \text{const. } Ar\sqrt{t_0})$ to get a contradiction.

75. II.7.3

To justify II.(7.6), if the claim is not true then for some $w > 0$, there are sequences (x_i, t_i) with $t_i \rightarrow \infty$, $\rho(x_i, t_i) t_i^{-1/2} \rightarrow 0$ and $\text{vol}(B(x_i, t_i, \rho(x_i, t_i))) \geq w\rho^3(x_i, t_i)$. In order to apply Corollary II.6.8, we want to know that for large i , $\rho(x_i, t_i) \geq \theta^{-1}(w)h(t_i)$. Suppose that this is not the case. There is some point $x'_i \in B(x_i, t_i, \rho(x_i, t_i))$ with a sectional curvature equal to $-\rho^{-2}(x_i, t_i) \leq -\theta^2(w)h(t_i)^{-2}$. By II.(5.1), $R(x'_i, t_i)\rho^2(x_i, t_i) \rightarrow \infty$, so $R(x'_i, t_i)h^2(t_i) \rightarrow \infty$. Thus for large i , x'_i is in a canonical neighborhood. For any $w' > 0$, we can take ϵ sufficiently small so that a canonical neighborhood of the type (a) or (b) in II.1.5 has $\text{vol}(B) \leq w'r^3$. We obtain a sequence of such canonical neighborhoods $B(x'_i, r'_i) \subset B_i \subset B(x'_i, 2r'_i)$ with $\frac{r'_i}{\rho(x_i, t_i)} \rightarrow 0$. Applying the Bishop-Gromov inequality gives a contradiction to the fact that $\text{vol}(B(x_i, t_i, \rho(x_i, t_i))) \geq w\rho^3(x_i, t_i)$.

Thus we can apply Corollary II.6.8 to get a smooth solution on $P(x_i, t_i, \rho(x_i, t_i)/4, -\tau\rho^2(x_i, t_i))$, with $R < K_0\rho(x_i, t_i)^{-2}$ there. Applying Lemma II.6.3(c) at time $t_i - \tau\rho^2(x_i, t_i)$ gives an estimate $R \leq K_2\rho(x_i, t_i)^{-2}$ on $B(x_i, t_i, \rho(x_i, t_i))$. But then for large i , II.(5.1) gives $\text{Rm} > -\rho(x_i, t_i)^{-2}$ on $B(x_i, t_i, \rho(x_i, t_i))$, which is a contradiction.

As in [9], we define the thin part $M^-(w, t)$ to be the points $x \in M$ so that

$$(75.1) \quad \text{vol}(B(x, t, \rho^*(x, t))) < w(\rho^*(x, t))^3,$$

where $\rho^*(x, t) = \min(\rho(x, t), \frac{1}{2}\sqrt{t})$. The thick part is $M^+(w, t) = M - M^-(w, t)$. We claim that for large t , Lemma II.7.2 applies to points in the thick part. To see this, if $x \in M^+(w, t)$ and $\rho(x, t) \leq \frac{1}{2}\sqrt{t}$ then we can apply Lemma II.7.2 directly with $r = t^{-1/2}\rho(x, t)$. If $x \in M^+(w, t)$ and $\rho(x, t) > \frac{1}{2}\sqrt{t}$ then as the sectional curvature is bounded below by $-\rho(x, t)^{-2}$ on the ball $B(x, t, \rho(x, t))$, it is bounded below by $-(\frac{1}{2}\sqrt{t})^{-2}$ on the ball $B(x, t, \frac{1}{2}\sqrt{t})$. Thus we can apply Lemma II.7.2 with $r = \frac{1}{2}$.

76. II.7.4

For an example where Theorem II.7.4 applies, consider a compact 3-dimensional nilmanifold that evolves under the Ricci flow. As in [19], let $\theta_1, \theta_2, \theta_3$ be affine-parallel 1-forms on M which lift to Maurer-Cartan forms on the Heisenberg group. The solution to the Ricci flow equation is

$$(76.1) \quad g(t) = A_0(1 + 2A_0^2t)^{-1/2}\theta_1^2 + B_0(1 + 2A_0^2t)^{1/2}\theta_2^2 + C_0(1 + 2A_0^2t)^{1/2}\theta_3^2,$$

with sectional curvatures $-\frac{3}{4}A_0(1 + 2A_0^2t)^{-1}$, $\frac{1}{4}A_0(1 + 2A_0^2t)^{-1}$ and $\frac{1}{4}A_0(1 + 2A_0^2t)^{-1}$. The rescaled metric

$$(76.2) \quad t^{-1}g(t) = A_0t^{-1}(1 + 2A_0^2t)^{-1/2}\theta_1^2 + B_0t^{-1}(1 + 2A_0^2t)^{1/2}\theta_2^2 + C_0t^{-1}(1 + 2A_0^2t)^{1/2}\theta_3^2.$$

has diameter that goes like $t^{-1/4}$ and lowest sectional curvature $-\frac{3}{4}A_0t(1 + 2A_0^2t)^{-1}$. If we take $\rho \sim t^{-1/2}$ then for large t , $B(x, \rho)$ is a circle bundle over a flat 2-ball of radius ρ ,

with circle lengths that go like $t^{-3/4}$. The sectional curvature on $B(x, \rho)$ is bounded below by $-\rho^{-2} \sim -t$, and $\rho^{-3} \text{vol}(B(x, \rho)) \sim t^{-1/4}$.

We remark that although the solution on M is collapsing in the sense of Theorem II.7.4, there is no contradiction with the κ -noncollapsing results of I. In terms of $g(t)$, with reference to I.4.1, if we take $r \sim t^{1/3}$ then for large t it is true that r^2/t is bounded, $|Rm| \leq r^{-2}$ on $B(x, r)$, and $r^{-3} \text{vol}(B(x, r)) \rightarrow 0$. However, Theorem I.4.1 only rules out local collapsing for a given solution on a finite time interval.

The fact that condition (3) in Theorem II.7.4 is satisfied for Ricci flow solutions comes from combining Corollary II.6.8 and derivative bounds on the curvature [15, Section 13].

To see that Theorem II.7.4 essentially describes all possibilities for $M^-(w, t)$, if we cannot apply Theorem II.7.4 to $M^-(w, t)$, with the rescaled metric $t^{-1}g$, then there is some point $x \in M^-(w, t)$ with $\min(t^{-1/2}\rho(x, t), \frac{1}{2}) > d(t)$. Here $d(t)$ is the diameter of M in the rescaled metric, a dimensionless quantity. As a boundary component of $M^-(x, t)$ has a collar of length one, $M^-(x, t)$ must be boundaryless. Furthermore, the sectional curvatures of M are bounded below by $-\frac{1}{t^{-1/2}\rho(x, t)}$, and so are bounded below by $-\frac{1}{d(t)^2}$. If $t^{-1/2}\rho(x, t)/d(t)$ is uniformly bounded above then $\text{vol}(M) = \text{vol}(B(x, \rho(x, t))) \leq w\rho(x, t)^3 \leq \text{const. } wd(t)^3$ and we can apply Theorem II.7.4 with $\rho = d(t)$. Thus we may assume that $t^{-1/2}\rho(x, t)/d(t) \rightarrow \infty$. If $V(t)/d^3(t) \rightarrow 0$ then we can apply Theorem II.7.4 with $\rho = d(t)$. Thus we may assume that $V(t)/d^3(t)$ is bounded away from zero. After rescaling the metric to make the diameter one, we are in a noncollapsing situation with the lower sectional curvature bound going to zero. Hence M carries a Ricci flow solution with nonnegative but strictly positive sectional curvature, and so admits a flat metric.

77. II.8.2

To give an argument for II.8.2 under the pretense of smooth flows : the equation

$$(77.1) \quad \frac{d}{dt}\lambda(t) \geq \frac{2}{3}\lambda^2(t)$$

comes from the equation on the top of page 6 of I. The fact that $\lambda(t)V^{\frac{2}{3}}(t)$ is nondecreasing when it is nonpositive comes from I.2.3.

For any metric g , there are inequalities

$$(77.2) \quad \min R \leq \lambda \leq \frac{\int_M R d \text{vol}}{\text{Vol}(M, g)},$$

where the first inequality comes from the nonnegativity of -4Δ and the second inequality comes from using 1 as a test function in the Rayleigh quotient.

If $\lambda > 0$ for some metric g then from (77.1), the Ricci flow starting from g will become extinct in finite time. If M does not admit any metric with $\lambda > 0$, let $\bar{\lambda}$ be the supremum of $\lambda V^{\frac{2}{3}}$ over all metrics on M .

If M is a graph manifold then M volume-collapses with bounded curvature, so (77.2) implies that $\bar{\lambda} = 0$.

Suppose that the geometric decomposition of M has a piece that is diffeomorphic to a finite-volume manifold of constant curvature $-\frac{1}{4}$, along with a possible graph piece. Let V_{hyp} be the volume of this hyperbolic manifold; it is well-defined by Mostow-Prasad rigidity. For any $\epsilon > 0$, by collapsing the graph manifold part, there is a metric g_ϵ on M with $R \geq -6 \cdot \frac{1}{4} - \epsilon$ and $Vol(M) \leq V_{hyp} + \epsilon$. Then (77.2) implies that $\lambda(g_\epsilon) V(g_\epsilon)^{\frac{2}{3}} \geq -\frac{3}{2} V_{hyp}^{2/3} - \text{const. } \epsilon$. Thus $\bar{\lambda} \geq -\frac{3}{2} V_{hyp}^{2/3}$. Next, take an arbitrary metric g on M and consider the Ricci flow $g(t)$ with initial metric g . For large t , $(M, g(t))$ has an approximately hyperbolic piece and a possible graph piece. Taking an appropriate test function (which is one on the thick part of the approximately hyperbolic piece and decays to zero on its ends) gives $\lambda(t) \geq -\frac{3}{2} - \epsilon_1(t)$, with $\lim_{t \rightarrow \infty} \epsilon_1(t) = 0$. We also know that $V(t) \geq V_{hyp} - \epsilon_2(t)$, with $\lim_{t \rightarrow \infty} \epsilon_2(t) = 0$. If the Ricci flow were always smooth (which of course is not the case) then the monotonicity of $\lambda(t) V^{2/3}(t)$ would imply that $\lambda(0) V^{2/3}(0) \leq -\frac{3}{2} V_{hyp}^{2/3}$, and so $\bar{\lambda} \leq -\frac{3}{2} V_{hyp}^{2/3}$. To handle flows with surgery, it suffices to show that for any $\epsilon > 0$ there is a flow with $\delta(t)$ -cutoff starting from g so that $\lambda(0) V^{2/3}(0) \leq \lambda(t) V^{2/3}(t) - \epsilon$ for all t .

If a cross-section of a hyperbolic cusp were compressible in M then there would be a metric on M with $R \geq -\frac{3}{2}$ and $\text{vol} < V_{hyp}$ [1, Pf. of Theorem 2.9]. Using (77.2), one would obtain a contradiction to the fact that $\bar{\lambda} = -\frac{3}{2} V_{hyp}^{2/3}$.

APPENDIX A. ALEXANDROV SPACES

We recall some facts about Alexandrov spaces (see [5], [4, Chapter 10]). Given points p, x, y in a nonnegatively curved Alexandrov space, we let $\tilde{\angle}_p(x, y)$ denote the comparison angle at p , i.e. the angle of the Euclidean comparison triangle at the vertex corresponding to p .

The Toponogov splitting theorem says that if X is a proper nonnegatively curved Alexandrov space which contains a line, then X splits isometrically as a product $X = \mathbb{R} \times Y$, where Y is a proper, nonnegatively curved Alexandrov space [4, Theorem 10.5.1].

Let M be an n -dimensional Alexandrov space with nonnegative curvature, $p \in M$, and $\lambda_k \rightarrow 0$. Then the sequence $(\lambda_k M, p)$ of pointed spaces Gromov-Hausdorff converges to the Tits cone $C_T M$ (the Euclidean cone over the Tits boundary $\partial_T M$) which is a nonnegatively curved Alexandrov space of dimension $\leq n$ [2, p. 58-59]. If the Tits cone splits isometrically as a product $\mathbb{R}^k \times Y$, then M itself splits off a factor of \mathbb{R}^k : using triangle comparison, one finds k orthogonal lines passing through a basepoint, and applies the Toponogov splitting theorem.

If $x_k \in M$ is a sequence with $d(x_k, p) \rightarrow \infty$, we can find a sequence $y_k \in M$ such that $\frac{d(y_k, x_k)}{d(x_k, p)} \rightarrow 1$, and $\tilde{\angle}_{x_k}(p, y_k) \rightarrow \pi$. Now suppose $r_k \rightarrow \infty$ and $\frac{r_k}{d(p, x_k)} \rightarrow 0$. Then the sequence $(\frac{1}{r_k} M, x_k)$ subconverges to a pointed Alexandrov space (N_∞, x_∞) . Observe that for any $\rho < \infty$, we can find sequences $p_k \in \overline{px_k}$, $z_k \in \overline{x_k y_k}$ such that $\frac{d(x_k, p_k)}{r_k} \rightarrow \rho$, $\frac{d(x_k, z_k)}{r_k} \rightarrow \rho$, and by monotonicity of comparison angles [4, Chapter 4.3] we will have $\tilde{\angle}_{x_k}(p_k, z_k) \rightarrow \pi$. Passing to the Gromov-Hausdorff limit, we find $p_\infty, z_\infty \in N_\infty$ such that $d(p_\infty, x_\infty) = d(z_\infty, x_\infty) = \rho$ and $\tilde{\angle}_{x_\infty}(p_\infty, z_\infty) = \pi$. Since this construction applies for all ρ , it follows that N_∞ contains

a line passing through x_∞ . Hence, by the Toponogov splitting theorem, it is isometric to a metric product $\mathbb{R} \times N'$ for some Alexandrov space N' .

If M is a nonnegatively curved manifold, and $C \subset M$ is a compact connected domain with weakly convex boundary, then the subsets $C_t := \{x \in C \mid d(x, \partial C) \geq t\}$ are convex in C [7, Chapter 8]. If the second fundamental form of ∂C is $\geq \frac{1}{r}$ at each point of ∂C , then for all $x \in C$ we have $d(x, \partial C) \leq r$, since the first focal point of ∂C along any inward pointing normal geodesic occurs at distance $\leq r$.

APPENDIX B. FINDING BALLS WITH CONTROLLED CURVATURE

Lemma B.1. *Let X be a Riemannian manifold, and suppose $\overline{B(x, 5r)}$ is a compact subset of X . Then there is a ball $B(y, \bar{r}) \subset B(x, 5r)$, $\bar{r} \leq r$, such that $R(z) \leq 2R(y)$ for all $z \in B(y, \bar{r})$ and $R(y)\bar{r}^2 \geq R(x)r^2$.*

Proof. Define sequences $x_i \in B(x, 5r)$, $r_i > 0$ inductively as follows. Let $x_1 = x$, $r_1 = r$. For $i > 1$, let $x_{i+1} = x_i$, $r_{i+1} = r_i$ if $R(z) \leq 2R(x_i)$ for all $z \in B(x_i, r_i)$; otherwise let $r_{i+1} = \frac{r_i}{\sqrt{2}}$, and let $x_{i+1} \in B(x_i, r_i)$ be a point such that $R(x_{i+1}) > 2R(x_i)$. The sequence of balls $B(x_i, r_i)$ is contained in $B(x, 5r)$, so the sequences x_i , r_i are eventually constant, and we can take $y = x_i$, $\bar{r} = r_i$ for large i . \square

There is an evident spacetime version of the lemma.

APPENDIX C. ϕ -ALMOST NONNEGATIVE CURVATURE

In three dimensions, the Ricci flow equation implies that

$$(C.1) \quad \frac{dR}{dt} = \Delta R + \frac{2}{3}R^2 + 2 \left| R_{ij} - \frac{R}{3}g_{ij} \right|^2.$$

The maximum principle implies that $\inf_M \frac{R}{1 + \frac{2}{3}tR}$ is nondecreasing in t . In particular, $tR(\cdot, t) > -\frac{3}{2}$ for all $t \geq 0$ (compare [16, Section 2]).

We remark that the definition of ϕ -almost nonnegative curvature as given in I.12 may be slightly confusing, as the domain of ϕ must be specified. (For example, if one interpreted ϕ to be defined on all of \mathbb{R} then one would conclude that R is always nonnegative, which is certainly not the case.) A more precise statement, as given in [16, Theorem 4.1], is as follows.

Assume that at $t = 0$ the eigenvalues $r_1 \geq r_2 \geq r_3$ of the curvature operator at each point satisfy $r_3 \geq -1$. (One can always achieve this by rescaling.) Given a point (x, t) , put $X = -r_3$. If $X > 0$ then

$$(C.2) \quad R(x, t) \geq X (\ln X + \ln(1 + t) - 3),$$

or equivalently,

$$(C.3) \quad tR(x, t) \geq tX \left(\ln(tX) + \ln\left(\frac{1+t}{t}\right) - 3 \right).$$

The content of this equation is that for any $s \in \mathbb{R}$, if $tR(\cdot, t) \leq s$ then there is a lower bound $t \operatorname{Rm}(\cdot, t) \geq \operatorname{const.}(s, t)$. Of course, this is a vacuous statement if $s \leq -\frac{3}{2}$. We will say that a Ricci flow satisfies the *Hamilton-Ivey pinching condition* if (C.3) holds at every point in space-time.

Fix $t_0 > 0$. Then (C.3) gives a decreasing function $\phi : (0, \infty) \rightarrow \mathbb{R}$ such that $\phi(s) \sim \frac{1}{\ln s}$ for large s , with the property that if $t \geq t_0$ and $tR \leq s$ (for s positive) then

$$(C.4) \quad t \operatorname{Rm}(\cdot, t) \geq -s\phi(s).$$

This bound has the most consequence when s is large.

We note that for the original unscaled Ricci flow solution, the precise bound that we obtain depends on t_0 and the time-zero metric, through its lower curvature bound.

Suppose that at some point, $R > 0$. A condition $\operatorname{Rm} \geq -R\phi(R)$ is obviously a lower sectional curvature bound, but it also implies an upper sectional curvature bound from the fact that the sectional curvatures add up to R . Namely,

$$(C.5) \quad -R\phi(R) \leq \operatorname{Rm} \leq R \left(1 + \left(\frac{n(n-1)}{2} - 1 \right) \phi(R) \right).$$

APPENDIX D. CONVERGENT SUBSEQUENCES OF RICCI FLOW SOLUTIONS

Theorem D.1. [14] *Let $\{g_i(t)\}_{i=1}^{\infty}$ be a sequence of Ricci flow solutions on connected pointed manifolds (M_i, m_i) , defined for $t \in [A, B]$, and complete for each i and t , with $-\infty \leq A < 0 \leq B \leq \infty$. Suppose that the following two conditions are satisfied :*

1. *For each $r > 0$ and $t \in [A, B]$, there is an $N_{r,t} < \infty$ so that for all i , $\sup_{B_r(m_i)} |\operatorname{Rm}(g_i(t))| \leq N_{r,t}$, and*
2. *The time-0 injectivity radii $\{\operatorname{inj}(g_i(0))(m_i)\}_{i=1}^{\infty}$ are uniformly bounded below by a positive number.*

Then a subsequence of the solutions converges (modulo diffeomorphisms) to a Ricci flow solution $g_{\infty}(t)$ on a connected pointed manifold (M_{∞}, m_{∞}) , defined for $t \in [A, B]$. That is, for any compact subset $I \subset [A, B]$ and any $r > 0$, the solutions g_i on $I \times B_r(m_i) \subset [A, B] \times M_i$ converge smoothly to g_{∞} on $I \times B_r(m_{\infty}) \subset [A, B] \times M_{\infty}$.

Given the sectional curvature bounds, the lower bound on the injectivity radii is equivalent to a lower bound on the volumes of balls around m_0 [7, Theorem 5.8].

Note that similar arguments allow one to extract convergent subsequences in more general situations. For instance, if the curvature bounds only hold when $r < r_0 < \infty$ then one can still extract a subsequence which converges on (compact subsets of) the r_0 -balls.

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