Perelman 1 3/21/03

Ricci flow. $(g_{ij})_t = -2R_{ij}$

introduced by Hamilton (82):

- short-time existence of smooth sol'ns
- $n=3$, $R_{ij} > 0$ at $t=0$ then
  singularity in finite time but
  normalization (vol = const) converges exp fast to const curv
  $\Rightarrow M \cong S^3/\Gamma$.

  only known proof of this - $R_{ij} > 0 \Rightarrow M \cong S^3/\Gamma$.
- main technique - maximum principle arguments.
- analogy w/ heat eqn. - several heuristic arg's -

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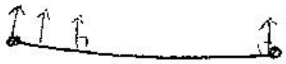


vary along orthog. parallel vector field

1^{st} variation $= 0$

2^{nd} - $\int K$ K = sec curv in planes

avg. over orthog parallel fields is $\int R$

* evol. of curvature tensor -

$$(R_{\dots})_t = \Delta R_{\dots} + Q_{\dots}$$

↑ Quadratic expression.

↓

scalar curv. satisfies

$$R_t = \Delta R + 2|R_{ij}|^2$$

so infimum is increasing.

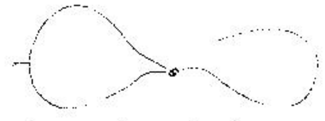
then showed eigenvalues of R_{ij} approach each other when curvature grows.

dim 2: normalized Ricci flow always converges (Hamilton)

In general: expect <u>some set</u> where flow converges,
limit metric should be a <u>stationary sol'n</u>,
<u>Einstein metrics</u> - in dim 3 ⟺ constant curvature

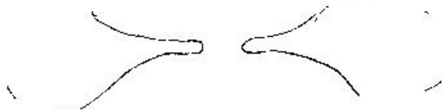
M^3 connected sum:

expect the S^2 to <u>shrink</u> faster, so



want to <u>continue</u> flow through the singularity

<u>Hamilton '96</u>: in dim 4, with a priori assumption on pos. of curvature
almost succeeded in continuing the flow -
remove small nbhd & glue something else (3-ball)



& continue flow. Repeat...
↓
(expect): <u>connected sum decomposition</u>

<u>Topologists know</u> - still not enough - expect decomposition along tori
so have to analyze flow when non singular for all time

Hamilton 99 - if ∃ for all time, nonsingular, bounds on a
certain normalized curvature ⟹
~~[illegible]~~ flow gives geometrization conjecture of
Thurston.

New technique

in a sense complements max principle technique

Integral monotonicity formulas (max principle gives ptwise monotonicity)

Ricci comes from a gradient flow in a certain sense:

functional: $$\mathcal{F}(g_{ij}, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV$$

$f : M \to \mathbb{R}$

g_{ij} metric

variational formula

$$\delta\mathcal{F}(v_{ij}, h) = \int_M \left[-v_{ij}(R_{ij} + \nabla_i\nabla_j f) + \left(\tfrac{1}{2}v - h\right)\left(2\Delta f - |\nabla f|^2 + R\right)\right] e^{-f}\, dV$$

$v_{ij} = \delta g_{ij}$

$v = g^{ij} v_{ij}$

$h = \delta f$

if $\boxed{\tfrac{1}{2}v - h = 0}$. (*)

$$\delta\mathcal{F} = \int_M -v_{ij}(R_{ij} + \nabla_i \nabla_j f)$$

So gradient flow would be

(**) $$(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f) \qquad = \text{Ricci} + \text{extra term}.$$

differ from Ricci flow only by diffeomorphism (flow is dual to ∇f)

about (*):

rewrite $$\mathcal{F}(g_{ij}) = \int_M \left(R + |\nabla f|^2\right) dm$$

m is some measure, fixed

f defined by $dm = e^{-f} dV$

for *this* variation, with dm fixed, $\underline{\frac{1}{2}V - h \equiv 0}$.

so (**) is gradient flow for *this* F, fixing dm.
different choice of dm gives metrics differing by *diffeo*
(like choice of gauge in physics)

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in general this flow doesn't exist even for short time
because $f_t = -\Delta f - R$ *backwards* heat eqn.
so sol'ns only for restricted f.

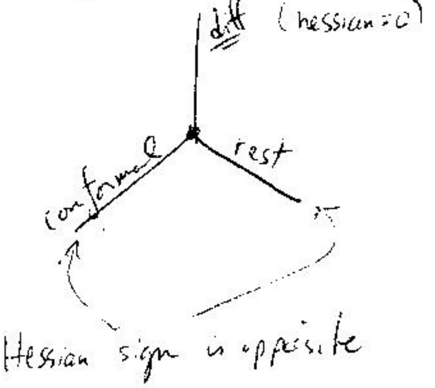
But Ricci flow *does* exist so it is the
diffeo part that may or may not exist.

Compare to Einstein-Hilbert function:

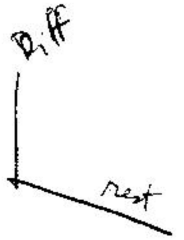
$$EH = \int_M R\, dV \qquad (g_{ij})_t = R g_{ij} - 2R_{ij}$$

gradient flow, *doesn't* exist
for short time for most metrics

because - EH inv't under difeo
so tangent space splits into:



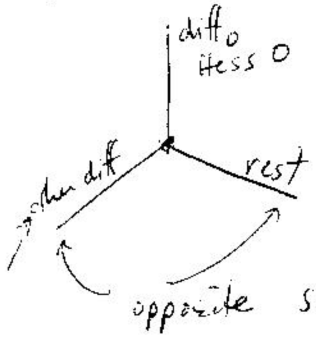
$\mathcal{F}$:



Hess has well-def sign.   seems like miracle or cheating...

but $\mathcal{F}$ not inv't wrt all diffeos –
only measure - preserving ones –

so



analogous
to conformal transformation.

Monotonicity formula:

if
$$(g_{ij})_t = -2(R_{ij} + \nabla_i\nabla_j f)$$
$$f_t = -\Delta f - R$$

$$\to \quad \frac{d}{dt}\mathcal{F} \geq 0.$$

so for Ricci –
$$(g_{ij})_t = -2R_{ij} \longrightarrow \frac{d}{dt}\mathcal{F} \geq 0$$
$$f_t = -\Delta f + |\nabla f|^2 - R$$
(f captures the diffeo information)

substitution –
$$u = e^{-f}$$
$$u_t = -\Delta u + \underset{\substack{\uparrow \\ \text{measures change} \\ \text{of volume form under Ricci}}}{Ru}$$

"adjoint heat eqn"
because if $h_t = \Delta h$,
$\int uh = \text{const}$

monotonicity without mentioning f:

$$\lambda(g_{ij}) = \inf_{f\in C^\infty} \mathcal{F}(g_{ij}, f)\,, \quad \int e^{-f}\,dV = 1$$

$\Rightarrow \quad \frac{d}{dt}\lambda \geq 0$.

because variation wrt f is 0 at minimum so f_t can be arbitrary.

$$w = e^{-\frac{1}{2}f}$$

$$\mathcal{F} = \int_M Rw^2 + \frac{1}{4}|\nabla w|^2$$

λ = lowest eigenvalue for $-4\Delta + R$.

↑

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elementary observation & not terribly useful –

λ not scale-invariant.

so don't rule out – metric coming back to itself up to homothety.

$\bar{\lambda} = \lambda V^{2/n}$ scale-invariant.

$\bar{\lambda}$ is monotonic but not always:

$\frac{d}{dt}\bar{\lambda} \geq 0$ if $\bar{\lambda} \leq 0$. (corresp. to neg. curvature)

need new formula for negative λ .

$$W(g_{ij}, f, \tau) = \int_M (4\pi\tau)^{-\frac{n}{2}} e^{-f}\,dV\,\left[\tau\left(R+|\nabla f|^2\right)+f-n\right]$$

$\tau > 0$ scale parameter

scale-inv wrt simultaneous scaling of g_{ij} & τ.

→ ~~[illegible]~~ evolution equation:

$$\left.\begin{aligned} (g_{ij})_t &= -2R_{ij} \\ u_t &= -\Delta u + Ru \\ \tau_t &= -1 \\ u &= (4\pi\tau)^{-\frac{n}{2}} e^{-f} \end{aligned}\right\} \longrightarrow \frac{d}{dt}W \geq 0 .$$

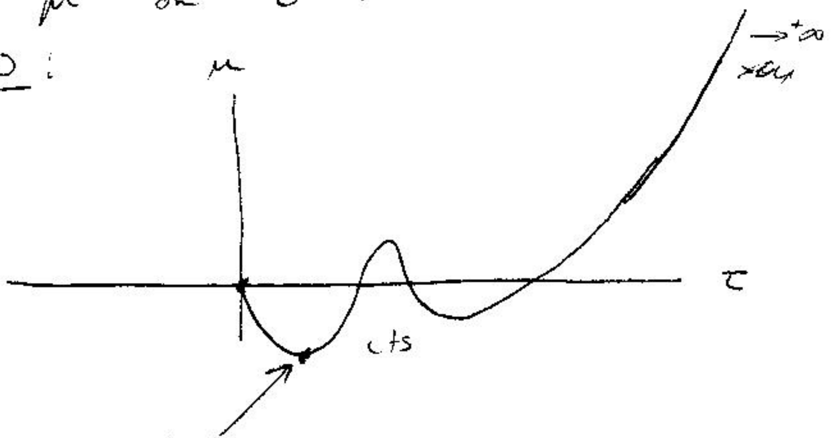
now to filter out f :

$$\mu(g_{ij}, \tau) = \inf_{\substack{f \\ \int u = 1}} W(g_{ij}, f, \tau)$$

$$\left.\begin{array}{l} (g_{ij})_t = -2R_{ij} \\ \tau_t = -1 \end{array}\right\} \longrightarrow \frac{d}{dt}\mu \geqslant 0 .$$

μ is eigenvalue of a <u>non</u> <u>linear</u> operator.
so not obv (but true) that this inf is obtained
(also Rothhouse '81)
?(Rothaus?)

dependence of μ on τ :
when <u>$\lambda > 0$</u> :



bot value is non-decreasing under Ricci.

<u>Rules</u> <u>out</u> cycles modulo homothety.

<u>How to prove this</u>: §

$\mu < 0$ for small τ: exhibit f that makes $W < 0$.

$\overline{g_{ij}} = g_{ij}(0)$ – run Ricci flow for $[0, T]$

let $u|_{t=T} = \delta$

$\tau = 0$ at $t = T$

then $W = 0$ at $t = T$

so at times 0, $\tau = T$, $W \leq 0$
by monotonicity

get f at time near 0 by running backward heat eqn backwards!

scale invariance -

as $\tau \to 0$ rescale τ & g_{ij} to make $\tau > 0$ fixed say $1/2$. Then g_{ij} tends to <u>flat</u> (heuristically)
so value of W as $\tau \to 0$ reduces to Euc Qn for Euclidean metrics with $\tau = 1/2$.

$$⍟ \quad \int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-f} \left[\tfrac{1}{2} |\nabla f|^2 + f - n \right]$$

<u>minimize</u> subject to $\int_{\mathbb{R}^n} (2\pi)^{-n/2} e^{-f} = 1$

<u>using</u>
sharp logarithmic Sobolev inequality (Leonard Gross, ~30 yrs ago)

Gaussian meas. $d\mu = (2\pi)^{-n/2} \exp\left(-\tfrac{1}{2}|x|^2\right)$

$$\int \varphi \, d\mu = 1 \implies \int |\nabla \varphi|^2 d\mu \geq \frac{1}{2} \int \varphi^2 \log \varphi^2 \, d\mu$$

after simple substitution - ⍟ min is non-neg.

~~and = 0~~

(heuristically) justifies the picture of $\mu(\tau)$.

<u>Heuristic interp. of W</u> - related to statistical mechanics
(just analogy)

"Canonical Ensemble"

temperature τ $\beta = \tau^{-1}$

partition function $Z = \int \exp(-\beta E) \, dw(E)$ E is energy

$$\langle E \rangle = -\frac{\partial}{\partial \beta} \log Z$$

entropy $S = \beta \langle E \rangle + \log Z$ $= -\sum p_n \log p_n$ if dw discrete distrib.

energy fluctuation $\sigma = \langle E - \langle E \rangle \rangle = \dfrac{\partial^2}{(\partial \beta)^2} \log Z$.

chg t to τ – recall $\tau_t = -1$ so $\tau = -t + \text{const}$.

so for us –

$$(g_{ij})_\tau = 2\left(R_{ij} + \nabla_i \nabla_j f\right) \qquad dm = u\,dV \qquad u = (4\pi\tau)^{-n/2} e^{-f}$$

$$\log Z = \int\left(-f + \frac{n}{2}\right) dm \qquad (\text{define } Z \text{ this way})$$

$$\langle E \rangle = \tau^2 \int_M \left(R + |\nabla f|^2 - \frac{n}{2\tau}\right) dm$$

$$S = -\int_M \left(\tau\left(R + |\nabla f|^2\right) + f - n\right) dm \qquad (= -W)$$

$$\sigma = 2\tau^4 \int_M \left|R_{ij} + \nabla_i \nabla_j f - \frac{1}{2\tau} g_{ij}\right|^2 dm$$

If $u = \delta$ fcn at $\tau = 0$ then $\langle E \rangle > 0$:

$$\frac{d}{dt} \mathcal{F} \geqslant \frac{2}{n} \mathcal{F}^2$$

; not following...

if $\sigma = 0$: under Ricci flow metric is same under diffeo & homothety ("Ricci soliton")

if S bdd, $\sigma \to 0$ since σ like ∇S

so is it true we get a Ricci soliton?

in general no.

(true in dim = 2)

Remarks about physics –

recall $\mathcal{F}(g_{ij}, f) = \int_M \left(R + |\nabla f|^2\right) e^{-f}\, dV$.

Well-known to physicists (string theory), "Low energy effective action"

f = "dilaton field".

they add extra field – antisymmetric 2-form. But so far no place for it here.

1985 Friedan - Renormalization gp flow ... ("2-dim non linear Σ model")

metaphysics - phys system viewed at difft. scales - difft. coupling constants

Tues informal discussion

Singularities of flow -

- $t \to T$ and $R(x,t) \to \infty$ everywhere (simple case)

use - Hamilton -

$$Rm \geq -\varphi(R)\, R - \text{const}$$

↑ Riem Tensor

so upper bds on R give lower bds on Rm
& upper bds on Rm - large eigenvalues would force large R.

a) $Rm > 0$ as in Hamilton S^3/Γ

b) $Rm \not> 0$

S^3, $\mathbb{R}P^3$, $\mathbb{R}P^3 \# \mathbb{R}P^3$

$S^2 \times S^1$

Haven't explained yet how.

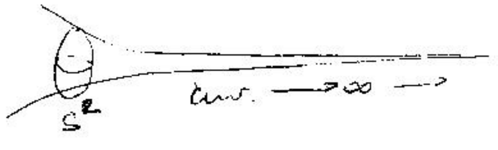
in this case - no sol'n left after time T.

$t \to T$ curv $\to \infty$ somewhere.

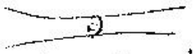
$\exists$ max'l domain $\Omega \subset M$, $\Omega \neq \emptyset$, where R bdd.

on Ω extract a limit -
control higher derivs of curvature
$\Rightarrow C^\infty$ limit.

ends of Ω "horns"
finitely many ends per comp't

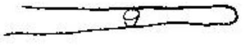


large curv. in "necks"



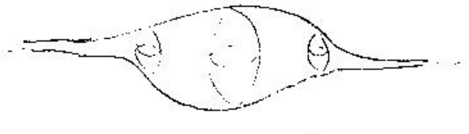
radius of spheres changes slowly.

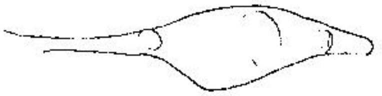
also -



don't know if Ω has finitely many components.
but only finitely many which are not necks

i.e.





(non trivial that such Ω exists)

Try to cap off Ω
& run the Ricci flow
on this. Repeat.

can the surgery times accumulate?

if done carefully with parameter

$\Rightarrow$ certain portion of volume lost each time.
$\exists$ lower bd on volume
and $\frac{d}{dt}V = -\int R$ so Vol increases only slowly —
$\Rightarrow$ can't have accumulating surgeries

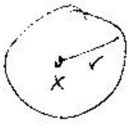
~~Where~~ as $t \to \infty$ ~~some component has no~~

$\exists\, T_0$ s.t. $\forall\, t > T_0$ each comp that is not yet extinct ~~is~~ of one of these types:

① $\mathbb{H}^3/\Gamma$ - almost isometric to closed hyp mfd
② graph mfd (topologically)
③ $\Omega \subset M$ complete noncpt hyp of finite vol, curvature almost const < 0.
$M \setminus \Omega$ graph mfd
$\partial\Omega$ = incompressible tori.

note - the graph mfd pieces don't necessarily geometrically have an F-structure -
different kind of collapse -
Alexandrov space used -

outside Ω: $\forall x\ \exists\, r(x)$



$Rm \geq -r^{-2}$
$\text{vol} \leq \epsilon r^3$.

$\Downarrow$

graph mfd (thm from 10 yrs ago...)
Yamaguchi - proved if r indept of x.

note - new necks can still form here...
but topological conclusion

Recall

Monotonicity formula

$$W(g_{ij}, f, \tau) = \int_M \underbrace{(4\pi\tau)^{-\frac{n}{2}} e^{-f}}_{u} dV \left[\tau(R + |\nabla f|^2) + f - n \right]$$

$$\textcircled{\ast} \quad \left.\begin{aligned} (g_{ij})_t &= -2R_{ij} \\ u_t &= -\Delta u + Ru \\ \tau_t &= -1 \end{aligned}\right\} \quad \frac{dW}{dt} \geq 0$$

Geometric application

Collapsing: suppose: $B(x,r)$ metric ball, $|Rm| \leq r^{-2}$ curvature bd (scale inv't)

then it is *K-collapsed* if $vol(B(x,r)) < \kappa r^n$ $\quad n = \dim M$

e.g. if inject. radius << r, get κ-collapsed for small κ.

"bounded curvature collapse".

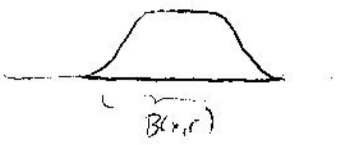
THEOREM:

Given M^n, $g_{ij}(0)$ - run Ricci flow - then:

$\forall\, T \quad \exists\, \kappa \quad$ if sol'n exists on $[0,T]$,
there cannot be a κ-collapsed ball in $g_{ij}(T)$ with $r < \sqrt{T}$

(κ depends on curvature, dim & vol bounds of $g_{ij}(0)$, and on T)

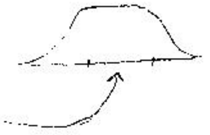
Suppose we have a strongly collapsed ball $B(x,r)$.
prescribe u sol'n of $\textcircled{\ast}$ s.t. at time T
let $\tau = r^2$



want $\int u = 1$ since $(4\pi\tau)^{-n/2} \sim r^{-n}$, $vol(B) << r^{-n}$ get $f \sim -\log \kappa$

now estimate W :

it shift f so = 0 in smaller ball



? $\int e^{-f}|\nabla f|^2 = |\nabla\sqrt{u}|^2$

$W \sim \log K$

solve on $[0,T)$ and look at time 0.

$\tau = T + r^2$ at $t=0$

Sobolev inequality

W at 0 bdd below.

since W increasing - get bd $\log K > \dots$

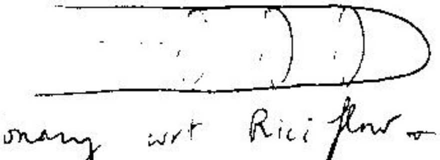
$\Rightarrow$ $\boxed{K > c,\ \text{some pos bd.}}$

this rules out "cigar" singularities:

rather cigar × $\mathbb{R}$

where cigar is $\mathbb{R}^2$, $ds^2 = \frac{dx^2+dy^2}{1+x^2+y^2}$

was a conceivable singularity model (after blowups)



stationary wrt Ricci flow ~ mod diffeo

non-collapsing couldn't happen like this in finite time.

Local Version of Monotonicity

$(g_{ij})_t = -2R_{ij}$

$u_t = -\Delta u + Ru$

$u|_{t=T} = \delta$

$\tau = T - t$

$V = [\tau(2\Delta f - |\nabla f|^2 + R) + f - n]u$

$u = (4\pi\tau)^{-\frac{n}{2}} e^{-f}$

$V_t = -\Delta V + RV + 2\tau u|R_{ij} + \nabla_i\nabla_j f - \frac{1}{2\tau} g_{ij}|^2$

? ($\Delta u = 0$ $\quad \Delta \frac{|\nabla u|^2}{u^{n/n-2}} \geq 0$ if $R_{ij} \geq 0$ (from Bochner's formula

$\Rightarrow$ $\int u$ = const because differentiate $\int -\Delta u + Ru - u(-R)$

$= \int -\Delta u = 0$

↑ deriv of mean under Ricci

$\int v \nearrow$ increasing

but because same comp leaves pos term.

why "local":

because $u = \delta$ at $t = T$, so v is mostly localized near some pt.

max principle to eqn for v_t tells:

max v/u $\nearrow$ (says "min" in paper, p. 22 (or 9.2

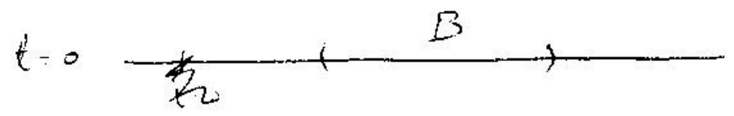
so $v \le 0$ in $[0,T]$ because

$v \to 0$ at $t \to T$ (computation? various singular terms ~~[illegible]~~ cancel out)

application

Pseudo locality theorem

$\exists \varepsilon > 0$ (dep's on n) s.t. if g_{ij} Ricci ~~[illegible]~~ solution $[0,T]$



$= B(x, 1)$ where $|Rm| < \varepsilon$.

$Vol(B) > (1-\varepsilon)\omega_n$

Conclusion: for $B(x,\varepsilon)$, $t \in [0, T \wedge \varepsilon^2]$

$|Rm| < \varepsilon^{-1}$

Another monotonicity formula

less straightfwd connection w/entropy formula

again the analogy from canonical ensemble -

embed system with some degrees of freedom into larger one, a "thermostat"

$(g_{ij})_\tau = 2 R_{ij}$ in M^n $\qquad$ τ is backwards time

S^N $\quad$ N large

sphere of radius $\sim \sqrt{N}$

its Ricci curvature $= \frac{1}{2}$ for $\tau = 1$

"standard" sphere" $\overline{g_{\alpha\beta}}$

Ricci flow here is ~~std~~

$g_{\alpha\beta} = \tau \overline{g_{\alpha\beta}}$

consider $\quad$ τ is the param here

$$\tilde{M} = M \times S^N \times \underset{\downarrow}{\mathbb{R}} \quad \text{with metric}$$

$$\tilde{g}_{ij} = g_{ij} \qquad \tilde{g}_{\alpha\beta} = \tau \bar{g}_{\alpha\beta} \qquad \tilde{g}_{00} = \frac{N}{2\tau} + R$$

M comp.

This metric is (close to:) Ricci - flat (?)

$\tilde{Ric} = O(N^{-1})$ — compared to metric tensor (which is large in $\mathbb{R}$ & S^N directs) but also wrt fixed coordinates (?)

Bishop-Gromov -

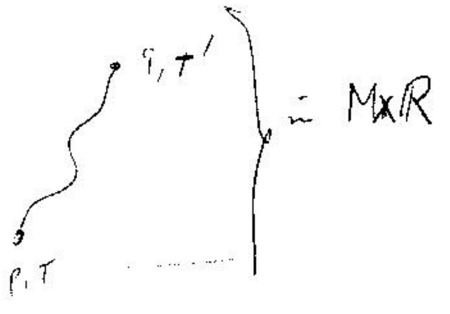
$$B(x,r) - \quad \frac{\mathrm{vol}(\partial B(x,r))}{r^{n-1}} \searrow \text{ in } r \qquad \text{for ricci flat or ricci} \geq 0$$

for this mfd $\tilde{M}$ can formulate a version

fix (p,T) $\quad \tau = T - t$

& for any (q,T') $T' < T$ consider curves γ

$$\mathcal{L}(\gamma) = \int^{T-T'} \sqrt{\tau}\,(R + |\dot{\gamma}(\tau)|^2)\, d\tau$$



normal length in $\tilde{M}$ would be

$$\int \sqrt{\frac{N}{2\tau} + R + |\dot{\gamma}(\tau)|^2}$$

$$= \int \underbrace{\sqrt{\frac{N}{2\tau}}}_{\text{const}} + \underbrace{\frac{1}{2}(R + |\dot{\gamma}(\tau)|^2)\sqrt{\frac{2\tau}{N}}}_{\text{leading term}} + \cdots$$

so $\mathcal{L}$ should be thought of as "length".

~~reduced length~~ ~~$\ell(q) =$~~ $L(q) = \inf_{\gamma} \mathcal{L}(\gamma)$

inf exists, smooth curve.

$$\ell(q) = \frac{1}{2\sqrt{\tau}} L(q)$$

$$\boxed{\int_M \tau^{-n/2} \exp(-\ell)\, dq \quad \text{decreases in } \tau} \qquad T' = T - t$$

$$\underset{V(\tau)}{\parallel} \qquad \text{fix } p$$

$|Rm| \le r^{-2}$

$Vol < \kappa r^n$

$\tau = T - t$

claim for $\tau \sim \kappa^{\frac{1}{2n}} r^2$
$V(\tau)$ is <u>small</u>.

over only $B(x, r)$ ℓ nonnegative - add below ? ~~still~~ b
~~go~~ so $\int$ is small.

Outside the Ball - ℓ is very large

at T V is smaller. by monotonicity

$\min l \le \frac{n}{2}$ (using some max principle)

in the ball?

get contradiction to monotonicity (how?)

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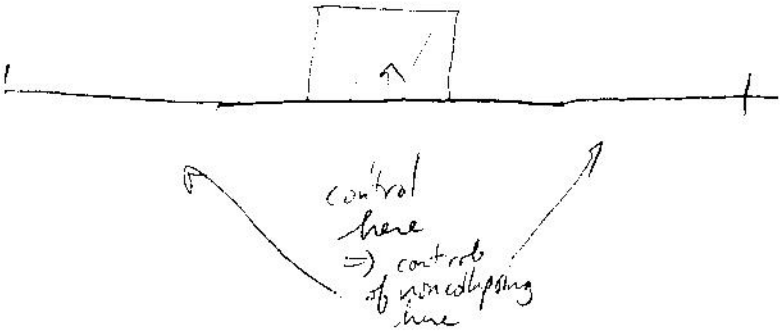
another . . .

$\forall A \ \exists \kappa$ s.t. if flow exists on $[T, T+1]$

and in some $B(x,1)$ & $\forall t \in [T, T+1]$

$|Rm| \le 1$, $\mathrm{Vol}\, B \geq A^{-1}$

<u>Then:</u> no κ-collapsed balls at $T+1$ inside $B(x,A)$



proof runs the same
| monotonicity

Hamilton stuff:

Pinching estimates for curvature & differential Harnack inequality

& recall $\left\{ R_t = \Delta R + 2|Ric|^2 \right.$ $\Rightarrow \min R \nearrow$ $\left\{ \begin{array}{l} \text{and} \\ Rt \geq -C \end{array} \right.$

$$(g_{ij})_t = -2R_{ij}$$

$$(R_{ijkl})_t = \Delta R_{ijkl} + Q_{ijkl}$$

↑ quadratic expression in curvature

Hamilton: if $R_{ijkl} u_{ij} u_{kl} \geq 0$ (for any antisym 2-form u)
$\Rightarrow$ $Q \geq 0$.

Therefore by max principle –
none-negativity of curvature preserved by flow.

in dim = 3 $R_{ij} \geq 0$ is also preserved.
can normalize so at time $t = 0$ $|Rm| \leq 1$ (by scaling)
then at any t –

$$t\, Rm \geq -\varphi(Rt)\, Rt - C$$

C explicit cont
R - scalar curv.

(Hamilton, Ivey)

so Rt bdd $\Rightarrow$ Rm bdd from below
& hence from above.

$\left(\varphi \sim \frac{1}{\log} \right)$

→ if Rt large, negative eigenv's are
small compared to pos eigenvalues or scalar curv

need to understand singularities – where curvature becomes unbdd
blow up limit around ~~such pt~~ pts where curvature grows w/out bd.

this $\Rightarrow$ in blow up limit – curvatures are non-negative.

e.g. complete, noncpt mfd w/ non neg sect curv.
Gromoll-Meyer, Cheeger-Gromoll – structure theory for these

Diff. Harnack (Lee-Yau-Hamilton) inequality

Hamilton 1993:

supp Ricci flow solution w/ $R_{ijkl} \geq 0$

and $$\left[\Delta R_{ij} - \frac{1}{2} \nabla_i \nabla_j R + 2 R_{ikjl} R_{kl} - R_{ik} R_{jk} + \frac{R_{ij}}{2t}\right] u_i u_j$$

$$+ \; 2 \left(\nabla_i R_{kl} - \nabla_k R_{il}\right) u_i V_{kl} + R_{abcd} V_{ab} V_{cd} \geq 0$$

u 1-form
V 2-form.

Geom interpretation - Chow - Chu
later Perelman in terms of $M \times S^N \times \mathbb{R}$ construction.

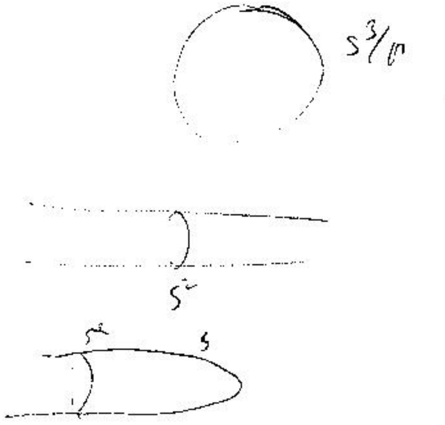
some traces of this 4 give:

$\forall X$ $$\frac{R}{t} + R_t + 2\langle \nabla R, X\rangle + 2 \, Ric(X,X) \geq 0$$

when $X=0$: $$\frac{R}{t} + R_t \geq 0$$
$$\Rightarrow tR_{\cancel{t}} \nearrow$$

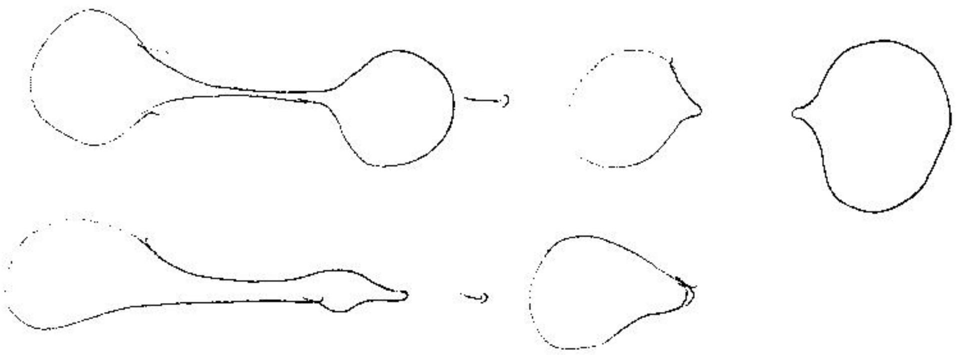
$(Rt)'$
$= R_t t + R$
$= t(R_t + \frac{R}{t}) \geq 0$. ✓

this + noncollapsing from before -
possible blowups -



that's all,
use pinching
Harnack
& noncollapsing

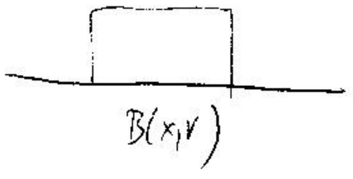
surgery -



Can show at most finitely many such surgeries for any finite time.

long-time behavior

Can prescribe for each surgery - lose a piece with definite amt of volume



suppose soln on

$B(x,r) \times [T-r^2, T]$ $Rm \geq -r^2$

Cond. in smaller $B(x,\frac{1}{2}r)$ ~~&~~ $\times [T-\frac{1}{2}r^2, T]$ $|Rm| \leq K r^{-2}$

also if say $Rm \geq -r^{-2}$ $\mathrm{vol}\, B \geq \alpha r^3$ $K = K(\alpha)$

got lost...

now use these singularity models to extend Hamilton's picture of long-time sol'ns.

Hamilton '99: if soln $\exists$ for $0 < t < \infty$

$$|Rm\, V^{2/3}| < C \quad \text{for all } t$$

then: for suff large t M admits thick-thin decomp

thick $\cong$ hyperbolic

thin $\sim$ collapsed w/ bdd curv. $\Rightarrow$ $\mathcal{F}$-structure, graph mfld

∂ thick $= \partial$ thin $=$ incompressible tori.

Main pts of his arg's & how to extend:

$$R_t = \Delta R + 2|Ric|^2 = \Delta R + \tfrac{2}{3}R^2 + 2|\overset{\circ}{Ric}|^2$$

$$\Rightarrow \quad \frac{d}{dt} R_{min} \geqslant \frac{2}{3} R_{min}^2$$

if $R_{min} > 0$, blows up in finite time

So must have $\underline{R_{min} < 0}$.

get connected sum of components becoming extinct $\Rightarrow S^2 \times S^1$, S^3/Γ.

so $\quad R_{min} \geqslant -\frac{3}{2} \dfrac{1}{t + \text{const}}$

$$V_t = -\int R$$

so $\quad V(t + \text{const})^{-3/2} \searrow$

(can normalize so $V \searrow$)

$$\hat{R} = R_{min} V^{2/3}$$

$$\frac{d}{dt}\hat{R} \geqslant \frac{2}{3} \underset{\overset{\wedge}{0}}{\hat{R}} \fint \Big(\underset{\overset{|\wedge}{0}}{R_{min} - R}\Big)$$

so $\hat{R} \nearrow$

$$\frac{d}{dt} \log|\hat{R}| \geqslant -\frac{2}{3} \fint R_{min} - R$$

if $\hat{R} \to$ nonzero lim $\Big)$ integrable in time

so must be $\sim \frac{1}{t}$

$R \sim R_{min}$ on most of mfd

moreover

if $t_i \to \infty$

$B(x_i, \sqrt{t_i})$ of $\mathrm{Vol} > c\sqrt{t_i}^{\,3}$ ← fastest volume growth in this setting

metrics $t_i^{-1} g_{ij}(t_i)$ ← become unit balls. suppose ~~in the~~ $|Rm| < Ct_i^{-1}$

~~with~~ → const scalar curvature

: hyperbolic metrics.

follows from Hamilton's a priori assumption

---

now use the previous estimates to remove Hamilton's assumption

$\boxed{R_{min} < 0}$ case

$\forall x \in M$, time t, $\exists\ \rho = \rho(x\ \&\ t)$ s.t. on $B(x,\rho)$ $\min Rm = -\rho^{-2}$

(triv since as $\rho \to 0$ $-\rho^{-2} \to -\infty$ ~~on~~ for very large ρ goes other way — assuming $R_{min} < 0$.

From before:

if $B(x,r) \times [T-r^2, T]$, $Rm \geq -r^2$, $\mathrm{vol}\, B \geq \alpha r^3$

Conclusion: on $B(x,\frac{1}{2}r) \times [T-\frac{1}{2}r^2, T]$

$|Rm| \leq K r^{-2}$, $K = K(\alpha)$.

if assumptions hold,

$\rho > \mathrm{diam}\, M$ more simple case..

$\rho << \sqrt{t}$ more difficult case

~~So get~~

this bd $\Rightarrow Rm \geq -\varepsilon r^2$ if $r << \sqrt{t}$

by pinching estimate $t Rm \geq -\varphi(tR)tR - \mathrm{const}$

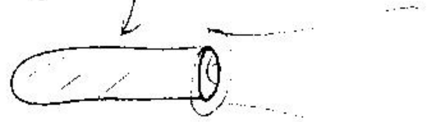
contradicts $\min R_m = -\rho^{-2}$.

So $\rho << \sqrt{t}$ $\Rightarrow \mathrm{vol}(B) << \rho^3$. rescale to unit size - $R_m \geq 1$ & small volume → collapse w/ lower curv bds.

if $\rho \not< \sqrt{t}$ use prev. to get a bound...

almost hyperbolic as before (use of Mostow Rigidity type argument)

incompressibility of tori – minimal surface argument
(Hamilton) – minimal compressing disk



area ↓ to 0 in finite time.

also goes through ~~[illegible]~~ unaltered – doesn't require curv. bds,
unaffected by surgeries
the disk won't pass through surgery locus,
& ∃ map from the pre to post surgery that is contracting

Compl. of hyp is locally collapsed w/ lower curv bds

$$\Downarrow \qquad \xleftarrow{\text{ka}}$$

graph mfd

by other techniques, not written yet (from 10 yrs ago – Alexandrov spaces)

so for large times: Mfd admits thick – thin decomp
thick → hyperbolic
thin → graph mfd
tori are incompressible

⟹ geometrization

avoidance of min surfaces, rigidity :

$\bar{\lambda} = \lambda V^{2/3}$ 1st eigenv of $-4\Delta + R$

$\bar{\lambda} \nearrow$ for smooth

but also can arrange for monotonicity through the surgeries
if $\bar{\lambda} \leq 0$,
because definite volume lost through surgery (& λ changes only a little

if $\lambda > 0$ $\frac{d}{dt}\lambda \geq \frac{2}{3}\lambda^2$ $\Rightarrow$ extinct in finite time

if $\sup \bar{\lambda} = 0$ over all metrics
get Graph Manifold

conversely Graph mfd $\Rightarrow \sup \bar{\lambda} = 0$
by construction of collapsing metrics

∴ "clear" – else run Ricci,
get hyperbolic piece,
vol $> \varepsilon > 0$ –
get some contradiction...

$\sup \bar{\lambda} < 0$ $\bar{V} = \left(-\frac{2}{3}\bar{\lambda}\right)^{3/2}$ $\bar{V}$ = min V: M = # (R>0 pieces)
M' other pieces
~~# complete hyp~~
$\cup$
hyp mfd
~~ca~~ $K = -1/4$
volume V
w/ incompress. cusps

again by running Ricci flow
+ Anderson's cusp closing them.

Informal Session . .

Singularity models.

$\varepsilon > 0$ $[0,T]$ <u>sol'n</u> $\underline{\text{smooth}}$

$\exists\, r > 0$ ~~$\exists R$~~ $\forall$ (x,t) where $\overset{Q:}{\underline{R(x,t) \geq r^{-2}}}$

dep in initial data

t not too small

partic a hds

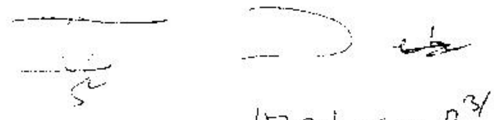
$B(x, \varepsilon^{-1} Q^{-1/2})$

$|Rm| \leq 1$ $\operatorname{vol} B(1) \geq 1$ at $t = 0$.

ε-close = Lip sense to $B(\bar{x}, \varepsilon^{-1})$ in an ancient sol'n.

after scaling, by Q metric tensor

for ancient solns.



$\eta > 0$ $|\nabla R| < \eta R^{3/2}$ $|R_t| < \eta R^2$

if ~~such r~~ $\nexists r$ $\exists$ sequence of counter examples

$r_k \longrightarrow 0$, sol'ns $[0, T_k)$ satisf. $|Rm| \leq 1$ $\operatorname{vol} B(1) \geq 1$

M_k, t_k

t_k first time where violated:

$\exists\, x_k$ s.t.

misprint in §12 says smallest

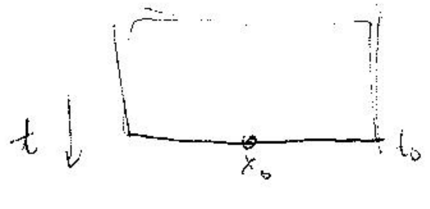
$\underline{\text{largest}}$ ~~minimal~~ $\underset{Q_k}{\underset{\shortparallel}{R(x_k, t_k)}} \geq r_k^{-2}$ ~~[illegible]~~ such that conclusion fails

so if $R(x, t_k) > Q_k$ - conclusion holds

$R(x,t) \geq r_k^{-2}$, $t < t_k \Rightarrow$ concl. holds.

So

$\Rightarrow$ get η - control of R_t & ∇R



(x_0, t_0) $Q_0 = R(x_0, t_0) + r^{-2}$

$R(x,t) \leq 8 Q_0$ in $B(x_0, \frac{1}{2}\eta^{-1} Q_0^{-1/2})$

$t \in (t_0 - \frac{1}{8}\eta^{-1} Q_0^{-1}, t_0)$

Monday 4/26

$\boxed{12.1}$ fix $\varepsilon > 0$. dim $= 3$

suppose $[0,T)$ smooth sol'n.

$\exists\, r > 0$ (dep on initial data & T) $\quad \forall\,(x,t)\quad t \in [0,T)$ s.t. $R(x,t) = r_x^{-2} \quad r_x \le r$

can assume $|Rm| \le 1$, $\mathrm{vol}(B(1)) \ge \frac{1}{2}\omega_3$

$\Rightarrow B(x, \varepsilon^{-1} r_x) \approx$ Ball in an ancient sol'n

rescaled to $\hat{g}_{ij} = r_x^{-2} g_{ij}$

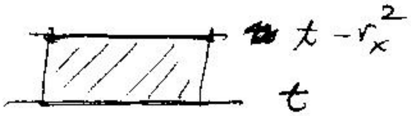
$\exists$ diffeo metrics ε-close in ~~for some~~ $C^{(\varepsilon^{-1})}$

a in §11: def'd on $(-\infty, 0]$, $Rm \ge 0$, Complete, $|Rm| \le C(t)$, K-noncollapsed

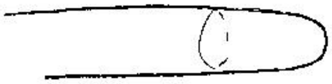
also can control a parabolic nbhd

$$P(x, \varepsilon^{-1} r_x, -r_x^2) = \{(y,t') : t - r_x^2 \le t' \le t,\ y \in B(x, \varepsilon^{-1} r_x, t)\}$$

& similar control "$\approx$"



Recall what kind of ancient sol'ns occur:



↑ estimates on this via the compactness thm for ancient sol'ns

diam bdd if $R = 1$ somewhere

exceptional $- S^3/\Gamma$

$\exists\, \eta > 0$

$|\nabla R| \le \eta R^{3/2}$

$|R_t| \le \eta R^2$

recall

$R_m \ge -\varphi(R) R - \text{const}$

$\varphi \sim 1/\log$

Proof by contradict. — assume no such r.

seq of $r \to 0$, solns that violate the st'mnt

(x_0, t_0) where largest curvature & don't satisfy conclusion —

i.e. (x,t) : either $t < t_0$ or $t = t_0$, $R(x,t) > R(x_0, t_0)$

then (x,t) has canonical nbhd.

Let $\boxed{r_0^{-2} = R(x_0, t_0) \quad r_0 \le r}$

$\gtrsim$ is open condition so such a pt exists.

$r_0 < r$ not too bad — for t near t_0 get canonical nbhds ...

$r_0 = r$

rescale, extract a limit as $r \to 0$.

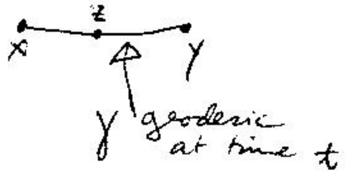
may have collapsing, or curvature $\to 0$.

Lemma $[0, t_0]$ suppose that $\boxed{R > r_0^{-2} \Longrightarrow \text{canonical nbhd}}$ for this sol'n.

$\forall D^0 \exists A^0 \exists \xi > 0$ s.t.

suppose
$Q = R(x,t) > r_0^{-2}$
$R(y,t) > AQ$ (large A) & $\varphi(r_0^{-1}) < \xi$

time t:



$$\Longrightarrow \boxed{\mathrm{dist}_t(x,y) > DQ^{-1/2}}$$

part of assumption

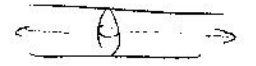
$$\boxed{R(z,t) > r_0^{-2} \quad \forall z \in \gamma}$$

So in natural scale where $R(x,t) = 1$,
y with high curvature must be far away
[like a Lip bound for curvature]

Proof. by contradiction ~~-~~ again

fix D, $A \to \infty$ $\xi \to 0$, counterexamples – take a limit

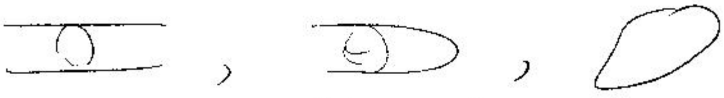
every pt z in γ has canon nbhd by ~~ξ~~ assumption)

claim if $R(z,1)$ much ~~[illegible]~~ more than at x, less than at y –
its canon nbhd is of type  ,

Else -if  

any shortest geod passing thru z
in one direction must end in cpct piece - where curvature
comparable to curvature at z.

(using: the canonical nbhd contains
all of one of the std pictures -

)

(havent seen this completely yet)

so

segment where canon nbhd is cylinder 

~~@~~ ratio of curvatures at beginning & end is large (∃ such interval)

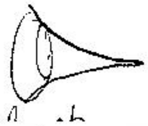
so get some neck of varying radius 

(not saying it's monotone)

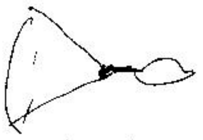
scale so ~~radius~~ curvature at left end is 1.

so by φ-pinching – curv. is almost non negative

~~smt~~ d(x,y) bdd by contradictory assumption

so get -  rad→0 at finite

nonneg sect curv at

maybe 

but nbhd almost rotationally invariant because of canonical nbhds
($Rm \geq 0$ by pinching)

near the singular pt - still canon nbhd,

$\Longrightarrow$ cpctified space still $Rm \geq 0$ in Alexandrov sense

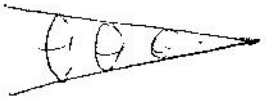
because: no shortest path has the sing. pt in its interior

because ⟵

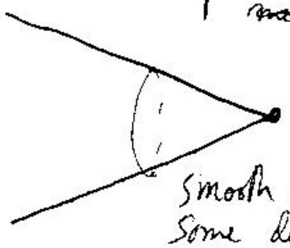
General fact:
loc cpct complete metric X
closed ? $K \subset X$ s.t. ~~if~~ no shortest paths in X have interior in K
& $X \setminus K$ Alex. ≥ 0 then so is X.

gen'l fact for Alexandrov ⇓

$\boxed{\text{tangent cone at sing pt is 3-dim'l}}$ ⟵ i.e. not

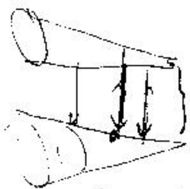
could be 

take blowup - ~~we~~ ~~we~~ get tangent cone which is a metric cone



Smooth away from vertex
Some definite aperture.

Canonical nbhd was in a time-nbhd too.



$\} \Delta t \sim 1$ ⟵ whole sol'n has $Rm \geq 0$

now apply strong max principle:

$(R_{ijkl}) \geq 0$

$$(R_{ijkl})_t = \Delta R_{ijkl} + Q_{ijkl}$$

≥ 0 if $Rm \geq 0$ (Hamilton)

so cone has 0 curv's in certain direction at final time

2-plane U_{ij} where R_{ijkl} is 0
non neg at all prev times

$\Rightarrow (R_{ijkl})_t \leq 0$ at this time in this direction

also $\Delta R_{ijkl} \geq 0$ on U_{ij}
by same arg essentially

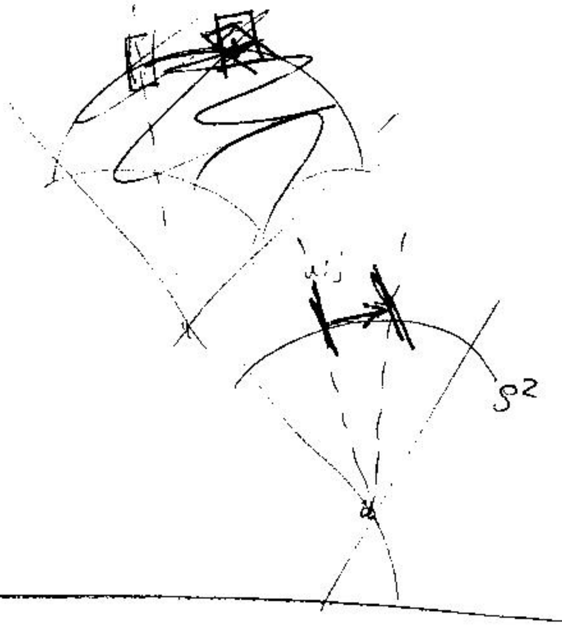
so in fact both sides 0.

$\Rightarrow$ curv. still 0 ~~to~~ 2nd order in space along u_{ij}

not true for metric cone:
if parallel translate U_{ij} in sphere direction, curvature does change to 2nd order.

Contradiction

proves the lemma.



didn't explain:
why in blow up get ~~some~~ smooth lim for Some nbhd of left end

but this is easier.

Back to theorem:

by choice of (x_0, t_0) & Lemma,
after blow up to get $R(x_0, t_0) = 1$, at bdd distances curvature is bounded.

now apply §7 argument to get <u>non collapsing</u>
(need first curv control in some <u>time</u> interval –
given by estimates on $|R_t|$, ~~for~~ $|\nabla R|$ for
~~ancient sol~~ canonical nbhds)

hence can extract a limit –
complete mfd of $Rm \geq 0$.

<u>Curvature bounded</u>:
compact OK
non cpt

pts far away –
if curv $\to \infty$
get <u>necks</u>

because curv $> R(x_0, t_0)$
so by choice of x_0
<u>have</u> canon. nbhds

but non-neg
curvature forbids this.

~~get complete~~

can shift time param to 0:
get sol'n on <u>$(-\delta, 0]$</u>. ← arg. why get <u>some</u> time interval
at 0 – this complete ~~B,a~~ $Rm \geq 0$
metric

so –
extend $-\delta$ to $-\infty$ – all rest will be automatic.

unif curv bd at $-\delta$ – can extend a little backwards.
so – show ~~can~~ curv bd <u>stays bounded</u>

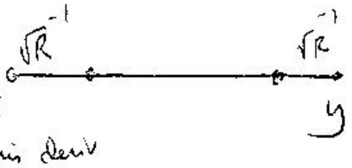
Harnack ⇒ $R_t + \frac{R}{t} \geq 0$.

so if curv control deteriorates at $-T$:

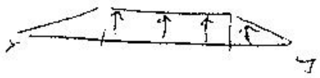
Harnack ⇒ ~~get~~ $R(t) \leq \frac{C}{t+T}$

Hamilton: $\frac{d}{dt} dist_t(x,y) \geq -C\sqrt{R_{max}(t)}$

because

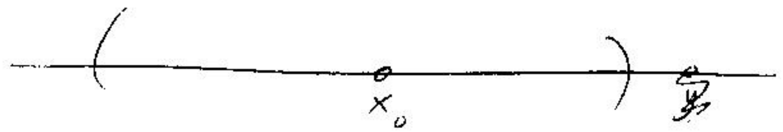


$\sqrt{R}$ controls this deriv

use to construct a variation  that decreases distance if $\frac{d}{dt}$ ~~[illegible]~~ too neg

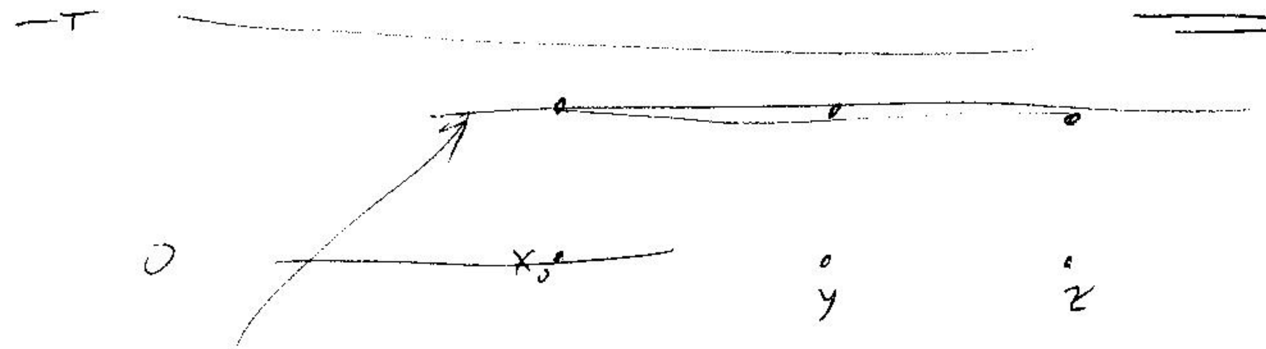
---

$\int \sqrt{R}\, dt$ converges so distances from 0 to $-T$ do not change more than large additive constant.

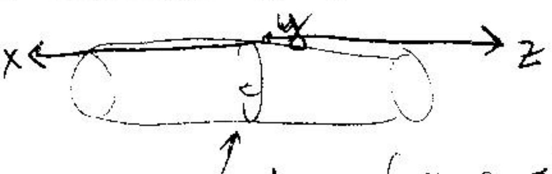


take large ball so outside it & in $(0, -T]$ curv $\leq C$

Why: else - for y far ~~A~~ take z
so $d(x,y) \sim d(y,z) \sim \frac{1}{2} d(x,z)$
because $\exists$ asymptotic cone



here $(x-z) - |x-y| - |y-z)$ relatively small.

if curv(y) large - it has a canonical nbhd of cylinder type 

separates (pos curvature)

downstairs this sphere is even smaller but some small nbhd of y can't separate x from z.

now use the lemma to get bds for all
time <u>inside</u> the C-ball
starting w/ point on outside.

Case where mfld is ~~bdd~~ cpct at time 0 –
similar ...
diam finite so at time T
get <u>some</u> a priori bd on diam
(depending on Harnack ineq &
Hamilton $\overset{\&}{\text{II}}$ distance bound)

→ get some curv. bd at time T
(not going to ∞ for finite T)

↓

<u>get an ancient sol'n.</u>

Wed 4/30 (missed Tues on 12.2)

Corollary 11.6, then 12.3 & 12.4 (roughly corresp to 6.4 & 6.8 in 2nd paper)

Recall:

Ancient sol'ns nonneg curv, κ-non collapsed has 0 volume growth – $\frac{\text{vol}\, B(p,r)}{r^n} \searrow 0$ as $r \to \infty$

◉ $\forall \varepsilon > 0 \ \exists A < \infty$ s.t. given sol'ns to Ricci, $R_m \geq 0$ $\left(\begin{matrix}\text{can assume} \\ R_m \geq -\varepsilon_k Q_k \\ \varepsilon_k \to 0\end{matrix}\right)$

on $M_k \times [t_k, 0]$

$\overset{\Cup}{B(x_k, r_k, 0)}$. $R(x_k, 0) = Q_k$

& $R(x,t) \leq 2Q_k$ in this region

$t_k Q_k \longrightarrow -\infty$

$r_k^2 Q_k \to \infty$

$$\Rightarrow \quad \text{vol}\, B(x_k, A Q_k^{-1/2}, 0) < \left(\varepsilon Q^{-1/2}\right)^3$$

~~Cor~~ Corollary of 0 vol. growth – else pass to limit which is – non-collapsed non-neg curv. ancient sol'n ⇒ contradiction

collapsing at or befor limit – then vol ineq automatic.

◉ $\forall W > 0 \ \exists B(W) > 0, C(W) > 0$ s.t. if soln on $M \times [t_0, 0]$ $R_m \geq 0$

$\overset{\Cup}{B(x_0, r_0, 0)}$ $(\geq -r_0^{-2} \text{ ok})$

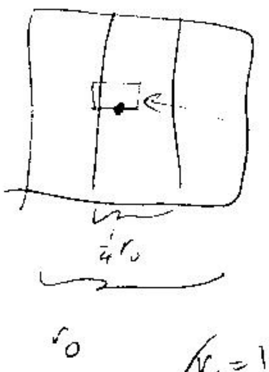
and $\text{Vol}\, B(x_0, r_0, t) \geq W r_0^3$

$$\Rightarrow \quad \text{in } B(x_0, \tfrac{1}{4} r_0, t) \quad R(x,t) \leq C r_0^{-2} + \frac{B}{t - t_0}$$

Pf blowup arg - if concl. fails at a pt for $\overset{a}{\text{seq}}$ $B^{\alpha}, C^{\alpha} \to \infty$ (same W)

$\exists A^{\alpha} \to \infty$

s.t.

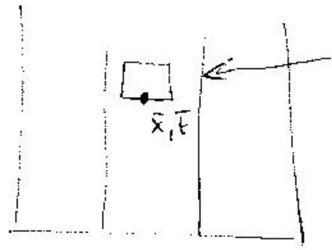


parabolic nbhd where pt has almost max curv:

can find $(\bar{x},\bar{t})$ $\bar{x} \in B(x_0, \frac{1}{3}r_0, \bar{t})$

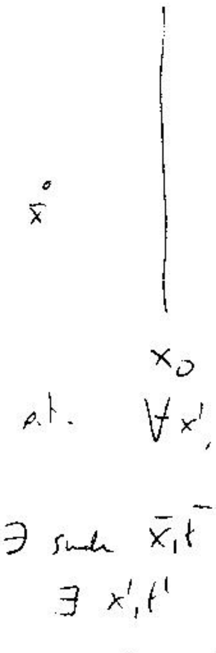
$\begin{pmatrix} r_0 = 1 \\ \text{scaling} \end{pmatrix}$ $Q = R(\bar{x},\bar{t}) > C + \frac{B}{\bar{t}-t_0}$

& $\forall (x',t')$ s.t. $\left\{ \begin{array}{l} \bar{t} - AQ^{-1} \le t' \le \bar{t} \\ x' \in B(\bar{x}, AQ^{-1/2}, \bar{t}) \end{array} \right.$



$R(x',t') < 2Q$

finding $\bar{x},\bar{t}$ (similarly to ~~b~~ arguments made before)



first find $(\bar{x},\bar{t})$ s.t. $\forall x',t'$ with $dist_{t'}(x',x_0) < dist_{\bar{t}}(\bar{x},x_0) + AQ^{-1/2}$

the above condition holds. $(R(x',t') < 2Q)$

either $\exists$ such $\bar{x},\bar{t}$

<u>or</u> $\exists$ x',t' bad ↗ so replace by x',t'.

curvature doubles each time .. formal argument.

if $\exists$ x',t' - claim it satisfies ~~dsm~~ desired condition

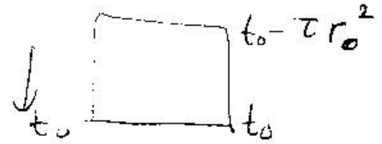
(~~[illegible]~~ (didn't follow this point)

Now given $\bar{x}, \bar{t}$ apply previous statement to this
get upper bd on volume of little ball -
but lower bd on vol of big ball
contradicts Bishop - Gromov comparison thm.

In fact don't need vol $\geq w r_0^3$ for <u>every</u> t.
suffices ~~$Vol(B(x_0, r_0, 0)) \geq w r_0^3$~~.
$\exists\ \tau(w) > 0$ s.t. if $t_0 \leq \tau(w_0) r_0^2$
and vol $B(x_0, r_0, 0) \geq w r_0^3 \Rightarrow$
$R(x, t) \leq C r_0^{-2} + \frac{B}{t - t_0}$

12.3 (discussion)

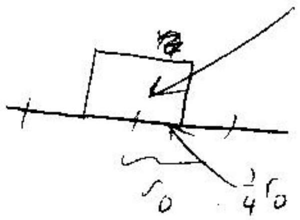
This was for volume control at $\overset{\text{final time}}{t=0}$ but positivity for all $t \in [t_0, 0]$.
now -
suppose $Rm \geq -r_0^2$, vol $\geq w r_0^3$ at t_0
$B(x_0, r_0, t_0)$.



Conclusion: $|Rm| \leq K r_0^{-2}$

also $r_0 < \bar{F} \sqrt{t_0}$

<u>Quantification</u> - $\forall w\ \exists\ \varepsilon_0 \bar{F}, \tau, K$
s.t. this situation
yields conclusion
$|Rm| \leq K r_0^{-2}$



if false - $\exists w$,

seq flows

$\bar{r}^\alpha \to 0$

$\tau^\alpha \to 0$

$K^\alpha \to \infty$

pts & counterexamples

look for earliest time & smallest ball giving counterexamples.

(for smooth sol'ns - no problem
for sol'ns w/<u>surgeries</u> - use canonical neighborhood control)

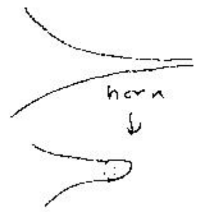
$\Bigg\{$ got lost.

5/1/03 Thurs.

2nd paper

similar ideas but consider sol'ns with surgeries.
(& some corrections)
(not discussing 1st section)

2nd section. surgery -



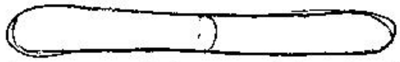
rescale the picture -

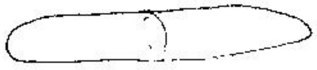
in the limit -  ~~co~~ $\frac{1}{2}$-cyl + smooth cap

spherically symmetric

have to study sol'n to flow that starts from this data
in order to understand continuation of flow after surgery.

add assumption - curvature bdd at each time -
so then sol'n is unique.

can double a long piece 
and consider solution to this (short time)
limit with basept on end ~~[illegible]~~
(with curvature bounds)
gives sol'n to infinite cylinder

if $R = 1$ (normalize)
sol'n exists for time $[0,1)$;
get it for short times as lim
of sol'ns for 

(true ~~also~~ orig. for doubly infi. cyl)
where it is exactly 1
(limit is $\mathbb{R}$)

if singularities develop for $T < 1$; T largest s.t. $\exists$ unif estimates for all $[0, t]$ $t < T$
take $T' < T$, close to T
have unif est for $[0, T']$

so if go far enough (given T') - s'ln is close to cylinder

curvature ~ curv of cylinder = $\frac{1}{1-T'}$,

curv & vol control

apply <u>pseudo locality theorem</u>:

extend w/ curv & vol ctrl to interval depending on $\frac{1}{1-T'} < \frac{1}{1-T}$ i.e. on T

$\Delta t = \varepsilon(1-T)$

can ~~make~~ choose $T' - T < \varepsilon(1-T)$

so in limit at time T get something smooth away from the tips. But no smooth lim at time t.

non neg curv $\Rightarrow$ conical pt

ruled out by strong max principle (as in pf of 12.1)

why $T>0$:

because $\frac{d}{dt}\sup|R_m| \le c|R_m|^2$

<u>generally</u> -

so bound on curvature implies $\exists$ sol'n on all cpct approximation - for <u>definite</u> time & all L.

<u>recap:</u> start w/ $R=1$

consider $\{t$ s.t. sol'n exists on $[0,t]$ for all L large & $\exists K(t)$ s.t. $R \le K\}$

~~tt~~ claim $T = \sup t = 1$

~~then [illegible]~~

if $T<1$ if $T'<T$ close

chosen L s.t. outside L-nbhd sol'n is ε-close to cylinder sol'n.

& pseudo locality $\Rightarrow$ control <u>outside</u> L-nbhd of ends.

~~then get to~~ but rule out cone pt

$\Rightarrow$ OK

want - parabolic nbhds, i.e. with some time back $[t, 0]$ ...
have this away from the cap, at the cylinder pts
(don't understand this) at the cap - no such possibility.

§ 3 sol'n smooth on $[T^-, T^+)$
and curv $\nearrow \infty$ at T^+.

$$\Omega = \{ x : R(x,t) \text{ bdd as } t \to T^+ \}$$

claim - Ω open
and $\forall\, y \in M \setminus \Omega \quad R(y,t) \to \infty$ as $t \to T^+$.

using 12.1 :

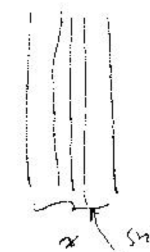
if $R(x,t) \le K$ in segment $(\,, T^+)$
* if $R \ge r^{-2}$ $\exists$ canon nbhd,

$$|\nabla R| \le \eta R^{3/2}$$
$$|R_t| \le \eta R^2$$

so can extend curv bds to a nbhd of definite size.

if $R < r^{-2}$
no canon nbhd
but then -
nearby pts $\le C \cdot r^{-2}$
else grad bds
give control

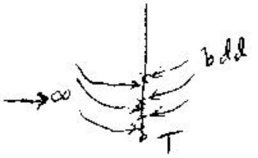
$\rightarrow$



still small nbhd of x stays in this size by Lipschitz control

$\Rightarrow \Omega$ open.

now if curv. alternates - $\nearrow \infty$ bdd along y.t
use same grad bds (R_t).

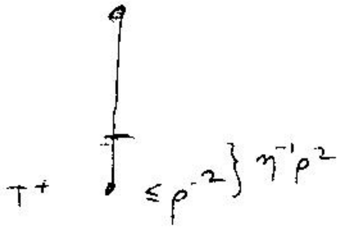


on Ω - get also bds on derivs of curvature (Hsu? Shi?)

$\rho<r$

$\Omega_\rho = \{x\in\Omega : R(x,T^+) \le \rho^{-2}\}$

claim Ω_ρ compact.



get proportional bd $100\cdot\rho^{-2}$

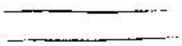
$\Rightarrow$ map from time T^- to T^+ at x is bilipschitz. (unif)

$\therefore \Rightarrow$ ~~get com~~ Ω_ρ cpct.

<u>ends of Ω</u>

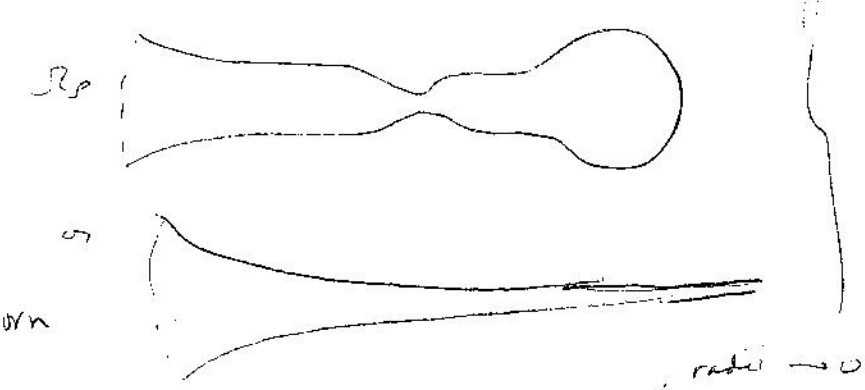
$x\in\Omega$, $R(x,T^+) > \rho^{-2} > r^{-2}$

so has a canonical nbhd

neck 

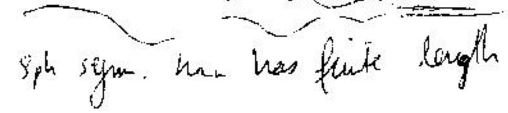
finger 

if neck - continue



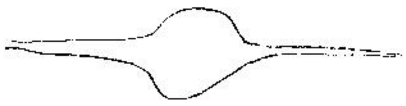
finitely many such pieces - because ~~[illegible]~~ each has definite volume.

radii $\to 0$ (volume bound)



conj - every sph sym. horn has finite length

components of Ω containing no pt. of Ω_ρ -
double horn



or

fix $\delta>0$
$(\rho<\delta r)$ — just $\rho=\delta r$
choose components of Ω that touch Ω_ρ
each has finitely many (horn) ends

cut at radius h (chosen later)

& attach cap

⋮

---

§4 - choosing $h>0$ (dep on δ & r)
such that $\forall$ component of Ω meeting Ω_ρ
$\forall$ horn of Ω & x in horn with

$$R(x,T^+) \geq h^{-2}$$

x has a canonical nbhd which is a strong δ-neck
& ~~for~~ same for time $\overset{\overset{r_x^2}{\shortparallel}}{R^{-1}(x,T^+)}$ before

(defines r_x)

i.e. parab. nbhd

$P(x, \delta^{-1} r_x, T^+, -r_x^2)$ after scaling δ-close to round cylinder

$\boxed{\text{Stmnt - } \exists \text{ such an } h}$

(r depends on initial data & time, originally now assume it is given)

strong δ-neck
↓
almost round cylinder, δ-close after rescaling to std round cylinder of rad 1 length 8 δ^{-1}
↓
with many derivs (δ^{-1})

pf - by blowup.

we already have canonical nbhd of quality ~~δ~~ ε -

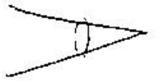
need to improve to δ,

check cylinder not ~~cap~~ finger

* no collapse

curvature bounds (after rescaling) -

as in 12.1.

needed to rule out cone  ? but not nec here.

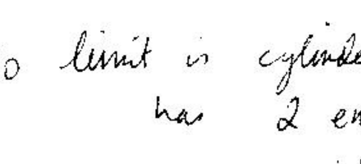
~~eg~~ cylinder ⇒ has parabolic nbhd.

finger - then can't be a cone (they are different)

⇒ get curvature control.

so in limit - noneg curved complete mfd

by curvature pinching since $h \to 0$.



but came from a horn -

in → direction

curv → ∞

rules out the two latter cases.

So limit is cylinder

has 2 ends

Toponogov thm - it splits $\mathbb{R} \times S^2$

↑ ε-round but not known to be δ-round.

extend backwards in time - sim to before - need to check curvature bdd for finite backwd time intervals
↓
sol'n on ancient time interval that splits -
so S^2 is <u>round</u>.

end of pf by contradiction.

---

two technical pts:

after surgery - need to check <u>pinching</u> estimate:

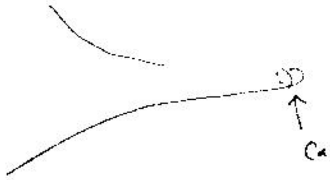
$$(Rm)_t = \Delta Rm + Q$$

implies pinching propagates forward

so just check it doesn't get worse <u>at</u> the surgery.

(done by Hamilton in harder situation)

idea



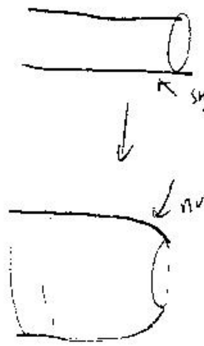
can put metric in explicit form by using a conformal factor times original metric - ≥1 on part we keep, <1 on cap.

?

$R + \text{Hess} f + \text{smaller}$ [illegible] e^{-f} conf. factor

again -

truncate horn



so pinching improves here

then glue, cap by hand

Conclusion -
r, δ, h depend on <u>time</u>
indept of # surgeries
⇓ finitely many surgeries in <u>finite</u> time by volume consideration.

Perelman

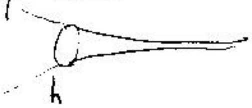
Fri 3/2

Most technically complicated part of paper 2:
Justification of apriori assumpt large curvature $\Rightarrow$ canon. nbhds,
<u>even in presence of surgeries</u>. (analogous to **12.1** in 1st paper)

$\varepsilon > 0$ fixed
$\exists\ r(t),\ \kappa(t),\ \bar{\delta}(t) > 0$ (decreasing) s.t.

if: at $t=0$ $|Rm| \le 1$ $\quad \mathrm{vol}\, B(1) \ge \frac{1}{2}\omega_3$
Ricci flow, $\to$ surgery at time t with <u>parameter</u> $0 < \delta(t) \le \bar{\delta}(t)$
$\to$ continue, etc.

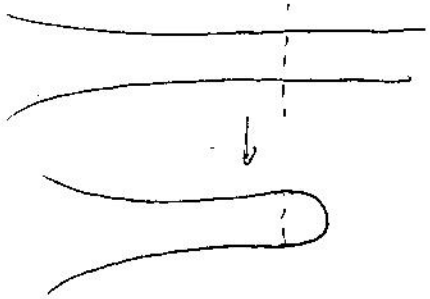
(controls upper radius h of horn



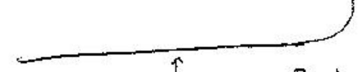
then this "sol'n" $\exists$ for all time, at time t
is $\kappa(t)$-noncollapsed, and if $R(x,t) > r(t)^{-2}$
then (x,t) has a canonical nbhd with quality ε

ability to choose
$\delta < \bar{\delta}$
used in §8
as with monotonicity of $\bar{\lambda}$

Recall
surgery -



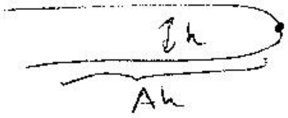
Yesterday explained sol'n on "ideal finger"



exists on $[0,1)$
exactly

as $t \to 1$
$\inf R \ge \frac{C}{1-t}$

<u>at time T_0</u>
on cylinder is δ-close to round cylinder
want to fix $B(p, Ah, T_0)$
for large A on $[T_0, (1-\theta)h^2 + T_0]$



$\forall \delta' > 0 \quad \exists \hat{\delta} > 0$ s.t. if $\delta < \hat{\delta}$ then sol'n after the surgery is δ'-close to scaled std sol'n.

possibly – another surgery [illegible] before $(1-\theta)h^2 + T_0$ on this ball.
might remove all a part of this finger?

[if not – estimate is OK]

or – at some T_1 $\exists$ surgery affecting this ball:
either disjoint or cuts off whole finger

for $[T_0, \eta^{-1}h^2 + T_0]$ ~~we are OK~~,

get curvature bds → repeat

$2\eta^{-1}h^2 + T_0$ } not following justification } ?

etc, bdd # steps to T_1

canon nbhd assumption – if surgery happens after one step – can't be in the region we're looking at.

now need:

estimate (reduced) length of curves in space-time that run into fingers

$\gamma(t)$ $\quad$ $\gamma(T_0)$ close to cap

$$\mathrm{dist}(\gamma(T_0), p) < 100\, h(T_0).$$

① $\exists T' < T_1$ $\quad \gamma(T') \in \partial B(p, Ah, T')$

② $[T_0, [\underbrace{(1-\theta)h^2 + T_0}_{T_\gamma}]$ $\quad \gamma(T_0 + (1-\theta)h^2) \in B(p, Ah, T)$

interested in $\displaystyle\int_{T_0}^{T_\gamma} \left(R(\gamma(t),t) + |\dot\gamma(t)|^2\right) dt > \ell$

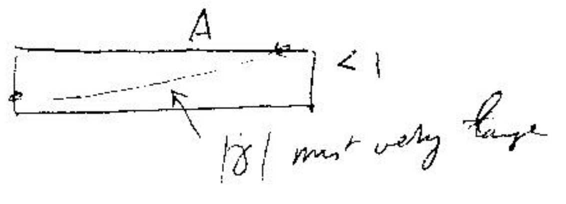
$\forall \ell \ \exists A, \theta$ s.t. ↑

recall $\inf R \geq \frac{c}{1-t}$ as $t \to 1$ (scale so $h=1$)

$\int_0^\theta \frac{c}{1-t}\,dt = -c\log(1-\theta)$ can be made as large as want

so ② can be made to hold by choice of θ.

Choose A very large



Reduced length ~~$\sqrt{t}\int$~~ $\frac{1}{\sqrt{t}}\int\sqrt{\tau'}\,(R+|\dot{\gamma}|^2)\,d\tau'$.

?

Pf of the result is in §5 :

$t \in [0,\varepsilon]$

$I_j = [2^j\varepsilon, 2^{j+1}\varepsilon]$ construct $r(t), \kappa(t), \bar{\delta}(t)$ ~~for~~ inductively on these intervals.

i.e. $r_j, \kappa_j, \bar{\delta}_j$ on I_j

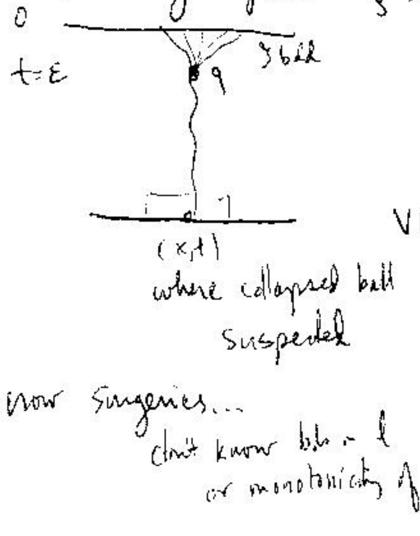
then next r_{j+1} κ_{j+1} $\bar{\delta}_{j+1}$

~~δ~~ $\bar{\delta}$ redefined once...

i.e. $\bar{\delta}_j$ doesn't work on I_j only $I_{<j}$?

So on all prev. intervals κ_j-noncollapsing.

use arg from §7.3, first paper:



$\ell(q, t_0-\varepsilon) \leq 3/2$

$V(\tau)$ reduced volume small

now surgeries...

don't know bds on ℓ

or monotonicity of reduced volume,