Uniqueness of standard solutions in the work of Perelman

Peng Lu and Gang Tian

The short time existence of Ricci flow on complete noncompact Riemannian manifolds with bounded curvature is proven by Shi [Sh1]. The uniqueness of such solutions is a difficult problem. Hsu studies this problem in dimension two [Hs], otherwise there is not any result about this problem. In the fundamental paper [Pe2] Perelman discussed a special family of solutions of Ricci flow on $\mathbb{R}^3$, the so-called **standard solutions**, the solutions are used to construct the geometric-topological surgeries and their uniqueness are used to construct the longtime existence and to study the properties of the **Ricci flow with surgery**. A particular nice feature about these solutions is that at space infinity these solutions are asymptotic to round infinity cylinder.

In [Pe2] §2 Perelman gives a proof of the uniqueness of the standard solutions. The idea is to reduce the Ricci flow equation to

$$\begin{aligned} f_t &= f'' + a_1 f' + b_1 g' + c_1 f + d_1 g \\ g_t &= a_2 f' + b_2 g' + c_2 f + d_2 g \end{aligned}$$

by using the rotational symmetry of the solutions, then prove the uniqueness of solutions of above equations for given initial data. However there is difficulty to bound the coefficients in above equations near the origin of $\mathbb{R}^3$, the first named author thanks John Lott for discussions in understanding this difficulty.

In this paper we give a proof of the uniqueness of standard solutions through the uniqueness of the DeTurk-Ricci flow. The general idea of using DeTurk-Ricci flow to prove the uniqueness of the Ricci flow is due to Hamilton [H95b] §6. Our proof uses the rotationally symmetry of the standard solutions which is used to prove the short time existence and the certain asymptotic behavior of harmonic map flow.

We make two remarks. First through a private communication we have learned that Perelman has similar idea of using the DeTurk-Ricci flow to prove the uniqueness; Second properties of standard solutions are proven in [Pe2] §2, for completeness we include here proofs of those properties needed to show the uniqueness.

1 Standard solutions

Let integer $n \geq 3$. Denote $S^{n-1}(r)$ the round $(n-1)$-sphere of radius r and let $d\sigma$ be the standard metric on $S^{n-1}(1)$. Let $\mathcal{G}_n$ be the set of rotationally symmetric metric g_0 on $\mathbb{R}^n$ which satisfies the following condition:

(i) The curvature operator of g_0 is nonnegative and is positive at some point;

(ii) The curvature $|Rm_{g_0}|$ and its derivatives $\left|\nabla^i Rm_{g_0}\right|, i = 1, 2, 3, 4$ are bounded;

(iii) There is a sequence of points $y_i \to \infty$ in $\mathbb{R}^n$, $(\mathbb{R}^n, g_0, y_i)$ converges to $\mathbb{R} \times S^{n-1}\left(\sqrt{2(n-2)}\right)$ in pointed C^3 Cheeger-Gromov topology.

We construct a rotationally symmetric metric to show that $\mathcal{G}_n$ is nonempty. Let $\left(\theta^1, \cdots, \theta^{n-1}\right)$ be normal coordinates on $S^{n-1}(1)$, then $\left(\theta^1, \cdots, \theta^{n-1}, r\right)$ are local coordinates on $\mathbb{R}^n$. Consider rotationally symmetric complete metric $dr^2 + f(r)^2 d\sigma$ on $\mathbb{R}^n$, the curvatures are given by

$$\begin{aligned} R_{ijji} &= f^2 - f^2 (f')^2 & \qquad R_{ijkn} &= 0 \\ R_{innj} &= 0 & \qquad R_{inni} &= -ff'' \end{aligned} \tag{1}$$

where $1 \leq i \neq j, k \leq n-1$ (see, for example, [BW] §9). We choose a smooth convex function $f_0(r)$ satisfies

$$f_0(r) = \begin{cases} \sin r & \text{if } 0 \leq r \leq \frac{\pi}{100} \\ \sqrt{2(n-2)} & \text{if } r \geq \sqrt{3n} \end{cases}.$$

Then clearly metric $g_* \doteqdot dr^2 + f_0(r)^2 d\sigma$ has nonnegative sectional curvature. Using (1) it is easy to check that g_* satisfies (i), condition (ii) and (iii) also hold and hence $\mathcal{G}_n$ is not empty.

Given $g_0 \in \mathcal{G}_n$, since g_0 is complete and has bounded curvature tensor, by Shi's existence theorem ([Sh1]) there is a solution $g_*(t), t \in [0, T_*], T_* > 0$ of the Ricci flow

$$\frac{\partial g}{\partial t} = -2R_{ij}, \;\; g(0) = g_0 \tag{2}$$

on $\mathbb{R}^n$. $g_*(t)$ has uniform bounded curvature $\sup_{\mathbb{R}^n \times [0,T_*]} \left|Rm_{g_*(t)}(x)\right| < +\infty$ and $g_*(t), t > 0$ has positive curvature operator ([Sh2] Theorem 4.14). We call any solution $g(t), t \in [0, T], T > 0$ of (2) with $\sup_{\mathbb{R}^n \times [0,T]} \left|Rm_{g(t)}(x)\right| < +\infty$ a **standard solution**, again $g(t), t > 0$ has positive curvature operator ([Sh2] Theorem 4.14). The **uniqueness problem** of the standard solutions is to show that $g(t) = g_*(t)$ on $t \in [0, \min\{T_*, T\}]$.

To prove the uniqueness, we need to establish a few properties of standard solutions.

1.1 Asymptotic behavior of standard solutions $g(t)$ at space infinity

Let $y_i \to \infty$ be the sequence of points in $\mathbb{R}^n$ such that $(\mathbb{R}^n, g_0, y_i)$ converges to $\mathbb{R} \times S^{n-1}\left(\sqrt{2(n-2)}\right)$ in pointed C^3 Cheeger-Gromov topology.

Lemma 1 *There is a subsequence of* $(\mathbb{R}^n, g(t), y_i), t \in [0,T]$ *which converges in pointed* C^3 *Cheeger-Gromov topology to the round cylinder solution*

$$dr^2 + 2(n-2)(1-t)\,d\sigma$$

on $\mathbb{R} \times S^{n-1}$*. In particular* $T < 1$*.*

Proof. We will apply Hamilton's Cheeger-Gromov type compactness theorem [H95a], but we need to make some modification of the compactness theorem since time 0 is not an interior point of $[0,T]$. In general if there is a sequence of complete solution of the Ricci flow $(M_k^n, g_k(t), p_k), t \in [0,T], k \in \mathbb{N}$ satisfying

$$\sup_k \sup_{M_k^n} \left|\nabla^i Rm_{g_k(0)}(x)\right| < +\infty, \quad 0 \le i \le i_0 + 1 \text{ for some } i_0 \in \mathbb{N}$$
$$\sup_k \sup_{M_k^n \times [0,T]} \left|Rm_{g_k(t)}(x)\right| < +\infty,$$

one can improve Shi's derivative estimate and get

$$\sup_k \sup_{M_k^n \times [0,T]} \left|\nabla^i Rm_{g_k(t)}(x)\right| < +\infty \quad i \le i_0 + 1.$$

We will give a proof of this estimate in the appendix at the end. This estimate implies that in normal coordinates the C^{i_0+1}-norm of metric tensor $g_k(t)$ are bounded independent of k, t. Suppose we have injectivity radius bound $i_{g_k(0)}(p_k) \ge \delta > 0$, then one can follow the proof of compactness theorem in [H95a] to conclude the following. There is a subsequence $(M_{k_j}^n, g_{k_j}(t), p_{k_j}), t \in [0,T]$ which converges in pointed C^{i_0} Cheeger-Gromov topology to an complete solution of the Ricci flow $(M_\infty^n, g_\infty(t), p_\infty), t \in [0,T]$, with bounded curvature tensor and $i_{g_\infty(0)}(p_\infty) \ge \delta$.

Applying this compactness statement to $(\mathbb{R}^n, g(t), y_i), t \in [0,T]$, we conclude that there is a subsequence y_{i_j} (still denoted by y_i) such that

$$(\mathbb{R}^n, g(t), y_i) \to (M_\infty, g_\infty(t), y_\infty), \qquad t \in [0,T]$$

in the C^3-topology. Note that by assumption (iii) $(M_\infty, g_\infty(0))$ is isometric to round cylinder $\mathbb{R} \times S^{n-1}\left(\sqrt{2(n-2)}\right)$. $(M_\infty, g_\infty(t))$ has nonnegative curvature operator because $g(t)$ has nonnegative curvature operator.

Since M_∞ has two ends, $(M_\infty, h_\infty(t))$ has a line for each $t \in [0,T]$. From the local version of Hamilton's strong maximal principle ([**?**] §8), the line direction is preserved by the Ricci flow. By the Toponogov splitting theorem the metric $g_\infty(t), t \in [0,T]$ splits and has the following form

$$g_\infty(t) = dr^2 + g_{S^{n-1}}(t),$$

where $g_{S^{n-1}}(t)$ is the solution of the Ricci flow on sphere S^{n-1} with initial metric $2(n-2)\,d\sigma$. It follows from the uniqueness of Ricci flow solution on closed manifold that $g_{S^{n-1}}(t) = 2(n-2)(1-t)\,d\sigma$. ■

1.2 Standard solutions are rotationally symmetric

Lemma 2 *Let X be a vector field evolving by*

$$\frac{\partial}{\partial t}X^i = \Delta X^i + R^i_k X^k \tag{3}$$

and $V_{ij} \doteqdot \nabla_i \left(g_{jk}X^k\right) = g_{jk}\nabla_i X^k$, *then V evolves by*

$$\frac{\partial}{\partial t}V_{ij} = \Delta V_{ij} - 2R_{ikjl}V_{kl} - R_{ik}V_{kj} - R_{jk}V_{ik}. \tag{4}$$

Proof. The evolution equation of $X_i \doteqdot g_{ik}X^k$ is

$$\begin{aligned}\frac{\partial}{\partial t}X_i &= -2R_{ik}X^k + g_{ik}\frac{\partial}{\partial t}X^k \\ &= -2R_{ik}X^k + g_{ik}\left(\Delta X^k + R^k_j X^j\right) \\ &= \Delta X_i - R_{ik}X_k.\end{aligned}$$

We compute

$$\begin{aligned}\frac{\partial}{\partial t}V_{ij} = \frac{\partial}{\partial t}\left(\nabla_i X_j\right) &= -\left(\frac{\partial}{\partial t}\Gamma^l_{ij}\right)X_l + \nabla_i\left(\frac{\partial}{\partial t}X_j\right) \\ &= \left(-\nabla_l R_{ij} + \nabla_i R_{jl} + \nabla_j R_{il}\right)X_l + \nabla_i\left(\Delta X_j - R_{jk}X_k\right).\end{aligned}$$

From

$$\begin{aligned}\nabla_i\left(\Delta X_j\right) &= \nabla_k\nabla_i\nabla_k X_j - R_{ikkl}\nabla_l X_j - R_{ikjl}\nabla_k X_l \\ &= \nabla_k\left(\nabla_k\nabla_i X_j - R_{ikjl}X_l\right) - R_{il}\nabla_l X_j - R_{ikjl}\nabla_k X_l \\ &= \Delta V_{ij} - X_l\nabla_k R_{ikjl} - 2R_{ikjl}V_{kl} - R_{il}V_{lj},\end{aligned}$$

we get

$$\begin{aligned}\frac{\partial}{\partial t}V_{ij} = {} & \Delta V_{ij} - 2R_{ikjl}V_{kl} - R_{ik}V_{kj} - R_{jk}V_{ik} \\ & - X_l\nabla_l R_{ij} + X_l\nabla_j R_{il} - X_l\nabla_k R_{ikjl}.\end{aligned}$$

(4) follows from the second Bianchi identity $0 = \nabla_k R_{ikjl} + \nabla_j R_{iklk} + \nabla_l R_{ikkj} = \nabla_k R_{ikjl} - \nabla_j R_{il} + \nabla_l R_{ij}$. ■

Let $h_{ij} \doteqdot V_{ij} + V_{ji}$. It follows from (4) that

$$\frac{\partial}{\partial t}h_{ij} = \Delta_L h_{ij}, \tag{5}$$

where $\Delta_L h_{ij} \doteqdot \Delta h_{ij} + 2R_{iklj}h_{kl} - R_{ik}h_{kj} - R_{jk}h_{ki}$ is the **Lichnerowicz Laplacian**. A simple calculation shows that there is a constant $C > 0$ such that

$$\left(\frac{\partial}{\partial t} - \Delta\right)\left|h_{ij}\right|^2 = -2\left|\nabla_k h_{ij}\right|^2 + 4R_{ijkl}h_{jk}h_{il} \tag{6}$$

$$\frac{\partial}{\partial t}\left|h_{ij}\right|^2 \leq \Delta\left|h_{ij}\right|^2 - 2\left|\nabla_k h_{ij}\right|^2 + C\left|h_{ij}\right|^2. \tag{7}$$

Note $X^i(t)$ is Killing vector fields for $g(t)$ if and only if $h_{ij}(t) = 0$. For any given Killing vector field $X^i(0)$ for metric $g(0)$, (3) has a bounded solution $X^i(t)$ for $t \in [0, T)$. Then $|h_{ij}(t)|^2$ is a bounded function and satisfying (7) and $|h_{ij}|^2(0) = 0$. Since the metric $g(t)$ has bounded sectional curvature, we can apply the maximum principle to $|h_{ij}|^2$ on complete manifolds ([Sh2] Theorem 4.6) to conclude that $h_{ij}(t) = 0$ for all $t \geq 0$. So $X^i(t)$ is Killing vector fields for $g(t)$.

The following is a very nice observation of Bennett Chow, we thank him for allowing us to use it here. From $h_{ij} = 0$ we have $\nabla_j X^i + \nabla_i X^j = 0$. Taking ∇_j derivative and summing over j we get $\Delta X^i + R^i_k X^k = 0$ for all t. Hence (3) gives $\frac{\partial}{\partial t} X^i = 0$ and $X^i(t) = X^i(0)$, i.e., the rotation group $O(n)$ of $\mathbb{R}^n$ are contained in the isometry group of $g(t)$. We conclude

Lemma 3 *The standard solution $g(t), t \in [0, T]$ are rotationally symmetric.*

2 From Ricci flow to DeTurk-Ricci flow

In this section we discuss the DeTurk-Ricci flow and the harmonic map flow.

2.1 Converting Ricci flow solutions to Deturk-Ricci flow solutions

Let $(M^n, h(t)), t \in [0, T]$ be a solution of the Ricci flow and let $\psi_t : M \to M, t \in [0, T_1]$ be a solution of harmonic map flow

$$\frac{\partial \psi_t}{\partial t} = \Delta_{h(t), h(0)} \psi_t, \quad \psi_0 = Id. \tag{8}$$

In local coordinates (x^i) on domain M and (y^α) on target M, the harmonic map flow (8) can be written as

$$\left(\frac{\partial}{\partial t} - \Delta_{h(t)}\right) \psi^\alpha(x, t) = h^{ij}(x, t) \Gamma^\alpha_{\beta\gamma}(\psi(x, t)) \frac{\partial \psi^\beta(x, t)}{\partial x^i} \frac{\partial \psi^\gamma(x, t)}{\partial x^j} \tag{9}$$

where $\Gamma^\alpha_{\beta\gamma}$ is the Christoffel symbols of $h(0)$.

Suppose $\psi(x, t)$ is a solution with bounded $|\nabla \psi|_{C^2}$ norm, then $\psi(t), t \in [0, T_1]$ are diffeomorphisms when $T_1 > 0$ is small. For $0 \leq t \leq T_1$, define $\hat{h}(t) \doteqdot \left(\psi_t^{-1}\right)^* h(t)$, then $\hat{h}(t)$ is a solution of **Ricci–DeTurck flow** by (see [D], [H95b] or [CK] Chapter 3 for details)

$$\frac{\partial}{\partial t} \hat{h}_{ij} = -2\hat{R}_{ij} + \hat{\nabla}_i W_j + \hat{\nabla}_j W_i \qquad \hat{h}(0) = h(0),$$

where $\hat{R}_{ij}$ and $\hat{\nabla}_i$ are the Ricci curvature and Levi-Civita connection of $\hat{h}$ respectively and the time-dependent 1-form $W = W\left(\hat{h}\right)$ is defined by

$$\left(W(\hat{h})\right)_j \doteqdot \hat{h}_{jk}\hat{h}^{pq}\left(\hat{\Gamma}^k_{pq} - \Gamma^k_{pq}\left(h(0)\right)\right).$$

In local coordinates the Ricci-DeTurk flow takes the following form ([Sh1] Lemma 2.1)

$$\begin{aligned}\frac{\partial \hat{h}_{ij}}{\partial t} &= \hat{h}^{kl}\nabla_k\nabla_l\hat{h}_{ij} - \hat{h}^{kl}h(0)_{ip}\hat{h}^{pq}R_{jkql}\left(h(0)\right) - \hat{h}^{kl}h(0)_{jp}\hat{h}^{pq}R_{ikql}\left(h(0)\right) \\ &+ \frac{1}{2}\hat{h}^{kl}\hat{h}^{pq}\left[\begin{array}{c}\nabla_i\hat{h}_{pk}\nabla_j\hat{h}_{ql} + 2\nabla_k\hat{h}_{jp}\nabla_q\hat{h}_{il} \\ -2\nabla_k\hat{h}_{jp}\nabla_l\hat{h}_{iq} - 2\nabla_j\hat{h}_{pk}\nabla_l\hat{h}_{iq} - 2\nabla_i\hat{h}_{pk}\nabla_l\hat{h}_{jq}\end{array}\right]. \end{aligned} \tag{10}$$

where ∇ is the Levi-Civita connection of $h(0)$. This is a parabolic system.

Recall the following derivative estimate for solution $\hat{h}(t)$ of the Ricci-DeTurck flow from [Sh1] Lemmas 4.1 and 4.2, one can check easily that the γ dependence of constant C below is not necessary.

Proposition 4 *Let* $\left(M^n, \hat{h}\left(t\right)\right)$, $t \in [0,T)$, *be a solution of the Ricci-DeTurck flow. For a given* $m \in \mathbb{N}$, *suppose*

$$\sup_{x \in B_{\tilde{g}}\left(x_0, \gamma + \frac{\delta}{m+1}\right), i \le k} |\nabla^i_{\hat{h}(0)} Rm_{\hat{h}(0)}|_{\hat{h}(0)} \le B_k.$$

There exists a constant $C = C\left(n, m, \delta, T, B_k\right)$ *depending only on* n, m, δ, T, B_k *such that if*

$$\left(1 - \frac{1}{256000n^{10}}\right)\hat{h}(0) \le \hat{h}\left(t\right) \le \left(1 + \frac{1}{256000n^{10}}\right)\hat{h}(0), \quad 0 \le t \le T$$

then

$$\left|\tilde{\nabla}^m\hat{h}\right|_{\hat{h}(0)} \le C \quad \text{in } B_{\hat{h}(0)}\left(x_0, \gamma + \frac{\delta}{m+1}\right) \times [0,T) \tag{11}$$

2.2 Solutions of harmonic map flow

In this section we study the existence of harmonic flow (8) and its asymptotic behavior at the space infinity when $h(t) = g(t)$ is a standard solutions. Here we use of the rotationally symmetric property and asymptotic property at infinity of $g(t)$.

Let $\theta = (\theta^1, \cdots, \theta^{n-1})$ be local coordinates on the round $(n-1)$-sphere of radius 1, and let $d\sigma$ the volume form on the sphere. Since $g(t)$ is rotationally symmetric and $n \ge 3$, we can write

$$g(t) = dr^2 + f(r,t)^2 d\sigma \qquad g_0 = dr^2 + f_0(r)^2 d\sigma \tag{12}$$

where r be the radial coordinate on $\mathbb{R}^n$ depending on time t, i.e. $\frac{\partial r}{\partial t} \neq 0$. It is clear that $f(r,0) = f_0(r)$. We want to solve (8) by maps of form

$$\phi(t) : \mathbb{R}^n \to \mathbb{R}^n \qquad (r,\theta) \to (\rho(r,t), \theta). \tag{13}$$

2.2.1 The harmonic map flow equation. Using (12) and (13) it is easy to calculate the energy functional

$$
\begin{aligned}
E(\phi(t)) &\doteqdot \frac{1}{2}\int_{\mathbb{R}^n} |\nabla\phi(t)|_{g(t),g_0}\, dV_{g(t)} \\
&= \frac{1}{2}\int_{\mathbb{R}^n} \left[\left(\frac{\partial\rho}{\partial r}\right)^2 + (n-1)f_0^2(\rho)f^{-2}(r,t)\right] dV_{g(t)}.
\end{aligned}
$$

If we have a compact-supported variation $\delta\rho = w$, then

$$
\begin{aligned}
&\delta E(\phi(t))(w) = \frac{1}{2}\int_{\mathbb{R}^n} \left[2\frac{\partial\rho}{\partial r}\frac{\partial w}{\partial r} + 2(n-1)f_0(\rho)\frac{\partial f_0}{\partial\rho}f^{-2}(r,t)w\right] dV_g(t) \\
&= \int_0^{+\infty} \left[f^{n-1}(r,t)\frac{\partial\rho}{\partial r}\frac{\partial w}{\partial r} + (n-1)f_0(\rho)\frac{\partial f_0}{\partial\rho}f^{n-3}(r,t)w\right] dr \cdot \int_{S^{n-1}} dV_{d\sigma} \\
&= \int_{\mathbb{R}^n} \left[-f^{1-n}\frac{\partial}{\partial r}\left(\frac{\partial\rho}{\partial r}f^{n-1}\right) + (n-1)f_0(\rho)\frac{\partial f_0}{\partial\rho}f^{-2}(r,t)\right] w dV_g(t).
\end{aligned}
$$

Hence for rotationally symmetric maps the harmonic map flow equation (8) has the following form

$$
\begin{aligned}
\frac{d\rho}{dt} &= \frac{1}{f^{n-1}(r,t)}\frac{\partial}{\partial r}\left(f^{n-1}(r,t)\frac{\partial\rho}{\partial r}\right) - (n-1)f^{-2}(r,t)f_0(\rho)\frac{\partial f_0}{\partial\rho} \\
\frac{\partial\rho}{\partial t} &= \frac{\partial^2\rho}{\partial r^2} + \frac{n-1}{f(r,t)}\frac{\partial f}{\partial r}\frac{\partial\rho}{\partial r} - \frac{n-1}{f^2(r,t)}f_0(\rho)\frac{\partial f_0}{\partial\rho} - \frac{\partial\rho}{\partial r}\frac{dr}{dt}.
\end{aligned} \tag{14}
$$

Note that coordinate r on domain $\mathbb{R}^n$ depends on time t, this leads us to take full derivative of $\rho(r,t)$ with respect to t in above formula, $\frac{d\rho}{dt} = \frac{\partial\rho}{\partial t} + \frac{\partial\rho}{\partial r}\frac{dr}{dt}$.

2.2.2 An equation equivalent to the harmonic map flow. We want to make change of variables and turn (14) to an equation more easily to solve. Let

$$
f(r,t) = re^{\tilde{f}(r^2,t)} \qquad f_0(\rho) = \rho e^{\tilde{f}_0(\rho^2)}.
$$

Note that $\tilde{f}(w,0) = \tilde{f}_0(w)$. We claim that $\tilde{f}(w,t)$ and $\tilde{f}_0(w)$ are both smooth functions of $w \geq 0$ and t. Let $f(r,t) = r\hat{f}(r^2,t)$, to see this claim we only need to show that $\hat{f}(w,t)$ is a smooth function of w,t and $\hat{f}(0,t) \neq 0$. Write metric $g(t) = g_{ij}dx^i dx^j$ and let $x^1 = \hat{r}\cos\theta^1, x^2 = \hat{r}\sin\theta^1\cos\theta^2, \cdots, x^n = \hat{r}\sin\theta^1\cdots\sin\theta^{n-1}$, we compute $f(r,t)$ by choosing $\hat{r} = x^1$ and $\theta^1 = \cdots = \theta^{n-1} = 0$, i.e., $x^2 = \cdots = x^n = 0$.

$$
g(t) = g_{11}(\hat{r},0,\cdots,0,t)d\hat{r}^2 + g_{22}(\hat{r},0,\cdots,0,t)\hat{r}^2 d\sigma.
$$

So

$$
r = \int_0^{\hat{r}} \sqrt{g_{11}(\hat{s},0,\cdots,0,t)}d\hat{s} = \hat{r}\int_0^1 \sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds
$$

and

$$\hat{f}(r^2,t)=\frac{\hat{r}\sqrt{g_{22}(\hat{r},0,\cdots,0,t)}}{\int_0^{\hat{r}}\sqrt{g_{11}(\hat{s},0,\cdots,0,t)}d\hat{s}}=\frac{\sqrt{g_{22}(\hat{r},0,\cdots,0,t)}}{\int_0^1\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds}.$$

For any $k>0$ suppose $F(\hat{r},t)$ is an even function in $\hat{r}$ and is differentiable up to order $2k$, it is clear that function $w^kF(\sqrt{w},t), w\geq 0$ is differentiable up to order k in $w\geq 0$ and its (left-)derivatives up to order $k-1$ at $w=0$ are 0. Since by rotational symmetry $g_{22}(\hat{r},0,\cdots,0,t)$ is even in $\hat{r}$ and by Taylor series we can write for any $k>0$

$$g_{22}(\hat{r},0,\cdots,0,t)=c_0(t)+c_1(t)\hat{r}^2+\cdots+c_{k-1}(t)\hat{r}^{2k-2}+\hat{r}^{2k}F(\hat{r},t)$$

for some smooth even function $F(\hat{r},t)$. Let $\hat{g}_{22}(\hat{r}^2,0,\cdots,0,t)=g_{22}(\hat{r},0,\cdots,0,t)$. Since $\hat{r}^{2k}F(\hat{r},t)$ has k-derivative, $\hat{g}_{22}(w,0,\cdots,0,t)$ is a smooth function of $w\geq 0,t$. Similarly $\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}$ is a smooth function of $\hat{r}^2,t$. Hence we conclude that $\frac{\sqrt{g_{22}(\hat{r},0,\cdots,0,t)}}{\int_0^1\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds}$ and $r^2=\hat{r}^2\int_0^1\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds$ is a smooth function of $\hat{r}^2,t$. Furthermore r^2 is a smooth invertible function of $\hat{r}^2$. We now conclude that $\hat{f}(w,t)$ is a smooth function of $w\geq 0,t$.

Since

$$r^{-1}\frac{dr}{dt}=\frac{\int_0^1\frac{\partial}{\partial t}\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds}{\int_0^1\sqrt{g_{11}(\hat{r}s,0,\cdots,0,t)}ds},$$

a simple consequence of above arguments is that $\tilde{r}(w,t)$ is a smooth function of $w\geq 0,t$ for $\tilde{r}(r^2,t)\doteqdot r^{-1}\frac{dr}{dt}$. Let $B(w,t)\doteqdot\frac{1}{2}\int_0^{r^2}\tilde{r}(w,t)dw$. Clearly $\frac{dr}{dt}=\frac{\partial}{\partial r}B(r^2,t)$ and $B(w,t)$ is a smooth function of $w\geq 0,t$.

We will solve (14) for solutions of form

$$\rho(r,t)=re^{\tilde{\rho}(r,t)}.$$

Then some straight forward calculation shows that (14) becomes

$$\begin{aligned}\frac{1}{r}\frac{dr}{dt}+\frac{\partial\tilde{\rho}}{\partial t}=&\frac{\partial^2\tilde{\rho}}{\partial r^2}+\frac{n+1}{r}\frac{\partial\tilde{\rho}}{\partial r}+(n-1)\frac{\partial\tilde{f}}{\partial r}\left(r^2,t\right)\frac{\partial\tilde{\rho}}{\partial r}+\left(\frac{\partial\tilde{\rho}}{\partial r}\right)^2\\&+\frac{n-1}{r^2}\left[1-e^{2\tilde{f}_0\left(\rho^2\right)-2\tilde{f}\left(r^2,t\right)}\right]+2\left(n-1\right)\frac{\partial\tilde{f}}{\partial w}\left(r^2,t\right)\\&-2\left(n-1\right)e^{2\tilde{f}_0\left(\rho^2\right)+2\tilde{\rho}-2\tilde{f}\left(r^2,t\right)}\frac{\partial\tilde{f}_0}{\partial w}(\rho^2)-\frac{1}{r}\frac{dr}{dt}-\frac{dr}{dt}\frac{\partial\tilde{\rho}}{\partial r}.\end{aligned}$$

Note that from the definition of $\tilde{f}\left(0,t\right)=0$ we can write $\tilde{f}\left(w,t\right)=w\tilde{f}^*(w,t)$ and $\tilde{f}_0\left(w\right)=w\tilde{f}_0^*(w)$ where both $\tilde{f}^*(w,t)$ and $\tilde{f}_0^*(w)$ are smooth functions. So

$$\frac{n-1}{r^2}\left[1-e^{2\tilde{f}_0\left(\rho^2\right)-2\tilde{f}\left(r^2,t\right)}\right]=\frac{n-1}{r^2}\left[1-e^{2r^2\left[e^{2\tilde{\rho}}\tilde{f}_0^*\left(\rho^2\right)-\tilde{f}^*\left(r^2,t\right)\right]}\right]$$

which is a smooth function of $\tilde{\rho}, r^2, t$. Let

$$
\begin{aligned}
G_1(r^2,\tilde{\rho},t) \doteqdot & \frac{n-1}{r^2}\left[1-e^{2\tilde{f}_0(\rho^2)-2\tilde{f}(r^2,t)}\right]+2\,(n-1)\,\frac{\partial\tilde{f}}{\partial w}\left(r^2,t\right) \\
& -2\,(n-1)\,e^{2\tilde{f}_0(\rho^2)+2\tilde{\rho}-2\tilde{f}(r^2,t)}\frac{\partial\tilde{f}_0}{\partial w}(\rho^2)-\frac{2}{r}\frac{dr}{dt}.
\end{aligned}
\tag{15}
$$

$G(w,\tilde{\rho},t)$ is a smooth function.

Recall $\frac{dr}{dt}=\frac{\partial}{\partial r}B(r^2,t)$. (14) can be written as

$$
\frac{\partial\tilde{\rho}}{\partial t}=\frac{\partial^2\tilde{\rho}}{\partial r^2}+\frac{n+1}{r}\frac{\partial\tilde{\rho}}{\partial r}+\left[(n-1)\frac{\partial\tilde{f}}{\partial r}-\frac{\partial B}{\partial r}\right]\left(r^2,t\right)\frac{\partial\tilde{\rho}}{\partial r}+\left(\frac{\partial\tilde{\rho}}{\partial r}\right)^2+G_1(r^2,\tilde{\rho},t).
$$

Now we think $\tilde{\rho}$ as a rotational symmetric function defined on $\mathbb{R}^{n+2}$ and let $G(x,\tilde{\rho},t)\doteqdot G_1(\sum_{i=1}^{n+2}(x^i)^2,\tilde{\rho},t)$. Then the equation above can be written as

$$
\frac{\partial\tilde{\rho}}{\partial t}=\Delta\tilde{\rho}+\nabla[(n-1)\tilde{f}-B]\cdot\nabla\tilde{\rho}+|\nabla\tilde{\rho}|^2+G(x,\tilde{\rho},t)
\tag{16}
$$

where ∇ and Δ are the Levi-Civita connection and Laplacian defined by Euclidean metric on $\mathbb{R}^{n+2}$ respectively. Note that $\tilde{f}=\tilde{f}(\sum_{i=1}^{n+2}(x^i)^2,t), B=B(\sum_{i=1}^{n+2}(x^i)^2,t)$ are smooth functions on $\mathbb{R}^{n+2}$.

From the nonnegativity of the curvature operator of $g(t)$ and Lemma 1, we have the following properties of $\tilde{f},\tilde{f}_0$ for large r,ρ.

$$
\begin{aligned}
& e^{\tilde{f}(r^2,t)}\sim\frac{1}{(1-2t)r} \quad \frac{\partial\tilde{f}}{\partial w}(r^2,t)\sim\frac{1}{r^2} \\
& e^{\tilde{f}_0(\rho)}\sim\frac{1}{\rho} \quad \frac{\partial\tilde{f}_0}{\partial w}(\rho)\sim\frac{1}{\rho^2} \\
& r^{-1}\frac{\partial r}{\partial t}\sim\frac{1}{r^2} \quad \frac{\partial B}{\partial r}(r^2,t)\sim\frac{1}{r}.
\end{aligned}
\tag{17}
$$

2.2.3 The short time existence. We will show that equation (16) with initial condition $\tilde{\rho}(x,0)=0$ has a solution on time interval $[0,T]$. Let x,y be two points in $\mathbb{R}^{n+1}$ and

$$
H(x,y,t)=\frac{1}{(4\pi t)^{(n+2)/2}}e^{-\frac{|x-y|^2}{4t}}
$$

be the heat kernel of $\mathbb{R}^{n+1}$. We solve (16) by successive approximation [LT]. Define

$$
F(x,\tilde{\rho},\nabla\tilde{\rho},t)\doteqdot\nabla\left[(n-1)\tilde{f}-B\right]\cdot\nabla\tilde{\rho}+|\nabla\tilde{\rho}|^2+G(x,\tilde{\rho},t).
$$

Let $\tilde{\rho}_0(x,t)=0$. For $i\geq 1$ we define $\tilde{\rho}_i$ inductively by

$$
\tilde{\rho}_i=\int_0^t\int_{\mathbb{R}^{n+2}}H(x,y,t-s)F(y,\tilde{\rho}_{i-1},\nabla\tilde{\rho}_{i-1},s)dyds
\tag{18}
$$

which solves

$$\frac{\partial \tilde{\rho}_i}{\partial t} = \Delta \tilde{\rho}_i + F(x, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, t) \qquad \tilde{\rho}_i(x, 0) = 0. \tag{19}$$

To show the existence of $\tilde{\rho}_i$ by induction, it suffices to prove the following statement: for any $i \geq 1$ if $|\tilde{\rho}_{i-1}|, |\nabla \tilde{\rho}_{i-1}|$ are bounded, then $\tilde{\rho}_i$ exists and $|\tilde{\rho}_i|, |\nabla \tilde{\rho}_i|$ are bounded. Assume $|\tilde{\rho}_{i-1}| \leq C_1, |\nabla \tilde{\rho}_{i-1}| \leq C_2$ are bounded on $\mathbb{R}^{n+2} \times [0, T]$, then it follows from (17) that $G(x, \tilde{\rho}_{i-1}, t)$ is bounded on $\mathbb{R}^{n+2} \times [0, T]$

$$|G(x, \tilde{\rho}_{i-1}, t)| \leq C_3$$

where C_3 depends on C_1, C_2. $F(x, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, t)$ is bounded

$$|F(x, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, t)| \leq \Big[(n-1) \sup |\nabla \tilde{f}| + \sup |\nabla B|\Big] C_2 + C_2^2 + C_3 \doteqdot C_4.$$

Hence $\tilde{\rho}_i$ exists.

The bounds of $|\tilde{\rho}_i|$ and $|\nabla \tilde{\rho}_i|$ follow from the following estimates

$$|\tilde{\rho}_i| \leq \int_0^t \int_{\mathbb{R}^{n+2}} H(x, y, t-s) C_4 dy ds \leq C_4 t,$$

and

$$\begin{aligned}
|\nabla \tilde{\rho}_i| &= |\int_0^t \int_{\mathbb{R}^{n+2}} [\nabla_x H(x, y, t-s)] F(y, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, s) dy ds| \\
&\leq \int_0^t \int_{\mathbb{R}^{n+2}} |\nabla_x H(x, y, t-s)| C_4 dy ds \\
&= \int_0^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{(n+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \frac{|x-y|}{2(t-s)} C_4 dy ds \\
&\leq \frac{(n+2)C_4}{\sqrt{\pi}} \int_0^t \frac{1}{\sqrt{t-s}} ds = \frac{2(n+2)C_4}{\sqrt{\pi}} \sqrt{t}.
\end{aligned}$$

If we define $T_1 \doteqdot \min\{\frac{C_1}{C_4}, \frac{\pi C_2^2}{4(n+2)^2 C_4^2}\}$, then for $0 \leq t \leq T_1$ we have for all i

$$|\tilde{\rho}_i| \leq C_1 \qquad |\nabla \tilde{\rho}_i| \leq C_2. \tag{20}$$

We prove the convergence of $\tilde{\rho}_i$ to a solution of (16) by showing that it is a Cauchy sequence in C^1-norm. We assume $0 \leq t \leq T_1$. Note that $\tilde{\rho}_i - \tilde{\rho}_{i-1}$ satisfies

$$\begin{aligned}
&\frac{\partial(\tilde{\rho}_i - \tilde{\rho}_{i-1})}{\partial t} = \Delta(\tilde{\rho}_i - \tilde{\rho}_{i-1}) + F(x, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, t) - F(x, \tilde{\rho}_{i-2}, \nabla \tilde{\rho}_{i-2}, t) \\
&(\tilde{\rho}_i - \tilde{\rho}_{i-1})(x, 0) = 0.
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
&F(x, \tilde{\rho}_{i-1}, \nabla \tilde{\rho}_{i-1}, t) - F(x, \tilde{\rho}_{i-2}, \nabla \tilde{\rho}_{i-2}, t) \\
=&[(n-1)\nabla \tilde{f} - \nabla B + \nabla(\tilde{\rho}_{i-1} + \tilde{\rho}_{i-2})] \cdot \nabla(\tilde{\rho}_{i-1} - \tilde{\rho}_{i-2}) \\
&+ G(x, \tilde{\rho}_{i-1}, , t) - G(x, \tilde{\rho}_{i-2}, t)
\end{aligned}$$

By lengthy but straight-forward calculations one can verify the Lipschitz property of $G(x, \tilde\rho, t)$

$$|G(x, \tilde\rho_{i-1}, t) - G(x, \tilde\rho_{i-2}, t)| \leq C_5 \cdot |\tilde\rho_{i-1} - \tilde\rho_{i-2}|$$

where C_5 depends on C_1. This and (20) implies

$$\begin{aligned} &|F(x, \tilde\rho_{i-1}, \nabla\tilde\rho_{i-1}, t) - F(x, \tilde\rho_{i-2}, \nabla\tilde\rho_{i-2}, t)| \\ \leq &C_5 \cdot |\tilde\rho_{i-1} - \tilde\rho_{i-2}| + C_6 \cdot |\nabla\tilde\rho_{i-1} - \nabla\tilde\rho_{i-2}| \end{aligned} \tag{22}$$

where $C_6 \doteqdot [(n-1)\sup|\nabla\tilde f| + \sup|\nabla B| + 2C_2)$.

Let

$$\begin{aligned} A_i(t) &= \sup_{0\leq s\leq t, x\in\mathbb{R}^{n+2}} |\tilde\rho_i - \tilde\rho_{i-1}|(x, s) \\ B_i(t) &= \sup_{0\leq s\leq t, x\in\mathbb{R}^{n+2}} |\nabla(\tilde\rho_i - \tilde\rho_{i-1})|(x, s). \end{aligned}$$

From (21) and (22) we can estimate $|\tilde\rho_i - \tilde\rho_{i-1}|$ and $|\nabla(\tilde\rho_i - \tilde\rho_{i-1})|$ in the same way as we estimate $|\tilde\rho_i|$ and $|\nabla\tilde\rho_i|$ above, we conclude

$$\begin{aligned} A_i(t) &\leq [C_5 A_{i-1}(t) + C_6 B_{i-1}(t)] \cdot t \\ B_i(t) &\leq \frac{2(n+2)[C_5 A_{i-1}(t) + C_6 B_{i-1}(t)]}{\sqrt{\pi}} \cdot \sqrt{t}. \end{aligned}$$

Let $C_7 \doteqdot \max\{C_5, C_6\}$, then we get

$$A_i(t) + B_i(t) \leq \left(C_7 t + \frac{2(n+2)C_7\sqrt{t}}{\sqrt{\pi}}\right) \cdot (A_{i-1}(t) + B_{i-1}(t)).$$

If we choose $T_2 \in (0, T_1]$ so that $C_7 T_2 + \frac{2(n+2)C_7\sqrt{T_2}}{\sqrt{\pi}} \leq \frac{1}{2}$, then

$$A_i(t) + B_i(t) \leq \frac{1}{2}(A_{i-1}(t) + B_{i-1}(t)),$$

so $\tilde\rho_i$ is a Cauchy sequence in $C^1(\mathbb{R}^{n+2})$.

Let $\lim_{i\to+\infty} \tilde\rho_i = \tilde\rho_\infty$. Then $\nabla\tilde\rho_i \to \nabla\tilde\rho_\infty$ and $F(x, \tilde\rho_{i-1}, \nabla\tilde\rho_{i-1}, t) \to F(x, \tilde\rho_\infty, \nabla\tilde\rho_\infty, t)$ uniformly. Hence we get from (18)

$$\tilde\rho_\infty = \int_0^t \int_{\mathbb{R}^{n+2}} H(x, y, t-s) F(y, \tilde\rho_\infty, \nabla\tilde\rho_\infty, s) dy ds \tag{23}$$

The argument below is similar to the argument in [LT] p.21. Since $\tilde\rho_i$ is a smooth solution of (19), $\tilde\rho_i(x, 0) = 0$ and both $\tilde\rho_i$ and $F(x, \tilde\rho_{i_1}, \nabla\tilde\rho_{i-1}, t)$ are uniformly bounded on $\mathbb{R}^{n+2} \times [0, T_2]$, by Theorem 1.11 [LSU] p.211 and Theorem 12.1 [LSU] p.223, for any compact $K \subset \mathbb{R}^{n+2}$ and any $0 < t_* < T_2$, there is C_8 and $\alpha \in (0, 1)$ independent of i such that

$$|\nabla\tilde\rho_i(x, t) - \nabla\tilde\rho_i(y, s)| \leq C_8 \cdot \left(|x-y|^\alpha + |t-s|^{\alpha/2}\right)$$

where $x, y \in K$ and $0 \le t < s \le t_*$. Let $i \to \infty$ we get

$$|\nabla \tilde{\rho}_\infty(x,t) - \nabla \tilde{\rho}_\infty(y,s)| \le C_8 \cdot \left(|x-y|^\alpha + |t-s|^{\alpha/2} \right). \tag{24}$$

Hence $\nabla \tilde{\rho}_\infty$ is Hölder continuous. From (23) we conclude that $\tilde{\rho}_\infty$ is a solution of (16) on $\mathbb{R}^{n+2} \times [0, T_2]$ with $\tilde{\rho}_\infty(x, 0) = 0$.

2.2.4 The asymptotic behavior of the solutions. In the rest of this subsection we study the asymptotic behavior of solution $\tilde{\rho}(x,t)$ as $x \to \infty$. First we prove inductively that there is a constant λ and $T_3 \in (0, T_2]$ such that for $x \in \mathbb{R}^{n+2}, t \in [0, T_3]$

$$|\tilde{\rho}_i(x,t)| \le \frac{\lambda}{(1+|x|)^2} \qquad |\nabla \tilde{\rho}_i(x,t)| \le \frac{\lambda}{(1+|x|)^2} \tag{25}$$

Clearly these estimates holds for $i = 0$. It follows from (20) and (17) that there is a constant C_9 independent of i such that

$$\begin{aligned} |G(x, \tilde{\rho}_i, t)| &\le \frac{C_9}{(1+|x|)^2} \\ \left[(n-1)|\nabla \tilde{f}| + |\nabla B| \right](x,t) &\le \frac{C_9}{1+|x|}. \end{aligned}$$

Now we assume the estimates hold for i, then for $0 \le t \le T_2$

$$\begin{aligned} |\tilde{\rho}_i(x,t)| &\le \int_0^t \int_{\mathbb{R}^{n+2}} H(x,y,t-s) \left[\frac{C_9 \lambda}{(1+|y|)^2} + \frac{\lambda^2}{(1+|y|)^2} + \frac{C_9}{(1+|y|)^2} \right] dyds \\ &= \int_0^t \int_{\mathbb{R}^{n+2}} \frac{1}{(4\pi(t-s))^{(n+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\frac{C_9\lambda + \lambda^2 + C_9}{(1+|y|)^2} \right] dyds \\ &\le (C_9 \lambda + \lambda^2 + C_9) \cdot \frac{C(n) t}{(1+|x|)^2}. \end{aligned}$$

Also we have

$$\begin{aligned} |\nabla \tilde{\rho}_i(x,t)| &\le \int_0^t \int_{\mathbb{R}^{n+2}} |\nabla_x H(x,y,t-s)| \left[\frac{C_9\lambda + \lambda^2 + C_9}{(1+|y|)^2} \right] dyds \\ &= \int_0^t \int_{\mathbb{R}^{n+2}} \frac{|x-y|}{2(t-s)} \frac{1}{(4\pi(t-s))^{(n+2)/2}} e^{-\frac{|x-y|^2}{4(t-s)}} \left[\frac{C_9\lambda + \lambda^2 + C_9}{(1+|y|)^2} \right] dyds \\ &\le (C_9\lambda + \lambda^2 + C_9) \cdot \frac{C(n)\sqrt{t}}{(1+|x|)^2}. \end{aligned}$$

If we choose $T_3 \in (0, T_2]$ such that

$$(C_9\lambda + \lambda^2 + C_9) \cdot C(n) T_3 \le \lambda \quad \text{and} \quad (C_9\lambda + \lambda^2 + C_9) \cdot C(n)\sqrt{T_3} \le \lambda,$$

then (25) hold for all i. From the definition of $\tilde{\rho}_\infty$ we conclude

$$|\tilde{\rho}_\infty(x,t)| \le \frac{\lambda}{(1+|x|)^2} \qquad |\nabla \tilde{\rho}_\infty(x,t)| \le \frac{\lambda}{(1+|x|)^2} \tag{26}$$

Now we consider $\tilde{\rho}_\infty$ as a solution of the following linear equation

$$\begin{aligned}
&\frac{\partial v}{\partial t} = \Delta v + \nabla[(n-1)\tilde{f} - B + \tilde{\rho}_\infty] \cdot \nabla v + G(x, \tilde{\rho}_\infty, t) \\
&v(x, 0) = 0.
\end{aligned}$$

From (24) and (17) we know that $\nabla[(n-1)\tilde{f} - B + \tilde{\rho}_\infty]$ has $C^{\alpha,\alpha/2}$-Holder-norm bound. By some lengthy calculation we get

$$|G(x, \tilde{\rho}_\infty, t)|_{C^{\alpha,\alpha/2}} \leq \frac{C_{10}}{(1+|x|)^2}.$$

By standard interior Schauder estimate for parabolic equation we conclude

$$|\tilde{\rho}_\infty|_{C^{2+\alpha,1+\alpha/2}} \leq \frac{C_{11}}{(1+|x|)^2}.$$

Using this estimate one can further show by calculation

$$\begin{aligned}
&|\nabla^2[(n-1)\tilde{f} - B + \tilde{\rho}_\infty]|_{C^{\alpha,\alpha/2}} \leq C_{12} \\
&|\nabla G(x, \tilde{\rho}_\infty, t)|_{C^{\alpha,\alpha/2}} \leq \frac{C_{13}}{(1+|x|)^2}.
\end{aligned}$$

By high order interior Schauder estimates for parabolic equation we conclude

$$|\nabla \tilde{\rho}_\infty|_{C^{2+\alpha,1+\alpha/2}} \leq \frac{C_{13}}{(1+|x|)^2}.$$

We have proved the following

Proposition 5 *For standard solution $(\mathbb{R}^n, g(t))$, there is a rotationally symmetric solution $\phi(t)(x) = xe^{\tilde{\rho}(x,t)}$ to the harmonic map flow*

$$\frac{\partial \phi(t)}{\partial t} = \Delta_{g(t),g(0)} \qquad \phi(0)(x) = x,$$

and $|\nabla^i \tilde{\rho}|(x,t) \leq \frac{C_{14}}{(1+|x|)^2}$ for $0 \leq i \leq 3$.

3 The uniqueness of standard solutions

Let $\phi(t)$ be the solution of harmonic map flow from §2.2, in this section we use the asymptotic behaviors of $g(t)$ and of $\phi(t)$ to prove the uniqueness of $\hat{g}(t) = \left(\phi_t^{-1}\right)^* g(t)$. Then the uniqueness of standard solutions follows easily.

3.1 The uniqueness for the solutions of Deturk-Ricci flow

We prove the following general uniqueness result for Deturk-Ricci flow on open manifolds.

Proposition 6 *Let $G(t)$ and $H(t), 0 \leq t \leq T$ be two bounded solution of the Deturk-Ricci flow on complete and noncompact manifold M^n with initial metric $G(0) = H(0) = \tilde{g}$ and for some $\delta \in (0,1)$*

$$\begin{aligned}
&(1-\delta)\tilde{g} \leq G(t) \leq (1+\delta)\tilde{g} \\
&(1-\delta)\tilde{g} \leq H(t) \leq (1+\delta)\tilde{g} \\
&\|G(t)\|_{C^2(\Omega_b),\tilde{g}} < +\infty \\
&\|H(t)\|_{C^2(\Omega_b),\tilde{g}} < +\infty.
\end{aligned}$$

Suppose $G(t)$ and $H(t)$ has the same sequenctial asymptotical behavior at ∞ in the sense that there is a sequence of exhausting submanifolds of $\Omega_k \subset M$ with $\Omega_k \subset \Omega_{k+1}$ and $\cup \Omega_k = M$. For any $\epsilon > 0$, there is a k_0 such that

$$|G(t) - H(t)|_{C^1(\partial\Omega_{k_0}),\tilde{g}} \leq \epsilon,$$

Then $G(t) = H(t)$, i.e., the Deturk-Ricci flow has uniqueness in above allowable family of solutions.

Proof. Using the orthonormal frame of $\tilde{g}$ and the Ricci-Deturk flow (10) for G and H we compute $\frac{\partial}{\partial t}(G_{ij}(t) - H_{ij}(t))$ and then estimate

$$\begin{aligned}
&\frac{\partial}{\partial t}|G(t) - H(t)|^2_{\tilde{g}} = 2\left\langle \frac{\partial}{\partial t}(G(t) - H(t)), G(t) - H(t) \right\rangle_{\tilde{g}} \\
&\leq 2G^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta (G_{ij} - H_{ij})(G_{ij}(t) - H_{ij}(t)) \\
&+ C_{14}|G(t) - H(t)|^2_{\tilde{g}} + C_{14}|G(t) - H(t)|^2_{\tilde{g}} \\
&+ \Big(C_{14}|G(t) - H(t)|_{\tilde{g}} + C_{14}|\nabla G(t) - \nabla H(t)|_{\tilde{g}}\Big)|G(t) - H(t)|_{\tilde{g}} \\
&\leq 2G^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta |G(t) - H(t)|^2_{\tilde{g}} - 4G^{\alpha\beta}\tilde{\nabla}_\alpha (G(t) - H(t)) \cdot \tilde{\nabla}_\beta (G(t) - H(t)) \\
&+ C_{14}|G(t) - H(t)|^2_{\tilde{g}} + C_{14}|\nabla (G(t) - H(t))|_{\tilde{g}} |G(t) - H(t)|_{\tilde{g}} \\
&\leq 2G^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta |G(t) - H(t)|^2_{\tilde{g}} - 4(1-\delta)|\nabla (G(t) - H(t))|^2_{\tilde{g}} \\
&+ C_{14}|G(t) - H(t)|^2_{\tilde{g}} + (1-\delta)|\nabla (G(t) - H(t))|^2_{\tilde{g}} + \frac{C_{14}^2}{4(1-\delta)}|G(t) - H(t)|^2_{\tilde{g}}.
\end{aligned}$$

We have proved

$$\frac{\partial}{\partial t}|G(t) - H(t)|^2_{\tilde{g}} \leq 2G^{\alpha\beta}\tilde{\nabla}_\alpha\tilde{\nabla}_\beta |G(t) - H(t)|^2_{\tilde{g}} + C_{15}|G(t) - H(t)|^2_{\tilde{g}} \quad (27)$$

pointwise on Ω_a with C_{15} depends only on n, δ, $\|G(t)\|_{C^2(\Omega_b),\tilde{g}}$ and $\|H(t)\|_{C^2(\Omega_b),\tilde{g}}$.

If $G(t) \neq H(t)$, then there is a point x_0 such that $|G(t_0) - H(t_0)|^2_{\tilde{g}}(x_0) > \epsilon_0$ for some $t_0 > 0$ and some $\epsilon_0 > 0$. We choose a k_0 such that $x_0 \in \Omega_{k_0}$ and

$$\sup_{x \in \partial\Omega_b} |g(t) - h(t)|^2_{\tilde{g}}(x) \leq \epsilon \quad (28)$$

where $\epsilon > 0$ is a constant to be chosen later. Recall we have initial condition $|G(0) - H(0)|_{\bar{g}}^2 = 0$. Applying maximum principle to $|G(t) - H(t)|_{\bar{g}}^2$ in (27) on domain Ω_{k_0}, we get

$$e^{-C_{15}t} |G(t) - H(t)|_{\bar{g}}^2 (x) \leq \epsilon. \text{ for all } x \in \Omega_b.$$

This is a contradiction if we choose $\epsilon \leq \epsilon_0 e^{-C_{15}T}$. The proposition is proved. ■

Let $g(t)$ and $g_*(t)$ be two standard solutions with same initial condition. By Proposition 4 there are $\phi(t)$ for $g(t)$ and $\phi_*(t)$ for $g_*(t)$ which are two solutions of the harmonic map flow, $0 \leq t \leq T_3$. Let $\hat{G}(t) \doteqdot (\phi^{-1}(t)) * g(t)$ and $\hat{H}(t) \doteqdot (\phi_*^{-1}(t)) * g_*(t)$. Then $\hat{G}(t)$ and $\hat{H}(t)$ are two solutions of Ricci-Deturk flow with $\hat{G}(0) = \hat{H}(0)$. Choose $T_4 \in (0, T_3]$ such that $\hat{G}(t)$ and $\hat{H}(t)$ is δ-close to $\hat{G}(0)$ as required in Proposition 5. It follows from Lemma 1 and the decay estimate in Proposition 4 that $\hat{G}(t)$ and $\hat{H}(t)$ are bounded solutions and they have same sequential asymptotic behavior at infinity. We can apply Proposition 5 to conclude $\hat{G}(t) = \hat{H}(t)$ on $0 \leq t \leq T_4$. We have proved

Lemma 7 *The Ricci-Deturk solutions $\hat{G}(t)$ and $\hat{H}(t)$ constructed from standard solutions $g(t)$ and $g_*(t)$ with $g(0) = g_*(0)$ are the same, $\hat{G}(t) = \hat{H}(t)$ for $t \in [0, T_4]$.*

Remark 8 *Another way to prove the uniqueness of Ricci-DeTurk flow is to use maximum principle on open manifolds, then we do not need using the asymptotic behavior.*

Proposition 9 *Let $\hat{h}(0)$ be a metric on complete manifold with injectivity radius lower bound $\delta_1 > 0$ and curvature bound*

$$|\nabla^i \hat{h}(0) Rm_{\hat{h}(0)}|_{\hat{h}(0)} \leq C \text{ for } i = 0, 1, 2.$$

Let $\hat{h}_1(t)$ and $\hat{h}_2(t)$, $0 \leq t \leq T$ are two solutions of the Ricci-Deturk flow with $\hat{h}_1(0) = \hat{h}_2(0) = \hat{h}(0)$. Suppose

$$\left(1 - \frac{1}{256000 n^{10}}\right) \hat{h}(0) \leq \hat{h}(t) \leq \left(1 + \frac{1}{256000 n^{10}}\right) \hat{h}(0), \quad 0 \leq t \leq T$$

Then $h_1(t) = h_2(t), 0 \leq t \leq T$.

Proof. *Let $u(x,t) \doteqdot \left| \hat{h}_1(t) - \hat{h}_2(t) \right|_{\hat{h}_0}^2$. By (11) and the computation of (27) we obtain that u is bounded and*

$$\frac{\partial}{\partial t} u \leq 3 \Delta_{\hat{h}(0)} u + C_{15} u \qquad u(x,0) = 0$$

If one check the proof of [Sh2] Theorem 4.6, it is clear the positive sectional curvature requirement in Assumption (A) can be replaced by lower bounded of sectional curvature. The requirement is used for constructing cut-off function on [Sh2] p310 which can be replaced by the injectivity radius assumption. We can apply [Sh2] Theorem 4.6 to conclude $u(x,t) = 0$ ■

3.2 The uniqueness of standard solutions

Recall the following procedure of converting the solution $\hat{h}(t)$ of Deturk-Ricci flow in §2.1 back to solution $h(t)$ of the Ricci flow (see [CK], p.89-90). Given two metrics g and g_*, we define 1-form

$$W(g, g_*)_j \doteqdot g_{jk} g^{pq} \left(\Gamma_{pq}^k (g) - \Gamma_{pq}^k (g_*) \right).$$

Define a family diffeomorphisms φ_t by solving the following ODE

$$\begin{aligned} \frac{d}{dt} (\varphi_t)^i &= \hat{h}^{ij}(t) W\left(\hat{h}(t), h(0)\right)_j, \\ \varphi_0 &= Id. \end{aligned}$$

Then $h(t) = \varphi^*(t)\hat{h}(t)$.

For the rest of this subsection, we adopt the notation at the end of §3.1. Define diffeomorphisms $\varphi_1(t)$ and $\varphi_2(t)$ by

$$\begin{aligned} \frac{d}{dt} (\varphi_1(t))^i &= \hat{G}^{ij}(t) W\left(\hat{G}(t), \hat{G}(0)\right)_j \quad \varphi_1(0) = Id \\ \frac{d}{dt} (\varphi_2(t))^i &= \hat{H}^{ij}(t) W\left(\hat{H}(t), \hat{H}(0)\right)_j \quad \varphi_2(0) = Id. \end{aligned}$$

Since $\hat{G}(t) = \hat{H}(t)$ by Lemma 6, $W\left(\hat{G}, \hat{G}(0)\right) = W\left(\hat{H}, \hat{H}(0)\right)$ and hence $\varphi_1(t) = \varphi_2(t)$. We have $g(t) = \varphi_1^*(t)\hat{G}(t) = \varphi_2^*(t)\hat{H}(t) = g_*(t)$ for $0 \le t \le T_4$.

To show $g(t) = g_*(t)$ for $0 \le t \le T$, we repeat above argument using new initial time T_4. This proves the uniqueness of the standard solutions.

Theorem 10 *Let $g(t)$ and $g_*(t), 0 \le t \le T$ be two standard solutions of the Ricci flow as defined in §1. Suppose $g(0) = g_*(0)$, then $g(t) = g_*(t)$ for $0 \le t \le T$.*

4 Appendix: Shi's local derivative estimate when initial metrics have higher regularity

If initial metric has better curvature bound, we can improve Shi's local derivative estimates as following.

Theorem 11 *For any α, K, K_1, r, $l \ge 0, n$ and $m \in \mathbb{N}$, there exists $C = C(\alpha, K, K_l, r, l, n, m)$ depending only on α, K, K_l, r, l, n and m such that if $\mathcal{M}^n$ is a manifold, $p \in \mathcal{M}$, and $g(t)$, $t \in [0, \tau]$, $0 < \tau \le \alpha/K$, is a solution to the Ricci flow on an open neighborhood U of p containing $\bar{B}_{g(0)}(p, r)$ as a compact subset, and if*

$$\begin{aligned} |\mathrm{Rm}(x,t)| &\le K \text{ for all } x \in U \text{ and } t \in [0, \tau], \\ \left|\nabla^\beta \mathrm{Rm}(x,0)\right| &\le K_l \text{ for all } x \in U \text{ and } \beta \le l \end{aligned}$$

then

$$|\nabla^m \operatorname{Rm}(y,t)| \leq \frac{C}{t^{\max\{m-l,0\}/2}}$$

for all $y \in B_{g(0)}(p, r/2)$ and $t \in (0,\tau]$. In particular if $m \leq l$ we have

$$|\nabla^m \operatorname{Rm}(y,t)| \leq C.$$

Proof. Below the constant C may change from line to line and depends on some or all α, K, K_l, r, n, l and m.

If $l = 0$, this is Shi's local estimates for higher derivatives.

Consider

$$F_m = \left(C + t^{\max\{m-l,0\}} |\nabla^m \operatorname{Rm}|^2\right) t^{\max\{m-l+1,0\}} \left|\nabla^{m+1} \operatorname{Rm}\right|^2,$$

where C is to be chosen. The main calculation is given by

Lemma 12

$$\left(\frac{\partial}{\partial t} - \Delta\right) F_m \leq -\frac{c}{t^{sign\{\max\{m-l+1,0\}\}}} (F_m)^2 + \frac{C}{t^{sign \max\{m-l+1,0\}}}.$$

We can easily obtain the theorem from the lemma. Let η be a cutoff function with $\eta = 1$ on $B_{g(0)}\left(p, r/2^{m+1}\right)$ and support in $B_{g(0)}(p, r/2^m)$. When $sign\{\max\{m-l+1,0\}\} \leq 0$, then we have

$$\left(\frac{\partial}{\partial t} - \Delta\right) F_m \leq -c(F_m)^2 + C.$$

We compute

$$\left(\frac{\partial}{\partial t} - \Delta\right)(\eta F_m) \leq \eta\left(-c(F_m)^2 + C\right) - \Delta\eta \cdot F - 2\nabla\eta \cdot \nabla F.$$

Let (x_0, t_0) be the point where ηF_m attains its maximum in $B_{g(0)}(p, r/2^m)$. The maximum is finite by the assumption of the theorem. Then if $t_0 = 0$, the estimate follows. If $t_0 > 0$, then a simple maximum principle argument shows that ηF_m is bounded.

When $sign\{\max\{m-l+1,0\}\} > 0$, again we use a maximum principle argument. We compute the evolution inequality for ηF_m and conclude that ηF_m is bounded. The theorem then follows from induction on m.

Proof of the lemma. Given l, we argue by induction, assume that for $j = 1, \ldots, m$ there exist constants C_j depending only on α, K, K_l, r, n, l and m. such that for $x \in B_{g(0)}\left(p, r/2^j\right)$ and $t \in [0,\tau]$,

$$t^{\max\{j-l,0\}/2} \left|\nabla^j \operatorname{Rm}(x,t)\right| \leq C_j.$$

Recall that

$$\frac{\partial}{\partial t}\left|\nabla^k \operatorname{Rm}\right|^2 \leq \Delta\left|\nabla^k \operatorname{Rm}\right|^2 - 2\left|\nabla^{k+1} \operatorname{Rm}\right|^2 + \sum_{\ell=0}^{k}\left|\nabla^\ell \operatorname{Rm}\right|\left|\nabla^{k-\ell} \operatorname{Rm}\right|\left|\nabla^k \operatorname{Rm}\right|. \tag{29}$$

Hence

$$\frac{\partial}{\partial t}\left(t^{\max\{k-l,0\}}\left|\nabla^k \operatorname{Rm}\right|^2\right) \leq \Delta\left(t^{\max\{k-l,0\}}\left|\nabla^k \operatorname{Rm}\right|^2\right)$$
$$-2t^{\max\{k-l,0\}}\left|\nabla^{k+1} \operatorname{Rm}\right|^2 + t^{\max\{k-l,0\}}\sum_{i=0}^{k}\left|\nabla^i \operatorname{Rm}\right|\left|\nabla^{k-i} \operatorname{Rm}\right|\left|\nabla^k \operatorname{Rm}\right|$$
$$+\max\{k-l,0\}\, t^{\max\{k-l,0\}-1}\left|\nabla^k \operatorname{Rm}\right|^2.$$

In particular, using our induction hypothesis that $t^{\max\{j-l,0\}/2}\left|\nabla^j \operatorname{Rm}(x,t)\right| \leq C_j$ on $B_{g(0)}(p, r/2^m)\times[0,\tau]$ for $j=1,\ldots,m,$, we have with $m_l \doteqdot \max\{m-l+1,0\}$

$$\frac{\partial}{\partial t}\left(t^{m_l}\left|\nabla^{m+1} \operatorname{Rm}\right|^2\right) \leq \Delta\left(t^{m_l}\left|\nabla^{m+1} \operatorname{Rm}\right|^2\right) - 2t^{m_l}\left|\nabla^{m+2} \operatorname{Rm}\right|^2$$
$$+t^{m_l}\sum_{i=0}^{m+1}\left|\nabla^i \operatorname{Rm}\right|\left|\nabla^{m-i} \operatorname{Rm}\right|\left|\nabla^{m+1} \operatorname{Rm}\right| + m_l t^{m_l-1}\left|\nabla^{m+1} \operatorname{Rm}\right|^2$$
$$\leq \Delta\left(t^{m_l}\left|\nabla^{m+1} \operatorname{Rm}\right|^2\right) - 2t^{m_l}\left|\nabla^{m+2} \operatorname{Rm}\right|^2$$
$$+Ct^{m_l/2}\left|\nabla^{m+1} \operatorname{Rm}\right| + (2t\left|\operatorname{Rm}\right| + m_l)\, t^{m_l-1}\left|\nabla^{m+1} \operatorname{Rm}\right|^2$$
$$\leq \Delta\left(t^{m_l}\left|\nabla^{m+1} \operatorname{Rm}\right|^2\right) - 2t^{m_l}\left|\nabla^{m+2} \operatorname{Rm}\right|^2$$
$$+Ct^{m_l-1}\left|\nabla^{m+1} \operatorname{Rm}\right|^2 + \frac{C}{t^{sign\{m_l\}}}.$$

Let $\hat{m}_l \doteqdot \max\{m-l,0\}$. From (29) and the induction hypothesis, we have

$$\frac{\partial}{\partial t}\left(t^{\hat{m}_l}\left|\nabla^m \operatorname{Rm}\right|^2\right) \leq \Delta\left(t^{\hat{m}_l}\left|\nabla^m \operatorname{Rm}\right|^2\right)$$
$$-2t^{\hat{m}_l}\left|\nabla^{m+1} \operatorname{Rm}\right|^2 + \hat{m}_l t^{\hat{m}_l-1}\left|\nabla^m \operatorname{Rm}\right|^2 + C.$$

Hence if C is chosen so that $4t^{\max\{m-l,0\}}\left|\nabla^m \mathrm{Rm}\right|^2 \leq C$, then

$$\begin{aligned}
&\left(\frac{\partial}{\partial t}-\Delta\right)\left[\left(C+t^{\hat{m}_l}\left|\nabla^m \mathrm{Rm}\right|^2\right)t^{m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2\right] \\
&\leq \left(C+t^{\hat{m}_l}\left|\nabla^m \mathrm{Rm}\right|^2\right)\left(-2t^{m_l}\left|\nabla^{m+2}\mathrm{Rm}\right|^2+Ct^{m_l-1}\left|\nabla^{m+1}\mathrm{Rm}\right|^2+\frac{C}{t^{sign\{m_l\}}}\right) \\
&+\left(-2t^{\hat{m}_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2+\hat{m}_l t^{\hat{m}_l-1}\left|\nabla^m \mathrm{Rm}\right|^2+C\right)t^{m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2 \\
&-2t^{\hat{m}_l+m_l}\nabla\left|\nabla^m \mathrm{Rm}\right|^2\cdot\nabla\left|\nabla^{m+1}\mathrm{Rm}\right|^2 \\
&\leq -10t^{\hat{m}_l+m_l}\left|\nabla^m \mathrm{Rm}\right|^2\left|\nabla^{m+2}\mathrm{Rm}\right|^2 \\
&-8t^{\hat{m}_l+m_l}\left|\nabla^m \mathrm{Rm}\right|\left|\nabla^{m+1}\mathrm{Rm}\right|^2\left|\nabla^{m+2}\mathrm{Rm}\right|-2t^{\max\{m-l,0\}+m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^4 \\
&+\left(C+t^{\hat{m}_l}\left|\nabla^m \mathrm{Rm}\right|^2\right)\left(Ct^{\hat{m}_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2+\frac{C}{t^{sign\{m_l\}}}\right) \\
&+\left(\max\{m-l,0\}\,t^{\hat{m}_l-1}\left|\nabla^m \mathrm{Rm}\right|^2+C\right)t^{m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2 \\
&\leq -\frac{2}{5t^{sign\{m_l\}}}\left(t^{m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2\right)^2+C\left(1+\tau\right)t^{\hat{m}_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2+\frac{C}{t^{sign\{m_l\}}} \\
&\leq -\frac{c}{t^{sign\{m_l\}}}\left[\left(C+t^{\hat{m}_l}\left|\nabla^m \mathrm{Rm}\right|^2\right)t^{m_l}\left|\nabla^{m+1}\mathrm{Rm}\right|^2\right]^2+\frac{C}{t^{sign\{m_l\}}}.
\end{aligned}$$

The lemma follows.

Peng Lu, Dept of Math, University of Oregon, Eugene, OR 97403

Gang Tian, Dept of Math, Princeton University, Princeton, NJ 08544

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