PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 145, Number 12, December 2017, Pages 5329–5340
http://dx.doi.org/10.1090/proc/13655
Article electronically published on July 10, 2017

AN INTRINSIC PARALLEL TRANSPORT IN WASSERSTEIN SPACE

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(Communicated by Guofang Wei)

Abstract. If M is a smooth compact connected Riemannian manifold, let $P(M)$ denote the Wasserstein space of probability measures on M. We describe a geometric construction of parallel transport of some tangent cones along geodesics in $P(M)$. We show that when everything is smooth, the geometric parallel transport agrees with earlier formal calculations.

1. Introduction

Let M be a smooth compact connected Riemannian manifold without boundary. The space $P(M)$ of probability measures of M carries a natural metric, the Wasserstein metric, and acquires the structure of a length space. There is a close relation between minimizing geodesics in $P(M)$ and optimal transport between measures. For more information on this relation, we refer to Villani's book [13].

Otto discovered a formal Riemannian structure on $P(M)$, underlying the Wasserstein metric [10]. One can do formal geometric calculations for this Riemannian structure [6]. It is an interesting problem to make these formal considerations into rigorous results in metric geometry.

If M has nonnegative sectional curvature, then $P(M)$ is a compact length space with nonnegative curvature in the sense of Alexandrov [8, Theorem A.8], [12, Proposition 2.10]. Hence one can define the tangent cone $T_\mu P(M)$ of $P(M)$ at a measure $\mu \in P(M)$. If μ is absolutely continuous with respect to the volume form dvol_M, then $T_\mu P(M)$ is a Hilbert space [8, Proposition A.33]. More generally, one can define tangent cones of $P(M)$ without any curvature assumption on M, using Ohta's 2-uniform structure on $P(M)$ [9]. Gigli showed that $T_\mu P(M)$ is a Hilbert space if and only if μ is a "regular" measure, meaning that it gives zero measure to any hypersurface which, locally, is the graph of the difference of two convex functions [3, Corollary 6.6]. For examples of tangent cones at nonregular measures, if S is an embedded submanifold of M, and μ is an absolutely continuous measure on S, then $T_\mu P(M)$ was computed in [7, Theorem 1.1].

If $\gamma : [0,1] \to M$ is a smooth curve in a Riemannian manifold, then one can define the (reverse) parallel transport along γ as a linear isometry from $T_{\gamma(1)}M$ to $T_{\gamma(0)}M$. If X is a finite-dimensional Alexandrov space, then the replacement of a tangent space is a tangent cone. If one wants to define a parallel transport along a curve $c : [0,1] \to X$, as a map from $T_{c(1)}X$ to $T_{c(0)}X$, then there is the problem that the tangent cones along c may not look much alike. For example, the curve c

Received by the editors August 9, 2016 and, in revised form, January 6, 2017.
2010 *Mathematics Subject Classification.* Primary 51K10, 58J99.
This research was partially supported by NSF grant DMS-1207654 and a Simons Fellowship.

may pass through various strata of X. One can deal with this problem by assuming that c is in the interior of a minimizing geodesic. In this case, Petrunin proved the tangent cones along c are mutually isometric, by constructing a parallel transport map [11]. His construction of the parallel transport map was based on passing to a subsequential limit in an iterative construction along c. It is not known whether the ensuing parallel transport is uniquely defined, although this is irrelevant for Petrunin's result.

In the case of a smooth curve $c : [0,1] \to P^\infty(M)$ in the space of smooth probability measures, one can do formal Riemannian geometry calculations on $P^\infty(M)$ to write down an equation for parallel transport along c [6, Proposition 3]. It is a partial differential equation in terms of a family of functions $\{\eta_t\}_{t\in[0,1]}$. Ambrosio and Gigli noted that there is a weak version of this partial differential equation [1, (5.9)]. By a slight extension, we will define weak solutions to the formal parallel transport equation; see Definition 2.4.

Petrunin's construction of parallel transport cannot work in full generality on $P(M)$, since Juillet showed that there is a minimizing Wasserstein geodesic c with the property that the tangent cones at measures on the interior of c are not all mutually isometric [5]. However one can consider applying the construction on certain convex subsets of $P(M)$. We illustrate this in two cases. The first and easier case is when c is a Wasserstein geodesic of δ-measures (Proposition 3.1). The second case is when c is a Wasserstein geodesic of absolutely continuous measures, lying in the interior of a minimizing Wasserstein geodesic, and satisfying a regularity condition. Suppose that $\nabla\eta_1 \in T_{c(1)}P(M)$ is an element of the tangent cone at the endpoint. Here $\nabla\eta_1 \in L^2(TM, dc(1))$ is a square-integrable gradient vector field on M and η_1 is in the Sobolev space $H^1(M, dc(1))$. For each sufficiently large integer Q, we construct a triple

$$\begin{aligned}(1.1)\qquad &(\nabla\eta_Q, \nabla\eta_Q(0), \nabla\eta_Q(1))\\ &\in L^2([0,1]; L^2(TM, dc(t))) \oplus L^2(TM, dc(0)) \oplus L^2(TM, dc(1))\end{aligned}$$

with $\nabla\eta_Q(1) = \nabla\eta_1$, which represents an approximate parallel transport along c.

Theorem 1.1. *Suppose that M has nonnegative sectional curvature. Then a subsequence of $\{(\nabla\eta_Q, \nabla\eta_Q(0), \nabla\eta_Q(1))\}_{Q=1}^\infty$ converges weakly to a weak solution $(\nabla\eta_\infty, \nabla\eta_{\infty,0}, \nabla\eta_{\infty,1})$ of the parallel transport equation with $\nabla\eta_{\infty,1} = \nabla\eta_1$. If c is a smooth geodesic in $P^\infty(M)$, η_1 is smooth, and there is a smooth solution η to the parallel transport equation* (2.6) *with $\eta(1) = \eta_1$, then*

$$\lim_{Q\to\infty} (\nabla\eta_Q, \nabla\eta_Q(0), \nabla\eta_Q(1)) = (\nabla\eta, \nabla\eta(0), \nabla\eta(1))$$

in norm.

Remark 1.2. In the setting of Theorem 1.1, we can say that $\nabla\eta_{\infty,0}$ is the parallel transport of $\nabla\eta_1$ along c to $T_{c(0)}P(M)$.

Remark 1.3. We are assuming that M has nonnegative sectional curvature in order to apply some geometric results from [11]. It is likely that this assumption could be removed.

Remark 1.4. A result related to Theorem 1.1 was proven by Ambrosio and Gigli when $M = \mathbb{R}^n$ [1, Theorem 5.14], and extended to general M by Gigli [4, Theorem 4.9]. As explained in [1,4], the construction of parallel transport there can be considered to be extrinsic, in that it is based on embedding the (linear) tangent cones into a Hilbert space and applying projection operators to form the approximate parallel transports. Although we instead use Petrunin's intrinsic construction, there are some similarities between the two constructions; see Remark 3.3. We use some techniques from [1], especially the idea of a weak solution to the parallel transport equation.

Remark 1.5. Besides its inherent naturality, the intrinsic construction of parallel transport given here is likely to allow for extensions. For example, using the results of [7], it seems likely that Petrunin's construction could be extended to define parallel transport along Wasserstein geodesics of absolutely continuous measures on submanifolds of M. In the present paper we have done this when the submanifolds have dimension zero or codimension zero.

The structure of this paper is as follows. In Section 2 we discuss weak solutions to the parallel transport equation. In Section 3 we prove Theorem 1.1.

2. Weak solutions to the parallel transport equation

Let M be a compact connected Riemannian manifold without boundary. Put

$$P^\infty(M) = \{\rho \ \mathrm{dvol}_M \ : \ \rho \in C^\infty(M), \rho > 0, \int_M \rho \ \mathrm{dvol}_M \ = \ 1\}. \tag{2.1}$$

Given $\phi \in C^\infty(M)$, define a vector field V_ϕ on $P^\infty(M)$ by saying that for $F \in C^\infty(P^\infty(M))$,

$$(V_\phi F)(\rho \, \mathrm{dvol}_M) \ = \ \frac{d}{d\epsilon}\Big|_{\epsilon=0} F\left(\rho \, \mathrm{dvol}_M \ - \epsilon \, \nabla^i(\rho \nabla_i \phi) \, \mathrm{dvol}_M\right). \tag{2.2}$$

The map $\phi \to V_\phi$ passes to an isomorphism $C^\infty(M)/\mathbb{R} \to T_{\rho \, \mathrm{dvol}_M} P^\infty(M)$. Otto's Riemannian metric on $P^\infty(M)$ is given [10] by

$$\begin{aligned} \langle V_{\phi_1}, V_{\phi_2} \rangle(\rho \, \mathrm{dvol}_M) \ &= \ \int_M \langle \nabla \phi_1, \nabla \phi_2 \rangle \ \rho \ \mathrm{dvol}_M \\ &= \ - \ \int_M \phi_1 \nabla^i(\rho \nabla_i \phi_2) \ \mathrm{dvol}_M \, . \end{aligned} \tag{2.3}$$

In view of (2.2), we write $\delta_{V_\phi} \rho \ = \ - \nabla^i(\rho \nabla_i \phi)$. Then

$$\langle V_{\phi_1}, V_{\phi_2} \rangle(\rho \, \mathrm{dvol}_M) \ = \ \int_M \phi_1 \, \delta_{V_{\phi_2}} \rho \ \mathrm{dvol}_M \ = \ \int_M \phi_2 \, \delta_{V_{\phi_1}} \rho \ \mathrm{dvol}_M \, . \tag{2.4}$$

To write the equation for parallel transport, let $c : [0, 1] \to P^\infty(M)$ be a smooth curve. We write $c(t) \ = \ \mu_t \ = \ \rho(t) \ \mathrm{dvol}_M$ and define $\phi(t) \in C^\infty(M)$, up to a constant, by $\frac{dc}{dt} \ = \ V_{\phi(t)}$. This is the same as saying

$$\frac{\partial \rho}{\partial t} + \nabla^j \left(\rho \nabla_j \phi\right) = 0. \tag{2.5}$$

Let $V_{\eta(t)}$ be a vector field along c, with $\eta(t) \in C^\infty(M)$. The equation for V_η to be parallel along c [6, Proposition 3] is

$$\nabla_i \left(\rho \left(\nabla^i \frac{\partial \eta}{\partial t} \ + \ \nabla_j \phi \, \nabla^i \nabla^j \eta \right) \right) \ = \ 0. \tag{2.6}$$

Lemma 2.1 ([6, Lemma 5])**.** *If $\eta, \overline{\eta}$ are solutions of* (2.6)*, then $\int_M \langle \nabla \eta, \nabla \overline{\eta} \rangle \, d\mu_t$ is constant in t.*

Lemma 2.2. *Given $\eta_1 \in C^\infty(M)$, there is at most one solution of* (2.6) *with $\eta(1) = \eta_1$, up to time-dependent additive constants.*

Proof. By linearity, it suffices to consider the case when $\eta_1 = 0$. From Lemma 2.1, $\nabla \eta(t) = 0$ and so $\eta(t)$ is spatially constant. □

For consistency with later notation, we will write $C^\infty([0,1]; C^\infty(M))$ for $C^\infty([0,1] \times M)$.

Lemma 2.3 (cf. [1, (5.8)])**.** *Given $f \in C^\infty([0,1]; C^\infty(M))$, if η satisfies* (2.6), *then*

$$\frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t = \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \, d\mu_t + \int_M \mathrm{Hess}_f(\nabla \eta, \nabla \phi) \, d\mu_t. \tag{2.7}$$

Proof. We have

$$\begin{aligned} \frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t =& \frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \, \rho \, \mathrm{dvol}_M \\ =& \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \, \rho \, \mathrm{dvol}_M + \int_M \langle \nabla f, \nabla \frac{\partial \eta}{\partial t} \rangle \, \rho \, \mathrm{dvol}_M \\ &+ \int_M \langle \nabla f, \nabla \eta \rangle \, \frac{\partial \rho}{\partial t} \, \mathrm{dvol}_M \, . \end{aligned} \tag{2.8}$$

Then

$$\begin{aligned} &\frac{d}{dt} \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t - \int_M \langle \nabla \frac{\partial f}{\partial t}, \nabla \eta \rangle \, d\mu_t \\ &= \int_M (\nabla_i f) \left(\nabla^i \frac{\partial \eta}{\partial t} \right) \rho \, \mathrm{dvol}_M - \int_M (\nabla_i f) \, (\nabla^i \eta) \, \nabla^j \, (\rho \nabla_j \phi) \, \mathrm{dvol}_M \\ &= - \int_M f \, \nabla_i \left(\rho \nabla^i \frac{\partial \eta}{\partial t} \right) \mathrm{dvol}_M - \int_M (\nabla_i f) \, (\nabla^i \eta) \, \nabla^j \, (\rho \nabla_j \phi) \, \mathrm{dvol}_M \\ &= \int_M f \nabla_i \left(\rho (\nabla_j \phi) \, (\nabla^i \nabla^j \eta) \right) \mathrm{dvol}_M + \int_M \nabla^j ((\nabla_i f) \, (\nabla^i \eta)) \, (\nabla_j \phi) \, \rho \, \mathrm{dvol}_M \\ &= - \int_M (\nabla_i f) \, (\nabla_j \phi) \, (\nabla^i \nabla^j \eta) \, \rho \, \mathrm{dvol}_M \\ &+ \int_M \nabla^j ((\nabla_i f) \, (\nabla^i \eta)) \, (\nabla_j \phi) \, \rho \, \mathrm{dvol}_M \\ &= \int_M (\nabla^j \nabla_i f) \, (\nabla^i \eta) \, (\nabla_j \phi) \, \rho \, \mathrm{dvol}_M \\ &= \int_M \mathrm{Hess}_f(\nabla \eta, \nabla \phi) \, d\mu_t. \end{aligned} \tag{2.9}$$

This proves the lemma. □

We now weaken the regularity assumptions. Let $P^{ac}(M)$ denote the absolutely continuous probability measures on M with full support. Suppose that $c : [0,1] \to P^{ac}(M)$ is a Lipschitz curve whose derivative $c'(t) \in T_{c(t)} P(M)$ exists for almost all t. We can write $c'(t) = V_{\phi(t)}$ with $\nabla \phi(t) \in L^2(TM, dc(t))$. By the Lipschitz assumption, the essential supremum over $t \in [0,1]$ of $\| \nabla \phi(t) \|_{L^2(TM, dc(t))}$ is finite. As before, we write $c(t) = \mu_t$.

Definition 2.4. Let $c : [0,1] \to P^{ac}(M)$ be a Lipschitz curve whose derivative $c'(t) \in T_{c(t)}P(M)$ exists for almost all t. Given $\nabla\eta_0 \in L^2(TM, d\mu_0)$, $\nabla\eta_1 \in L^2(TM, d\mu_1)$ and $\nabla\eta \in L^2([0,1]; L^2(TM, d\mu_t))$, we say that $(\nabla\eta, \nabla\eta_0, \nabla\eta_1)$ is a weak solution of the parallel transport equation if

$$\begin{aligned} &\int_M \langle \nabla f(1), \nabla\eta_1\rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla\eta_0\rangle \, d\mu_0 \\ &= \int_0^1 \int_M \left(\left\langle \nabla \frac{\partial f}{\partial t}, \nabla\eta \right\rangle + \mathrm{Hess}_f(\nabla\eta, \nabla\phi) \right) d\mu_t \, dt \end{aligned} \tag{2.10}$$

for all $f \in C^\infty([0,1]; C^\infty(M))$.

Remark 2.5. In what follows, there would be analogous results if we replaced $C^\infty([0,1]; C^\infty(M))$ everywhere by $C^0([0,1]; C^2(M)) \cap C^1([0,1]; C^1(M))$. We will stick with $C^\infty([0,1]; C^\infty(M))$ for concreteness.

From Lemma 2.3, if c is a smooth curve in $P^\infty(M)$ and $\eta \in C^\infty([0,1]; C^\infty(M))$ is a solution of (2.6), then $(\nabla\eta, \nabla\eta(0), \nabla\eta(1))$ is a weak solution of the parallel transport equation. We now prove the converse.

Lemma 2.6. *Suppose that c is a smooth curve in $P^\infty(M)$. Given $\eta_0, \eta_1 \in C^\infty(M)$ and $\eta \in C^\infty([0,1]; C^\infty(M))$, if $(\nabla\eta, \nabla\eta_0, \nabla\eta_1)$ is a weak solution of the parallel transport equation, then η satisfies* (2.6), *$\eta(0) = \eta_0$ and $\eta(1) = \eta_1$ (modulo constants).*

Proof. In this case, equation (2.10) is equivalent to

$$\begin{aligned} &\int_M \langle \nabla f(1), \nabla\eta_1\rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla\eta_0\rangle \, d\mu_0 \\ &= \int_M \langle \nabla f(1), \nabla\eta(1)\rangle \, d\mu_1 - \int_M \langle \nabla f(0), \nabla\eta(0)\rangle \, d\mu_0 \\ &\quad + \int_0^1 \int_M f \nabla_i \left(\nabla^i \frac{\partial \eta}{\partial t} + \nabla_j \phi \nabla^i \nabla^j \eta \right) d\mu_t \, dt. \end{aligned} \tag{2.11}$$

Taking $f \in C^\infty([0,1]; C^\infty(M))$ with $f(0) = f(1) = 0$, it follows that (2.6) must hold. Then taking all $f \in C^\infty([0,1]; C^\infty(M))$, it follows that $\nabla\eta_0 = \nabla\eta(0)$ and $\nabla\eta_1 = \nabla\eta(1)$. Hence $\eta(0) = \eta_0$ and $\eta(1) = \eta_1$ (modulo constants). $\square$

Lemma 2.7. *Suppose that c is a smooth curve in $P^\infty(M)$. Given $\nabla\eta_0 \in L^2(TM, d\mu_0)$, $\nabla\eta_1 \in L^2(TM, d\mu_1)$, $\nabla\eta \in L^2([0,1]; L^2(TM, d\mu_t))$ and $f \in C^\infty([0,1]; C^\infty(M))$, suppose that*

(1) *$(\nabla\eta, \nabla\eta_0, \nabla\eta_1)$ is a weak solution to the parallel transport equation,*
(2) *f satisfies* (2.6),
(3) *$\nabla f(1) = \nabla\eta_1$,*
(4)

$$\int_M |\nabla\eta_0|^2 \, d\mu_0 \le \int_M |\nabla\eta_1|^2 \, d\mu_1 \tag{2.12}$$

and

(5)

$$\int_0^1 \int_M |\nabla\eta|^2 \, d\mu_t \, dt \le \int_M |\nabla\eta_1|^2 \, d\mu_1. \tag{2.13}$$

Then $\nabla f(0) = \nabla \eta_0$, *and* $\nabla f(t) = \nabla \eta(t)$ *for almost all* t.

Proof. From (2.6) (applied to f) and (2.10), we have

$$\int_M \langle \nabla f(0), \nabla \eta_0 \rangle \, d\mu_0 = \int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1. \tag{2.14}$$

From Lemma 2.1,

$$\int_M \langle \nabla f(0), \nabla f(0) \rangle \, d\mu_0 = \int_M \langle \nabla f(1), \nabla f(1) \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1. \tag{2.15}$$

Then

$$\int_M |\nabla(\eta_0 - f(0))|^2 \, d\mu_0 = \int_M |\nabla \eta_0|^2 \, d\mu_0 - \int_M |\nabla \eta_1|^2 \, d\mu_1 \leq 0. \tag{2.16}$$

Thus $\nabla f(0) = \nabla \eta_0$ in $L^2(TM, d\mu_0)$.

Next, replacing f by tf in (2.10) gives

$$\int_0^1 \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t \, dt = \int_M \langle \nabla f(1), \nabla \eta_1 \rangle \, d\mu_1 = \int_M \langle \nabla \eta_1, \nabla \eta_1 \rangle \, d\mu_1. \tag{2.17}$$

Then

$$\begin{aligned} & \int_0^1 \int_M |\nabla f - \nabla \eta|^2 \, d\mu_t \, dt \\ &= \int_0^1 \int_M |\nabla f|^2 \, d\mu_t \, dt - 2 \int_0^1 \int_M \langle \nabla f, \nabla \eta \rangle \, d\mu_t \, dt + \int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt \\ &= \int_M |\nabla f(1)|^2 \, d\mu_1 - 2 \int_M |\nabla \eta_1|^2 \, d\mu_1 + \int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt \\ &= \int_0^1 \int_M |\nabla \eta|^2 \, d\mu_t \, dt - \int_M |\nabla \eta_1|^2 \, d\mu_1 \leq 0. \end{aligned} \tag{2.18}$$

Thus $\nabla f(t) = \nabla \eta(t)$ in $L^2(TM, d\mu_t)$, for almost all t. □

3. Parallel transport along Wasserstein geodesics

3.1. **Parallel transport in a finite-dimensional Alexandrov space.** We recall the construction of parallel transport in a finite-dimensional Alexandrov space X.

Let $c : [0,1] \to X$ be a geodesic segment that lies in the interior of a minimizing geodesic. Then $T_{c(t)}X$ is an isometric product of $\mathbb{R}$ with the normal cone $N_{c(t)}X$. We want to construct a parallel transport map from $N_{c(1)}X$ to $N_{c(0)}X$.

Given $Q \in \mathbb{Z}^+$ and $0 \leq i \leq Q - 1$, define $c_i : [0,1] \to X$ by $c_i(u) = c\left(\frac{i+u}{Q}\right)$. We define an approximate parallel transport $P_i : N_{c_i(1)}X \to N_{c_i(0)}X$ as follows. Given $v \in N_{c_i(1)}X$, let $\gamma : [0, \epsilon] \to X$ be a minimizing geodesic segment with $\gamma(0) = c_i(1)$ and $\gamma'(0) = v$. For each $s \in (0, \epsilon]$, let $\mu_s : [0,1] \to X$ be a minimizing geodesic with $\mu_s(0) = c_i(0)$ and $\mu_s(1) = \gamma(s)$. Let $w_s \in N_{c_i(0)}X$ be the normal projection of $\frac{1}{s}\mu_s'(0) \in T_{c_i(0)}X$. After passing to a sequence $s_i \to 0$, we can assume that $\lim_{i \to \infty} w_{s_i} = w \in N_{c_i(0)}X$. Then $P_i(v) = w$. If X has nonnegative Alexandrov curvature, then $|w| \geq |v|$.

In [11], the approximate parallel transport from an appropriate dense subset $L_Q \subset N_{c(1)}X$ to $N_{c(0)}X$ was defined to be $P_0 \circ P_1 \circ \ldots \circ P_{Q-1}$. It was shown that by taking $Q \to \infty$ and applying a diagonal argument, in the limit one obtains an isometry from a dense subset of $N_{c(1)}X$ to $N_{c(0)}X$. This extends by continuity to an isometry from $N_{c(1)}X$ to $N_{c(0)}X$.

If X is a smooth Riemannian manifold, then P_i is independent of the choices and can be described as follows. Given $v \in N_{c_i(1)}X$, let $j_v(u)$ be the Jacobi field along c with $j_v(0) = 0$ and $j_v(1) = v$. (It is unique since c is in the interior of a minimizing geodesic.) Then $P_i(v) = j_v'(0)$.

3.2. **Construction of parallel transport along a Wasserstein geodesic of delta measures.** Let M be a compact connected Riemannian manifold without boundary. Let $\gamma : [0,1] \to M$ be a geodesic segment that lies in the interior of a minimizing geodesic. Let $\Pi : T_{\gamma(1)}M \to T_{\gamma(0)}M$ be (reverse) parallel transport along γ. Put $c(t) = \delta_{\gamma(t)} \in P(M)$. Then $\{c(t)\}_{t\in[0,1]}$ is a Wasserstein geodesic that lies in the interior of a minimizing geodesic. We apply Petrunin's construction to define parallel transport directly from the tangent cone $T_{c(1)}P(M)$ to the tangent cone $T_{c(0)}P(M)$ (instead of the normal cones). From [7, Theorem 1.1], we know that $T_{c(t)}P(M) \cong P_2(T_{\gamma(t)}M)$.

Proposition 3.1. *The parallel transport map from $T_{c(1)}P(M) \cong P_2(T_{\gamma(1)}M)$ to $T_{c(0)}P(M) \cong P_2(T_{\gamma(0)}M)$ is the map $\mu \to \Pi_* \mu$.*

Proof. Given $Q \in \mathbb{Z}^+$ and $0 \le i \le Q-1$, define $\gamma_i : [0,1] \to M$ by $\gamma_i(u) = \gamma\left(\frac{i+u}{Q}\right)$ and $c_i : [0,1] \to P(M)$ by $c_i(u) = \delta_{\gamma_i(u)}$. We define an approximate parallel transport $P_i : T_{c_i(1)}P(M) \to T_{c_i(0)}P(M)$ as follows.

Given $s \in \mathbb{R}^+$ and a real vector space V, let $R_s : V \to V$ be multiplication by s. Let ν be a compactly-supported element of $P(T_{\gamma_i(1)}M)$. For small $\epsilon > 0$, there is a Wasserstein geodesic $\sigma : [0,\epsilon] \to P(M)$, with $\sigma(0) = c_i(1)$ and $\sigma'(0)$ corresponding to $\nu \in T_{c_i(1)}PM$, given by $\sigma(s) = (\exp_{\gamma_i(1)} \circ R_s)_* \nu$. Given $s \in (0,\epsilon]$, let $\mu_s : [0,1] \to P(M)$ be a minimizing geodesic with $\mu_s(0) = c_i(0) = \delta_{\gamma_i(0)}$ and $\mu_s(1) = \sigma(s)$. There is a compactly-supported measure $\tau_s \in P_2(T_{\gamma_i(0)}M) = T_{c_i(0)}P(M)$ so that for $v \in [0,1]$, we have $\mu_s(v) = (\exp_{\gamma_i(0)} \circ R_v)_* \tau_s$. If Q is large and ϵ is small, then all of the constructions take place well inside a totally convex ball, so τ_s is unique and can be written as $\tau_s = \left(\exp^{-1}_{\gamma_i(0)} \circ \exp_{\gamma_i(1)} \circ R_s\right)_* \nu$. Then $\lim_{s\to 0} \frac{1}{s}(\tau_s - \tau_0)$ exists and equals $(d\exp_{\gamma_i(0)})^{-1}_* \nu$. Thus $P_i = (d\exp_{\gamma_i(0)})^{-1}_*$.

Now

$$
\begin{aligned}
(3.2) \qquad & P_0 \circ P_1 \circ \ldots \circ P_{Q-1} \\
&= \left((d\exp_{\gamma_0(0)})^{-1} \circ (d\exp_{\gamma_1(0)})^{-1} \circ \ldots \circ (d\exp_{\gamma_{Q-1}(0)})^{-1}\right)_*.
\end{aligned}
$$

Taking $Q \to \infty$, this approaches Π_*. ∎

3.3. **Construction of parallel transport along a Wasserstein geodesic of absolutely continuous measures.** Let M be a compact connected boundaryless Riemannian manifold with nonnegative sectional curvature. Then $(P(M), W_2)$ has nonnegative Alexandrov curvature.

Let $c : [0,1] \to P^{ac}(M)$ be a geodesic segment that lies in the interior of a minimizing geodesic. Write $c'(t) = V_{\phi(t)}$. Since $\phi(t)$ is defined up to a constant, it will be convenient to normalize it by $\int_M \phi(t)\, d\mu_t = 0$. We assume that

$$
(3.3) \qquad \sup_{t\in[0,1]} \|\phi(t)\|_{C^2(M)} < \infty.
$$

In particular, this is satisfied if c lies in $P^\infty(M)$.

Let $N_{c(t)}P(M)$ denote the normal cone to c at $c(t)$. We want to construct a parallel transport map from $N_{c(1)}P(M)$ to $N_{c(0)}P(M)$.

Given $Q \in \mathbb{Z}^+$ and $0 \le i \le Q-1$, define $c_i : [0,1] \to P(M)$ by $c_i(u) = c\left(\frac{i+u}{Q}\right)$. Correspondingly, write $\mu_{i,u} = \mu_{\frac{i+u}{Q}}$. We define an approximate parallel transport $P_i : N_{c_i(1)}P(M) \to N_{c_i(0)}P(M)$, using Jacobi fields, as follows.

Let us write $c_i'(u) = V_{\phi_i(u)}$, i.e., $\phi_i(u) = \frac{1}{Q}\phi\left(\frac{i+u}{Q}\right)$. The curve c_i is given by $c_i(u) = (F_{i,u})_* c_i(0)$, where $F_{i,u}(x) = \exp_x(u \nabla_x \phi_i(0))$. That is, for any $f \in C^\infty(M)$,

$$\int_M f \, dc_i(u) = \int_M f(F_{i,u}(x)) \, d\mu_{i,0}(x). \tag{3.4}$$

If σ_i is a variation of $\phi_i(0)$, i.e., $\delta\phi_i(0) = \sigma_i$, then taking the variation of (3.4) gives

$$\begin{aligned}\int_M f \, d\delta c_i(u) &= \int_M \langle \nabla f, d\exp_{u\nabla_x \phi_i(0)}(u \nabla_x \sigma_i)\rangle_{F_{i,u}(x)} \, d\mu_{i,0}(x) \\ &= u \int_M \langle \nabla f, W_{\sigma_i}(u) \rangle \, d\mu_{i,u}.\end{aligned} \tag{3.5}$$

Here

$$(W_{\sigma_i}(u))_y = d\exp_{u\nabla_x \phi_i(0)}(\nabla_x \sigma_i), \tag{3.6}$$

with $y = F_{i,u}(x)$. The corresponding tangent vector at $c_i(u)$ is represented by $L_{\sigma_i}(u) = \Pi_{c_i(u)} W_{\sigma_i}(u)$, where $\Pi_{c_i(u)}$ is orthogonal projection on $\overline{\operatorname{Im}\nabla} \subset L^2(TM, d\mu_{i,u})$. We can think of $J_{\sigma_i}(u) = u L_{\sigma_i}(u)$ as a Jacobi field along c_i. If $v = J_{\sigma_i}(1) = L_{\sigma_i}(1) = \Pi_{c_i(1)} W_{\sigma_i}(1)$, then its approximate parallel transport along c_i is represented by $w = J_{\sigma_i}'(0) = L_{\sigma_i}(0) = \nabla \sigma_i \in \overline{\operatorname{Im}\nabla} \subset L^2(TM, d\mu_{i,0})$.

Next, using (3.6), for $f \in C^\infty(M)$ we have

$$\begin{aligned}&\frac{d}{du}\int_M \langle V_f, L_{\sigma_i}\rangle \, d\mu_{i,u} = \frac{d}{du}\int_M \langle V_f, W_{\sigma_i}\rangle \, d\mu_{i,u} \\ &= \frac{d}{du}\int_M \langle \nabla f, d\exp_{u\nabla_x\phi_i(0)}(\nabla_x \sigma_i)\rangle_{F_{i,u}(x)} \, d\mu_{i,0}(x) \\ &= \int_M \operatorname{Hess}_{F_{i,u}(x)}(f)\left(d\exp_{u\nabla_x\phi_i(0)}(\nabla_x\phi_i(0)), d\exp_{u\nabla_x\phi_i(0)}(\nabla_x\sigma_i)\right) \, d\mu_{i,0}(x) \\ &\quad + \int_M \langle \nabla f, D_{\partial_u} d\exp_{u\nabla_x\phi_i(0)}(\nabla_x\sigma_i)\rangle_{F_{i,u}(x)} \, d\mu_{i,0}(x) \\ &= \int_M \operatorname{Hess}(f)\left(\nabla\phi_i(u), W_{\sigma_i}(u)\right) \, d\mu_{i,u} + \int_M \langle \nabla f, D_{\partial_u} W_{\sigma_i}(u)\rangle \, d\mu_{i,u}.\end{aligned} \tag{3.7}$$

Here ∂_u is the vector at $F_{i,u}(x)$ given by

$$\partial_u = \frac{d}{du} F_{i,u}(x) = d\exp_{u\nabla_x\phi_i(0)}(\nabla_x \phi_i(0)). \tag{3.8}$$

If instead $f \in C^\infty([0,1]; C^\infty(M))$, then

$$\begin{aligned}(3.9)\qquad \frac{d}{du}\int_M \langle V_f, L_{\sigma_i}\rangle \, d\mu_{i,u} &= \int_M \left\langle \nabla \frac{\partial f}{\partial u}, L_{\sigma_i} \right\rangle \, d\mu_{i,u} \\ &+ \int_M \mathrm{Hess}(f)\left(\nabla\phi_i(u), W_{\sigma_i}(u)\right) \, d\mu_{i,u} \\ &+ \int_M \langle \nabla f, D_{\partial_u} W_{\sigma_i}(u)\rangle \, d\mu_{i,u}.\end{aligned}$$

We will need to estimate $\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u}$.

Lemma 3.1. *For large Q, there is an estimate*

$$\begin{aligned}(3.10)\qquad &\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \\ &\leq \text{const.} \, \| \mathrm{Hess}(\phi_i(\cdot))\|^2_{L^\infty([0,1]\times M)} \|L_{\sigma_i}(0)\|^2_{L^2(TM, d\mu_{i,0})}.\end{aligned}$$

Here, and hereafter, const. *denotes a constant that can depend on the fixed Riemannian manifold (M, g).*

Proof. Since $\Pi_{c_i(u)}$ is projection onto $\overline{\mathrm{Im}(\nabla)} \subset L^2(TM, d\mu_{i,u})$, and $\nabla(\sigma_i \circ F^{-1}_{i,u}) \in \mathrm{Im}(\nabla)$, we have

$$\begin{aligned}(3.11)\qquad \int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} &\leq \int_M |W_{\sigma_i}(u) - \nabla(\sigma_i \circ F^{-1}_{i,u})|^2_g \, d\mu_{i,u} \\ &= \int_M |(dF_{i,u})^{-1}_* W_{\sigma_i}(u) - \nabla\sigma_i|^2_{F^*_{i,u} g} \, d\mu_{i,0}.\end{aligned}$$

(Compare with [1, Proposition 4.3].) Defining $T_{i,t,x} : T_x M \to T_x M$ by

$$(3.12)\qquad T_{i,t,x}(z) = (dF_{i,u})^{-1}_* \left(d\exp_{u\nabla_x \phi_i(0)}(z)\right),$$

we obtain

$$\begin{aligned}(3.13)\qquad &\int_M |W_{\sigma_i}(u) - L_{\sigma_i}(u)|^2 \, d\mu_{i,u} \\ &\leq \left(\sup_{x\in M} \|dF^*_{i,u} dF_{i,u}(x)\| \cdot \|T_{i,u,x} - I\|^2\right) \|L_{\sigma_i}(0)\|^2_{L^2(TM, d\mu_{i,0})}.\end{aligned}$$

Since $\sup_{t\in[0,1]} \|\nabla\phi(t)\|_{C^0(M)} < \infty$, if Q is large, then $\|\nabla\phi_i(0)\|_{C^0(M)}$ is much smaller than the injectivity radius of M. In particular, the curve $\{F_{i,u}(x)\}_{u\in[0,1]}$ lies well within a normal ball around x. Now $T_{i,t,x}$ can be estimated in terms of $\mathrm{Hess}(\phi_i)$. In general, if a function h on a complete Riemannian manifold satisfies $\mathrm{Hess}(h) = 0$, then the manifold isometrically splits off an $\mathbb{R}$-factor and the optimal transport path generated by ∇h is translation along the $\mathbb{R}$-factor. In such a case, the analog of $T_{i,t,x}$ is the identity map. If $\mathrm{Hess}(h) \neq 0$, then the divergence of a short optimal transport path from being a translation can be estimated in terms of $\mathrm{Hess}(h)$. Putting in the estimates gives (3.10). ∎

Using Lemma 3.1, we have

$$\begin{aligned}(3.14)\qquad &\left| \int_M \mathrm{Hess}(f)\left(\nabla\phi_i(u), W_{\sigma_i}(u)\right) \, d\mu_{i,u} - \int_M \mathrm{Hess}(f)\left(\nabla\phi_i(u), L_{\sigma_i}(u)\right) \, d\mu_{i,u} \right| \\ &\leq \text{const.} \, \| \mathrm{Hess}(f)\|_{C^0(M)} \| \mathrm{Hess}(\phi_i(\cdot))\|_{L^\infty([0,1]\times M)} \\ &\cdot \|\nabla\phi_i(u)\|_{L^2(TM, d\mu_{i,0})} \|L_{\sigma_i}(0)\|_{L^2(TM, d\mu_{i,0})}.\end{aligned}$$

Next, given $x \in M$, consider the geodesic

$$\gamma_{i,x}(u) = F_{i,u}(x). \tag{3.15}$$

Put

$$j_{\sigma_i,x}(u) = u(W_{\sigma_i}(u))_{\gamma_{i,x}(u)} \in T_{\gamma_{i,x}(u)}M. \tag{3.16}$$

Then $j_{\sigma_i,x}$ is a Jacobi field along $\gamma_{i,x}$, with $j_{\sigma_i,x}(0) = 0$ and $j'_{\sigma_i,x}(0) = \nabla_x \sigma_i$. Jacobi field estimates give

$$\| D_{\partial_u} W_{\sigma_i}(u) \|_{L^2(TM, d\mu_{i,u})} \le \text{const.} \| \nabla \sigma_i \|_{L^2(TM, d\mu_{i,u})} \| \nabla \phi_i(\cdot) \|^2_{L^\infty([0,1] \times M)}, \tag{3.17}$$

again for Q large.

Lemma 3.2. *Define* $A_i : \left(\overline{\mathrm{Im}(\nabla)} \subset L^2(TM, d\mu_{i,0}) \right) \to \left(\overline{\mathrm{Im}(\nabla)} \subset L^2(TM, d\mu_{i,1}) \right)$ *by*

$$A_i(\nabla \sigma_i) = L_{\sigma_i}(1). \tag{3.18}$$

Then for large Q*, the map* A_i *is invertible for all* $i \in \{0, \ldots, Q-1\}$*.*

Proof. Define $B_i : \left(\overline{\mathrm{Im}(\nabla)} \subset L^2(TM, d\mu_{i,1}) \right) \to \left(\overline{\mathrm{Im}(\nabla)} \subset L^2(TM, d\mu_{i,0}) \right)$ by

$$B_i(\nabla f) = \nabla(f \circ F_{i,1}). \tag{3.19}$$

Then whenever $\nabla f \in L^2(TM, d\mu_{i,1})$, we have

$$(A_i B_i)(\nabla f) = A_i(\nabla(f \circ F_{i,1})) = L_{f \circ F_{i,1}}(1), \tag{3.20}$$

so whenever $\nabla f' \in L^2(TM, d\mu_{i,1})$, for large Q we have

$$\begin{aligned} &\langle \nabla f', (A_i B_i - I)(\nabla f) \rangle_{L^2(TM, d\mu_{i,1})} \\ &= \langle \nabla f', W_{f \circ F_{i,1}}(1) - \nabla f \rangle_{L^2(TM, d\mu_{i,1})} \\ &\le \text{const.} \| \mathrm{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)} \| \nabla f' \|_{L^2(TM, d\mu_{i,1})} \| \nabla f \|_{L^2(TM, d\mu_{i,1})}. \end{aligned} \tag{3.21}$$

Hence $\| A_i B_i - I \| = o(Q)$, so for large Q the map $A_i B_i$ is invertible and a right inverse for A_i is given by $B_i (A_i B_i)^{-1}$. This implies that A_i is surjective.

Now suppose that $\nabla \sigma \in \mathrm{Ker}(A_i)$ is nonzero, with $\sigma \in H^1(M, d\mu_{i,0})$. After normalizing, we may assume that $\nabla \sigma$ has unit length. Then

$$\begin{aligned} 0 =& \langle \nabla(\sigma \circ F_{i,1}), A_i(\nabla \sigma) \rangle_{L^2(TM, d\mu_{i,1})} = \langle \nabla(\sigma \circ F_{i,1}), L_\sigma(1) \rangle_{L^2(TM, d\mu_{i,1})} \\ =& \langle \nabla(\sigma \circ F_{i,1}), W_\sigma(1) \rangle_{L^2(TM, d\mu_{i,1})} = \langle \nabla \sigma, (dF_{i,1})^{-1} W_\sigma(1) \rangle_{L^2(TM, d\mu_{i,0})} \\ =& 1 - \langle \nabla \sigma, \nabla \sigma - (dF_{i,1})^{-1} W_\sigma(1) \rangle_{L^2(TM, d\mu_{i,0})} \\ \ge& 1 - \text{const.} \| \mathrm{Hess}(\phi_i(\cdot)) \|_{L^\infty([0,1] \times M)}, \end{aligned} \tag{3.22}$$

for large Q. If Q is sufficiently large, then this is a contradiction, so A_i is injective. □

Fix $\mathcal{V}_1 \in N_{c(1)} P(M)$. If $\mathcal{V}_1 \neq 0$, then after normalizing, we may assume that it has unit length. For $Q \in \mathbb{Z}^+$ large and $t \in [0,1]$, define $\mathcal{V}_Q(t) \in N_{c(t)} P(M)$ as follows. First, using Lemma 3.2, find σ_{Q-1} so that $\mathcal{V}_1 = L_{\sigma_{Q-1}}(1)$. For $t \in \left[\frac{Q-1}{Q}, 1 \right]$, put

$$\mathcal{V}_Q(t) = L_{\sigma_{Q-1}}(Qt - (Q-1)). \tag{3.23}$$

Doing backward recursion, starting with $i = Q - 2$, using Lemma 3.2 we find σ_i so that $L_{\sigma_i}(1) = L_{\sigma_{i+1}}(0) = \nabla \sigma_{i+1}$. For $t \in \left[\frac{i}{Q}, \frac{i+1}{Q}\right]$, put

$$\mathcal{V}_Q(t) = L_{\sigma_i}(Qt - i). \tag{3.24}$$

Decrease i by one and repeat. The last step is when $i = 0$.

From the argument in [11, Lemma 1.8],

$$\lim_{Q \to \infty} \sup_{t \in [0,1]} |\|\mathcal{V}_Q(t)\| - 1| = 0. \tag{3.25}$$

We note that the proof of [11, Lemma 1.8] only uses results about geodesics in Alexandrov spaces, it so applies to our infinite-dimensional setting. It also uses the assumption that c lies in the interior of a minimizing geodesic. After passing to a subsequence, we can assume that

$$\lim_{Q \to \infty} (\mathcal{V}_Q, \mathcal{V}_Q(0), \mathcal{V}_Q(1)) = (\mathcal{V}_\infty, \mathcal{V}_{\infty,0}, \mathcal{V}_{\infty,1}) \tag{3.26}$$

in the weak topology on $L^2([0,1]; L^2(TM, d\mu_t)) \oplus L^2(TM, d\mu_0) \oplus L^2(TM, d\mu_1)$. Note that $\mathcal{V}_{\infty,1} = \mathcal{V}_1$.

From (3.9), (3.14) and (3.17), for a fixed $f \in C^\infty([0,1]; C^\infty(M))$, on each interval $\left[\frac{i}{Q}, \frac{i+1}{Q}\right]$ we have

$$\begin{aligned} \frac{d}{dt} \int_M \langle V_f, \mathcal{V}_Q \rangle \, d\mu_t = & \int_M \left\langle \nabla \frac{\partial f}{\partial t}, \mathcal{V}_Q(t) \right\rangle d\mu_t \\ & + \int_M \operatorname{Hess}(f)(\nabla \phi(t), \mathcal{V}_Q(t)) \, d\mu_t + o(Q). \end{aligned} \tag{3.27}$$

It follows that $(\mathcal{V}_\infty, \mathcal{V}_{\infty,0}, \mathcal{V}_{\infty,1})$ is a weak solution of the parallel transport equation. As the limiting vector fields are gradient vector fields, we can write $(\mathcal{V}_\infty, \mathcal{V}_{\infty,0}, \mathcal{V}_{\infty,1}) = (\nabla \eta_\infty, \nabla \eta_{\infty,0}, \nabla \eta_{\infty,1})$ for some

$$(\eta_\infty, \eta_{\infty,0}, \eta_{\infty,1}) \in L^2([0,1]; H^1(M, d\mu_t)) \oplus H^1(M, d\mu_0) \oplus H^1(M, d\mu_1)).$$

Suppose that c is a smooth geodesic in $P^\infty(M)$, that $\mathcal{V}_1$ (and hence $\eta_{\infty,1}$) is smooth and that there is a smooth solution η to the parallel transport equation (2.6) with $\nabla \eta(1) = \nabla \eta_{\infty,1}$. By Lemma 2.1, $\|\nabla \eta(t)\|$ is independent of t. By Lemma 2.7, $(\nabla \eta_\infty, \nabla \eta_{\infty,0}, \nabla \eta_{\infty,1}) = (\nabla \eta, \nabla \eta(0), \nabla \eta(1))$. We claim that

$$\lim_{Q \to \infty} (\nabla \eta_Q, \nabla \eta_Q(0), \nabla \eta_Q(1)) = (\nabla \eta, \nabla \eta(0), \nabla \eta_{\infty,1}) \tag{3.28}$$

in the norm topology on $L^2([0,1]; L^2(TM, d\mu_t)) \oplus L^2(TM, d\mu_0) \oplus L^2(TM, d\mu_1)$. This is because of the general fact that if $\{x_i\}_{i=1}^\infty$ is a sequence in a Hilbert space H with $\lim_{i \to \infty} |x_i| = 1$, and there is some unit vector $x_\infty \in H$ so that every weakly convergent subsequence of $\{x_i\}_{i=1}^\infty$ has weak limit x_∞, then $\lim_{i \to \infty} x_i = x_\infty$ in the norm topology.

In particular,

$$\lim_{Q \to \infty} \nabla \eta_Q(0) = \nabla \eta(0) \tag{3.29}$$

in the norm topology on $L^2(TM, d\mu_0)$.

This proves Theorem 1.1.

Remark 3.3. The construction of parallel transport in [1, Section 5] and [4, Section 4] is also by taking the limit of an iterative procedure. The underlying logic in [1,4] is different from what we use, which results in a different algorithm. The iterative construction in [1,4] amounts to going forward along the curve c applying certain maps $\mathcal{P}_i$, instead of going backward along c using the inverses of the A_i's as we do. In the case of $\mathbb{R}^n$, the map $\mathcal{P}_i$ is the same as A_i, but this is not the case in general. The map $\mathcal{P}_i$ is nonexpanding, which helps the construction in [1,4]. In contrast, A_i^{-1} is not nonexpanding. In order to control its products, we use the result (3.25) from [11].

Acknowledgments

The author thanks Takumi Yokota and Nicola Gigli for references to the literature.

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