ON 3-MANIFOLDS WITH POINTWISE PINCHED NONNEGATIVE RICCI CURVATURE

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Abstract. There is a conjecture that a complete Riemannian 3-manifold with bounded sectional curvature, and pointwise pinched nonnegative Ricci curvature, must be flat or compact. We show that this is true when the negative part (if any) of the sectional curvature decays quadratically.

1. Introduction

Let (M, g) be a complete connected Riemannian 3-manifold. Suppose that $\mathrm{Ric}(M, g) \geq 0$. At a point $m \in M$, the Ricci tensor on $T_m M$ can be diagonalized relative to $g(m)$. Let $r_1 \leq r_2 \leq r_3$ be its eigenvalues. Given $c \in (0, 1]$, we say that (M, g) is *c-Ricci pinched* if at all $m \in M$, we have $r_1 \geq c r_3$.

Conjecture 1.1. *Let (M, g) be a complete connected Riemannian manifold of dimension three, with bounded sectional curvature and nonnegative Ricci curvature. Suppose that (M, g) is c-Ricci pinched for some $c \in (0, 1]$. Then (M, g) is flat or M is compact.*

Using basic properties of Ricci flow, one can show that Conjecture 1.1 is equivalent to the following conjecture.

Conjecture 1.2. *Let (M, g) be a complete connected Riemannian manifold of dimension three, with bounded sectional curvature and positive Ricci curvature. Suppose that (M, g) is c-Ricci pinched for some $c \in (0, 1]$. Then M is compact.*

We will think of Conjectures 1.1 and 1.2 interchangeably. They are apparently due to Hamilton, who proved a result similar to Conjecture 1.2 for hypersurfaces in Euclidean space [12]. Conjecture 1.2 can be considered to be a scale-invariant version of the Bonnet-Myers theorem. The latter says that if a complete Riemannian n-manifold (M, g) has $\mathrm{Ric} \geq (n-1) k^2 g$, with $k > 0$, then M is compact with diameter at most $\frac{\pi}{k}$. In Conjecture 1.2, rather than an explicit bound for the diameter, the claim is that the diameter is finite.

To get a feeling why Conjecture 1.1 might be true, consider a Riemannian manifold (M, g) with nonnegative Ricci curvature that is strictly conical outside of a compact subset. The Ricci curvature vanishes in the radial direction of the cone. The c-Ricci pinching then implies that M is Ricci-flat on the conical region and hence flat there, since the dimension is three. Then the link of the cone consists of copies of round S^2's and $\mathbb{R}P^2$'s. From the splitting theorem, the link must be connected. Since it bounds a compact 3-manifold,

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it must be S^2. The global nonnegativity of the Ricci curvature now implies that M is isometric to $\mathbb{R}^3$. This intuition will enter into the proof of Theorem 1.4 below.

One could ask about generalizations of Conjectures 1.1 and 1.2 without the uniform curvature bound, or in higher dimension. The higher dimensional analog of Conjecture 1.1 would be to say that the manifold is Ricci-flat or compact. However, in this paper we stick with three dimensions and bounded sectional curvature.

We show that Conjectures 1.1 and 1.2 are true under an extra curvature assumption.

Theorem 1.3. *Conjecture 1.1 is true if*
a. (M, g) has nonnegative sectional curvature, or
b. (M, g) has quadratic curvature decay.

Theorem 1.3(a) was proven earlier in [6].

Theorem 1.4. *Conjecture 1.1 is true if there is some $A < \infty$ so that the sectional curvatures of (M, g) satisfy $K(m) \geq -\frac{A}{d(m,m_0)^2}$, where m_0 is some basepoint.*

Theorem 1.4 implies Theorem 1.3 but we state it separately, since the proof of Theorem 1.4 uses results from the research announcement [17].

Besides the particular results in Theorems 1.3 and 1.4, we prove more general results that may lead to a proof of Conjecture 1.1. The next proposition says that if (M, g_0) is noncompact and satisfies the hypotheses of Conjecture 1.1 then the ensuing Ricci flow exists for all positive time and is type-III.

Proposition 1.5. *Given (M, g_0) as in Conjecture 1.1 with M noncompact, there is a smooth Ricci flow solution $(M, g(\cdot))$ with $g(0) = g_0$ that exists for all $t \geq 0$. There is a constant $C < \infty$ so that $\parallel \mathrm{Rm}(g(t)) \parallel_\infty \leq \frac{C}{t}$ for all $t \geq 0$.*

The main technical result of this paper is that a three dimensional Ricci flow solution $(M, g(t))$ with positive Ricci curvature, that satisfies the conclusion of Proposition 1.5, admits a three-dimensional blowdown limit.

Proposition 1.6. *Let (M, g_0, m_0) be a complete connected pointed Riemannian manifold of dimension three, with bounded sectional curvature and positive Ricci curvature. Suppose that the ensuing Ricci flow exists for all $t \geq 0$, and that there is some $C < \infty$ so that $\parallel \mathrm{Rm}(g(t)) \parallel_\infty \leq \frac{C}{t}$ for all $t \geq 0$. For $s > 0$, put $g_s(t) = s^{-1} g(st)$. Then for some sequence $s_i \to \infty$, there is a limit $\lim_{i\to\infty} g_{s_i}(\cdot) = g_\infty(\cdot)$ in the pointed Cheeger-Hamilton topology. The Ricci flow solution $g_\infty(u)$ lives on a three dimensional manifold and is defined for $u > 0$.*

The issue in proving Proposition 1.6 is to rule out collapsing at large time. Examples of Proposition 1.6 come from expanding gradient solitons, for which the tangent cone at infinity can be the cone over any two-sphere with Gaussian curvature greater than one [9]. Of course, these are not c-Ricci pinched (Lemma 4.6).

Using distance distortion estimates, Proposition 1.6 has the following implication about the initial metric.

Corollary 1.7. *Under the hypotheses of Proposition 1.6, the Riemannian manifold* (M, g_0) *has cubic volume growth.*

The proof of Theorem 1.3(a) then uses a Ricci flow result of Simon-Schulze [25]. To prove Theorem 1.3(b) we apply a spatial rescaling argument to a time slice of the blowdown Ricci flow solution.

The proof of Theorem 1.4 uses Corollary 1.7 and results of [17] about weak convergence of curvature operators. Assuming that M is noncompact, we apply a spatial rescaling to the original metric (M, g_0) to get an locally Alexandrov three dimensional tangent cone at infinity. If (M, g_0) is nonflat then the weak convergence of curvature operators, along with the c-Ricci pinching, forces the tangent cone at infinity of (M, g_0) to be $\mathbb{R}^3$, which contradicts the nonflatness assumption.

The structure of the paper is the following. In Section 2 we prove Proposition 1.5 and give some distance distortion estimates. In Section 3 we prove Proposition 1.6. Section 4 has the proof of Corollary 1.7. In Section 5 we prove Theorem 1.3 and in Section 6 we prove Theorem 1.4. More detailed descriptions are at the beginnings of the sections.

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2. Long-time existence and curvature decay

In this section we prove Proposition 1.5. We first show that the Ricci flow exists for all $t > 0$. The proof is similar to an argument in Hamilton's original Ricci flow paper [11] about what could possibly happen at a curvature blowup under the Ricci pinching assumption. When applied to long-time solutions, essentially the same argument is used to rule out type-II solutions, thereby proving the curvature bound in Proposition 1.5. Using the curvature bound, we give some distance distortion estimates that will be important in Section 3.

We begin by recalling some facts from Ricci flow. Let (M, g_0) be a Riemannian manifold as in the statement of Conjecture 1.1. Let $(M, g(\cdot))$ denote the unique maximal Ricci flow solution with initial time slice $g(0) = g_0$, having complete time slices and bounded curvature on compact time intervals. The condition $\mathrm{Ric} \geq 0$ is preserved under Ricci flow. Using the weak maximum principle, one can show that being c-Ricci pinched is preserved under Ricci flow. Using the strong maximum principle, if (M, g_0) is nonflat then for $t > 0$, the Ricci curvature is positive. Hence we can assume that (M, g_0) has positive Ricci curvature. This shows the equivalence between Conjecture 1.1 and Conjecture 1.2.

Under the hypotheses of Conjecture 1.2, to argue by contradiction, hereafter we also assume that M is noncompact. Then it is diffeomorphic to $\mathbb{R}^3$ [24].

Proposition 2.1. *The Ricci flow solution* $(M, g(\cdot))$ *exists for all* $t \geq 0$*.*

Proof. We have

$$r_1 \geq c r_3 \Rightarrow r_1 \geq \frac{1}{2} c (r_2 + r_3) \Rightarrow \left(1 + \frac{1}{2} c\right) r_1 \geq \frac{1}{2} c (r_1 + r_2 + r_3). \tag{2.2}$$

Hence $\mathrm{Ric} \geq \rho R$, where $\rho = \frac{c}{2+c} \in \left(0, \frac{1}{3}\right]$, and R denotes the scalar curvature. Put $\sigma = \rho^2$.

Suppose that the maximal Ricci flow solution is on a finite time interval $[0, T)$. We claim first that for all $t \in [0, T)$, we have

$$R^{\sigma-2} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2 \leq \left(\frac{3}{2t} \right)^{\sigma} \tag{2.3}$$

everywhere on M. To prove this, we combine methods from [1, Pf. of Proposition 3] and [6, Pf of Lemma 6.1]. Put

$$f = R^{\sigma-2} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2 . \tag{2.4}$$

From the bounded curvature assumption, f is uniformly bounded above at time zero. From [1, p. 539] and [6, Eqn. (76)], which are based on [11, Lemma 10.5],

$$\left(\frac{\partial}{\partial t} - \triangle \right) f \leq 2(1-\sigma) \left\langle \frac{\nabla R}{R}, \nabla f \right\rangle - \sigma(1-\sigma) R^{\sigma-4} \left| \mathrm{Ric} - \frac{1}{3} R g(t) \right|^2 |\nabla R|^2 - \frac{2}{3} \sigma f^{1+\frac{1}{\sigma}}. \tag{2.5}$$

If M were compact then we could immediately derive (2.3) using the weak maximum principle, as in [1, Proposition 3]. If M is noncompact then the possible unboundedness of $\frac{\nabla R}{R}$ is an issue. To get around this, using

$$2 \left\langle \frac{\nabla R}{R}, \nabla f \right\rangle \leq \sigma f \left| \frac{\nabla R}{R} \right|^2 + \frac{|\nabla f|^2}{\sigma f}, \tag{2.6}$$

we obtain

$$\left(\frac{\partial}{\partial t} - \triangle \right) f \leq \frac{1-\sigma}{\sigma} \frac{|\nabla f|^2}{f} - \frac{2}{3} \sigma f^{1+\frac{1}{\sigma}}. \tag{2.7}$$

Equivalently,

$$\left(\frac{\partial}{\partial t} - \triangle \right) f^{\frac{1}{\sigma}} \leq - \frac{2}{3} f^{\frac{2}{\sigma}} \tag{2.8}$$

in the barrier sense. From the weak maximum principle,

$$\sup_{m \in M} f^{\frac{1}{\sigma}}(m, t) \leq \frac{3}{2t}, \tag{2.9}$$

which proves the claim.

There is a sequence $\{t_i\}_{i=1}^{\infty}$ of times increasing to T, and points $\{m_i\}_{i=1}^{\infty}$ in M so that $\lim_{i \to \infty} |\mathrm{Rm}(m_i, t_i)| = \infty$ and $|\mathrm{Rm}(m_i, t_i)| \geq \frac{1}{2} \sup_{(m,t) \in M \times [0, t_i]} |\mathrm{Rm}(m, t)|$. Put $Q_i = |\mathrm{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1} u)$. Then g_i is a Ricci flow solution with curvature norm equal to one at $(m_i, 0)$, and curvature norm uniformly bounded above by two for $u \in [-Q_i t_i, 0]$.

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \, \mathrm{inj}_{g(t_i)}(m_i)^2 \geq i_0$. (This does not follow from Perelman's no local collapsing result, since we do not assume that the initial metric has positive injectivity radius.) After passing to a subsequence, there is a pointed Cheeger-Hamilton limit

$$\lim_{i \to \infty} (M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty), \tag{2.10}$$

where $g_\infty(u)$ is defined for $u \in (-\infty, 0]$. The property of having nonnegative Ricci curvature passes to the limit. By construction, g_∞ has curvature norm one at $(m_\infty, 0)$. Hence g_∞ has positive scalar curvature at $(m_\infty, 0)$. By the strong maximum principle, it follows that g_∞ has positive scalar curvature everywhere.

Given $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $\{(m'_i, 0)\}_{i=1}^\infty$ with $\lim_{i \to \infty} R_{g_i}(m'_i, 0) = R_{g_\infty}(m', 0) > 0$. As $\lim_{i \to \infty} Q_i = \infty$, after undoing the rescaling it follows that $\lim_{i \to \infty} R_g(m'_i, t_i) = \infty$. As $\lim_{i \to \infty} t_i = T$, we also have $\lim_{i \to \infty} t_i R_g(m'_i, t_i) = \infty$. Applying (2.3) to g_i and taking the limit as $i \to \infty$, it follows that the metric $g_\infty(0)$ satisfies $\mathrm{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature at $(m_\infty, 0)$, it follows that M_∞ is a spherical space form. Then M is compact, which is a contradiction.

Even if there is no uniform positive lower bound for $Q_i \, \mathrm{inj}_{g(t_i)}(m_i)^2$, after passing to a subsequence, there is a pointed limit

$$\lim_{i \to \infty} (M, g_i(\cdot), m_i) = (\mathcal{G}_\infty, g_\infty(\cdot), \mathcal{O}_{x_\infty}). \tag{2.11}$$

Here $\mathcal{G}_\infty$ is a three dimensional closed Hausdorff étale groupoid and $g_\infty(\cdot)$ is a family of invariant Riemannian metrics on the unit space of $\mathcal{G}_\infty$ [18, Section 5]. Let X_∞ denote he orbit space of $\mathcal{G}_\infty$; then $\mathcal{O}_{x_\infty} \in X_\infty$ is a basepoint. The Ricci flow $g_\infty(u)$ is defined for $u \in (-\infty, 0]$. For each u, the metric $g_\infty(u)$ induces a metric on X_∞ that makes it into a complete metric space. As before, $\lim_{i \to \infty} R_g(m_i, t_i) = \infty$ and (2.3) again implies that the metric $g_\infty(0)$ satisfies $\mathrm{Ric} - \frac{1}{3} R g_\infty(0) = 0$. As $g_\infty(0)$ has positive scalar curvature along the orbit $\mathcal{O}_{x_\infty}$ in the unit space, the metric $g_\infty(0)$ has constant positive Ricci curvature. The argument for the Bonnet-Myers theorem implies that X_∞ is compact; c.f. [14, Section 2.9]. Then M is compact, which is a contradiction. □

Remark 2.12. One could avoid the use of étale groupoids by first looking at the pullback flows on $T_{m_i} M$ and taking a limit, to argue that for large i, the metric $g(t_i)$ has almost constant positive sectional curvature on $B\left(m_i, R(m_i, t_i)^{-\frac{1}{2}}\right)$. One could then shift basepoints and repeat the argument, to obtain that for any $A < \infty$ and for large i, the metric $g(t_i)$ has almost constant positive sectional curvature on $B\left(m_i, AR(m_i, t_i)^{-\frac{1}{2}}\right)$. From Bonnet-Myers, one concludes that M is compact, which is a contradiction.

Proposition 2.13. *There is some $C < \infty$ so that for all $t \in [0, \infty)$, we have $\| \mathrm{Rm}(g(t)) \|_\infty \leq \frac{C}{t}$.*

Proof. Suppose that the proposition is not true. After doing a type-II point picking [7, Chapter 8, Section 2.1.3], there are points (m_i, t_i) so that $\lim_{i \to \infty} t_i |\mathrm{Rm}(m_i, t_i)| = \infty$ and $|\mathrm{Rm}| \leq 2|\mathrm{Rm}(m_i, t_i)|$ on $M \times [a_i, b_i]$, with $\lim_{i \to \infty} |\mathrm{Rm}(m_i, t_i)|(t_i - a_i) = \lim_{i \to \infty} |\mathrm{Rm}(m_i, t_i)|(b_i - t_i) = \infty$. Put $Q_i = |\mathrm{Rm}(m_i, t_i)|$ and $g_i(x, u) = Q_i g(x, t_i + Q_i^{-1} u)$.

Suppose first that for some $i_0 > 0$ and all i, we have $Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 \geq i_0$. After passing to a subsequence, we get a limiting Ricci flow solution $\lim_{i\to\infty}(M, g_i(\cdot), m_i) = (M_\infty, g_\infty(\cdot), m_\infty)$ defined for times $u \in \mathbb{R}$. Here M_∞ is a 3-manifold and $|\operatorname{Rm}(m_\infty, 0)| = 1$. As in the proof of Proposition 2.1, for each $m' \in M_\infty$, the point $(m', 0)$ is the limit of a sequence of points $(m'_i, 0)$ with $\lim_{i\to\infty} t_i R_g(m'_i, t_i) = \infty$, where the latter statement now comes from the type-II rescaling. From (2.3), we get $\operatorname{Ric} - \frac{1}{3} R g_\infty = 0$. Then (M_∞, g_∞) has constant positive curvature time slices, which implies that M_∞ is compact. Then M is also compact, which is a contradiction.

If $\liminf_{i\to\infty} Q_i \operatorname{inj}_{g(t_i)}(m_i)^2 = 0$, we can still take a limit as in (2.11). As in the argument after (2.11), we again conclude that M is compact, which is a contradiction. □

Corollary 2.14. *There are numbers $\{A_k\}_{k=0}^\infty$ that for all $t \in [0, \infty)$ and all multi-indices I, we have $\| \nabla^I \operatorname{Rm} \|_{g(t)} \leq A_{|I|} t^{-\frac{|I|}{2} - 1}$.*

Proof. This follows from Proposition 2.13, along with derivative estimates for the Ricci flow [7, Theorem 6.9]. □

Let $d_t : M \times M \to \mathbb{R}$ be the distance function on M with respect to the Riemannian metric $g(t)$. In particular, d_0 be the distance function with respect to g_0.

Lemma 2.15. *There is some $C' < \infty$ so that whenever $0 \leq t_1 \leq t_2 < \infty$, we have*

$$d_{t_1} - C' \left(\sqrt{t_2} - \sqrt{t_1} \right) \leq d_{t_2} \leq d_{t_1}. \tag{2.16}$$

Proof. This follows from distance distortion estimates for Ricci flow, as in [15, Remark 27.5 and Corollary 27.16]. □

Fix $m_0 \in M$. Given $s > 0$, put $g_s(u) = s^{-1} g(su)$. Then $(M, g_s(\cdot))$ is also a Ricci flow solution, with $\| \operatorname{Rm}(g_s(u)) \| \leq \frac{C}{u}$ and $\| \nabla^I \operatorname{Rm} \|_{g_s(u)} \leq A_{|I|} u^{-\frac{|I|}{2} - 1}$. Its distance function at time u is $\widehat{d}_{s,u} = s^{-\frac{1}{2}} d_{su}$. From (2.16), we have

$$\frac{1}{\sqrt{s}} d_0 - C' \sqrt{u} \leq \widehat{d}_{s,u} \leq \frac{1}{\sqrt{s}} d_0. \tag{2.17}$$

Given $\rho > 0$, it follows that

$$B_{\widehat{d}_{s,u}}(m_0, \rho - C'\sqrt{u}) \subset B_{d_0}(m_0, \rho\sqrt{s}) \subset B_{\widehat{d}_{s,u}}(m_0, \rho) \tag{2.18}$$

Also, if $0 \leq s_1 \leq s_2 < \infty$ then

$$\sqrt{\frac{s_1}{s_2}} \widehat{d}_{s_1,u} - C' \left(1 - \sqrt{\frac{s_1}{s_2}} \right) \sqrt{u} \leq \widehat{d}_{s_2,u} \leq \sqrt{\frac{s_1}{s_2}} \widehat{d}_{s_1,u}. \tag{2.19}$$

Given $\rho > 0$, it follows that

$$B_{\widehat{d}_{s_2,u}} \left(m_0, \sqrt{\frac{s_1}{s_2}} \rho - C' \left(1 - \sqrt{\frac{s_1}{s_2}} \right) \sqrt{u} \right) \subset B_{\widehat{d}_{s_1,u}} (m_0, \rho) \subset B_{\widehat{d}_{s_2,u}} \left(m_0, \sqrt{\frac{s_1}{s_2}} \rho \right). \tag{2.20}$$

Given a sequence $\{s_i\}_{i=1}^\infty$ tending to infinity and $u > 0$, after passing to a subsequence we can assume that there is a limit of $\lim_{i\to\infty}(M, g_{s_i}(u), m_0)$ in the pointed Gromov-Hausdorff topology. We claim that we can choose the subsequence so that the limit exists simultaneously for each u, and as u varies the limiting metric spaces are all biLipschitz equivalent to each other. To see this, after passing to a subsequence we can assume that there is a limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0) = (\mathcal{G}_\infty, g_\infty(\cdot), \mathcal{O}_{x_\infty})$. Here $g_\infty(\cdot)$ is a Ricci flow solution on the étale groupoid $\mathcal{G}_\infty$, that exists for $u > 0$. As u varies, the pointed Gromov-Hausdorff limit $\lim_{i\to\infty}(M, g_{s_i}(u), m_0)$ always has the same underlying pointed topological space, namely the pointed orbit space (X_∞, x_∞) of $\mathcal{G}_\infty$. The metric on the limit depends on u, and is the quotient metric $\widehat{d}_{\infty,u}$ coming from $g_\infty(u)$. It follows that the various quotient metrics, as u varies, are biLipschitz to each other.

Since M is noncompact, X_∞ is also noncompact. In particular, $\dim(X_\infty) > 0$.

3. Noncollapsing at large time

In this section we show that the Ricci flow solution from Section 2 is noncollapsed for large time, in a scale-invariant sense. More precisely, we show that there is a blowdown limit on a three dimensional manifold, where the emphasis is on the three dimensionality.

We recall that the Ricci flow solution from Section 2 has positive Ricci curvature and lives on a noncompact manifold, which is necessarily then diffeomorphic to $\mathbb{R}^3$. After passing to a subsequence, we can extract a blowdown limit X_∞ (corresponding to a fixed rescaled time) in the sense of pointed Gromov-Hausdorff convergence. The issue is to show that $\dim(X_\infty) = 3$. Since X_∞ is noncompact, we must exclude that $\dim(X_\infty)$ is one or two. This is done in Subsections 3.1 and 3.2. The argument goes by showing that if $\dim(X_\infty) < 3$ then the collapsing structure at large time can be extended in the sense of rough geometry to time zero. This will eventually give a contradiction to the fact that the original manifold is diffeomorphic to $\mathbb{R}^3$. We note that $\mathbb{R}^3$ can collapse with bounded sectional curvature [3, Example 1.4] due to a graph manifold structure, so the contradiction is not immediate.

The following statement is the main result of this section.

Proposition 3.1. *There is some sequence $\{s_i\}_{i=1}^\infty$ tending to infinity so that the pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ exists as a Ricci flow $(M_\infty, g_\infty(\cdot), m_\infty)$ on a pointed 3-manifold (M_∞, m_∞).*

Proof. Suppose that the proposition is not true. Fix a time parameter $u > 0$. Then for any $\epsilon > 0$, there is some $\widehat{s} = \widehat{s}(\epsilon) < \infty$ so that for all $s \geq \widehat{s}$, the metric space $(M, \widehat{d}_{s,u}, m_0)$ has pointed Gromov-Hausdorff distance at most ϵ from a complete pointed metric space (X_∞, x_∞) of dimension one or two.

3.1. **One dimensional limits.**

We first show that if ϵ is small enough then X_∞ cannot be one dimensional. If $\dim(X_\infty) = 1$ then, as mentioned above, we can find a large s so that the pointed metric space $(M, \widehat{d}_{s,u}, m_0)$ is almost one dimensional. We will show that upon increasing s, the metric space evolves into something two dimensional. Looking at the transition region, we obtain a contradiction for topological reasons.

We begin with a couple of geometric lemmas. The first lemma says in a quantitative way that if a three dimensional pointed Riemannian manifold is sufficiently Gromov-Hausdorff close to a two dimensional space then there is a metric ball around the basepoint that can be slightly deformed to a solid torus.

Lemma 3.2. *Given*

1. *A collection* $\mathcal{X}$ *of pointed two dimensional complete Alexandrov spaces* $(X, d_X, \star_X)$ *that is compact in the pointed Gromov-Hausdorff topology,*
2. *Positive numbers* $\{A_k\}_{k=0}^{\infty}$*, and*
3. $R < \infty$*,*

there are some $\epsilon' > 0$ *and* $r << R$ *with the following property. Suppose that* $(M, g, \star_M)$ *is a complete pointed orientable connected three dimensional Riemannian manifold, with* $\|\nabla^I \operatorname{Rm}\|_\infty \leq A_{|I|}$ *for all multi-indices* I *(including* $I = \emptyset$*). Suppose that* $(M, g, \star_M)$ *is* ϵ'*-close to some* $(X, d_X, \star_X) \in \mathcal{X}$ *in the pointed Gromov-Hausdorff topology. Then there are*

1. *A complete pointed Riemannian* 2*-orbifold* $(X', d_{X'}, \star_{X'})$*,*
2. *Connected open subsets* $U_r, U_R \subset M$ *and* $V_r, V_R \subset X'$ *with*

$$\begin{aligned} B(\star_M, .9r) &\subset U_r \subset B(\star_M, 1.1r), \\ B(\star_M, .9R) &\subset U_R \subset B(\star_M, 1.1R), \\ B(\star_{X'}, .9r) &\subset V_r \subset B(\star_{X'}, 1.1r), \\ B(\star_{X'}, .9R) &\subset V_R \subset B(\star_{X'}, 1.1R), \end{aligned} \tag{3.3}$$

and

3. *An almost Riemannian submersion* $p : U_R \to V_R$ *(in the orbifold sense) that is a* const. ϵ'*-Gromov-Hausdorff approximation so that*
4. $U_r = p^{-1}(V_r)$ *is diffeomorphic to a solid torus.*

Proof. Suppose that the lemma fails. Then there is a sequence of pointed Riemannian manifolds $\left\{(M_j, g_{M_j}, \star_{M_j})\right\}_{j=1}^{\infty}$ that satisfy the hypotheses with $\epsilon' = \frac{1}{j}$, but which together provide a counterexample. After passing to a subsequence, we can assume that $\lim_{j\to\infty} \left(M_j, g_{M_j}, \star_{M_j}\right) = (X', d_{X'}, \star_{X'})$ in the pointed Gromov-Hausdorff topology, for some complete pointed two dimensional Alexandrov space X'.

In terms of the orthonormal frame bundles FM_j, after passing to a subsequence we can assume that there is a SO(3)-equivariant Gromov-Hausdorff limit $\lim_{j\to\infty} FM_j = \mathcal{M}$, where $\mathcal{M}$ is a smooth five dimensional manifold on which SO(3) acts locally freely, with $X' = \mathcal{M}/\operatorname{SO}(3)$ being a two dimensional Riemannian orbifold; see [10, Proposition 11.5 and Theorem 12.8]. As M_j is orientable, the orbifold has isolated singular points.

Given a compact codimension-zero submanifold-with-boundary K_∞ of $\mathcal{M}$, for large j there is a compact codimension-zero submanifold-with-boundary K_j of FM_j and an SO(3)-equivariant circle fibering $K_j \to K_\infty$ that is an almost Riemannian submersion. Quotienting by SO(3) gives a singular fibration $p_j : K_j/\operatorname{SO}(3) \to K_\infty/\operatorname{SO}(3)$, with $K_j/\operatorname{SO}(3) \subset M_j$ and $K_\infty/\operatorname{SO}(3) \subset X'$. Taking K_∞ sufficiently large, we let V_R be an approximation to $B(\star_{X'}, R) \subset X'$ and put $U_R = p_j^{-1}(V_R)$.

For sufficiently small $r' > 0$, the ball $B(\star_{X'}, 1.1r') \subset X'$ has no singular points or a single singular point at $\star_{X'}$. In either case, we take V_r to be an approximation to $B(\star_{X'}, r')$ and put $U_r = p_j^{-1}(V_r)$, a solid torus. This gives a contradiction. □

The next lemma describes the local geometry and topology of a pointed Riemannian 3-manifold that is Gromov-Hausdorff close to a one dimensional space.

Lemma 3.4. *Given $K, L < \infty$, there is some $\widehat{\epsilon} = \widehat{\epsilon}(K, L) > 0$ with the following property. Suppose that (M, m_0) is a complete pointed Riemannian 3-manifold diffeomorphic to $\mathbb{R}^3$ with sectional curvatures bounded in absolute value by K, so that (M, m_0) has pointed Gromov-Hausdorff distance at most $\widehat{\epsilon}$ from a complete pointed one dimensional length space (X, x_0). Then there is a pointed singular fibration $\pi : B(m_0, L) \to B(x_0, L)$ with the following properties.*

1. *The generic fiber of the fibration is T^2.*
2. *If $B(x_0, L)$ is an open interval then $B(m_0, L)$ is diffeomorphic to $T^2 \times (-1, 1)$. If $B(x_0, L)$ is a half open interval then $B(m_0, L)$ is diffeomorphic to $B^2 \times S^1$.*
3. *If x_0 is an endpoint of $B(x_0, L)$ then $\pi^{-1}(x_0)$ is a circle, while if x_0 is not an endpoint of $B(x_0, L)$ then $\pi^{-1}(x_0)$ is a 2-torus. In either case, the diameter of $\pi^{-1}(x_0)$ is bounded above by const. $\widehat{\epsilon}$.*
4. *The inclusion $\pi^{-1}(x_0) \to B(m_0, L)$ induces a nontrivial map on π_1.*

Proof. This follows from [2, Theorem 1.7]. In our case, the relevant nilpotent Lie groups N to describe the local geometry near a point $m \in M$, from [2, p. 331], are $\mathbb{R}^2$ and $\mathbb{R}$.

If $N = \mathbb{R}^2$ then the local covering group Λ must be $\mathbb{Z}^2$ or $\mathbb{Z}^2 \rtimes \mathbb{Z}_2$. If $\Lambda = \mathbb{Z}^2$ then the local topology is an interval times T^2. If $\Lambda = \mathbb{Z}^2 \rtimes \mathbb{Z}_2$ then the local topology is $(I \times T^2)/\mathbb{Z}_2$, where the generator of $\mathbb{Z}_2$ reverses orientation on both I and T^2. In this case, there would be an embedded copy of the Klein bottle in M, which cannot occur since M is diffeomorphic to $\mathbb{R}^3$.

If $N = \mathbb{R}$ then Λ is virtually cyclic. The local geometry near m can be seen by rescaling so that the injectivity radius at m becomes one. If $\widehat{\epsilon}$ is small enough then the rescaled manifold is approximated by a flat orientable 3-manifold whose soul is a circle. This local model is $(\mathbb{R} \times \mathbb{R}^2)/\mathbb{Z}$, where a generator of $\mathbb{Z}$ acts by a small translation on $\mathbb{R}$ and by a small rotation on $\mathbb{R}^2$. Letting C be an approximate image of the soul in M, we can assume that near C, the map π is the distance from C.

Putting this together, the lemma follows. □

From Proposition 2.13, for all $s \geq 1$, the curvature of $g_s(u)$ is bounded in magnitude by $\frac{C}{u}$. Corollary 2.14 gives higher derivative bounds on the curvature. Choose ϵ (which we will adjust) as at the beginning of the proof of Proposition 3.1. Assuming that there is a one dimensional limit, choose $s_0 \geq \widehat{s}(\epsilon)$ so that the metric space $\left(M, \widehat{d}_{s_0,u}, m_0\right)$ has pointed Gromov-Hausdorff distance at most ϵ from a complete pointed metric space of (X_∞, x_∞) of dimension one. Choosing $L >> 1$, we can apply Lemma 3.4 with $K = \frac{C}{u}$, taking $\epsilon \leq \widehat{\epsilon}(K, L)$. Put $\mathcal{C} = \pi^{-1}(x_\infty)$. By Lemma 3.4, $\operatorname{diam}(\mathcal{C}, \widehat{d}_{s_0,u})$ is comparable to ϵ.

As M is diffeomorphic to $\mathbb{R}^3$, there is some $\sigma < \infty$ so that the inclusion $\mathcal{C} \to B_{d_0}(m_0, \sigma)$ is trivial on π_1. Let Δ be the infimum of such σ's. By (2.18), for any $s \geq 1$ and any $R > s^{-\frac{1}{2}}\Delta$, the inclusion $\mathcal{C} \to B_{\widehat{d}_{s,u}}(m_0, R)$ is trivial on π_1.

Let $\mu(s)$ be the infimum of the numbers l so that the inclusion $\mathcal{C} \to B_{\widehat{d}_{s,u}}(m_0, l)$ is trivial on π_1.

Lemma 3.5. *μ is continuous in s.*

Proof. This follows from (2.20). □

From Lemma 3.4, we have $\mu(s_0) \geq L$. From the above discussion, if s is sufficiently large then $\mu(s) \leq \frac{1}{2}$. Let s_1 be the smallest $s \geq s_0$ so that $\mu(s) = 1$.

As $\mu(s_1) = 1$, there is an *a priori* $\epsilon'' > 0$, independent of ϵ, so that $\left(M, \widehat{d}_{s_1,u}, m_0\right)$ has pointed Gromov-Hausdorff distance at least ϵ'' from a one dimensional space; otherwise the product structure coming from Lemma 3.4 would contradict the fact that $\mu(s_1)$ is exactly 1. Hence if ϵ is sufficiently small then, as $\left(M, \widehat{d}_{s_1,u}, m_0\right)$ is ϵ-close to a one or two dimensional space, it must have pointed Gromov-Hausdorff distance at most ϵ from a two dimensional complete pointed Alexandrov space (X'_∞, x'_∞). We apply Lemma 3.2 with $R = 2$ and $\mathcal{X}$ being the two dimensional complete Alexandrov spaces with curvature bounded below by $-\frac{2C}{u}$ and pointed Gromov-Hausdorff distance at least ϵ'' from a one dimensional space. Taking ϵ less than the ϵ' of Lemma 3.2, the lemma gives an $r << 2$ and an open set U_r, diffeomorphic to a solid torus, with $B_{\widehat{d}_{s_1,u}}(m_0, .9r) \subset U_r \subset B_{\widehat{d}_{s_1,u}}(m_0, 1.1r)$. It also gives a orbifold circle fibration $\pi' : U_2 \to V_2$ that is an almost Riemannian submersion.

Since M does not have any embedded Klein bottles, the circle fibration is orientable and so describes a Seifert fibration. Using the fact that V_2 is noncompact, from [27, Lemma 3.2] there is an exact sequence

$$1 \to \mathbb{Z} \to \pi_1(U_2) \to \pi_1(V_2) \to 1, \tag{3.6}$$

where the image of a generator of $\mathbb{Z}$ is represented by a regular fiber of the Seifert fibration, and $\pi_1(V_2)$ denotes the orbifold fundamental group. (Since the $\mathbb{Z}$-subgroup is central in $\pi_1(U_2)$, it is well defined independent of basepoint.) From (2.17), $\mathrm{diam}(\mathcal{C}, \widehat{d}_{s_1,u}) \leq \mathrm{diam}(\mathcal{C}, \widehat{d}_{s_0,u})$. Hence with reference to Lemma 3.2, if ϵ is sufficiently small then $\mathcal{C} \subset U_r$. As the inclusion $\mathcal{C} \to B_{\widehat{d}_{s_1,u}}\left(m_0, \frac{1}{2}\right)$ is nontrivial on π_1, the loop $\mathcal{C}$ represents a nontrivial element $[\mathcal{C}]$ of $\pi_1(U_r) \cong \mathbb{Z}$. Hence there is some $m \neq 0$ so that $[\mathcal{C}]^m$ is a power of the element of $\pi_1(U_r)$ represented by a regular fiber of the Seifert fibration. Then (3.6) implies that $[\mathcal{C}]^m$ is a nontrivial element of $\pi_1(U_2)$, which contradicts the fact that the inclusion $\mathcal{C} \to B_{\widehat{d}_{s_1,u}}(m_0, 1.5)$ is trivial on π_1, from the definition of μ.

3.2. **Two dimensional limits.** We now are reduced to the case when every limit (X_∞, x_∞) is two dimensional. We will show that the collapsing structure of $(M, g_s(u))$ as $s \to \infty$ implies that the original manifold (M, g_0) has arbitrarily large regions with a Seifert structure. In itself this is not a contradiction, as $\mathbb{R}^3$ is the union of an ascending chain of embedded solid tori. However, we will show that the Seifert structures on these large regions can be fitted together to give a Seifert structure on $\mathbb{R}^3$, which is a contradiction.

Lemma 3.7. *There is some $\rho > 0$ with the following property. Given $\widetilde{\epsilon} > 0$ and $K \in \mathbb{Z}^+$, there are some $s_0 > 0$ and $k \in \mathbb{Z}^+$ so that for all $s \geq s_0$,*

- *There is an open subset $M_s \subset M$ so that $B_{\widehat{d}_{s,u}}(m_0, \rho) \subset M_s \subset B_{\widehat{d}_{s,u}}(m_0, \rho(1+\widetilde{\epsilon}))$ and a metric $g_{s,\widetilde{\epsilon}}(u)$ on M_s that is $\widetilde{\epsilon}$-close in the pointed C^K-topology to $g_s(u)$, along with a Riemannian submersion $\pi_s : M_s \to X_s$ to a two dimensional Riemannian orbifold X_s (that can depend on s).*
- *The preimages of π_s are circles with diameter less than $\widetilde{\epsilon}$.*
- *The underlying space of X_s is a subset of a complete two dimensional Alexandrov space whose curvature is uniformly bounded below in s. The volume of X_s is uniformly bounded below in s by some positive constant.*

Proof. Suppose that the lemma is not true. Then there is a sequence $\{s_i\}_{i=1}^\infty$ tending to infinity so that for each i, the conclusion of the lemma is not satisfied for $s = s_i$. After passing to a subsequence, we can assume that $\lim_{i\to\infty}(M, \widehat{d}_{s_i,u}, m_0) = (X_\infty, \widehat{d}_{\infty,u}, x_\infty)$ in the pointed Gromov-Hausdorff topology.

The conclusions of the lemma now hold for sufficiently large i; see [5, Section 2], which is based on [2]. (For our purposes it would be enough to work with C^1-closeness, but we have higher derivative bounds from Corollary 2.14.) This is a contradiction. ∎

With reference to Lemma 3.7, put $M_0 = M_{s_0}$ and $X_0 = X_{s_0}$. Consider the Riemannian submersion $\pi_0 : (M_0, g_{s_0,u}) \to X_0$. Let $\Lambda \in (1, \infty)$ be a parameter to be determined.

Inductively, given $j \geq 0$, put $s_{j+1} = \Lambda s_j$. Let $M_{j+1} = M_{s_{j+1}}$ be the manifold from Lemma 3.7, with the Riemannian submersion $\pi_{j+1} : M_{j+1} \to X_{j+1}$. From (2.20), we have

$$B_{\widehat{d}_{s_{j+1},u}}\left(m_0, \Lambda^{-\frac{1}{2}}\rho - C'(1-\Lambda^{-\frac{1}{2}})\sqrt{u}\right) \subset M_j \subset B_{\widehat{d}_{s_{j+1},u}}\left(m_0, \Lambda^{-\frac{1}{2}}(1+\widetilde{\epsilon})\rho\right). \tag{3.8}$$

Lemma 3.9. *If Λ is sufficiently close to 1 then for large j, one can isotope the fibration $\pi_{j+1} : M_{j+1} \to X_{j+1}$ to a new fibration, which we relabel π_{j+1}, so that it agrees with π_j on $B_{\widehat{d}_{s_{j+1},u}}(m_0, \frac{1}{4}\rho)$.*

Proof. Given Λ, which we will adjust, if the lemma is not true then there is an infinite sequence $\{j_k\}_{k=1}^\infty$ for which the lemma fails.

Given $K < \infty$ and $\widetilde{\epsilon}' > 0$, by the Ricci flow equation and Corollary 2.14, if Λ is close enough to 1 then $g_{s_{j_k},u}$ and $g_{s_{j_{k+1}},u}$ are $\widetilde{\epsilon}'$-close in the C^K-topology on M_j. Also, if Λ is close enough to 1 then $\frac{1}{2}\rho < \Lambda^{-\frac{1}{2}}\rho - C'(1-\Lambda^{-\frac{1}{2}})\sqrt{u}$.

Given such Λ, after passing to a subsequence of $\{s_{j_k}\}_{k=1}^\infty$, we can apply Lemma 3.7 with $s = s_{j_k}$, and with parameter $\widetilde{\epsilon} = \widetilde{\epsilon}_k$, where $\lim_{k\to\infty}\widetilde{\epsilon}_k = 0$. Also, for large k, we have

$$B_{\widehat{d}_{s_{j+1},u}}\left(m_0, \frac{\rho}{2}\right) \subset M_j \subset B_{\widehat{d}_{s_{j+1},u}}\left(m_0, \Lambda^{-\frac{1}{4}}\rho\right) \subset B_{\widehat{d}_{s_{j+1},u}}(m_0, \rho) \subset M_{j+1}. \tag{3.10}$$

If $\widetilde{\epsilon}'$ is small enough then it follows that for large k, the conclusion of the lemma holds for $j = j_k$; see, e.g., [4, Lemma 1.4]. The ingredients are the inclusions of (3.10), the

$\widetilde{\epsilon}'$-closeness of the metrics and the precompactness of the X_j's in the pointed Gromov-Hausdorff topology.

This gives a contradiction. □

We now iterate the procedure in terms of the variable j. If $B_{d_0}(m_0, R)$ is a ball in the initial time slice then by (2.18), there is some $J = J(R)$ so that for all $j \geq J$, we have $B_{d_0}(m_0, R) \subset B_{\widehat{d}_{s_{j+1},u}}(m_0, \frac{1}{4}\rho)$. Hence the fibration on $B_{d_0}(m_0, R)$ is only changed a finite number of times. In the limit, we obtain a Seifert fibration of $\mathbb{R}^3$. However, this is impossible [28, p. 216-217]. This proves Proposition 3.1. □

4. Cubic volume growth

Proposition 4.1. *Under the hypotheses of Proposition 1.6, and with reference to Proposition 3.1, both (M, g_0) and $(M_\infty, g_\infty(u))$ have cubic volume growth. In addition, each tangent cone at infinity of $(M_\infty, g_\infty(u))$ is isometric to the tangent cone at infinity $T_\infty M = \lim_{i\to\infty} \left(M, m_0, s_i^{-\frac{1}{2}} d_0\right)$ of M.*

Proof. We know that the pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ exists as a Ricci flow $(M_\infty, g_\infty(\cdot), m_\infty)$ on a pointed 3-manifold (M_∞, m_∞). We claim first that (M, g_0) has cubic volume growth. Fix $u > 0$. Given $R > 0$, put $U_i = B_{\widehat{d}_{s_i,u}}(m_0, R)$ and $C_R = \mathrm{vol}(B(m_\infty, R), g_\infty(u))$. Then for large i, using (2.17) we have

$$s_i^{-\frac{3}{2}} \mathrm{vol}(U_i, d_0) = \mathrm{vol}(U_i, s_i^{-\frac{1}{2}} d_0) \geq \mathrm{vol}(U_i, \widehat{d}_{s_i,u}) \geq \frac{1}{2} C_R, \tag{4.2}$$

where vol denotes the 3-dimensional Hausdorff mass computed with the given metric. Also from (2.18), we have $U_i \subset B_{d_0}(m_0, s_i^{\frac{1}{2}}(R + C'\sqrt{u}))$. Hence

$$\mathrm{vol}(B_{d_0}(m_0, s_i^{\frac{1}{2}}(R + C'\sqrt{u}))) \geq \frac{1}{2} C_R s_i^{\frac{3}{2}}. \tag{4.3}$$

Since $r^{-3}\,\mathrm{vol}(B(m_0, r), g_0)$ is nonincreasing in r, it follows that there is some $v_0 > 0$ so that for all $r > 0$, we have $\mathrm{vol}(B(m_0, r), d_0) \geq v_0 r^3$.

Let d_∞ denote the metric on $T_\infty M$. Let $\widehat{d}_{\infty,u}$ denote the metric on $(M_\infty, g_\infty(u))$. From (2.17), we have

$$d_\infty - C'\sqrt{u} \leq \widehat{d}_{\infty,u} \leq d_\infty \tag{4.4}$$

on $T_\infty M - B_{d_\infty}(\star_\infty, C'\sqrt{u})$. Hence the tangent cone at infinity of $(M_\infty, g_\infty(u))$ is unique and is isometric to $(T_\infty M, d_\infty)$. □

Proposition 4.5. *If $\{s_i\}_{i=1}^\infty$ is any sequence tending to infinity then after passing to a subsequence, there is a pointed limit $\lim_{i\to\infty}(M, g_{s_i}(\cdot), m_0)$ as a Ricci flow on a pointed 3-manifold, defined for times $u \in (0, \infty)$.*

Proof. Put $v_\infty = \lim_{r\to\infty} r^{-3}\,\mathrm{vol}(B_{d_0}(m_0, r), g_0) > 0$, the asymptotic volume ratio of (M, g_0). Fix $u > 0$. For any $s > 1$, from (2.17) a tangent cone at infinity of $(M, g_s(u))$ is isometric to a tangent cone at infinity of (M, g_0). Hence the asymptotic volume ration of

$(M, g_s(u))$ is v_0. Given $R > 0$, the Bishop-Gromov inequality implies that $\mathrm{vol}(B_{\widehat{d}_{s,u}}(m_\infty, R), \widehat{d}_{s,u}) \geq v_\infty R^3$. As $|\operatorname{Rm}(g_s(u))| \leq \frac{C}{u}$, the claim follows from the Hamilton compactness theorem. □

The next lemma will be used in Section 5.

Lemma 4.6. *A three dimensional complete gradient expanding soliton (M, g) with bounded sectional curvature, c-pinched nonnegative Ricci curvature, and cubic volume growth, must be isometric to flat $\mathbb{R}^3$.*

Proof. If (M, g) is flat then because of the cubic volume growth, it must be isometric to $\mathbb{R}^3$. Hence we can assume that $\mathrm{Ric}(M, g) > 0$. From [21, Proposition 3.1], (M, g) has exponential curvature decay. Fix a basepoint m_0. We can find a sequence $\alpha_i \to \infty$ so that $\{(M, \alpha_i^{-2} g, m_0)\}_{i=1}^\infty$ converges in the pointed Gromov-Hausdorff topology to a tangent cone at infinity (X_∞, x_∞) of (M, g). In particular, (X_∞, x_∞) is a cone over a connected surface. Because of the quadratic curvature decay, after passing to a further subsequence we can assume that there is a $W^{2,p}$-regular Riemannian metric on $X_\infty - x_\infty$, along with convergence of metrics in the pointed weak $W^{2,p}_{loc}$-topology. From the weak $W^{2,p}_{loc}$-convergence and the exponential curvature decay of (M, g), the Riemannian metric on $X_\infty - x_\infty$ is flat. Hence X_∞ is a cone over the round S^2 or its $\mathbb{Z}_2$-quotient $\mathbb{R}P^2$. As M was orientable, the second possibility cannot occur, so X_∞ is the flat $\mathbb{R}^3$. Then by [8, Theorem 0.3], (M, g) is flat, which is a contradiction. □

Remark 4.7. Under the additional assumption of nonnegative sectional curvature, Lemma 4.6 was proven in [6].

5. Proof of Theorem 1.3

Proposition 5.1. *If (M, g_0) has nonnegative sectional curvature then Conjecture 1.1 holds.*

Proof. It is enough to prove that Conjecture 1.2 holds, so we will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. Using Proposition 4.1 and [25, Theorem 1.2], there is a blowdown limit $(M_\infty, g_\infty(\cdot), m_\infty)$ that is an gradient expanding soliton. From Lemma 4.6, it must be isometric to $\mathbb{R}^3$. Hence $T_\infty M$ is isometric to $\mathbb{R}^3$. By [8, Theorem 0.3], (M, g_0) is isometric to $\mathbb{R}^3$, which contradicts our assumption that $\mathrm{Ric}_M > 0$. □

Remark 5.2. To clarify a technical point, in [6] use is made of [13, Theorem 16.5] to say that $A = \limsup_{t \to \infty} t \| \operatorname{Rm}(g(t)) \|_\infty$ is positive. The proof of [13, Theorem 16.5] is based on [13, Theorem 16.4], which has a similar conclusion without an assumption of positivity of curvature, but whose proof is only valid in the compact case (since it invokes the diameter). With nonnegative curvature operator, the trace Harnack inequality directly implies that $A > 0$ for nonflat solutions.

Proposition 5.3. *If (M, g_0) has quadratic curvature decay then Conjecture 1.1 holds.*

Proof. We will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. Using pseudolocality [20], there is some $u_0 > 0$ so that for $u \in (0, u_0)$, the metric $g_\infty(u)$

has quadratic curvature decay; c.f. [19, Section 5.2]. Using Shi's local derivative estimate, for any such u, it follows that $|\nabla^I \operatorname{Rm}(m)|_{g_\infty(u)} = O\left(\widehat{d}_\infty(m_\infty, m)^{-|I|-2}\right)$.

Given $\alpha > 1$, consider the rescaled metric $\alpha^{-2} g_\infty(u)$. Using Proposition 4.1, there is a sequence $\{\alpha_i\}_{i=1}^\infty$ tending to infinity so that $\{\left(M_\infty, \alpha_i^{-2} g_\infty(u), m_\infty\right)\}_{i=1}^\infty$ converges to $T_\infty M = \lim_{i \to \infty} \left(M, \alpha_i^{-2} g_0, m_0\right)$ in the pointed Gromov-Hausdorff topology, with smooth convergence away from the basepoints. Hence $T_\infty M$ is a cone over a smooth connected manifold, with c-pinched Ricci curvature away from the vertex. However, if ∂_r denotes the radial vector field then from the cone structure, $\operatorname{Ric}(\partial_r, \partial_r) = 0$. Hence by the c-pinching, $T_\infty M$ is Ricci-flat away from the vertex. This means that it is flat, and so is a cone over the round S^2 or $\mathbb{R}P^2$. Since M is orientable, $T_\infty M$ must be a cone over the round S^2, and hence is isometric to $\mathbb{R}^3$. By [8, Theorem 0.3], (M, g_0) is isometric to $\mathbb{R}^3$, which contradicts our assumption that $\operatorname{Ric}_M > 0$. □

6. Proof of Theorem 1.4

To prove Theorem 1.4 we will use a rescaling argument as in the proof of Proposition 5.3. The rescalings no longer have uniform local double sided bounds on their curvatures, so we need a different convergence result. This will come from [17], which provides a weak convergence of curvature operators. It turns out that this is enough to obtain a contradiction.

We recall some results from [17]. Given an n-dimensional Riemannian manifold (M, g), let Riem be the curvature operator of M and let $\star_M : \Lambda^{n-2}(TM) \to \Lambda^2(TM)$ be Hodge duality. Given C^1-functions $\{f_j\}_{j=1}^{n-2}$ on M, put

$$\sigma = \star_M (\nabla f_1 \wedge \nabla f_2 \wedge \ldots \wedge \nabla f_{n-2}) \tag{6.1}$$

and define

$$r_M(f_1, \ldots, f_{n-2}) = \langle \sigma, \operatorname{Riem}(\sigma) \rangle \ \operatorname{dvol}_M, \tag{6.2}$$

a measure on M.

Suppose that $\{M_i, g_i\}_{i=1}^\infty$ is a sequence of compact n-dimensional pointed Riemannian manifolds with sectional curvatures uniformly bounded below, that converges to a compact n-dimensional pointed Alexandrov space X_∞ in the Gromov-Hausdorff topology. Given C^1-functions $\{f_i\}_{i=1}^\infty$, there is a notion of the sequence C^1-converging to a function f_∞ on X_∞. A function f_∞ on X_∞ is called Alexandrov smooth if it arises as the limit of such a sequence. Averaged distance functions are Alexandrov smooth.

The main result of [17] is the following. Suppose that for each i, $\{f_{i,j}\}_{1 \le j \le n-2}$ is a collection of C^1-functions on M_i. Suppose that for each j, there is a C^1-limit $\lim_{i \to \infty} f_{i,j} = f_{\infty,j}$, where $f_{\infty,j}$ is a function on X_∞. Then there is a weak limit

$$\lim_{i \to \infty} r_{M_i}(f_{i,1}, \ldots, f_{i,n-2}) = r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2}). \tag{6.3}$$

Furthermore, the measure $r_{X_\infty}(f_{\infty,1}, \ldots, f_{\infty,n-2})$ is intrinsic to X_∞. It vanishes on the strata of X_∞ with codimension greater than two, and has descriptions on the codimension-two stratum and the set of regular points. Similarly, there is a measure R_{X_∞} on X_∞ to which the scalar curvature measures converge, i.e. $\lim_{i\to\infty} R_{M_i}\,\mathrm{dvol}_{M_i} = R_{X_\infty}$ in the weak topology.

The preceding constructions can also be carried out locally.

Proposition 6.4. *If there is some $A < \infty$ so that the sectional curvatures of (M, g) satisfy $K(m) \geq -\frac{A}{d(m,m_0)^2}$, where m_0 is some basepoint, then Conjecture 1.1 holds.*

Proof. We will assume that $\mathrm{Ric}_M > 0$, with M noncompact, and derive a contradiction. From Proposition 4.1, there is a sequence $\{\alpha_i\}_{i=1}^\infty$ tending to infinity so that putting $g_i = \alpha_i^{-2} g_0$ and $M_i = (M, g_i)$, the sequence $\{(M_i, m_0)\}_{i=1}^\infty$ converges to a three-dimensional metric cone (X_∞, x_∞) in the pointed Gromov-Hausdorff topology. From the curvature assumption, the cone X_∞ has curvature bounded below by the function $-\frac{A}{d(x,x_\infty)^2}$ in the Alexandrov sense. As a locally Alexandrov space, the cone will have no boundary points, i.e. no codimension-one stratum. Let Σ_∞ denote the link of the cone, so that $X_\infty = \mathrm{cone}(\Sigma_\infty)$. Then Σ_∞ is a connected Alexandrov surface with curvature bounded below by $-A$. The underlying topological space of Σ_∞ is a 2-manifold Y without boundary, which hence admits a smooth structure. Let ω_Y denote the curvature measure on Y, in the sense of [22]. (If Y is a smooth Riemannian 2-manifold then $\omega_Y = K\,\mathrm{dvol}_Y$, where K is the Gaussian curvature.)

Lemma 6.5. *Let ∂_r denote the radial vector field on X_∞. Then*

$$r_{X_\infty}(f) = (\partial_r f)^2 dr \wedge (d\omega_Y - \mathrm{dvol}_Y), \tag{6.6}$$

where $d\omega_Y$ is the curvature measure of the Alexandrov surface Y and dvol_Y *is the two-dimensional Hausdorff measure of Y. Also,*

$$R_{X_\infty} = 2dr \wedge (d\omega_Y - \mathrm{dvol}_Y). \tag{6.7}$$

Proof. From [23, Section 1 and Appendix A], there is a 1-parameter family of smooth Riemannian metrics $\{h_s\}_{s\in(0,\epsilon)}$ on Y so that $\lim_{s\to 0}(Y, h_s) = \Sigma_\infty$ in the Gromov-Hausdorff topology, and the curvature of (Y, h_s) is bounded below by $-A$. For $s \in (0, \epsilon)$, let Y_s denote Y with the Riemannian metric h_s. We first compute $r_{\mathrm{cone}(Y_s)}$. Writing $\mathrm{cone}(Y_s) - \star = (0, \infty) \times Y_s$, if V is a vector field on Y_s then we can also consider it to be a vector field on $\mathrm{cone}(Y_s) - \star$. We have $\mathrm{Riem}(\partial_r \wedge V) = 0$. If V and W are vector fields on Y_s then $\mathrm{Riem}(V \wedge W) = \frac{K_s - 1}{r^2} V \wedge W$. Hence if f is the radial function on $\mathrm{cone}(Y_s)$ and K_s is the Gaussian curvature of Y_s then

$$\begin{aligned} r_{\mathrm{cone}(Y_s)}(f) =& \langle \star_{\mathrm{cone}(Y_s)} \partial_r, \mathrm{Riem}(\star_{\mathrm{cone}(Y_s)} \partial_r) \rangle \ \mathrm{dvol}_{\mathrm{cone}(Y_s)} \\ =& \frac{K_s - 1}{r^2} \langle \star_{\mathrm{cone}(Y_s)} \partial_r, \star_{\mathrm{cone}(Y_s)} \partial_r \rangle \, r^2 dr \wedge \mathrm{dvol}_{Y_s} \\ =& (K_s - 1) dr \wedge \mathrm{dvol}_{Y_s} = dr \wedge (K_s\, \mathrm{dvol}_{Y_s} - \mathrm{dvol}_{Y_s}). \end{aligned} \tag{6.8}$$

Then in general,

$$r_{\mathrm{cone}(Y_s)}(f) = (\partial_r f)^2 dr \wedge (K_s\, \mathrm{dvol}_{Y_s} - \mathrm{dvol}_{Y_s}). \tag{6.9}$$

As $s \to 0$, we have pointed Gromov-Hausdorff convergence $\lim_{s\to 0} \mathrm{cone}(Y_s) = X_\infty$. Working locally on X_∞, say on an annular region $a \leq r \leq A$, there is a weak limit

$\lim_{s\to\infty} r_{\mathrm{cone}(Y_s)} = r_{X_\infty}$. The construction of r_{X_∞} in [17] is done separately on the different strata. It vanishes on strata of codimension greater than two. The codimension-two stratum of X_∞ is the cone over the codimension-two stratum of Y. The restriction of r_{X_∞} to the codimension-two stratum is described in [17] using a blowup argument to reduce it to the case of a local product structure. The arguments show that (6.6) is correct when restricted to the codimension-two stratum of X_∞. There is no codimension-one stratum on X_∞. The construction of q_{X_∞} on the regular points uses local coordinates around a given regular point. Then (6.6) holds on the regular points is correct, as r_{X_∞} can be read off there as the limit of (6.6) as $s \to 0$.

As

$$R_{\mathrm{cone}(Y_s)} \,\mathrm{dvol}_{\mathrm{cone}(Y_s)} = \frac{2(K_s-1)}{r^2} r^2 dr \wedge \mathrm{dvol}_{Y_s} = 2(K-1) dr \wedge \mathrm{dvol}_{Y_s}, \tag{6.10}$$

equation (6.7) follows. ☐

If f is the radial function r on X_∞ then from (6.6) and (6.7)

$$R_{X_\infty} = 2 r_{X_\infty}(f). \tag{6.11}$$

Lemma 6.12. *If W_i is a unit tangent vector at $m_i \in M_i$ then*

$$\frac{R(m_i)}{2} - \mathrm{Ric}(W_i, W_i) \ge \frac{c}{3} R(m_i). \tag{6.13}$$

Proof. Let $\{e_j\}_{j=1}^3$ be an orthonormal basis of $T_{m_i}M_i$ consisting of eigenvectors for Ric_{m_i}, with corresponding eigenvalues $r_1 \le r_2 \le r_3$. Write $W_i = \sum_{j=1}^3 w_{i,j} e_j$. As $R(m_i) = 2(r_1 + r_2 + r_3)$, we have

$$\frac{R(m_i)}{2} - \mathrm{Ric}(W_i, W_i) = \sum_{j=1}^3 (1 - w_{i,j}^2) r_j \ge c \sum_{j=1}^3 (1 - w_{i,j}^2) r_3 = 2cr_3 \ge \frac{c}{3} R(m_i). \tag{6.14}$$

This proves the lemma. ☐

Lemma 6.15. *If V_i is a tangent vector at $m_i \in M_i$ then*

$$\frac{R(m_i)}{2} - \mathrm{Ric}(V_i, V_i) \ge \left(\frac{c}{3} + \left(\frac{1}{2} - \frac{c}{3} \right) \left(1 - |V_i|^2 \right) \right) R(m_i). \tag{6.16}$$

Proof. We can assume $V_i \neq 0$. Put $W_i = \frac{V_i}{|V_i|}$. From Lemma 6.12,

$$\begin{aligned} \frac{R(m_i)}{2} - \mathrm{Ric}(V_i, V_i) &= |V_i|^2 \left(\frac{R(m_i)}{2} - \mathrm{Ric}(W_i, W_i) \right) + \left(1 - |V_i|^2 \right) \frac{R(m_i)}{2} \\ &\ge \frac{c}{3} |V_i|^2 R(m_i) + \left(1 - |V_i|^2 \right) \frac{R(m_i)}{2} \\ &= \left(\frac{c}{3} + \left(\frac{1}{2} - \frac{c}{3} \right) \left(1 - |V_i|^2 \right) \right) R(m_i). \end{aligned} \tag{6.17}$$

This proves the lemma. ☐

Returning to the sequence $\{M_i\}_{i=1}^\infty$, let $f_i \in C^\infty(M_i)$ be a slight smoothing of the distance function from the basepoint m_0 [16, Section 3.6]. Put $V_i = \nabla f_i$. We can assume that for any $R > 0$, we have $\lim_{i\to\infty} \sup_{m \in B_{g_i}(m_0,R)} ||V_i| - 1| = 0$ and $\{f_i\}_{i=1}^\infty$ locally C^1-converges to the radial function on X_∞ in the sense of [17].

Given $x \in X_\infty - \star$, let $B(x,\epsilon)$ be a small ball around x. Let $\{m_i\}_{i=1}^\infty$ be a sequence, with $m_i \in M_i$, approaching x. From the weak convergence and (6.11), we have

$$\lim_{i\to\infty} \int_{B(m_i,\epsilon)} \left(\frac{R(m_i)}{2} - \mathrm{Ric}(V_i,V_i) \right) \mathrm{dvol}_{M_i} = 0. \tag{6.18}$$

From (6.16), we obtain

$$\lim_{i\to\infty} \int_{B(m_i,\epsilon)} R(m_i)\ \mathrm{dvol}_{M_i} = 0. \tag{6.19}$$

Hence $R_{X_\infty} = 0$. Equation (6.7) gives

$$d\omega_Y = \mathrm{dvol}_Y\ . \tag{6.20}$$

Integrating (6.20) over Y shows that the Euler characteristic of Y is positive. By Perelman stability, $\mathrm{cone}(X_\infty) - \star$ is orientable, so Y is a 2-sphere. As an Alexandrov surface, the Alexandrov geometry on Y comes from a Riemannian metric of the form $e^{2\phi}\,\mathrm{dvol}_{S^2}$ which is subharmonic in the sense of [22, Section 7]. As

$$d\omega_Y = (1 - \triangle_{S^2}\phi)\,\mathrm{dvol}_{S^2}, \tag{6.21}$$

ϕ is harmonic on S^2 and hence is constant. Thus Y is isometric to the round S^2. Hence X_∞ is the flat $\mathbb{R}^3$. By [8, Theorem 0.3], (M,g) is flat, which is a contradiction. □

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