The work of Grigory Perelman

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Grigory Perelman has been awarded the Fields Medal for his contributions to geometry and his revolutionary insights into the analytical and geometric structure of the Ricci flow.

Perelman was born in 1966 and received his doctorate from St. Petersburg State University. He quickly became renowned for his work in Riemannian geometry and Alexandrov geometry, the latter being a form of Riemannian geometry for metric spaces. Some of Perelman's results in Alexandrov geometry are summarized in his 1994 ICM talk [20]. We state one of his results in Riemannian geometry. In a short and striking article, Perelman proved the so-called Soul Conjecture.

Soul Conjecture (conjectured by Cheeger–Gromoll [2] in 1972, proved by Perelman [19] in 1994). Let M be a complete connected noncompact Riemannian manifold with nonnegative sectional curvatures. If there is a point where all of the sectional curvatures are positive then M is diffeomorphic to Euclidean space.

In the 1990s, Perelman shifted the focus of his research to the Ricci flow and its applications to the geometrization of three-dimensional manifolds. In three preprints [21], [22], [23] posted on the arXiv in 2002–2003, Perelman presented proofs of the Poincaré conjecture and the geometrization conjecture.

The Poincaré conjecture dates back to 1904 [24]. The version stated by Poincaré is equivalent to the following.

Poincaré conjecture. A simply-connected closed (= compact boundaryless) smooth 3-dimensional manifold is diffeomorphic to the 3-sphere.

Thurston's geometrization conjecture is a far-reaching generalization of the Poincaré conjecture. It says that any closed orientable 3-dimensional manifold can be canonically cut along 2-spheres and 2-tori into "geometric pieces" [27]. There are various equivalent ways to state the conjecture. We give the version that is used in Perelman's work.

Geometrization conjecture. If M is a connected closed orientable 3-dimensional manifold then there is a connected sum decomposition $M = M_1 \# M_2 \# \cdots \# M_N$ such that each M_i contains a 3-dimensional compact submanifold-with-boundary $G_i \subset M_i$ with the following properties:

1. G_i is a graph manifold.
2. The boundary of G_i, if nonempty, consists of 2-tori that are incompressible in M_i.

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3. $M_i - G_i$ admits a complete finite-volume Riemannian metric of constant negative curvature.

In the statement of the geometrization conjecture, G_i is allowed to be $\emptyset$ or M_i. (For example, if $M = S^3$ then we can take $M_1 = G_1 = S^3$.) The geometrization conjecture implies the Poincaré conjecture. Thurston proved that the geometrization conjecture holds for Haken 3-manifolds [27]. Background information on the Poincaré and geometrization conjectures is in [17].

Perelman's papers have been scrutinized in various seminars around the world. At the time of this writing, the work is still being examined. Detailed expositions of Perelman's work have appeared in [1], [16], [18].

1. The Ricci flow approach to geometrization

Perelman's approach to the geometrization conjecture is along the lines of the Ricci flow strategy developed by Richard Hamilton. In order to put Perelman's results in context, we give a brief summary of some of the earlier work. A 1995 survey of the field is in [12].

If M is a manifold and $\{g(t)\}$ is a smooth one-parameter family of Riemannian metrics on M then the Ricci flow equation is

$$\frac{dg}{dt} = -2\,\mathrm{Ric}. \tag{1}$$

It describes the time evolution of the Riemannian metric. The right-hand side of the equation involves the Ricci tensor Ric of $g(t)$. We will write $(M, g(\cdot))$ for a Ricci flow solution.

Ricci flow was introduced by Hamilton in 1982 in order to prove the following landmark theorem.

Positive Ricci curvature ([7]). Any connected closed 3-manifold M that admits a Riemannian metric of positive Ricci curvature also admits a Riemannian metric of constant positive sectional curvature.

A connected closed 3-dimensional Riemannian manifold with constant positive sectional curvature is isometric, up to scaling, to the quotient of the standard round 3-sphere by a finite group that acts freely and isometrically on S^3. In particular, if M is simply-connected and admits a Riemannian metric of positive Ricci curvature then M is diffeomorphic to S^3. The idea of the proof of the theorem is to run the Ricci flow, starting with the initial metric $g(0)$ of positive Ricci curvature. The Ricci flow will go singular at some finite time T, caused by the shrinking of M to a point. As time approaches T, if one continually rescales M to have constant volume then the rescaled sectional curvatures become closer and closer to being constant on M.

In the limit, one obtains a Riemannian metric on M with constant positive sectional curvature.

Based on Hamilton's result, Hamilton and S.-T. Yau developed a program to attack the Poincaré conjecture using Ricci flow. The basic idea was to put an arbitrary initial Riemannian metric on the closed 3-manifold, run the Ricci flow and analyze the evolution of the metric.

Profound results about Ricci flow were obtained over the years by Hamilton and others. Among these results are the Hamilton–DeTurck work on the existence and uniqueness of Ricci flow solutions [6], [7], Hamilton's maximum principle for Ricci flow solutions [8], the Hamilton–Chow analysis of Ricci flow on surfaces [4], [9], Shi's local derivative estimates [25], Hamilton's differential Harnack inequality for Ricci flow solutions with nonnegative curvature operator [10], Hamilton's compactness theorem for Ricci flow solutions [11] and the Hamilton–Ivey curvature pinching estimate for three-dimensional Ricci flow solutions [12, Theorem 24.4],[15]. We state one more milestone result of Hamilton, from 1999.

Nonsingular flows ([14]). *Suppose that the normalized Ricci flow on a connected closed orientable* 3*-manifold M has a smooth solution that exists for all positive time and has uniformly bounded sectional curvatures. Then M satisfies the geometrization conjecture.*

The normalized Ricci flow is a variant of the Ricci flow in which the volume is kept constant. The above result clearly showed that Ricci flow was a promising approach to the geometrization conjecture. The remaining issues were to remove the assumption that the Ricci flow solution is smooth for all positive time, and the *a priori* bound on the sectional curvature.

Regarding the smoothness issue, many 3-dimensional solutions of the Ricci flow equation (1) encounter a singularity within a finite time. One example of a singularity is a standard neckpinch, in which a cross-sectional 2-sphere $\{0\} \times S^2$ in a topological neck $((-1, 1) \times S^2) \subset M$ shrinks to a point in a finite time. Hamilton introduced the idea of performing a surgery on a neckpinch [13]. At some time, one removes a neighborhood $(-c, c) \times S^2$ of the shrinking 2-sphere and glues three-dimensional balls onto the ensuing boundary 2-spheres $\{-c\} \times S^2$ and $\{c\} \times S^2$. After the surgery operation the topology of the manifold has changed, but in a controllable way, since the presurgery manifold can be recovered from the postsurgery manifold by connected sums. One then lets the postsurgery manifold evolve by Ricci flow. If one encounters another neckpinch singularity then one performs a new surgery, lets the new manifold evolve, etc.

One basic issue was to show that if the Ricci flow on a closed 3-manifold M encounters a singularity then an entire connected component disappears or there are nearby 2-spheres on which to do surgery. To attack this, Hamilton initiated a blowup analysis for Ricci flow [12, Section 16]. It is known that singularities arise from curvature blowups [7]. That is, if a Ricci flow solution exists on a maximal time interval $[0, T)$, with $T < \infty$, then $\lim_{t \to T^-} \sup_{x \in M} |\mathrm{Riem}(x, t)| = \infty$, where Riem denotes the sectional curvatures. Suppose that $\{(x_i, t_i)\}_{i=1}^{\infty}$ is a sequence of spacetime points with $\lim_{i\to\infty} |\operatorname{Riem}(x_i, t_i)| = \infty$. In order to understand the geometry of the Ricci flow solution as one approaches the singularity time, one would like to spatially expand around (x_i, t_i) by a factor of $|\operatorname{Riem}(x_i, t_i)|^{\frac{1}{2}}$ and take a convergent subsequence of the ensuing geometries as $i \to \infty$. In fact, if one also expands the time coordinate by $|\operatorname{Riem}(x_i, t_i)|$ then one can consider taking a subsequence of rescaled Ricci flow solutions, that converges to a limit Ricci flow solution $(M_\infty, g_\infty(\cdot))$.

Hamilton's compactness theorem [11] gives sufficient conditions to extract a convergent subsequence. Roughly speaking, on the rescaled solutions one needs uniform curvature bounds on balls and a uniform lower bound on the injectivity radius at (x_i, t_i). One can get the needed curvature bounds by carefully choosing the blowup points (x_i, t_i). However, before Perelman's work, the needed injectivity radius bound was not available in full generality.

If the blowup limit $(M_\infty, g_\infty(\cdot))$ exists then it is a nonflat ancient solution, meaning that it is defined for $t \in (-\infty, 0]$. The manifold M_∞ may be compact or noncompact. In the three-dimensional case, Hamilton–Ivey pinching implies that for each $t \in (-\infty, 0]$, the time-t slice $(M_\infty, g_\infty(t))$ has nonnegative sectional curvature. Thus the possible blowup limits are very special. Hamilton gave detailed analyses of various singularity models [12, Section 26]. One troublesome possibility, the so-called $\mathbb{R} \times$ *cigar soliton* ancient solution, could not be excluded. If this particular solution occurred in a blowup limit then it would be problematic for the surgery program, as there would be no evident 2-spheres along which to do surgery. Hamilton conjectured [12, Section 26] that the $\mathbb{R} \times$ *cigar soliton* solution could be excluded by means of a suitable generalization of the "little loop lemma" [12, Section 15].

In addition, there was the issue of showing that any point of high curvature in the original Ricci flow solution on $[0, T)$ has a neighborhood that is indeed modeled by a blowup limit.

2. No local collapsing theorem

Perelman's first breakthrough in Ricci flow, the no local collapsing theorem, removed two major stumbling blocks in the program to prove the geometrization of three-dimensional manifolds using Ricci flow. It allows one to take blowup limits of finite time singularities and it shows that the $\mathbb{R} \times$ *cigar soliton* solution cannot arise as a blowup limit.

No local collapsing theorem ([21]). *Let M be a closed n-dimensional manifold. If $(M, g(\cdot))$ is a given Ricci flow solution that exists on a time interval $[0, T)$, with $T < \infty$, then for any $\rho > 0$ there is a number $\kappa > 0$ with the following property. Suppose that $r \in (0, \rho)$ and let $B_t(x, r)$ be a metric r-ball in a time-t slice. If the sectional curvatures on $B_t(x, r)$ are bounded in absolute value by $\frac{1}{r^2}$ then the volume of $B_t(x, r)$ is bounded below by κr^n.*

Perelman expresses the conclusion of the no local collapsing theorem by saying that the Ricci flow solution is "κ-noncollapsed at scales less than ρ". The theorem says that after rescaling the metric ball to have radius one, if the sectional curvatures of the rescaled ball are bounded in absolute value by one then the volume of the rescaled ball is bounded below by κ. This lower bound on the volume is a form of noncollapsing. The theorem is scale-invariant, except for the condition that r should be less than the scale ρ.

Perelman proves his no local collapsing theorem using new monotonic quantities for Ricci flows, which he calls the $\mathcal{W}$-functional and the reduced volume $\tilde{V}$. Expressions that are time-nondecreasing under the Ricci flow, loosely known as entropies, were known to be potentially useful tools; for example, such an entropy was used in the two-dimensional case in [9]. However, no relevant entropies were previously known in higher dimensions. Perelman's entropy functionals arise from a new and profound understanding of the underlying structure of the Ricci flow equation. The method of proof of the no local collapsing theorem is to show that a local collapsing contradicts the monotonicity of the entropy.

The significance of the no local collapsing theorem is that under a curvature assumption, it implies a lower bound on the injectivity radius at x, using [3]. This is what one needs in order to extract blowup limits. Any blowup limit $(M_\infty, g_\infty(t))$ will be a nonflat ancient solution which, from the no local collapsing theorem, is κ-noncollapsed at all scales. If such an ancient solution additionally has nonnegative curvature operator and bounded curvature (which will be the case for three-dimensional blowup limits) then Perelman calls it a *κ-solution*.

Hereafter we assume that M is an orientable three-dimensional manifold. Perelman gives the following classification of κ-solutions.

Three-dimensional κ-solutions ([21], [22]). *Any three-dimensional orientable κ-solution $(M_\infty, g_\infty(\cdot))$ falls into one of the following types:*

(a) *$(M_\infty, g_\infty(\cdot))$ is a finite isometric quotient of the round shrinking 3-sphere.*

(b) *M_∞ is diffeomorphic to S^3 or $\mathbb{R}P^3$.*

(c) *$(M_\infty, g_\infty(\cdot))$ is the standard shrinking $\mathbb{R} \times S^2$ or its $\mathbb{Z}_2$-quotient $\mathbb{R} \times_{\mathbb{Z}_2} S^2$.*

(d) *M_∞ is diffeomorphic to $\mathbb{R}^3$ and, after rescaling, each time slice is asymptotically necklike at infinity.*

In particular, Perelman shows that the $\mathbb{R} \times$ *cigar soliton* ancient solution cannot arise as a blowup limit (as there is no $\kappa > 0$ for which it is κ-noncollapsed at all scales), thereby realizing Hamilton's conjecture.

Perelman's main use of κ-solutions is to model the high-curvature regions of a Ricci flow solution. By means of a sophisticated version of the blowup analysis, he proves the following result, which we state in a qualitative form.

Canonical neighborhoods ([21]). *Given $T < \infty$, if $(M, g(\cdot))$ is a nonsingular Ricci flow on a closed orientable 3-manifold M that is defined for $t \in [0, T)$ then any region* *of high scalar curvature is modeled, after rescaling, by the corresponding region in a three-dimensional κ-solution.*

Perelman's first Ricci flow paper [21] concludes by showing that if the Ricci flow on a closed orientable 3-manifold M has a smooth solution that exists for all positive time then M satisfies the geometrization conjecture. There is no *a priori* curvature assumption. The proof of this result uses the long-time analysis described in Section 4.

3. Ricci flow with surgery

Perelman's second Ricci flow paper [22] is a technical tour de force. He constructs a surgery algorithm to handle Ricci flow singularities. There are several issues involved in setting up a surgery algorithm. The most basic issue is to know that if the Ricci flow encounters a singularity then a connected component disappears or there are 2-spheres along which one can perform surgery, with control on the topology of the excised regions. This is an issue about the geometry of the Ricci flow near a singularity. For the first singularity time, it is handled by the above canonical neighborhood theorem. A second issue is to perform the surgery so as to not ruin the Hamilton–Ivey pinching condition on the curvature. A third issue is to show that surgery times do not accumulate. If surgery times accumulate then one may never get to a sufficiently large time to draw any topological conclusions.

Hereafter we consider a Ricci flow $(M, g(\cdot))$ on a connected closed oriented 3-manifold. With T being the first singularity time (if there is one), Perelman defines Ω to be the points in M where the scalar curvature R stays bounded up to time T, i.e. $M - \Omega = \{x \in M \; : \; \lim_{t \to T^-} R(x,t) = \infty\}$. Here Ω is an open subset of M. The next result gives the topology of M if the scalar curvature blows up everywhere.

Components that go extinct ([22]). *If $\Omega = \emptyset$ then M is diffeomorphic to a finite isometric quotient S^3/Γ of the round 3-sphere, to $S^1 \times S^2$ or to $S^1 \times_{\mathbb{Z}_2} S^2 = \mathbb{R}P^3 \# \mathbb{R}P^3$.*

Now suppose that the scalar curvature blows up somewhere but not everywhere, i.e. $\Omega \neq \emptyset$. Perelman's surgery procedure involves going up to the singularity time T and then trimming off horns. More precisely, there is a limiting time-T metric $\bar{g}$ on Ω, with scalar curvature function $\bar{R}$. For a small number $\rho > 0$, the part of Ω where the scalar curvature is not too big is $\Omega_\rho = \{x \in \Omega \; : \; \bar{R}(x) \leq \rho^{-2}\}$, a compact subset of M. The connected components of Ω can be divided into those that intersect Ω_ρ and those that do not. The connected components of Ω that do not intersect Ω_ρ have uniformly large scalar curvature and are discarded. Using the canonical neighborhood theorem, Perelman shows that if a connected component of Ω intersects Ω_ρ then it has a finite number of ends, each being a so-called "ε-horn". The latter statement means that the scalar curvature goes to infinity as one exits the end, and in addition if x is a point in the ε-horn then after expanding the metric to make the scalar curvature at x equal to one, there is a neighborhood of x in Ω that is geometrically close to a cylinder $(-\varepsilon^{-1}, \varepsilon^{-1}) \times S^2$. (Here ε is a fixed small number.) The surgery procedure consists of cutting each such ε-horn along one of these cross-sectional 2-spheres and gluing in a 3-ball.

If one does the surgery in this way then one has control on how the topology changes. Indeed, the presurgery manifold is recovered from the postsurgery manifold by taking connected sums of components, along with some possible additional connected sums with a finite number of $S^1 \times S^2$ or $\mathbb{R}P^3$ factors. (The $S^1 \times S^2$ factors come from surgeries that do not disconnect M. The $\mathbb{R}P^3$ factors can arise from connected components of Ω that were thrown away.) One can guarantee that the surgery preserves the Hamilton–Ivey curvature pinching condition by carefully prescribing the geometric way that the 3-ball is glued, following [13].

One can then run the Ricci flow, starting from the postsurgery manifold, up to the next singularity time T' (if there is one). However, if one wants to do surgery at time T' then one must find the 2-spheres along which to cut. The main problem is that the earlier surgeries could invalidate the conclusion of the canonical neighborhood theorem on $[0, T')$. The proof of the canonical neighborhood theorem in turn relied on the no local collapsing theorem.

One ingredient of Perelman's resolution of this problem is to perform surgery sufficiently far down in the ε-horns. In effect, there is a self-improvement phenomenon as one goes down the horn. Perelman shows that for any $\delta > 0$, if a point x is in an ε-horn as before, and is sufficiently deep within the horn, then after rescaling to make the scalar curvature at x equal to one, there is a neighborhood of x that is geometrically close to a cylinder $(-\delta^{-1}, \delta^{-1}) \times S^2$. Hence one can ensure that the surgeries are done within cylinders that are very long relative to the cross-section. This turns out to be a key to extending the no local collapsing theorem to the case when there are intervening surgeries within the time interval. To summarize, Perelman proves the following technically difficult theorem.

Surgery algorithm ([22]). *The surgery parameters can be chosen so that there is a well-defined Ricci-flow-with-surgery.*

The statement means that the Ricci-flow-with-surgery exists for all time. (It is not excluded that at some finite time the remaining manifold becomes the empty set.) Perelman's proof is quite intricate and uses an induction on the time interval. In addition, for a given induction step, i.e. on a given time interval, he uses a contradiction argument to show that the surgery parameters can be chosen so as to ensure that versions of the no local collapsing theorem and the canonical neighborhood theorem hold on the time interval, despite possible intervening surgeries.

Volume considerations show that only a finite number of surgeries occur on a finite time interval; any surgery within the time interval removes a definite amount of volume, but there is only so much volume available for removal.

4. Long-time behavior

Once one has the Ricci-flow-with-surgery, in order to obtain topological information about the original manifold one needs to analyze the long-time behavior. One special case is when there is a finite extinction time, i.e. the manifold in the Ricci-flow-with-surgery becomes the empty set at some finite time. Using his characterization of components that go extinct and analyzing the topology change caused by surgeries, Perelman gives the possible topology of a manifold whose Ricci flow has a finite extinction time.

Finite extinction time ([22]). *If a Ricci-flow-with-surgery starting from M has a finite extinction time then M is diffeomorphic to a connected sum of finite isometric quotients of the round S^3 and copies of $S^1 \times S^2$.*

In his third Ricci flow paper [23], Perelman goes further and uses minimal disk arguments to give a condition that ensures a finite extinction time; see also [5].

No aspherical factors ([23]). *If the Kneser–Milnor prime decomposition of M does not have any aspherical factors then a Ricci-flow-with-surgery starting with any initial Riemannian metric on M has a finite extinction time.*

When put together, the above two steps give the topological possibilities for a connected closed orientable 3-manifold M whose prime decomposition does not have any aspherical factors. In particular, if M is simply-connected then the above two steps say that M is diffeomorphic to a connected sum of 3-spheres, and hence is diffeomorphic to the 3-sphere.

In the general case when the Ricci-flow-with-surgery may not have a finite extinction time, the goal is to show that as time goes on, one sees the desired decomposition of the geometrization conjecture. There could be an infinite number of total surgeries. At the time of this writing it is not known whether this actually happens. Perelman had the insight that one can draw topological conclusions nevertheless.

Let M_t be a connected component of the time-t manifold. (If t is a surgery time then we consider the postsurgery manifold.) If M_t admitted any metric with nonnegative scalar curvature then it would be flat or the corresponding Ricci flow would have finite extinction time, in which case the topology is understood. So we can assume that M_t carries no metric with nonnegative scalar curvature. Consider hereafter the metric $\hat{g}(t) = \frac{1}{t} g(t)$ on M_t. Perelman defines the "thick" part of M_t as follows. Given $x \in M_t$, let the intrinsic scale $\rho(x,t)$ be the radius ρ such that $\inf_{B(x,\rho)} \mathrm{Riem} = -\rho^{-2}$, where Riem is the sectional curvature of $\hat{g}(t)$. For any $w > 0$, the *w-thick part* of M_t is given by $M^+(w,t) = \{x \in M_t \,:\, \mathrm{vol}(B(x, \rho(x,t))) > w\rho(x,t)^3\}$. It is not excluded that $M^+(w,t) = \emptyset$ or $M^+(w,t) = M_t$.

By definition, one has a lower curvature bound on the ball $B(x, \rho(x,t))$. Perelman shows by a subtle argument that for large t, if x is in the w-thick part $M^+(w,t)$ then $B(x, \rho(x,t))$ actually has an effective upper curvature bound. Adapting arguments from [13], he then shows that for any $w > 0$, as time goes on, $M^+(w,t)$ approaches the

w-thick part of a hyperbolic manifold whose cuspidal tori, if any, are incompressible in M_t. On the other hand, if w is small and x is not in the w-thick part then the ball $B(x, \rho(x,t))$ has a lower curvature bound and a relatively small volume compared to $\rho(x,t)^3$. From Perelman's earlier work in collapsing theory, he knew that 3-manifolds which are locally volume collapsed, with respect to a lower curvature bound, are graph manifolds. Putting this together, Perelman is able to achieve the remarkable feat of realizing the hyperbolic/graph dichotomy, without making any *a priori* curvature assumptions.

Hyperbolic pieces ([22]). *Given the Ricci-flow-with-surgery, there are a finite collection* $\{(H_i, x_i)\}_{i=1}^k$ *of complete pointed finite-volume Riemannian* 3*-manifolds of constant sectional curvature* $-\frac{1}{4}$*, a decreasing function* $\alpha(t)$ *tending to zero and a family of maps* $f_t \colon \bigcup_{i=1}^k B\Big(x_i, \frac{1}{\alpha(t)}\Big) \to M_t$ *such that for large* t*,*

1. f_t *is* $\alpha(t)$*-close to being an isometry.*
2. *The image of* f_t *contains* $M^+(\alpha(t), t)$*.*
3. *The image under* f_t *of a cuspidal torus of* $\{H_i\}_{i=1}^k$ *is incompressible in* M_t*.*

That is, for large t, the $\alpha(t)$-thick part of M_t is well approximated by the corresponding subset of $\bigcup_{i=1}^k H_i$. The remainder of M_t is highly collapsed with respect to a local lower curvature bound.

Graph manifold pieces ([22], [26]). *Let* Y_t *be the truncation of* $\bigcup_{i=1}^k H_i$ *obtained by removing horoballs at distance approximately* $\frac{1}{2\alpha(t)}$ *from the basepoints* x_i*. Then for large* t*,* $M_t - f_t(Y_t)$ *is a graph manifold.*

The above two steps, along with the fact that the components that go extinct are graph manifolds, and the fact that presurgery manifolds can be reconstructed from postsurgery manifolds via connected sums, imply the geometrization conjecture.

Grigory Perelman has revolutionized the fields of geometry and topology. His work on Ricci flow is a spectacular achievement in geometric analysis. Perelman's papers show profound originality and enormous technical skill. We will certainly be exploring Perelman's ideas for many years to come.

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