Overview

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Notation : M a closed (= compact boundaryless) orientable 3-dimensional manifold

Conjecture. *(Poincaré, 1904)*

If M is simply connected then it is diffeomorphic to the three-sphere S^3.

Extension to non-simply-connnected case

Thurston's "Geometrization Conjecture"

Motivation : Recall that any closed surface carries a Riemannian metric of constant curvature 1, 0, or -1.

Question : Does every closed 3-manifold carry a constant curvature metric?

Answer : No.

Geometrization Conjecture (rough version)

Conjecture. *(Thurston, 1970's) M can be canonically cut into pieces, each of which carries one of eight magic geometries.*

Picture :

The eight magic geometries :

1. Constant curvature 0
2. Constant curvature 1
3. Constant curvature -1

etc.

Fact : Geometrization $\Longrightarrow$ Poincaré

More details in John Morgan's lectures.

Analytic approach to Geometrization Conjecture

Start with an arbitrary Riemannian metric g_0 on M.

Evolve g_0 so that as time goes on, one starts to "see" M's geometric decomposition.

Geometric Flow

A prescribed 1-parameter family of metrics $g(t)$, with $g(0) = g_0$. We want it to improve the original metric.

Ricci flow :

$$\frac{\partial g_{ij}}{\partial t} = -2\,R_{ij}.$$

Introduced by Hamilton in 1982.

Its properties will be explained by Ben Chow.

Here R_{ij} denotes the **Ricci tensor** of $g(t)$.

Somewhat like the heat equation

$$\frac{\partial f}{\partial t} = \nabla^2 f,$$

except *nonlinear*.

$$\frac{\partial g_{ij}}{\partial t} = -2\, R_{ij}.$$

Heat equation evolves a **function**.

Ricci flow evolves a **Riemannian metric**.

Recall : heat flow evolves an initial function f_0 towards a constant function.

Hope : the Ricci flow will evolve the metric g_0 so that we see the geometrization decomposition.

This works for surfaces!

Examples of Ricci flow solutions

1. **3-torus :** $g(t) = g_{T^3}$ for all t (flat metric)

2. **3-sphere :** $g(t) = r^2(t)\, g_{S^3}$,

$r^2(t) = -4t$ for all $t < 0$.

3. **hyperbolic 3-manifold :** $g(t) = r^2(t)\, g_{hyp}$,

$r^2(t) = r_0^2 + 4t$ for all $t \geq 0$.

4. **shrinking cylinder :** $g(t) = r^2(t) g_{S^2} + g_{\mathbb{R}}$,

$r^2(t) = -2t$ for all $t < 0$.

How to prove the Poincaré Conjecture

1. Start with simply-connected M. Put any Riemannian metric on it.

Run Ricci flow. What can happen?

(Overly) optimistic hope :

Maybe the solution shrinks to a point, and get rounder as it shrinks. If so, M has a metric of **constant positive sectional curvature**.

But M is simply connected, so then it must be topologically a 3-sphere!

Theorem. *(Hamilton, 1982) This works if the initial metric has positive Ricci curvature.*

But what if it doesn't?

With a general initial metric

New problem : the flow could go singular before it can shrink to a point. For example,

Neckpinch :

What to do now?

Idea (Hamilton) : Do **surgery** on a neckpinch.

Then continue the flow. If more neckpinches occur, do the same.

Basic problem : How do we know that the singularities are actually caused by neckpinches?

Solved by Perelman (2002).

Outline of an argument to prove the Poincaré conjecture

1. Start with M a simply-connected closed 3-manifold.

2. Put an arbitrary Riemannian metric g_0 on M.

3. **Claim (Perelman II) :**
There **is** a well-defined **Ricci-flow-with-surgery**.

4. Run flow up to first singularity time (if there is one).

a. **First case :** Entire solution disappears.

Claim (Perelman II) :
If an entire solution disappears then the manifold is diffeomorphic to
(i) S^3/Γ (with Γ a finite subgroup of $SO(4)$ acting freely on S^3), or
(ii) $S^1 \times S^2$, or
(iii) $(S^1 \times S^2)/\mathbb{Z}_2$.

b. **Next case :** If the entire solution doesn't disappear, say Ω is what's left at the singular time.

Form a **new manifold** M' by **surgering out the horns** in Ω.

Note : M' may be disconnected. Flow to next singularity time.

Remove high-curvature regions by **surgering out the horns** and by **throwing away isolated components**.

Continue.

5. **Claim (Perelman III, Colding-Minicozzi)**

After a finite time, there's nothing left. The entire solution went **extinct**.

Note : this uses the assumption that the original manifold was simply connected.

Will be explained by Toby Colding.

6. How to reconstruct the original manifold?

Pieces that went extinct, or were thrown away, were each diffeomorphic to

$$S^3/\Gamma,\ S^1 \times S^2 \text{ or } (S^1 \times S^2)/\mathbb{Z}_2.$$

Going from **after** a surgery to **before** a surgery amounts to performing **connected sums**.

(Possibly with some new $S^1 \times S^2$'s and $\mathbb{R}P^3$'s).

7. Conclusion

The original manifold M is diffeomorphic to a connected sum :

$(S^3/\Gamma_1)\#\ldots\#(S^3/\Gamma_k)\#(S^1\times S^2)\#\ldots\#(S^1\times S^2)$
(Here $(S^1 \times S^2)/\mathbb{Z}_2 = \mathbb{R}P^3\#\mathbb{R}P^3$.) By van Kampen's theorem,

$$\pi_1(M) = \Gamma_1 \star \ldots \star \Gamma_k \star \mathbb{Z} \star \ldots \star \mathbb{Z}.$$

But M is simply connected! So each Γ_i is trivial and there are no $S^1 \times S^2$ factors. Then

$$M = S^3\#\ldots\#S^3 = S^3.$$

This would prove the Poincaré Conjecture!

What if the starting manifold is not simply-connected?

The solution $g(t)$ could go on for all time t.

(Example : manifolds with constant negative sectional curvature.)

Shrink by a factor of t :

$$\widehat{g}(t) = \frac{g(t)}{t}.$$

Claim (Perelman II) : For large t, M_t decomposes into two pieces

$$M_t = M_{thick} \cup M_{thin}$$

(each possibly empty), where

a. The metric $\widehat{g}$ on M_{thick} is close to constant sectional curvature $-\frac{1}{4}$. The interior of M_{thick} admits a **complete finite-volume metric of constant negative sectional curvature**.

b. M_{thin} is a **graph manifold**. These are known to have a "geometric" decomposition.

c. The gluing of M_{thick} and M_{thin} is done along **incompressible 2-dimensional tori**.

Decomposition of M :

This would prove the Geometrization Conjecture!

Important earlier case : when there are no singularities and $\sup_M |\mathrm{Riem}(g_t)| = O(t^{-1})$ (Hamilton, 1999).

Back to claims :

Claim (Perelman II) :
There **is** a well-defined Ricci-flow-with-surgery.

Two statements here :

1. We know how to do surgery if we encounter a singularity.
2. The surgery times do not accumulate.

How do we do surgery if we encounter a singularity?

Fact : Singularities are caused by curvature blowup.

If the solution exists on the time interval $[0, T)$, but no further, then

$$\lim_{t \to T^-} \sup_M |\mathrm{Riem}(g_t)| = \infty.$$

(Here |Riem| denotes the largest sectional curvature at a point, in absolute value.)

To do surgery, we need to know that singularities are caused by tiny necks collapsing.

How do we know that this is the case?

Rescaling argument

Choose a sequence of times t_i and points x_i in M so that

1. $\lim_{i\to\infty} t_i = T$.
2. $\lim_{i\to\infty} |\mathrm{Riem}(x_i, t_i)| = \infty$.

Blowup

Zoom in to the spacetime point (x_i, t_i).

Define $r_i > 0$ by $r_i^{-2} = |\mathrm{Riem(x_i, t_i)}|$.

(The **intrinsic scale** at the point (x_i, t_i).)

Note $\lim_{i \to \infty} r_i = 0$.

Spatially expand M by a factor r_i^{-1} so that $|\mathrm{Riem(x_i, t_i)}|$ becomes $\mathbf{1}$.

Fact : If M is spatially expanded by a factor of r^{-1} then time has to be expanded by r^{-2}, in order to still have a Ricci flow solution.

Also shift time so that the old t_i-time becomes the new 0-time.

Get a **sequence** of Ricci flow solutions $g_i(t)$ on M, each centered at a spacetime point $(x_i, 0)$.

$$g_i(x,t) = \frac{1}{r_i^2}\, g\left(x, \frac{t-t_i}{r_i^2}\right).$$

The **new** Ricci flow solution (M, g_i) is defined on the time interval $\left[-\frac{t_i}{r_i^2}, 0\right]$.

Its curvature at $(x_i, 0)$ has $|\mathrm{Riem(x_i, 0)}| = 1$.

Idea : Take a "convergent subsequence" of the Ricci flow solutions (M_i, g_i).

Call limit Ricci flow solution (M_∞, g_∞) (if it exists).

The original time interval $[0, T)$ was expanded in length by the factors r_i^{-2}, so (M_∞, g_∞) exists on an **infinite** time interval $(-\infty, 0]$.

"Ancient solution"

Very special type of Ricci flow solution!

If we can find a **near-cylinder** $S^2 \times [L, -L]$ in $(M_\infty, g_\infty(0))$ then there were tiny necks in the **original** solution near the (x_i, t_i)'s and we're in business.

With a bit more argument, get that **any** high-curvature region in the original solution is modeled by a rescaled chunk of an ancient solution.

Two problems :

1. Why can we take a convergent subsequence of the Ricci flow solutions (M_i, g_i)'s?

2. Even if we can, why is there a near-cylinder in $(M_\infty, g_\infty(0))$?

Why can we take a convergent subsequence of the Ricci flow solutions (M_i, g_i)?

We know that $|\mathrm{Riem}(x_i, 0)| = 1$ for the solution (M_i, g_i). Want to take a **pointed** limit, i.e. so that $\lim_{i\to\infty} x_i = x_\infty$ in some sense.

Need two things to take a convergent subsequence of Ricci flow solutions (Cheeger, Hamilton)

1. Need to know that the curvature stays **uniformly** bounded.

Given $r > 0$ and t, need a constant $K_{r,t}$ so that

$$|\mathrm{Riem}(x, t)| \leq K_{r,t}$$

for all x in the time-t ball of radius r around x_i, for all i.

2. Need to know that the sequence of Ricci flow solutions doesn't "collapse",
i.e. that it stays uniformly 3-dimensional.

It's enough to have a **uniform lower bound** on the volume of the 1-ball around x_i at time zero.

I.e.

$$\mathrm{vol}(\mathrm{B}_1(\mathrm{x_i})) \geq \mathrm{v}_0 > 0$$

for all i.

We'll get the curvature bounds by choosing the blowup points (x_i, t_i) cleverly.

How to get lower volume bound?

Even if we can do this, why is there a near-cylinder in (M_∞, g_∞)?

Bad news possibility : $\mathbb{R} \times cigar\ soliton$

A particular ancient solution, which has nothing like a cylinder in it.

If this appears in a blowup limit then we're in trouble.

This was the pre-Perelman status, as developed by Hamilton and others.

Perelman's first big innovation in Ricci flow :

No Local Collapsing Theorem

Say we have a Ricci flow solution on a finite time interval $[0, T)$.

The theorem says that at a spacetime point (x, t), the solution looks **noncollapsed at the intrinsic scale of the spacetime point**.

(Recall : intrinsic scale is $|\mathrm{Riem}(x, t)|^{-1/2}$.)

At a spacetime point (x,t), the solution looks **noncollapsed at the intrinsic scale of the spacetime point**.

More precisely, given the Ricci flow solution on the interval $[0,T)$ and a scale $\rho > 0$, we can find a number $\kappa > 0$ so that the following holds:

Suppose that a metric ball B in some time slice has radius r (less than ρ). If

$$|\mathrm{Riem}| \le \mathrm{r}^{-2}$$

on B then

$$\mathrm{vol(B)} \ge \kappa\, \mathrm{r}^3.$$

Essentially scale-invariant, so it passes to a blowup limit!

In short,
Local curvature bound $\Longrightarrow$ local lower bound on volumes of balls

Of course, this is a statement about **Ricci flow solutions**.

With the **no local collapsing theorem**, we can take blowup limits. Furthermore, we won't get $\mathbb{R} \times \textit{cigar soliton}$ as a limit.

Still need to show that the blowup limit actually has a near-cylinder in it (if it's non-compact).

I.e. have to understand "κ-noncollapsed ancient solutions".

How to prove No Local Collapsing Theorem :

Find some "functional" $I(g)$ of metrics g with the following two properties :

1. If g is "locally collapsed" somewhere then $I(g)$ is very small.

2. If $g(t)$ is a Ricci flow solution then $I(g(t))$ is nondecreasing in t.

If we can find I then we're done!

Not at all clear that there is such a functional.

Perelman found **two** :

1. Entropy (I.1)
2. Reduced volume (I.7).

Next time, we'll start on the **entropy** functional.

But we'll actually prove the No Local Collapsing Theorem using the **reduced volume**.