Long-time behavior in geometric flows

John Lott
UC-Berkeley
http://math.berkeley.edu/~lott

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The big question

When does a geometric flow make a space more homogenous?

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When does a geometric flow make a space more homogenous?

Today: Two ways to evolve a three dimensional space.

1. If a three dimensional space admits a locally homogeneous structure, can we find it with the Ricci flow?

2. For a compact universe with no matter, will gravitational dynamics make it more homogeneous?
Outline of the talk

1. Homogeneous spaces and the geometrization conjecture

2. The geometrization conjecture and Ricci flow

3. Finiteness of the number of surgeries

4. Long-time behavior of Ricci flow

5. The Einstein flow
Long-time behavior

Homogeneous spaces and the geometrization conjecture

Geometrization conjecture and Ricci flow

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Einstein flow
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First, how do we understand three dimensional spaces?

In terms of homogeneous spaces.
A metric space $X$ is \textit{locally homogeneous} if all $x, y \in X$, there are neighbourhoods $U$ and $V$ of $x$ and $y$ and an isometric isomorphism $(U, x) \rightarrow (V, y)$. 
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The metric space $X$ is *globally homogeneous* if for all $x, y \in X$, there is an isometric isomorphism $\phi : X \rightarrow X$ that $\phi(x) = y$. 
Locally homogeneous Riemannian manifolds

Any Riemannian manifold $M$ gets a metric space structure.
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**Theorem**

*Singer 1960* If $M$ is a complete, simply connected Riemannian manifold which is locally homogeneous, then $M$ is globally homogeneous.

So passing to the universal cover turns “locally homogeneous” into “globally homogeneous”.

Locally homogeneous Riemannian manifolds
Globally homogeneous $S^2$, 
Globally homogeneous $S^2$, locally homogeneous
Two-dimensional geometries

Globally homogeneous $S^2$, locally homogeneous

Globally homogeneous $\mathbb{R}^2$, 
Two-dimensional geometries

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Three-dimensional Thurston geometries

$S^3, \mathbb{R}^3, H^3$

Warning: Unlike in two dimensions, not every compact three-dimensional manifold admits a geometric structure, i.e., admits a locally homogeneous Riemannian metric.
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\[ S^3, \mathbb{R}^3, H^3 \]

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If $M$ is a compact orientable 3-manifold then there is a way to split $M$ into canonical pieces, using certain embedded 2-spheres and 2-tori. (The collection of 2-spheres and 2-tori could be empty.)
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Conjecture (Thurston, 1982)

*The ensuing pieces have geometric structures, i.e. admit locally homogeneous metrics*
Cut along the 2-spheres and cap off the resulting pieces with 3-balls.
Geometric decomposition

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Cut along the 2-tori. The interiors of the ensuing pieces should admit complete locally homogeneous metrics.
Long-time behavior

Homogeneous spaces and the geometrization conjecture

Geometrization conjecture and Ricci flow

Finiteness of the number of surgeries

Long-time behavior of Ricci flow

Einstein flow
Ricci flow approach to geometrization

Hamilton’s Ricci flow equation

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\frac{dg}{dt} = -2 \text{ Ric}_g.
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The ordinary heat equation

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acts on functions \( f \) on a fixed (compact connected) Riemannian manifold \( M \). It takes an initial function \( f_0 \) and evolves it into something homogeneous (i.e. constant).
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Maybe the Ricci flow will evolve an initial Riemannian metric into something homogeneous.
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Surfaces

For the Ricci flow on a compact surface, after rescaling the metric approaches a locally homogeneous metric.
Some components may disappear, e.g. a round shrinking 3-sphere.
Neckpinch

A 2-sphere pinches off. (Drawn one dimension down.)
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Hamilton’s idea of surgery
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Role of singularities

Singularities are good because we know that in general, we have to cut along some 2-spheres to see the geometric pieces. They are also problematic because they may cause lots of topologically trivial surgeries. (Spitting out 3-spheres.)

Remark: the surgeries are done on 2-spheres, not 2-tori.
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Intuitive way to prove the geometrization conjecture using Ricci flow

Step 1: Show that one can perform surgery.
  a. Show that singularities are only caused by components disappearing or by 2-spheres pinching down.
  b. Show that the surgery times do not accumulate.

Step 2: Show that only a finite number of surgeries occur.

Step 3: Show that after the singularities are over, as time evolves, the locally homogeneous pieces in the Thurston decomposition asymptotically appear.

(Relevant geometries: $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\tilde{\text{SL}}(2, \mathbb{R})$, $\text{Sol}$, $\text{Nil}$.)
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Perelman’s work

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From Perelman’s second Ricci flow paper: This is a technical paper, which is a continuation of [I]. Here we verify most of the assertions, made in [I, §13]; the exceptions are ... the smoothness of the solution from some time on, which turned out to be unjustified, and, on the other hand, irrelevant for the other conclusions.
What Perelman actually showed

For any $t$, one can define a “thick-thin” decomposition of the time-$t$ manifold (assuming that it’s nonsingular). Then for large but finite $t$, the following properties hold.

1. The interior of the thick part carries a complete finite-volume hyperbolic metric. (This uses Ricci flow.)
2. The thin part is a “graph manifold”. (This doesn’t use Ricci flow. Stated by Perelman, proofs by Shioya-Yamaguchi, Morgan-Tian, Bessières-Besson-Boileau-Maillot-Porti and Kleiner-L.)
3. The interface between the thick and thin parts consists of “incompressible” 2-tori (Hamilton).

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Remark: Answering these questions has no topological implication. We already know that the geometrization conjecture holds. Rather, they are analytic questions about the Ricci flow.
Long-time behavior

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Einstein flow
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To be more precise, there is a parameter in Perelman’s Ricci-flow-with-surgery that determines the scale at which surgery is performed.
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To be more precise, there is a parameter in Perelman’s Ricci-flow-with-surgery that determines the scale at which surgery is performed.

The statement is that if this parameter is small enough (which can always be achieved) then there is a finite number of surgeries.
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The statement is that for large time, the rescaled metrics $\{\hat{g}(t)\}$ have *uniformly* bounded sectional curvatures.

This is good because we know lots about metrics with bounded sectional curvature (Cheeger-Fukaya-Gromov).
Bamler’s proof uses all of Perelman’s work, and more. Some of the new ingredients:

1. Localizing Perelman’s estimates and applying them to local covers of the manifold.

2. Use of minimal surfaces to control the geometry of the thin part.

3. Use of minimal embedded 2-complexes.
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The only case that we completely understand is when $M$ admits some hyperbolic metric. Then from Perelman’s work, for any initial metric on $M$, as $t \to \infty$ the rescaled Riemannian metric $\hat{g}(t)$ approaches the metric on $M$ of constant sectional curvature $-\frac{1}{4}$.
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**Question**: if $M$ doesn’t admit a hyperbolic metric, what are the candidate geometries for the long-time behavior?
The static solutions of the Ricci flow equation

$$\frac{dg}{dt} = -2 \text{Ric}_g$$

are Ricci-flat.
Quasistatic solutions

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The solutions that are *scale-invariant*, ie. static up to rescaling, are *Einstein metrics*: \( \text{Ric} = \text{const.} \ g \).
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The solutions that are \textit{self-similar}, i.e. static up to rescaling and diffeomorphisms are \textbf{Ricci solitons}: \( \text{Ric} = \text{const. } g + \mathcal{L}_V g \).
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Fact: On a compact 3-manifold, any self-similar solution has constant sectional curvature.
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The solutions that are *self-similar*, i.e. static up to rescaling and diffeomorphisms are *Ricci solitons*: $\text{Ric} = \text{const. } g + \mathcal{L}_V g$.

Fact: On a compact 3-manifold, any self-similar solution has constant sectional curvature.

Apparent paradox: What happens to the Ricci flow if our 3-manifold doesn’t admit a constant curvature metric?
Nil geometry

\[ Z = \begin{cases} \begin{pmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} : & a, b, c \in \mathbb{Z} \end{cases}. \]

Define \( \text{Nil} \mathbb{R} \) similarly.

Put \( M = \text{Nil} \mathbb{R} / \text{Nil} \mathbb{Z} \). It is the total space of a nontrivial circle bundle over \( T^2 \).

Run the Ricci flow. The base torus expands like \( O(t^{1/6}) \). The circle fibers shrink like \( O(t^{-1/6}) \).

With the rescaled metric \( \hat{g}(t) = g(t) t \), \((M, \hat{g}(t))\) shrinks to a point.
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Run the Ricci flow. The base torus expands like $O(t^{1/6})$. The circle fibers shrink like $O(t^{-1/6})$. With the rescaled metric $\hat{g}(t) = g(t)t$, $(M, \hat{g}(t))$ shrinks to a point.
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Run the Ricci flow. The base torus expands like $O \left( t^{\frac{1}{6}} \right)$. The circle fibers shrink like $O \left( t^{-\frac{1}{6}} \right)$.

With the rescaled metric $\hat{g}(t) = \frac{g(t)}{t}$, $(M, \hat{g}(t))$ shrinks to a point.
$M$ fibers over a circle with 2-torus fibers. The monodromy is a hyperbolic element of $SL(2, \mathbb{Z})$. 
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Is there a common pattern?

There is a common pattern, but to see it one must pass to the universal cover. That is, we are looking at the Ricci flow on a Thurston geometry of type $\mathbb{R}^3$, $\mathbb{H}^3$, $\mathbb{H}^2 \times \mathbb{R}$, $\text{Sol}$, $\text{Nil}$ or $\tilde{\text{SL}}_2(\mathbb{R})$.

Proposition (L. 2007) For any initial globally homogeneous metric on such a Thurston geometry, there is a limiting (blowdown) Ricci flow solution, which is an expanding soliton. There is one such soliton for each homogeneity type. It is a universal attractor.

$\text{Ric} + \frac{1}{2} \nabla \parallel g = -\frac{1}{2} t g.$

A subtlety: the limit is in the pointed sense. The soliton metric $g$ is homogeneous but the vector field $\nabla$ need not be homogeneous. Also, the homogeneity type may change in the limit.
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The limiting solitons

<table>
<thead>
<tr>
<th>Thurston type</th>
<th>Expanding soliton</th>
</tr>
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<tbody>
<tr>
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Remarks:
▶ By Bamler's result, the sectional curvatures are always \(O(t^{-1})\).
▶ The hypotheses imply that \(M\) admits a locally homogeneous metric.

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4. Local stability results for certain expanding Ricci solitons (due to Dan Knopf).
What happens to the Ricci flow on a 3-torus?

Theorem (L.-Sesum 2014) Let $g^0$ be a warped product metric on $T^3$, with respect to the circle fibering $T^3 \to T^2$ and any Riemannian metric on $T^2$. Then under the Ricci flow, $g(t)$ approaches a flat metric $g^\infty$ on $T^3$ exponentially fast.

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A more refined result

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Homogeneous spaces and the geometrization conjecture

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Finiteness of the number of surgeries

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Einstein flow
I’m interested in expanding vacuum spacetimes. What is the future behavior?
The setup

I’m interested in expanding vacuum spacetimes. What is the future behavior?

The spacetime is diffeomorphic to \((0, \infty) \times X\), where \(X\) is a compact three-dimensional manifold.
Einstein equations

The spacetime has a Lorentzian metric $g$. The Einstein equation of general relativity is

$$R_{\alpha\beta} - \frac{1}{2} R g_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{8\pi G}{c^4} T_{\alpha\beta}.$$ 

Here $R_{\alpha\beta}$ is the Ricci tensor and $R = \sum_{\alpha, \beta} g^{\alpha \beta} R_{\alpha\beta}$ is the scalar curvature function.

I will make the following simplifications:

1. The cosmological constant vanishes, i.e. $\Lambda = 0$.
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There has been lots of work on this, mostly under some symmetry assumptions for the spatial slices (e.g. locally homogeneous or $T^2$-symmetry). Are there more general results?
What is time?

Suppose that we have a foliation of the spacetime by compact hypersurfaces.
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Let’s assume that along any given hypersurface, the expansion factor is constant. This defines a constant mean curvature (CMC) foliation.
Using the foliation, the metric takes the form

\[ g = -L^2 dt^2 + h(t), \]

where \( L = L(t) \) is a function on \( X \) and \( h(t) \) is a Riemannian metric on \( X \).
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where \( L = L(t) \) is a function on \( X \) and \( h(t) \) is a Riemannian metric on \( X \). The Ricci-flat condition on \( g \) becomes

\[ \frac{\partial h_{ij}}{\partial t} = -2LK_{ij} \tag{3} \]

and

\[ \frac{\partial K_{ij}}{\partial t} = LH K_{ij} - 2L \sum_{k,l} h^{kl} K_{ik} K_{lj} - L_{;ij} + LR_{ij}, \tag{4} \]

along with certain time-independent “constraint” equations. Here the mean curvature \( H = \sum_{i,j} h^{ij} K_{ij} \) is spatially constant.
Monotonicity

With our conventions, *expanding* solutions have $H < 0$. There’s a corresponding time parameter, the Hubble time $t = -\frac{3}{H}$. 

Theorem (Fischer-Moncrief) If $(h(t), K(t), L(t))$ is an expanding CMC Einstein flow on a compact three-dimensional manifold $X$ then $t - \frac{3}{H} \text{vol}(X) h(t)$ is monotonically nonincreasing. It is constant if and only if the Einstein flow describes a compact quotient of the Milne universe, i.e. $g = -dt^2 + h_{\text{hyp}}$. The analogous statement in Ricci flow is that $t - \frac{3}{2} \text{vol}(X) h(t)$ is monotonically nonincreasing.
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**Theorem**

*(Fischer-Moncrief)*

*If $(h(t), K(t), L(t))$ is an expanding CMC Einstein flow on a compact three-dimensional manifold $X$ then $t^{-3} \text{vol}(X, h(t))$ is monotonically nonincreasing.*
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A Lorentzian metric $g$ is \textit{self-similar} if there’s a one-parameter group of diffeomorphisms $\{\phi_s\}$ so that $\phi_s^* g = e^{cs} g$, for some $c \in \mathbb{R}$. 

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Explicit solutions

Here are the simply-connected homogeneous self-similar solutions (that admit a spatially compact quotient):

1. The Milne spacetime is the interior of a forward lightcone in $\mathbb{R}^3$. It is foliated by hyperboloids. The metric is $g = -dt^2 + t^2 h_{\text{hyp}}$. It is scale-invariant. A spatially compact quotient is called a L"obell spacetime.

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4. The Kasner spacetimes live on $(0, \infty) \times \mathbb{R}^3$, with metric

$$g = -dt^2 + t^{2p_1} \, dx^2 + t^{2p_2} \, dy^2 + t^{2p_3} \, dz^2.$$ 

Here

$$p_1 + p_2 + p_3 = p_1^2 + p_2^2 + p_3^2 = 1.$$
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(L. 2018) Suppose that $(h(t), K(t), L(t))$ is an expanding CMC Einstein flow on a compact aspherical three dimensional manifold $X$. Suppose that the curvature is $O(t^{-2})$ in magnitude, and the diameter of $(X, h(t))$ is $O(t)$.

Then there are arbitrarily large future time intervals where the pullback of the flow to the universal cover $\tilde{X}$ is modelled by one of the homogeneous self-similar solutions.
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Then there are arbitrarily large future time intervals where the pullback of the flow to the universal cover \(\tilde{X}\) is modelled by one of the homogeneous self-similar solutions.

(If there is a lower volume bound \( \text{vol}(h(t)) \geq \text{const.} \ t^3 \) then the model space is the Milne spacetime. This case is due to Mike Anderson.)
Unlike in Ricci flow, there are expanding CMC Einstein flows that do not satisfy the scale-invariant curvature condition $\| R_{m\mu} \| = O(t^{-2})$. (Homogeneous examples are due to Hans Ringström.)
Type-II solutions

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It turns out to be flat.
An apparent paradox

In the blowdown analysis, we rescale so that $\| Rm_g(x_i, t_i) \| = 1$. How can the limit be flat?
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The limit of the metrics exists in the weak $W^{2,p}$-topology, for $1 \leq p < \infty$, and in the $C^{1,\alpha}$-topology for $0 < \alpha < 1$. This implies that the curvature tensors converge in the weak $L^p$-topology. The limit could well be zero. In effect, there are increasing curvature fluctuations that average out the curvature to zero. The rescaled metrics do converge to a flat metric in the $C^{1,\alpha}$-topology.
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Some questions

1. Suppose that the 3-manifold is not prime. The Ricci flow develops singularities. What happens under the Einstein flow?
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2. By Hawking’s singularity theorem, if we look backward in time, there is geodesic incompleteness, and often curvature blowup. (Big bang.)

Can one understand the geometric asymptotics as one approaches the singularity? (BKL conjectures.)