LONG-TIME BEHAVIOR OF RICCI FLOW ON SOME COMPLEX SURFACES

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Abstract. We give biLipschitz models for the Ricci flow on some 4-manifolds (minimal surfaces of general type), exhibiting a combination of expanding and static behavior.

1. Introduction

The long-time behavior of the Ricci flow, when it exists, gives a way to construct canonical metrics on manifolds. Such canonical metrics include Einstein metrics and, more generally, Ricci solitons. For higher genus surfaces, there is a limit $\lim_{t\to\infty} t^{-1}g(t)$ which is a metric of constant sectional curvature in the conformal class of $g(0)$. For compact 3-manifolds, there is a conjectural picture for the long-time behavior [11], although there are still open questions.

In this paper we look at a four dimensional Ricci flow in which the long-time model is a hybrid of expanding behavior and static behavior. We state the results and motivate them afterward.

If g_1 and g_2 are two Riemannian metrics on a manifold, and $K \geq 1$, then we say that the metrics are K-biLipschitz (relative to the identity map) if $K^{-1}g_1 \leq g_2 \leq Kg_1$. Explicit model flows $g_{mod}(t)$ and $g^{(k)}_{mod}(t)$ will be described later.

We will identify a Kähler metric g with its associated Kähler form ω.

Theorem 1. *Let M be a minimal complex surface of general type, with disjoint rational curves $\{E_i\}$ of self intersection -2. Let $[E_i] \in \mathrm{H}^{1,1}(M;\mathbb{R})$ be the cohomology class dual to the homology class of E_i. Given positive numbers $\{b_i\}$, there is a model flow of Kähler metrics $g_{mod}(t)$ on M, defined for large t, so that*

- *As cohomology classes, $[\omega_{mod}(t)] = \sum_i b_i^{-1}[E_i] - 2\pi t c_1(M)$, and*
- *For all sufficiently large T, the Ricci flow solution $\{g(t)\}_{t\geq T}$ with initial condition $g(T) = g_{mod}(T)$ is such that for all $\epsilon > 0$, $g(t)$ is $K(t)$-biLipschitz to $g_{mod}(t)$, where $K(t) = 1 + O\left(t^{-1+\epsilon}\right)$.*

Corollary 1. *In the setting of Theorem 1, if $m \in E_i$ then $\lim_{s\to\infty}(M, g(s+\cdot), m)$ exists in the pointed Lipshitz Cheeger-Hamilton topology. The limit is the static flow of b_i^{-1} times the Eguchi-Hanson metric.*

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We note that to get a limit in Corollary 1, one must perform s-dependent diffeomorphisms.

Contracting each curve E_i in M to a point gives an orbifold X that admits a Kähler-Einstein metric of negative Einstein constant. If X is complex hyperbolic then there is an improved convergence statement.

Theorem 2. *Suppose that X is a complex hyperbolic orbifold. There is a model flow $g^{(k)}_{mod}(t)$ so that the result of Theorem 1 holds with $K(t) = 1 + O\left(t^{-2+\epsilon}\right)$.*

There is also a stability result in this case (Proposition 1).

To motivate the theorems, we begin with a general discussion of Kähler-Ricci flow. Let $(M, g(t))$ be a Kähler-Ricci flow on a compact complex manifold M of arbitrary dimension, with initial metric $g(0)$. Tian and Zhang showed that the flow is immortal, i.e. exists for all positive time, if and only if the canonical bundle K_M is nef, i.e. $c_1(K_M)$ is in the closure of the Kähler cone of M [20].

We restrict to immortal flows. Assuming the Abundance Conjecture (which is known for Kähler surfaces), Song and Tian showed that the scalar curvature R is $O(t^{-1})$ in magnitude as $t \to \infty$ [15]. Immortal Ricci flows are divided into types III and II, depending on whether or not the sectional curvatures decay in magnitude like $O(t^{-1})$. If the flow is type-II, i.e. if $\| \operatorname{Rm}(g(t))\|_\infty$ fails to be $O(t^{-1})$, then we can take a type-II rescaling limit. That is, we can find a sequence of spacetime points $\{(m_i, t_i)\}_{i=1}^\infty$ so that $|\operatorname{Rm}|$ achieves its maximal time-t_i value Q_i at m_i, with $\lim_{i\to\infty} t_i Q_i = \infty$, and so that there is a pointed smooth Cheeger-Hamilton limit $\lim_{i\to\infty}(M_i, Q_i g(t_i + Q_i^{-1} s), m_i) = (M_\infty, g_\infty(s), m_\infty)$ [4, Chapter 8.2.1.3]. Here the limit is an eternal Ricci flow, i.e. exists for all $s \in \mathbb{R}$, and is not flat. It lives on a manifold if one has the relevant injectivity radius lower bounds at $\{m_i\}_{i=1}^\infty$, and otherwise lives on an étale groupoid [10]. Because of the rescaling, the limiting scalar curvature $R(g_\infty)$ vanishes. Then from the evolution equation for scalar curvature $\frac{\partial R}{\partial t} = \triangle R + 2|\operatorname{Ric}|^2$, one concludes that $g_\infty(s)$ is Ricci flat and hence constant in s. In short, if an immortal Kähler-Ricci flow is type-II then one expects to extract a nontrivial Ricci flat space in a scaling limit.

In this paper we look at immortal Kähler-Ricci flows on compact complex manifolds M of complex dimension two. A survey is in [21]. For cohomological reasons, the volume growth is a polynomial in the time t. If it is constant in t then M is a Calabi-Yau manifold and Cao showed that the Kähler-Ricci flow approaches the Ricci flat metric in the given Kähler class [3]. If the volume is linear in t then M is an elliptic surface and the Kähler-Ricci flow was studied by Song and Tian [13]. We are concerned with the remaining case, when the volume is quadratic in t, which was studied by Tian and Zhang [20]. In this case the canonical bundle K_M is big in the sense that $c_1(K_M)^2 \neq 0$.

Equivalently, one could say that we are looking at the Kähler-Ricci flow on projective surfaces of general type. The Kähler-Ricci flow on such a surface may encounter singularities corresponding to undoing blowups, i.e. contracting rational curves of self intersection -1 to points. It is known how to flow through such singularities [16, 17], of which there is a finite number, so we are reduced to studying Kähler-Ricci flows on minimal projective surfaces of general type.

Such a manifold M has a canonical model X, which is an orbifold of complex dimension two with isolated singularities. There is a morphism $p : M \to X$, a resolution of singularities, so that if $x \in X$ is a regular point then $p^{-1}(x)$ is a point, while if x is a singular point then $p^{-1}(x)$ is a connected union of rational curves E of self intersection -2 [1, Chapter VII.5]. There is a unique Kähler-Einstein metric g_{KE} on X with $\mathrm{Ric}_{KE} = -\omega_{KE}$ [7].

If X is smooth, i.e. if M is already a canonical model, then the Kähler-Ricci flow on M, starting from any initial Kähler metric $g(0)$, has the property that $\lim_{t\to\infty} t^{-1}g(t) = g_{KE}$ smoothly [3]. On the other hand, if X is not smooth then the flow on M is type-II [22]. Tian and Zhang showed that $\lim_{t\to\infty} t^{-1}\omega(t) = \omega_{KE}$ as a current, with smooth convergence on compact subsets of $M - \bigcup_i E_i$, where we identify the latter with the regular part of X [20]. In particular, the geometry away from the -2 rational curves $\{E_i\}$ of M is asymptotically linearly expanding in t. However, the asymptotic behavior of the Kähler-Ricci flow on all of M is less clear.

In this paper we consider the case when the singular points of X all have isotropy group $\mathbb{Z}_2$. Equivalently, $\{E_i\}$ is a disjoint family. Then M can be reconstructed from X as follows. If x is an orbifold point of X then there is a neighborhood U_x that is analytically equivalent to $B(0,\delta)/\mathbb{Z}_2$, the $\mathbb{Z}_2$-quotient of a ball in $\mathbb{C}^2$. On the other hand, there is a morphism $q : T^*\mathbb{C}P^1 \to \mathbb{C}^2/\mathbb{Z}_2$, a resolution of the singularity at the vertex of the cone. It can be seen as the $\mathbb{Z}_2$-quotient of the blowdown $O(-1) \to \mathbb{C}^2$, where we identify the blowup of $\mathbb{C}^2$ at the origin with the $O(-1)$ line bundle on $\mathbb{C}P^1$, and identify $T^*\mathbb{C}P^1$ with $O(-2)$. We can remove U_x from X and glue in $q^{-1}(B(0,\delta)/\mathbb{Z}_2)$. Then M is the result of doing such an operation for each singular point of X.

Under the Kähler-Ricci flow, the area of each curve E_i is constant in time, since the adjunction formula implies that $\int_{E_i} c_1(K_M) = 0$. This is in contrast to the expanding behavior away from $\bigcup_i E_i$. Hence it is not so clear what the global model should be. To put it another way, there is a Ricci flat Kähler metric on $T^*\mathbb{C}P^1$, the Eguchi-Hanson metric. At spatial infinity, it is asymptotic to $\mathbb{C}^2/\mathbb{Z}_2$. One can construct a Kähler metric on M by taking a large piece of the Eguchi-Hanson metric, scaling it down and gluing it onto (X, g_{KE}) to replace the singularities. While one can do this at a given time, if one lets it evolve under the Kähler-Ricci flow then the Eguchi-Hanson region wants to remain static, while M wants to expand outside of $\bigcup_i E_i$. It isn't immediately clear how the evolution mediates between these two conflicting tendencies.

The solution to this problem comes from the fact that there is actually a two parameter family of Eguchi-Hanson metrics on $T^*\mathbb{C}P^1$. One parameter just comes from multiplicative rescaling. The other parameter comes from pulling back an Eguchi-Hanson metric by automorphisms of $T^*\mathbb{C}P^1$ that act by rescaling the cotangent fiber. In terms of the morphism $q : T^*\mathbb{C}P^1 \to \mathbb{C}^2/\mathbb{Z}_2$, these automorphisms fix the exceptional $\mathbb{C}P^1$ and push down to a rescaling on $\mathbb{C}^2/\mathbb{Z}_2$. While the pullback gives isometric metrics, they are different metrics on a fixed $T^*\mathbb{C}P^1$. Pulling back the Eguchi-Hanson metric by a rescaling $z \to \sqrt{bt}z$ on $\mathbb{C}^2/\mathbb{Z}_2$ and multiplying by b^{-1}, we obtain a 1-parameter family $g^{(0)}_{EH}(t)$ of Kähler metrics on $T^*\mathbb{C}P^1$ that are asymptotic to tg_{flat} at spatial infinity and for which the area of the exceptional $\mathbb{C}P^1$ is proportionate to b^{-1}, independent of t. In effect, we are making an artificially expanding family. This family can be glued to the expanding Kähler-Ricci flow solution tg_{KE} on X, to obtain a 0^{th}-order approximation to a Kähler-Ricci flow on M.

The Kähler potential for $g^{(0)}_{EH}(t)$ is given in (3.1). Of course, $g^{(0)}_{EH}(t)$ is not a Ricci flow solution; the time slices are Ricci flat but the solution is time dependent. One finds that the norm of the deviation from solving the Ricci flow equation decays in time, but unfortunately it does not decay fast enough. To this end, we iteratively find a sequence $\{g^{(k)}_{EH}\}_{k=1}^{\infty}$ of corrections to $g^{(0)}_{EH}(t)$ that are closer and closer to being Kähler-Ricci flow solutions. It turns out that $g^{(1)}_{EH}$ is good enough for Theorem 1; its Kähler potential is given in (3.18).

To construct the model flow $g_{mod}(t)$ for Theorem 1, we glue the approximate Kähler-Ricci flow $g^{(1)}_{EH}(t)$ on $T^*\mathbb{C}P^1$ to the exact Kähler-Ricci flow $g_X(t) = tg_{KE}$ on X. The curvature of $g^{(1)}_{EH}(t)$ is concentrated in the region $|z| \leq \text{const.}\, t^{-\frac{1}{2}}$. On the other hand, we do the gluing at a scale $|z| \sim t^{-a}$, with $a \in (0, \frac{1}{2})$, so that the gluing is done in the distant conical region of $(T^*\mathbb{C}P^1, g^{(1)}_{EH}(t))$. We perform the gluing at the level of Kähler potentials. Both $g^{(1)}_{EH}(t)$ and $g_X(t)$ are approximately conical in the gluing region but differ in lower orders, which makes the gluing delicate. It turns out that the best choice for a is $\frac{1}{4}$. To prove Theorem 1 we use a fixed point theorem as in the paper [2] of Brendle and Kapouleas. Since we are in the Kähler setting, we can reduce the Kähler-Ricci flow to an evolution equation for the Kähler potential, as is customary. This means that we are dealing with a scalar equation, which makes the analysis simpler than in [2].

To prove Theorem 2, we use the fact that as k increases, the approximate Kähler-Ricci flow solutions $g^{(k)}_{EH}(t)$ become better and better approximations, at large scale, to the evolution of a complex hyperbolic metric. We can do the gluing at a scale $|z| \sim t^{-a}$ with a arbitrarily small. Then the proof of Theorem 2 is similar to that of Theorem 1, with the freedom in the choice of a giving the improved convergence.

We mention some earlier related work. On the static side, one can form a Ricci flat metric on a K3 manifold, using the Kummer construction, as in the paper by Donaldson [6]. One glues Eguchi-Hanson metrics to the 16 singular points in a $\mathbb{Z}_2$-quotient of T^4. In the nonKähler case, Brendle and Kapouleas constructed an ancient Ricci flow on the result of performing the Kummer construction except reversing the orientation of half of the 16 Eguchi-Hanson spaces [2]. Our treatment of the analytic aspects is taken from [2].

While this paper was being written, Deruelle and Ozuch posted a preprint in which they construct ancient and immortal Ricci flow solutions in the four dimensional nonKähler case via gluing [5]. They consider oriented Riemannian orbifolds that have isolated singularities with certain isotropy groups, such as finite subgroups of $SU(2)$, and which satisfy a stability condition at the singular points. They glue in rescaled regions of Ricci flat ALE manifolds to construct an ancient or immortal Ricci flow solution, also following the analytic approach of [2], and get information about the curvature as time goes to $\pm\infty$. (We get information about the biLipschitz behavior of the metric because we start with the Kähler potential and get estimates about its second derivatives, i.e. the metric, while Deruelle and Ozuch start with the metric and get estimates about its second derivatives, i.e. the curvature.)

They give an example of an orbifold that satisfies their stability condition by reversing the orientation of a complex hyperbolic orbifold. The immortal solution obtained by gluing in the ALE spaces is nonKähler. The minimal 2-spheres in the ALE regions have areas that increase like $t^{\frac{2}{3}}$, whereas in the Kähler case the areas are constant in t.

Some further questions are:

(1) If one starts with any initial Kähler metric on M in the time-T Kähler class of Theorem 1, does the Kähler-Ricci flow approach the model flow?
(2) Are there analogous results for all initial Kähler classes on M? One would have to construct model flows using the Kähler-Ricci flow on X, rather than just uniformly expanding flows.
(3) Can one extend the methods to when the Kähler-Einstein orbifold X has isolated singularities of arbitrary isotropy group? One would glue in more general Ricci flat ALE Kähler manifolds [8], rather than just Eguchi-Hanson spaces.
(4) Can one extend the biLipschitz closeness in Theorems 1 and 2 to C^r-closeness?
(5) Can the methods be extended to elliptic fibrations? Some information about the flow is in [14].
(6) Are there analogous results in higher dimension or for finite time singularities?

The structure of the paper is as follows. Section 2 has some background information. In Section 3 we construct approximate Kähler-Ricci flows on $T^*\mathbb{C}P^1$ that are linearly expanding in time at large distances. We then construct the model flow on M in Section 4 by gluing the approximate Kähler-Ricci flow on $T^*\mathbb{C}P^1$ to the expanding Kähler-Ricci flow on the orbifold X. Theorem 1 is proved in Section 5 and Theorem 2 is proved in Section 6. More detailed descriptions appear at the beginnings of the sections.

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2. Background

In this section we review some facts about the Kähler-Ricci flow and the Eguchi-Hanson metric. We will use the Einstein summation convention freely.

2.1. **Kähler-Ricci flow and potential flow.**

Given a Kähler manifold M of complex dimension n, the Kähler form is a real $(1,1)$-form ω which can be expressed in holomorphic normal coordinates at a point p by $\omega(p) = \frac{\sqrt{-1}}{2}\sum_{i=1}^n dz^i \wedge d\overline{z}^i$. Writing $\omega = \sum_{i,j=1}^n \frac{\sqrt{-1}}{2} g_{i\overline{j}} dz^i \wedge d\overline{z}^j$ locally, the Ricci form is

$$\text{Ric} = -\sqrt{-1}\partial\overline{\partial}\log\frac{\omega^n}{\Omega} = -\sqrt{-1}\partial\overline{\partial}\log\det\left(g_{i\overline{j}}\right), \tag{2.1}$$

where $\Omega = n!(\frac{\sqrt{-1}}{2})^n dz^1 \wedge d\overline{z}^1 \wedge \ldots \wedge dz^n \wedge d\overline{z}^n$.

The Kähler-Ricci flow equation is

$$\frac{d\omega}{dt} = -\operatorname{Ric}(\omega). \tag{2.2}$$

The corresponding cohomology class satisfies $[\omega(t)] = [\omega(0)] - 2\pi t c_1(M) \in \mathrm{H}^{1,1}(M;\mathbb{R})$. In local coordinates, if we solve the potential flow equation

$$\frac{\partial u}{\partial t} = \log \det \left(\partial_i \overline{\partial}_j u \right) \tag{2.3}$$

then $\omega(t) = \sqrt{-1}\partial\overline{\partial u}$ is a solution of (2.2), provided that $\omega(t)$ is a positive $(1,1)$-form.

More globally, suppose that $\omega_{mod}(t)$ is a 1-parameter family of Kähler forms so that $[\omega_{mod}(t)] = [\omega_{mod}(0)] - 2\pi t c_1(M)$. By the $\partial\overline{\partial}$-lemma, if M is compact then we can solve

$$\frac{d\omega_{mod}}{dt} = -\operatorname{Ric}(\omega_{mod}) + \sqrt{-1}\partial\overline{\partial} f \tag{2.4}$$

for some smooth 1-parameter family of real-valued functions $f(t)$. If u is a solution to the potential flow equation

$$\frac{\partial u}{\partial t} = \log \frac{(\omega_{mod} + \sqrt{-1}\partial\overline{\partial} u)^n}{\omega_{mod}^n} - f \tag{2.5}$$

then $\omega(t) = \omega_{mod}(t) + \sqrt{-1}\partial\overline{\partial} u(t)$ is a solution to (2.2), provided that it is a positive $(1,1)$-current. Conversely, any solution of (2.2) arises in this way from a solution to (2.5).

If ω_{KE} is the Kähler form of a metric with $\operatorname{Ric}(\omega_{KE}) = -\omega_{KE}$ then there is a Kähler-Ricci flow solution $\omega(t) = t\omega_{KE}$. As a special case, the Kähler potential for the complex hyperbolic metric on $B\left(0, \sqrt{3}\right) \subset \mathbb{C}^2$, normalized so that $\operatorname{Ric}(\omega) = -\omega$, is $-3\log\left(1 - \frac{1}{3}|z|^2\right)$. A potential for the corresponding flow, solving (2.3), is $u(t) = 2(t\log t - t) - 3t\log\left(1 - \frac{1}{3}|z|^2\right)$.

2.2. **Eguchi-Hanson metric.** A reference is [12]. The Eguchi-Hanson metric is a Ricci flat Kähler metric on $T^*\mathbb{C}P^1$, i.e. on the total space of the $O(-2)$-bundle on $\mathbb{C}P^1$. The manifold is a resolution of the cone $\mathbb{C}^2/\mathbb{Z}_2$, i.e. there is an analytic map $T^*\mathbb{C}P^1 \to \mathbb{C}^2/\mathbb{Z}_2$ that is a biholomorphism from the complement of the zero section in $T^*\mathbb{C}P^1$ to the complement of the vertex $\star$ in $\mathbb{C}^2/\mathbb{Z}_2$.

Fixing the area of the exceptional $\mathbb{C}P^1$, there is actually a 1-parameter of Eguchi-Hanson metrics on $T^*\mathbb{C}P^1$ that differ by pullback under fiberwise rescalings of $T^*\mathbb{C}P^1$. Equivalently, the action is by rescalings of $\mathbb{C}^2/\mathbb{Z}_2 - \{\star\}$, and the identity on $\mathbb{C}P^1 \subset T^*\mathbb{C}P^1$. Of course, the elements of the 1-parameter family are mutually isometric, but they describe different metrics on the fixed manifold $T^*\mathbb{C}P^1$. We will normalize the Eguchi-Hanson metrics as follows. Restricting to $\mathbb{C}^2/\mathbb{Z}_2 - \{\star\}$, we can write a Kähler potential for the Eguchi-Hanson metric as a $\mathbb{Z}_2$-invariant function on $\mathbb{C}^2 - (0,0)$, with coordinates $\{z^1, z^2\}$. Putting $\rho = |z|^2$, the potentials are given by

$$\phi_{EH,c} = \sqrt{1 + c^2\rho^2} + \frac{1}{2}\log\frac{\sqrt{1+c^2\rho^2} - 1}{\sqrt{1+c^2\rho^2}+1} \tag{2.6}$$

for $c > 0$. Their derivatives are $\frac{d}{d\rho}\phi_{EH,c} = \frac{1}{\rho}\sqrt{1+c^2\rho^2}$.

Our normalization is such that as $\rho \to \infty$, the potential $\phi_{EH,c}$ is asymptotic to $c\rho = c|z|^2$. That is, the potential approaches c times that of a flat Euclidean cone.

The Eguchi-Hanson metric is

$$g^{EH}_{i\bar{j}} = \delta_{ij}\frac{\sqrt{1+c^2\rho^2}}{\rho} - \frac{z^{\bar{i}}z^j}{\rho^2\sqrt{1+c^2\rho^2}} = \frac{\sqrt{1+c^2\rho^2}}{\rho}\left(\delta_{ij} - \frac{z^{\bar{i}}z^j}{\rho}\right) + \frac{c^2\rho}{\sqrt{1+c^2\rho^2}}\frac{z^{\bar{i}}z^j}{\rho}, \tag{2.7}$$

where the latter expression gives an orthogonal decomposition of g^{EH} in terms of $(\mathbb{C}\vec{z})^\perp$ and $\mathbb{C}\vec{z}$. The inverse metric is

$$g^{\bar{j}i}_{EH} = \frac{\rho}{\sqrt{1+c^2\rho^2}}\left(\delta^{ij} - \frac{z_{\bar{i}}z_j}{\rho}\right) + \frac{\sqrt{1+c^2\rho^2}}{c^2\rho}\frac{z_{\bar{i}}z_j}{\rho} \tag{2.8}$$

3. Approximate Kähler-Ricci flow on the caps

In this section we describe potentials for an approximate solution to the potential flow on $T^*\mathbb{C}P^1$, with the property that it is linearly expanding in time at spatial infinity. There is a sequence of such approximate solutions that are closer and closer to being solutions to the potential flow. We estimate the deviation from being a solution.

We will want to glue the Eguchi-Hanson metric, on a neighborhood of $\mathbb{C}P^1 \subset T^*\mathbb{C}P^1$, to a neighborhood of a singular point in the Kähler-Einstein orbifold. We know that under the Kähler-Ricci flow on the glued manifold, the area of the $\mathbb{C}P^1$ subvariety is constant in time. On the other hand, under the Kähler-Ricci flow, the metric on the orbifold increases linearly in time. This motivates an initial approximate flow on $T^*\mathbb{C}P^1$ given by the potential

$$\begin{aligned}\phi^{(0)}_{EH}(t,z) =& 2(t\log t - t) + b^{-1}\phi_{EH,bt}\left(|z|^2\right)\\ =& 2(t\log t - t) + \frac{1}{b}\sqrt{1+b^2t^2|z|^4} + \frac{1}{2b}\log\frac{\sqrt{1+b^2t^2|z|^4}-1}{\sqrt{1+b^2t^2|z|^4}+1}\end{aligned} \tag{3.1}$$

in a deleted neighborhood of the vertex in $\mathbb{C}^2/\mathbb{Z}_2$. Here b is a positive constant that determines the area of the $\mathbb{C}P^1$ subvariety; the area is proportionate to b^{-1}.

The corresponding metric has Ricci flat time slices which are asymptotically flat, with a metric at infinity that is linearly increasing in t. The area of the zero section $\mathbb{C}P^1 \subset T^*\mathbb{C}P^1$ is constant in t. The curvature of the metric is concentrated in a region where $|z| \le \text{const.}\, t^{-\frac{1}{2}}$.

The $2(t\log t - t)$ term in (3.1) is arranged so that the potential flow equation (2.3) is satisfied to leading order. One finds that

$$\frac{\partial\phi^{(0)}_{EH}}{\partial t} - \log\det\left(\partial_i\overline{\partial}_j\phi^{(0)}_{EH}\right) = \frac{1}{bt}\sqrt{1+b^2t^2|z|^4}. \tag{3.2}$$

We will eventually want to consider (2.5) when $|z| = O\left(t^{-a}\right)$, with $a \in \left(0,\frac{1}{2}\right)$, in which case the right-hand side of (3.2) is $O\left(t^{-2a}\right)$. While this is decreasing in t, it is not decreasing fast enough and we need a better approximate solution.

To this end, we first write down what (2.3) becomes if we assume a $U(2)$-symmetry. If $u(z,t) = F(\rho,t)$ with $\rho = |z|^2$ then the Kähler form $\omega = \sqrt{-1}\partial\overline{\partial}u$ is associated to the metric

$$(g_{i\bar{j}}) = \begin{pmatrix} F_\rho + |z^1|^2 F_{\rho\rho} & z^1\overline{z}^2 F_{\rho\rho} \\ z^2\overline{z}^1 F_{\rho\rho} & F_\rho + |z^2|^2 F_{\rho\rho} \end{pmatrix} \tag{3.3}$$

and (2.3) becomes

$$F_t = \log\left(F_\rho(F_\rho + \rho F_{\rho\rho})\right). \tag{3.4}$$

Next, we do a change of variable to bring the curvature concentration region to unit scale. That is, we change variables from (t,ρ) to (s,η), where $s = t$ and $\eta = t\rho$. After renaming F to G, equation (3.4) becomes

$$G_s + \frac{1}{s}\eta G_\eta = \log\left(G_\eta(G_\eta + \eta G_{\eta\eta})\right) + 2\log s. \tag{3.5}$$

The approximate solution

$$\begin{aligned} G^{(0)}(s,\eta) =& 2(s\log s - s) + b^{-1}\phi_{EH,b}(\eta) \\ =& 2(s\log s - s) + \frac{1}{b}\sqrt{1+b^2\eta^2} + \frac{1}{2b}\log\frac{\sqrt{1+b^2\eta^2}-1}{\sqrt{1+b^2\eta^2}+1} \end{aligned} \tag{3.6}$$

satisfies

$$G^{(0)}_s = \log\left(G^{(0)}_\eta(G^{(0)}_\eta + \eta G^{(0)}_{\eta\eta})\right) + 2\log s. \tag{3.7}$$

We now write a formal solution

$$G(s,\eta) = G^{(0)}(s,\eta) + \frac{1}{s}G^{(1)}(\eta) + \frac{1}{s^2}G^{(2)}(\eta) + \ldots \tag{3.8}$$

and substitute it into (3.5) in order to find the terms $\{G^{(j)}\}_{j=1}^\infty$ iteratively by equating orders of s. One gets equations of the form

$$G^{(j)}_{\eta\eta} + \left(\frac{1}{\eta} + \frac{b^2\eta}{1+b^2\eta^2}\right)G^{(j)}_\eta = H^{(j)}, \tag{3.9}$$

where $H^{(j)}$ is a function of η constructed from $\{G^{(1)},\ldots,G^{(j-1)}\}$, which appear polynomially. The relevant solution to (3.9) is

$$G^{(j)}(\eta) = \int_0^\eta \frac{1}{\sigma\sqrt{1+b^2\sigma^2}}\int_0^\sigma \tau\sqrt{1+b^2\tau^2}\, H^{(j)}(\tau)\,d\tau\,d\sigma. \tag{3.10}$$

For example, $H^{(1)}(\eta) = 1$ and

$$G^{(1)}(\eta) = \frac{1}{3b^2}\left[\log\eta + \frac{1}{2}b^2\eta^2 - \frac{1}{2}\log\frac{\sqrt{1+b^2\eta^2}-1}{\sqrt{1+b^2\eta^2}+1}\right]. \tag{3.11}$$

Lemma 1. *$G^{(j)}(\eta) = \frac{1}{(j+1)3^j}\eta^{j+1} + O(\eta^j)$ as $\eta\to\infty$, with similar asymptotics for the derivatives of $G^{(j)}$.*

Proof. The large-η asymptotics of $G^{0)}$ are $G^{(0)}_{asymp}(\eta) = 2(s\log s - s) + \eta$. We can do a similar iteration procedure as in (3.5)-(3.10), starting with $G^{(0)}_{asymp}$, to obtain $\{G^{(j)}_{asymp}\}_{j=1}^{\infty}$. One finds that they are all polynomials in η. To say what polynomials they are, we can use the fact that an exact solution of (3.5) is given by the expanding complex hyperbolic Kähler-Ricci flow

$$G = 2(t\log t - t) - 3t\log\left(1 - \frac{1}{3}\rho\right) \tag{3.12}$$
$$= 2(s\log s - s) - 3s\log\left(1 - \frac{\eta}{3s}\right)$$
$$= 2(s\log s - s) + \sum_{j=0}^{\infty} \frac{1}{(j+1)3^j} s^{-j}\eta^{j+1}.$$

Equating terms with equal powers of s shows that $G^{(j)}_{asymp}(\eta) = \frac{1}{(j+1)3^j}\eta^{j+1}$. In view of the construction in (3.10), inductively the difference between $G^{(j)}_{asymp}$ and $G^{(j)}$ will be of the lower order $O(\eta^j)$. A similar argument works for the derivatives. □

Put

$$\widehat{G}^{(k)}(s,\eta) = G^{(0)}(s,\eta) + \sum_{j=1}^{k} s^{-j} G^{(j)}(\eta) \tag{3.13}$$

and in view of (3.5), put

$$\widehat{F}^{(k)} = \widehat{G}^{(k)}_s + \frac{1}{s}\eta\widehat{G}^{(k)}_\eta - \log\left(\widehat{G}^{(k)}_\eta(\widehat{G}^{(k)}_\eta + \eta\widehat{G}^{(k)}_{\eta\eta})\right) - 2\log s \tag{3.14}$$

Lemma 2. *Given $a \in \left(0, \frac{1}{2}\right)$, if s is large then in the interval $0 \le \frac{\eta}{s} \le s^{-2a}$, the terms $\widehat{G}^{(k)}_\eta$ and $\widehat{G}^{(k)}_\eta + \eta\widehat{G}^{(k)}_{\eta\eta}$ are positive, and there is a bound $|\widehat{F}^{(k)}| \le \mathrm{const.}\, s^{-(k+1)}\left(1+\eta^{k+1}\right)$.*

Proof. From (3.10), $G^{(j)}$ is regular at $\eta = 0$ if $j \ge 1$. Substituting (3.6) into (3.13) gives

$$\widehat{G}^{(k)}_\eta = \frac{1}{b\eta}\sqrt{1+b^2\eta^2} + \sum_{j=1}^{k} s^{-j} G^{(j)}_\eta \tag{3.15}$$

and

$$\widehat{G}^{(k)}_\eta + \eta\widehat{G}^{(k)}_{\eta\eta} = \frac{b\eta}{\sqrt{1+b^2\eta^2}} + \sum_{j=1}^{k} s^{-j}(G^{(j)}_\eta + \eta G^{(j)}_{\eta\eta}) \tag{3.16}$$

From Lemma 1, both $s^{-j}G^{(j)}_\eta$ and $s^{-j}(G^{(j)}_\eta + \eta G^{(j)}_{\eta\eta})$ are bounded in magnitude by const. $s^{-j}(1+\eta^j)$, from which the positivity claim follows.

Next,

$$
\begin{aligned}
\widehat{F}^{(k)} = & \, 2\log s - \sum_{j=1}^{k} j s^{-j-1} G^{(j)}(\eta) + \frac{1}{bs}\sqrt{1+b^2\eta^2} + \sum_{j=1}^{k} s^{-j-1}\eta G^{(j)}_{\eta} - \\
& \log\left[\left(\frac{1}{b\eta}\sqrt{1+b^2\eta^2} + \sum_{j=1}^{k} s^{-j} G^{(j)}_{\eta}\right)\right. \\
& \left.\left(\frac{b\eta}{\sqrt{1+b^2\eta^2}} + \sum_{j=1}^{k} s^{-j}(G^{(j)}_{\eta} + \eta G^{(j)}_{\eta\eta})\right)\right] - 2\log s \\
= & \frac{1}{bs}\sqrt{1+b^2\eta^2} + \sum_{j=1}^{k} s^{-j-1}(\eta G^{(j)}_{\eta} - jG^{(j)}(\eta)) - \log\left(1 + \sum_{j=1}^{k} s^{-j}\frac{b\eta}{\sqrt{1+b^2\eta^2}} G^{(j)}_{\eta}\right) - \\
& \log\left(1 + \sum_{j=1}^{k} s^{-j}\frac{\sqrt{1+b^2\eta^2}}{b\eta}(G^{(j)}_{\eta} + \eta G^{(j)}_{\eta\eta})\right).
\end{aligned} \tag{3.17}
$$

Looking at the terms in (3.17), each factor of s^{-l} has a coefficient that is a function of η, whose leading asymptotic as $\eta \to \infty$ is const. η^l. If we put $u = \frac{1}{s}$ and $v = \frac{\eta}{s}$ then we can write $\widehat{F}^{(k)}(s,\eta) = K^{(k)}(u,v)$ where $K^{(k)}$ is smooth near $(u,v) = (0,0)$. For any $\alpha > 0$, the line $v = \alpha u$ corresponds to $\eta = \alpha$. By the construction of $\widehat{G}^{(k)}$, as $u \to 0$, $K^{(k)}(u,\alpha u)$ is $O\left(u^{k+1}\right)$. Considering the Taylor series expansion of $K^{(k)}$ around $(0,0)$, it follows that $K^{(k)}(u,v) \le \text{const.}\left(u^{k+1} + u^k v + \ldots + uv^k + v^{k+1}\right) \le \text{const.}\left(u^{k+1} + v^{k+1}\right)$ in a neighborhood of $(0,0)$, from which the lemma follows. □

We now scale back and put $\phi^{(k)}_{EH}(t,z) = \widehat{G}^{(k)}(t, t|z|^2)$. Explicitly, if $k = 1$ then

$$
\begin{aligned}
\phi^{(1)}_{EH}(t,z) = & 2(t\log t - t) + \frac{1}{b}\sqrt{1+b^2t^2|z|^4} + \frac{1}{2b}\log\frac{\sqrt{1+b^2t^2|z|^4}-1}{\sqrt{1+b^2t^2|z|^4}+1} + \\
& \frac{1}{3b^2t}\left[\log(t|z|^2) + \frac{1}{2}b^2t^2|z|^4 - \frac{1}{2}\log\frac{\sqrt{1+b^2t^2|z|^4}-1}{\sqrt{1+b^2t^2|z|^4}+1}\right].
\end{aligned} \tag{3.18}
$$

Lemma 2 and (3.3) imply that $\phi^{(k)}_{EH}(t,\cdot)$ is strictly plurisubharmonic for large t in the region $|z| \le t^{-a}$. Defining $\omega^{(k)}_{EH} = \sqrt{-1}\partial\overline{\partial}\phi^{(k)}_{EH}$ and

$$
f^{(k)}_{EH}(t,z) = \frac{\partial\phi^{(k)}_{EH}}{\partial t} - \log\det\left(\partial_i\overline{\partial}_j\phi^{(k)}_{EH}\right), \tag{3.19}
$$

we have

$$
\sqrt{-1}\partial\overline{\partial} f^{(k)}_{EH} = \frac{d\omega^{(k)}_{EH}}{dt} + \mathrm{Ric}(\omega^{(k)}_{EH}). \tag{3.20}
$$

Lemma 3. *Given $a \in \left(0, \frac{1}{2}\right)$, if t is large then in the region $|z| \leq t^{-a}$ the magnitudes of $f^{(k)}_{EH}$ and $t\partial_t f^{(k)}_{EH}$ are bounded above by* const. $\left(|z| + t^{-\frac{1}{2}}\right)^{2(k+1)}$*, and the magnitude of $\nabla f^{(k)}_{EH}$ is bounded above by* const. $t^{-\frac{1}{2}} \left(|z| + t^{-\frac{1}{2}}\right)^{2k+1}$.

Proof. The bound on $f^{(k)}_{EH}$ follows from Lemma 2. In this region, $\omega^{(k)}_{EH}$ is uniformly biLipschitz equivalent to $\omega^{(0)}_{EH}$, so to bound $|\nabla f^{(k)}_{EH}|$ it is enough to estimate the sum $g^{(0),\bar{j}i}_{EH} \partial_{z^i} f^{(k)}_{EH} \partial_{\bar{z}^j} f^{(k)}_{EH}$. From the chain rule,

$$\frac{\partial f^{(k)}_{EH}}{\partial \bar{z}^j} = \frac{\partial \eta}{\partial \bar{z}^j} \frac{\partial \widehat{F}^{(k)}}{\partial \eta} = t z_j \frac{\partial \widehat{F}^{(k)}}{\partial \eta} = s z_j \frac{\partial \widehat{F}^{(k)}}{\partial \eta}. \tag{3.21}$$

From (2.8), we have

$$g^{(0),\bar{j}i}_{EH} z_{\bar{j}} z_i = \frac{\sqrt{1 + b^2\eta^2}}{bs^2}, \tag{3.22}$$

so

$$|\nabla f^{(k)}_{EH}|^2 = \frac{\sqrt{1 + b^2\eta^2}}{b} \left(\frac{\partial \widehat{F}^{(k)}}{\partial \eta}\right)^2. \tag{3.23}$$

By the argument in the proof of Lemma 2, in the given region we have $|\frac{\partial \widehat{F}^{(k)}}{\partial \eta}| \leq \text{const.}\, s^{-(k+1)}(1+\eta^k)$, from which the bound on $|\nabla f^{(k)}_{EH}|$ follows.

Next, by the chain rule,

$$\frac{\partial f^{(k)}_{EH}}{\partial t} = \frac{\partial \eta}{\partial t} \frac{\partial \widehat{F}^{(k)}}{\partial \eta} + \frac{\partial s}{\partial t} \frac{\partial \widehat{F}^{(k)}}{\partial s} = |z|^2 \frac{\partial \widehat{F}^{(k)}}{\partial \eta} + \frac{\partial \widehat{F}^{(k)}}{\partial s} = s^{-1}\eta \frac{\partial \widehat{F}^{(k)}}{\partial \eta} + \frac{\partial \widehat{F}^{(k)}}{\partial s}. \tag{3.24}$$

Estimating $|\frac{\partial \widehat{F}^{(k)}}{\partial s}|$ similarly and applying the previous estimate on $|\frac{\partial \widehat{F}^{(k)}}{\partial \eta}|$ gives the bound on $t|\frac{\partial f^{(k)}_{EH}}{\partial t}|$. □

4. Model flow

In this section we construct the model Kähler potential $\phi_{mod}(t)$ on a manifold M as in the statement of Theorem 1. The potential is obtained by gluing the potential $\phi^{(1)}_{EH}(t)$ from Section 3 to a Kähler potential $\phi_X(t)$ for the expanding flow on the orbifold X, in a neighborhood of each singular point of X. The only ambiguities in the construction of $\phi_{mod}(t)$ are the choice of a bump function σ and the number $a \in \left(0, \frac{1}{2}\right)$ that determines the scale $|z| \sim t^{-a}$ at which we do the gluing. To get the best estimates for Theorem 1, we will take a to be $\frac{1}{4}$.

We will estimate the deviation of $\phi_{mod}(t)$ from satisfying the potential flow equation. The individual terms $\phi^{(1)}_{EH}(t)$ and $\phi_X(t)$ satisfy good estimates for the deviation. To leading order, $\phi^{(1)}_{EH}(t)$ and $\phi_X(t)$ agree on the gluing region. However, the discrepancy between them introduces some errors to the result of gluing, in terms of satisfying the potential flow equation, that need to be controlled. There are two main sources of error. One source is a lower order term in $\phi^{(1)}_{EH}(t)$ that doesn't appear in $\phi_X(t)$. The other source is the difference between the order-t terms in $\phi^{(1)}_{EH}(t)$ and $\phi_X(t)$. Fortunately, both of these error terms are harmonic to leading order.

We now start the construction. Let X be a compact Kähler orbifold of complex dimension that admits a Kähler-Einstein orbifold metric ω_{KE} satisfying $\mathrm{Ric}\,(\omega_{KE}) = -\omega_{KE}$. We assume that X has isolated singular points with isotropy group $\mathbb{Z}_2$.

Example 1. [18, Section 10] Let Z_1 and Z_2 be hyperelliptic Riemann surfaces of genus at least two, with involutions i_1 and i_2, respectively. Let ω_{Z_i} be a constant curvature metric on Z_i, normalized so that $\mathrm{Ric}\,(\omega_{Z_i}) = -\omega_{Z_i}$. Let $p_i : Z_1 \times Z_2 \to Z_i$ be the projection operator. Then $\omega_{Z_1 \times Z_2} = p_1^* \omega_{Z_1} + p_2^* \omega_{Z_2}$ is a Kähler-Einstein metric on $Z_1 \times Z_2$, with $\mathrm{Ric}\,(\omega_{Z_1\times Z_2}) = -\omega_{Z_1 \times Z_2}$. Put $X = (Z_1 \times Z_2)/\mathbb{Z}_2$, the quotient by the diagonal $\mathbb{Z}_2$-action. Then X is a Kähler-Einstein orbifold with $\mathrm{Ric}\,(\omega_{KE}) = -\omega_{KE}$, having isolated singularities with $\mathbb{Z}_2$-isotropy groups.

There is an orbifold Kähler-Ricci flow ω_X on X with $\omega_X(t) = t\omega_{KE}$.

Given a singular point $x \in X$, some neighborhood U_x of x is analytically equivalent to $B(0,\delta)/\mathbb{Z}_2$, for some ball $B(0,\delta) \subset \mathbb{C}^2$. Letting $p : B(0,\delta) \to U_x$ be the quotient map, $p^*\omega_{KE}$ is a smooth Kähler-Einstein metric on $B(0,\delta)$. Let ϕ_{KE} be a potential for $p^*\omega_{KE}$, i.e. $p^*\omega_{KE} = \sqrt{-1}\partial\overline{\partial}\phi_{KE}$, satisfying the Kähler-Einstein equation $\log \frac{(p^*\omega_{KE})^n}{\Omega} = \phi_{KE}$. We can choose complex coordinates so that $\phi_{KE}(z) = |z|^2 + C_{a\bar{b}c\bar{d}} z^a \overline{z}^b z^c \overline{z}^d + O(|z|^6)$, and the $\mathbb{Z}_2$-action is $z \to -z$ [19, Proposition 1.6]. (A priori there is an $O(|z|^5)$ term, but this is excluded by the $\mathbb{Z}_2$ symmetry.) Here $C_{a\bar{b}c\bar{d}}$ is proportional to the curvature tensor at 0 and has the symmetries $C_{a\bar{b}c\bar{d}} = C_{c\bar{b}a\bar{d}} = C_{a\bar{d}c\bar{b}} = C_{c\bar{d}a\bar{b}}$. The Kähler-Einstein condition implies that $\sum_{a=1}^2 C_{a\bar{a}c\bar{d}} = \frac{1}{4}\delta_{cd}$.

We will implicitly use the same notation when the Kähler potential is descended from $B(0,\delta) \subset \mathbb{C}^2$ to U_x. A potential for the orbifold Kähler-Ricci flow on $B(0,\delta)$ is $\phi_X(t) = 2(t\log t - t) + t\phi_{KE}$.

For simplicity, we assume hereafter that X only has one singular point x. It is straightforward to extend to the case of more singular points. Let $\sigma : [0,\infty) \to [0,1]$ be a smooth nonincreasing function so that $\sigma\Big|_{[0,\frac{1}{2}]} = 1$ and $\sigma\Big|_{[1,\infty)} = 0$. Given $a \in (0,\frac{1}{2})$ and $|z| < \delta$, define the model potential by

$$\phi_{mod}(t,z) = \sigma\left(t^a|z|\right)\phi^{(1)}_{EH}(t,z) + \left(1 - \sigma\left(t^a|z|\right)\right)\phi_X(t,z). \tag{4.1}$$

Put

$$\omega_{mod}(t,z) = \sqrt{-1}\partial\overline{\partial}\phi_{mod}(t,z) \text{ if } |z| < \delta. \tag{4.2}$$

We extend it to the rest of X as $\omega_X(t)$. We obtain a model flow on M, the complex manifold that is the result of gluing a truncated copy of $T^\star\mathbb{C}P^1$ at the orbifold point of X.

Putting

$$f_{mod}(t,z) = \frac{\partial \phi_{mod}}{\partial t} - \log\det\left(\partial_i\overline{\partial}_j\phi_{mod}\right) \text{ if } |z| < \delta \tag{4.3}$$

and extending it by zero to M, we have

$$\sqrt{-1}\partial\overline{\partial} f_{mod} = \frac{d\omega_{mod}}{dt} + \mathrm{Ric}(\omega_{mod}). \tag{4.4}$$

In the rest of this section, we take $a = \frac{1}{4}$. For brevity, if a point is in $M - U_x$ then we set $|z|$ to be δ at that point.

Lemma 4. *Given $\alpha \in (0,1)$, for large t we have uniform bounds*

(1) $t^{3/2}\left(|z| + t^{-\frac{1}{2}}\right)^2 |f_{mod}(t,z)| \leq \text{const.}$, *and*

(2) $t^{3/2}\left(r + t^{-\frac{1}{2}}\right)^2 \left(1 + t^{\frac{1}{2}} r\right)^{2\alpha} \frac{|f_{mod}(m,t) - f_{mod}(m',t')|}{\left(d_t^2(m,m') + |t-t'|\right)^{\alpha}} \leq \text{const.}$

whenever (m',t') satisfies $|z|, |z'| \in \left[\frac{1}{2}r, r + t^{-\frac{1}{2}}\right]$ and $t \leq t' \leq t + \left(1 + t^{\frac{1}{2}} r\right)^2$ for some $r \in (0, 2\delta]$.

Proof. When $|z| \leq \frac{1}{2}t^{-a}$, the claim of part (1) follows from Lemma 3. Since $f_{mod}(t,z)$ vanishes when $|z| \geq t^{-a}$, the only region to check for part (1) is when $\frac{1}{2}t^{-a} < |z| < t^{-a}$.

In this region, we write

$$\phi_{mod}(t,z) = \phi^{(1)}_{EH}(t,z) + \left(1 - \sigma\left(t^a|z|\right)\right)\left(\phi_X(t,z) - \phi^{(1)}_{EH}(t,z)\right). \tag{4.5}$$

Then

$$\begin{aligned} f_{mod} =& f^{(1)}_{EH} + \frac{\partial}{\partial t}\left((1-\sigma)\left(\phi_X(t,z) - \phi^{(1)}_{EH}(t,z)\right)\right) - \\ & \mathrm{Tr}\log\left(I + \left(\omega^{(1)}_{EH}\right)^{-1}\sqrt{-1}\partial\overline{\partial}\left((1-\sigma)\left(\phi_X(t,z) - \phi^{(1)}_{EH}(t,z)\right)\right)\right), \end{aligned} \tag{4.6}$$

where the argument of σ is $t^a|z|$.

In this region, $\omega^{(1)}_{EH}$ is asymptotically $\frac{t}{2}\sqrt{-1}dz^i \wedge d\overline{z}^i$. To leading order, $G^{(0)}$ is asymptotic to

$$2(s\log s - s) + \eta - \frac{1}{2b^2\eta} = 2(t\log t - t) + t|z|^2 - \frac{1}{2b^2 t|z|^2}. \tag{4.7}$$

Also, $G^{(1)}$ is asymptotic to

$$\frac{1}{3b^2}\left(\frac{1}{2}b^2\eta^2 + \log\eta\right) = \frac{1}{3b^2}\left(\frac{1}{2}b^2t^2|z|^4 + \log(t|z|^2)\right). \tag{4.8}$$

Along with (3.13), it follows that $\phi^{(1)}_{EH}$ is asymptotic to

$$2(t\log t - t) + t|z|^2 - \frac{1}{2b^2t|z|^2} + \frac{1}{6}t|z|^4 + \frac{1}{3b^2t}\log(t|z|^2). \tag{4.9}$$

Hence $\phi_X - \phi^{(1)}_{EH}$ is asymptotic to

$$tC_{a\bar{b}c\bar{d}}z^a\overline{z}^b z^c\overline{z}^d - \frac{1}{6}t|z|^4 + \frac{1}{2b^2t|z|^2} - \frac{1}{3b^2t}\log(t|z|^2). \tag{4.10}$$

Since $a = \frac{1}{4}$, one can check from (4.6) that $|f_{mod}(t,z)|$ is $O\left(t^{-1}\right)$ in this region. Here we use the fact that $\partial_i \overline{\partial}_i \left(C_{a\bar{b}c\bar{d}} z^a \overline{z}^b z^c \overline{z}^d - \frac{1}{6}|z|^4 \right) = 0$, since ω_{KE} is a Kähler-Einstein metric. We also use the fact that $\partial_i \overline{\partial}_i \frac{1}{|z|^2} = 0$.

Hence we have the bound $t^{3/2} \left(|z| + t^{-\frac{1}{2}} \right)^2 |f_{mod}(t,z)| \leq$ const. in this region.

A similar argument applies to part (2), where we use the inequality

$$\begin{aligned} (4.11) \qquad & \frac{|f_{mod}(m,t) - f_{mod}(m',t')|}{(d_t^2(m,m') + |t-t'|)^{\alpha}} = \\ & |f_{mod}(m,t) - f_{mod}(m',t')|^{1-2\alpha} \left(\frac{|f_{mod}(m,t) - f_{mod}(m',t')|}{\sqrt{d_t^2(m,m') + |t-t'|}} \right)^{2\alpha} \leq \\ & (2 \max(|f_{mod}(m,t)|, |f_{mod}(m',t')|))^{1-2\alpha} \left(\frac{|f_{mod}(m,t) - f_{mod}(m',t')|}{\sqrt{d_t^2(m,m') + |t-t'|}} \right)^{2\alpha}, \end{aligned}$$

along with the derivative estimates in Lemma 3. □

5. Proof of Theorem 1

In this section we use the analytic setup of [2] to prove Theorem 1. We use a fixed point theorem to show the existence of a Kähler-Ricci flow that is $K(t)$-biLipschitz close to the model flow, where $K(t) = 1 + O\left(t^{-\frac{1}{2}+\epsilon}\right)$ as $t \to \infty$. Since many of the details are as in [2], we just outline the main steps. We then use the parabolic Schauder estimate to improve this to $K(t) = 1 + O\left(t^{-1+\epsilon}\right)$.

We first define certain weighted Hölder norms, where the weighting is crucial for the proof.

Definition 1. *Given $\alpha \in (0,1)$ and $\gamma, \sigma, \Lambda > 0$, let $\parallel u \parallel_{X^{0,\alpha}_{\gamma,\sigma,\Lambda}}$ be the supremum of*

$$(5.1) \qquad t^{\gamma} \left(r + t^{-\frac{1}{2}} \right)^{\sigma} \left[|u(m,t)| + \left(1 + t^{\frac{1}{2}} r \right)^{2\alpha} \frac{|u(m,t) - u(m',t')|}{(d_t^2(m,m') + |t-t'|)^{\alpha}}, \right]$$

where,

- $r \in (0, 2\delta]$,
- $\Lambda \leq t \leq t' \leq t + \left(1 + t^{\frac{1}{2}} r \right)^2$ *and*
- $|z|, |z'| \in \left[\frac{1}{2} r, r + t^{-\frac{1}{2}} \right]$.

Here $|z|$ denotes the magnitude of the z-coordinate of m if $m \in U_x$ and is δ otherwise, and similarly for z'. The distance $d_t(m,m')$ is measured with $g_{mod}(t)$. Put

$$(5.2) \qquad \parallel u \parallel_{X^{1,\alpha}_{\gamma,\sigma,\Lambda}} = \parallel u \parallel_{X^{0,\alpha}_{\gamma,\sigma,\Lambda}} + \parallel \nabla u \parallel_{X^{0,\alpha}_{\gamma,\sigma+1,\Lambda}} + \parallel \mathrm{Hess}\, u \parallel_{X^{0,\alpha}_{\gamma,\sigma+2,\Lambda}} + \parallel \partial_t u \parallel_{X^{0,\alpha}_{\gamma,\sigma+2,\Lambda}},$$

where pointwise norms are taken with respect to $g_{mod}(t)$ and we use time-t parallel transport along minimizing geodesics to define tensor differences for the Hölder norm.

The conditions $|z|, |z'| \in \left[\frac{1}{2}r, r + t^{-\frac{1}{2}}\right]$ and $t \leq t' \leq t + \left(1 + t^{\frac{1}{2}} r\right)^2$ mean that (m', t') lies in a quasiparabolic neighborhood of (m, t). The term $1 + t^{\frac{1}{2}} r$ appears because $1 + t^{\frac{1}{2}}|z|$ is comparable to the time-t injectivity radius at m for g_{mod}; see the proof of Lemma 5.

Let $X^{0,\alpha}_{\gamma,\sigma,\Lambda}$ and $X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ be the corresponding Banach spaces, where we impose the initial condition $u(\Lambda) = 0$ on elements $u \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$.

Using Lemma 4, for any $\epsilon > 0$ we can find $\alpha, \sigma > 0$ small so that putting $\gamma = \frac{3}{2} - \epsilon$, we have that $\|f_{mod}\|_{X^{0,\alpha}_{\gamma+\alpha,\sigma+2,\Lambda}}$ is $o(1)$ as $\Lambda \to \infty$. Here to bump σ up to be positive, we use the fact that the support of f_{mod} is in $\{(z,t) : |z| \leq t^{-a}\}$.

We now rewrite (2.5) as

$$\frac{\partial u}{\partial t} - \triangle_{g_{mod}} u = Q(u) - f_{mod}, \tag{5.3}$$

where

$$\begin{aligned} Q(u) &= \log \frac{(\omega_{mod} + \sqrt{-1}\partial\overline{\partial}u)^n}{\omega^n_{mod}} - \triangle_{g_{mod}} u \\ &= \operatorname{Tr} \log \left(I + \omega^{-1}_{mod}\sqrt{-1}\partial\overline{\partial}u\right) - \operatorname{Tr}\left(\omega^{-1}_{mod}\sqrt{-1}\partial\overline{\partial}u\right). \end{aligned} \tag{5.4}$$

Algebraically, there is some $C < 0$ so that $Q(u) \leq C|\omega^{-1}_{mod}\sqrt{-1}\partial\overline{\partial}u|^2$. If ϵ, α and σ are sufficiently small then whenever $\| u \|_{X^{1,\alpha}_{\gamma,\sigma,\Lambda}} \leq 1$, there is a uniform bound showing that $\| Q(u) \|_{X^{0,\alpha}_{\gamma+\alpha,\sigma+2,\Lambda}}$ is $o(1)$ as $\Lambda \to \infty$.

We now set up a fixed point problem, taking spatially constant functions into account, as follows. Given $v \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$, we form $Q(v) - f_{mod} \in X^{0,\alpha}_{\gamma+\alpha,\sigma+2,\Lambda}$ and solve

$$\frac{\partial u}{\partial t} - \triangle_{g_{mod}} u = Q(v) - f_{mod} + \phi \tag{5.5}$$

for $u \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ with $\int_M u(t)\, \mathrm{dvol}_{g_{mod}(t)} = 0$, where ϕ is a function just of t. As in [2], when adapted to the case of immortal flows, there is a unique solution and an estimate $\|u\|_{X^{1,\alpha}_{\gamma,\sigma,\Lambda}} \leq \text{const.}\, \|Q(v) - f_{mod}\|_{X^{0,\alpha}_{\gamma+\alpha,\sigma+2,\Lambda}}$.

(To compare with [2], it is convenient to rewrite (5.5) in terms of the normalized model flow $\widehat{g}_{mod}(t) = t^{-1} g_{mod}(t)$, with bounded diameter. Putting $\widehat{u}(t) = t^{-1}u(t)$, $\widehat{v}(t) = t^{-1}v(t)$ and $\widehat{t} = \log t$, equation (5.5) is equivalent to

$$\frac{\partial \widehat{u}}{\partial \widehat{t}} - \triangle_{\widehat{g}_{mod}} \widehat{u} + \widehat{u} = \widehat{Q}(\widehat{v}) - f_{mod} + \phi. \tag{5.6}$$

The only constraints to worry about come from constant functions.)

As in [2], the map from v to u gives a continuous map τ from the Banach space $X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ to itself. The unit ball in $X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ is a convex compact subset of the Banach space $X^{1,\widetilde{\alpha}}_{\widetilde{\gamma},\sigma,\Lambda}$, where $\widetilde{\alpha}$ is slightly less than α and $\widetilde{\gamma}$ is slightly less than γ. The map τ extends to a similarly defined map from $X^{1,\widetilde{\alpha}}_{\widetilde{\gamma},\sigma,\Lambda}$ to itself. For large Λ, the above estimates show that the unit ball in $X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ is sent by τ to itself. The Schauder fixed point theorem implies that there is some

$u \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$ so that $\tau(u) = u$. Applying $\partial\overline{\partial}$ to (5.5) with $v = u$ shows that $\omega_{mod} + \sqrt{-1}\partial\overline{\partial u}$ is a Kähler-Ricci flow solution on M that exists for all time greater than or equal to Λ. It equals $\omega_{mod}(T)$ at time Λ, as $u(\Lambda) = 0$.

Since $u \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$, we have

$$|\operatorname{Hess} u| \le \frac{\text{const.}}{t^{\gamma}(|z| + t^{-\frac{1}{2}})^{\sigma+2}} \le \frac{\text{const.}}{t^{\gamma - \frac{\sigma}{2} - 1}}, \tag{5.7}$$

which is almost $O\left(t^{-\frac{1}{2}}\right)$.

We can improve the convergence rate by using the fact that

$$|u| \le \frac{\text{const.}}{t^{\gamma}(|z| + t^{-\frac{1}{2}})^{\sigma}} \le \frac{\text{const.}}{t^{\gamma - \frac{\sigma}{2}}} \le \text{const.}\, t^{-1}. \tag{5.8}$$

Denote the norm in $X^{0,\alpha}_{0,0,\Lambda}$ by the C^{α}-norm. From the estimates in Lemma 3 and the proof of Lemma 4, when $|z| \le \frac{1}{2}t^{-a}$ we have $\|f_{mod}(t)\|_{C^{\alpha}} \le \text{const.} \max_{|z| \le \frac{1}{2}t^{-a}} \left(|z| + t^{-\frac{1}{2}}\right)^4 \le$ const. t^{-1} for large t, while if $\frac{1}{2}t^{-a} \le |z| \le t^{-a}$ then $\|f_{mod}(t)\|_{C^{\alpha}} \le \text{const.}\, t^{4a-2} = \text{const.}\, t^{-1}$. Hence $\|f_{mod}(t)\|_{C^{\alpha}}$ is $O\left(t^{-1}\right)$ on M. Also, since $u \in X^{1,\alpha}_{\gamma,\sigma,\Lambda}$, for any $\epsilon' > 0$ we can choose ϵ, α and σ so that $\|Q(u)(t)\|_{C^{\alpha}}$ is $O\left(t^{-1+\epsilon'}\right)$ on M. From (5.5), we have

$$0 = \frac{d}{dt}\int_M u(t)\, \text{dvol}_{g_{mod}(t)} = \int_M (Q(u) - f_{mod})\, \text{dvol}_{g_{mod}(t)} + \phi(t)\, \text{vol}_{g_{mod}(t)} + \int_M u(t) \frac{d}{dt}\, \text{dvol}_{g_{mod}(t)}. \tag{5.9}$$

Since $|\frac{d}{dt}\text{dvol}_{g_{mod}(t)}| \le \text{const.}\, \text{dvol}_{g_{mod}(t)}$, we conclude that $\phi(t)$ is $O\left(t^{-1+\epsilon'}\right)$. Using Lemma 5 below, we can apply the parabolic Schauder lemma [9, Theorem 4.9] to (5.5). It gives a local result of the form $\|u\|_{C^{2,\alpha}} \le \text{const.}\left(\|u\|_{C^0} + \|Q(u) - f_{mod} + \phi\|_{C^{\alpha}}\right)$, where local is in the sense of Lemma 5. Hence $\|u\|_{C^{2,\alpha}}$ is $O\left(t^{-1+\epsilon'}\right)$, so $\omega_{mod} + \sqrt{-1}\partial\overline{\partial u}$ is K-biLipschitz to ω_{mod} where $K = 1 + O(t^{-1+\epsilon'})$.

Lemma 5. *There are $\rho > 0$ and $C < \infty$ so that for all $(m,t) \in M \times [T,\infty)$, the region $B(m,\rho) \times [t, t+\rho^2]$, endowed with the Riemannian metric $g_{mod}(t) + dt^2$, is C-close in the C^3-topology to a Euclidean product region.*

Proof. The statement is clearly true for $|z| \ge \frac{1}{2}t^{-a}$. For $|z| \le \frac{1}{2}t^{-a}$, using (3.13) we will have uniform C^3-closeness between $g^{(1)}_{mod}(t) + dt^2$ and $g^{(0)}_{mod}(t) + dt^2$. Hence it is enough to verify the claim for $g^{(0)}_{mod}(t) + dt^2$.

Let α_{τ} be the automorphism of $T^*\mathbb{C}P^1$ which fixes the exceptional $\mathbb{C}P^1$ and acts on $\mathbb{C}^2/\mathbb{Z}_2$ by sending z to τz. Then $g^{(0)}_{mod}(t)$ is isometric to $b^{-1}\alpha^*_{\sqrt{bt}} g_{EH}$. From the bounded geometry of g_{EH}, and its conical structure at infinity, we can choose $\rho > 0$ and $C' < \infty$ so that for any $m \in T^*\mathbb{C}P^1$, the product region $B(m,\rho) \times [0,\rho^2]$ with the Riemannian metric $b^{-1}g_{EH} + dt^2$ is C'-close in the C^3-topology to a Euclidean product region. Given $t' \in [T,\infty)$, the same will be true for $B(m,\rho) \times [t', t'+\rho^2]$, endowed with the Riemannian metric $b^{-1}\alpha^*_{\sqrt{bt'}} g_{EH} + dt^2$. Since $b^{-1}\alpha^*_{\sqrt{bt'}} g_{EH} = b^{-1}\alpha^*_{\sqrt{\frac{t}{t'}}}\alpha^*_{\sqrt{bt'}} g_{EH}$ and $\frac{t}{t'} \in \left[1, 1+\frac{\rho^2}{t'}\right]$, there is some $C < \infty$ independent of t' so that the region $B(m,\rho) \times [t', t'+\rho^2]$, with the metric $g^{(0)}_{mod}(t) + dt^2$, is C-close in the C^3-topology to a Euclidean product region. □

6. Proof of Theorem 2

In this section we look at the case when the Kähler-Einstein orbifold X is complex hyperbolic. By doing the gluing at a scale $|z| \sim t^{-a}$ with a close to zero, we show that we can improve the biLipschitz closeness to $K(t) = 1 + O\left(t^{-2+\epsilon}\right)$. We then prove a stability result saying that for the potential flow (2.5), the C^0-norm of the time-t potential remains close to the C^0-norm of an initial time-T potential, if T is sufficiently large.

Suppose that ω_{KE} is a complex hyperbolic orbifold metric, again with isolated singularities and $\mathbb{Z}_2$ isotropy groups. Such orbifolds exist, as explained to me by Alan Reid. Given $k \geq 0$, $a \in \left(0, \frac{1}{2}\right)$ and $|z| < \delta$, for large t define

$$\phi^{(k)}_{mod}(t,z) = \sigma\left(t^a|z|\right)\phi^{(k)}_{EH}(t,z) + \left(1 - \sigma\left(t^a|z|\right)\right)\phi_X(t,z) \tag{6.1}$$

and

$$\omega^{(k)}_{mod}(t,z) = \sqrt{-1}\partial\overline{\partial}\phi^{(k)}_{mod}(t,z). \tag{6.2}$$

We extend it to the rest of X as $\omega_X(t)$. We obtain a model flow on M, the complex manifold that is the result of gluing a truncated copy of $T^\star\mathbb{C}P^1$ to the orbifold point of X.

Putting

$$f^{(k)}_{mod}(t,z) = \frac{\partial\phi^{(k)}_{mod}}{\partial t} - \log\det\left(\partial_i\overline{\partial}_j\phi^{(k)}_{mod}\right) \text{ if } |z| < t^{-a} \tag{6.3}$$

and extending it by zero to M, we have

$$\sqrt{-1}\partial\overline{\partial}f^{(k)}_{mod} = \frac{d\omega^{(k)}_{mod}}{dt} + \mathrm{Ric}(\omega^{(k)}_{mod}). \tag{6.4}$$

Lemma 6. *For any $\epsilon > 0$ and $\alpha \in (0,1)$, we can choose k and a so that for large t, there are uniform bounds*

(1) $t^{2-\epsilon}\left(|z| + t^{-\frac{1}{2}}\right)^2 |f^{(k)}_{mod}(t,z)| \leq \text{const.}$, *and*

(2) $t^{2-\epsilon}\left(|z| + t^{-\frac{1}{2}}\right)^2 \left(1 + t^{\frac{1}{2}}|z|\right)^{2\alpha} \frac{|f^{(k)}_{mod}(m,t) - f^{(k)}_{mod}(m',t')|}{\left(d^2_t(m,m') + |t-t'|\right)^\alpha} \leq \text{const.}$

whenever (m',t') satisfies $|z|, |z'| \in \left[\frac{1}{2}r, r + t^{-\frac{1}{2}}\right]$ and $t \leq t' \leq t + \left(1 + t^{\frac{1}{2}}r\right)^2$ for some $r \in (0, 2\delta]$.

Proof. For part (1), it is enough to look at the region $|z| < t^{-a}$, as $f^{(k)}_{mod}$ vanishes outside of the region. The region $|z| \leq \frac{1}{2}t^{-a}$ is covered by Lemma 3, so we can assume that $\frac{1}{2}t^{-a} < |z| < t^{-a}$. In this region, we can write

$$\phi^{(k)}_{mod}(t,z) = \phi^{(k)}_{EH}(t,z) + \left(1 - \sigma\left(t^a|z|\right)\right)\left(\phi_X(t,z) - \phi^{(k)}_{EH}(t,z)\right). \tag{6.5}$$

Then

$$f^{(k)}_{mod} = f^{(k)}_{EH} + \frac{\partial}{\partial t}\left((1-\sigma)(\phi_X - \phi^{(k)}_{EH})\right) - \mathrm{Tr}\log\left(I + \left(\omega^{(k)}_{EH}\right)^{-1}\sqrt{-1}\partial\overline{\partial}\left((1-\sigma)(\phi_X - \phi^{(k)}_{EH})\right)\right), \tag{6.6}$$

where the argument of σ is $t^a|z|$. Lemma 3 gives the bounds for $f^{(k)}_{EH}$.

Since ω_{KE} is complex hyperbolic, we have

$$\phi_X = 2(t\log t - t) - 3t\log\left(1 - \frac{1}{3}|z|^2\right). \tag{6.7}$$

The leading asymptotics of $G^{(0)}$ are given in (4.7). In view of Lemma 1 and (6.7), in the given region, $\phi_X - \phi^{(k)}_{EH}$ is asymptotic to $\frac{1}{2b^2t|z|^2} + \mathrm{const.}\, t|z|^{2(k+1)}$. Taking k large, we can neglect the const. $t|z|^{2(k+1)}$ term. Also, $\omega^{(k)}_{EH}$ is asymptotically $\frac{t}{2}\sqrt{-1}dz^i \wedge d\overline{z}^i$. Using the fact that $\partial_i\overline{\partial}_i\frac{1}{|z|^2} = 0$, one finds that $f^{(k)}_{mod}$ is $O\left(t^{4a-2}\right)$ in the given region, from which part (1) follows. The proof of part (2) is similar. □

We can set $a = \frac{1}{k}$ and take k large. The proof of Theorem 1 goes through in the complex hyperbolic setting, changing γ from $\frac{3}{2} - \epsilon$ to $2 - \epsilon$.

Finally, we give a stability result for the flow.

Proposition 1. *There are some $a > 0$ and $k \in \mathbb{Z}^+$ with the following property. Construct the model flow $g^{(k)}_{mod}$ with gluing at scale $|z| \sim t^{-a}$. Given $\epsilon > 0$, there is some $T < \infty$ so that if $u(t)$ is a solution to (2.5) on $[T, \infty)$ then $\|u(t)\|_{C^0} \le \|u(T)\|_{C^0} + \epsilon$ for all $t \ge T$.*

Proof. We first claim that for any $\mu > 0$, if a is sufficiently small and k is sufficiently large then $\|f^{(k)}_{mod}\|_{C^0}$ is $O\left(t^{-2+\mu}\right)$. In the region $\frac{1}{2}t^{-a} < |z| < t^{-a}$, the claim follows from Lemma 6. Lemma 3 implies that the claim is true in the region $|z| \le \frac{1}{2}t^{-a}$ if we take k large enough.

If μ is small then $\lim_{T\to\infty}\int_T^\infty t^{-2+\mu}\, dt = 0$. Applying the maximum principle to (2.5) gives $\frac{d}{dt}\max u(t) \le -\min f(t)$, where the inequality is understood in the sense of forward differences. The minimum principle gives $\frac{d}{dt}\min u(t) \ge -\max f(t)$. Hence

$$\max|u(t)| \le \max|u(T)| + \int_T^t \max|f(s)|\, ds. \tag{6.8}$$

We can take T large enough that $\int_T^\infty \max|f(s)|\, ds \le \epsilon$. □

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