The Ricci Flow Approach to 3-Manifold Topology

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<u>Two-dimensional topology</u>

All of the compact surfaces that anyone has ever seen :

These are *all* of the compact connected oriented surfaces without boundary.

How to distinguish the sphere from the other surfaces? It is simply-connected.

Recall : A topological space is *simply-connected* if any closed curve in the space can be continuously contracted to a point.

What about three dimensions?

Notation : M a compact connected orientable 3-dimensional manifold without boundary.

Basic Question : How can we distinguish the three-dimensional sphere from the other three-dimensional manifolds?

Conjecture. *(Poincaré, 1904)*

If M is simply connected then it is topologically equivalent to the three-sphere S^3.

Extension to non-simply-connnected case

Thurston's "Geometrization Conjecture"

Motivation : Recall that any compact surface carries a Riemannian metric of constant curvature 1, 0, or -1.

Question : Does every closed 3-manifold carry a constant curvature metric?

Answer : (Un)fortunately, no.

<u>Geometrization Conjecture (rough version)</u>

Conjecture. *(Thurston, 1970's) M can be canonically cut into pieces, each of which carries one of eight magic geometries.*

Picture :

The eight magic geometries :

1. Constant curvature 0
2. Constant curvature 1
3. Constant curvature -1

and five others.

Fact : Geometrization $\Longrightarrow$ Poincaré

Analytic approach to Geometrization Conjecture

Idea : Start with the manifold M in an arbitrary "shape".

Evolve the shape to smooth it out. Maybe, as time goes on, one will start to "see" M's geometric pieces.

Background from differential geometry

M an n-dimensional manifold.

By a "shape" of a manifold M, we mean a *Riemannian metric* g on M.

To each point $p \in M$, the Riemannian metric specifies an inner product on the tangent space T_pM.

Sectional curvature

Given the Riemannian metric g, one can compute its *sectional curvatures*.

For each point $p \in M$ and each 2-plane $P \subset T_pM$, one computes a number $K(P)$.

(Example : If M is a surface then there is only one 2-plane $P \subset T_pM$, and its sectional curvature $K(P)$ equals the Gaussian curvature at p.)

Ricci curvature

The *Ricci curvature* is an average sectional curvature :

Given a unit vector $\mathbf{v} \in T_pM$, let Ric$(\mathbf{v}, \mathbf{v})$ be $(n-1)$ times the average sectional curvature of all of the 2-planes P containing $\mathbf{v}$.

Fact : Ric$(\mathbf{v}, \mathbf{v})$ extends to a bilinear form on T_pM. This is the *Ricci tensor* on M.

Hamilton's Ricci flow

A prescribed 1-parameter family of metrics $g(t)$, with $g(0) = g_0$.

Goal : to smooth out the original metric g_0.

Idea : Let the "shape" evolve by its curvature.

Ricci flow equation :

$$\frac{dg}{dt} = -2\,\mathrm{Ric}(g(t)).$$

Both sides are the same type of object : at each point $p \in M$, a bilinear form on T_pM.

In terms of local coordinates,

$$\frac{\partial g_{ij}}{\partial t} = -2\,R_{ij}.$$

Introduced by Hamilton in 1982.

Somewhat like the heat equation

$$\frac{\partial f}{\partial t} = \nabla^2 f,$$

except *nonlinear*.

$$\frac{\partial g_{ij}}{\partial t} = -\,2\,R_{ij}.$$

Heat equation evolves a **function**.

Ricci flow evolves a **Riemannian metric**.

Recall : heat flow evolves an initial function f_0 towards a constant function.

Hope : for a <u>three-dimensional</u> manifold, the Ricci flow will evolve the metric g_0 so that we see the geometrization decomposition.

This works for surfaces!

Examples of Ricci flow solutions

1. **3-torus :** $g(t) = g_{T^3}$ for all t (flat metric)

2. **3-sphere :** $g(t) = r^2(t)\, g_{S^3}$,

$r^2(t) = -4t$ for all $t < 0$.

3. **hyperbolic 3-manifold :** $g(t) = r^2(t)\, g_{hyp}$,

$r^2(t) = r_0^2 + 4t$ for all $t \geq 0$.

4. **shrinking cylinder :** $g(t) = r^2(t) g_{S^2} + g_{\mathbb{R}}$,

$r^2(t) = -2t$ for all $t < 0$.

How to prove the Poincaré Conjecture

Start with simply-connected three-dimensional M. Put any Riemannian metric on it.

Run Ricci flow. What can happen?

(Overly) optimistic hope :

Maybe the solution shrinks to a point, and get rounder as it shrinks. If so, M has a metric of **constant positive sectional curvature**.

But M is simply connected, so then it must be topologically a 3-sphere!

Theorem. *(Hamilton, 1982) This works if the initial metric has positive Ricci curvature.*

But what if it doesn't?

With a general initial metric

New problem : the flow could go singular before it can shrink to a point. For example,

Neckpinch :

The positive curvature of the two-dimensional cross-section can make it squeeze to a point in finite time.

What to do now?

Idea (Hamilton) : Do **surgery** on a neckpinch.

Then continue the flow. If more neckpinches occur, do the same.

Basic problem : How do we know that the singularities are actually caused by neckpinches?

Solved by Perelman (2002).

Outline of an argument to prove the Poincaré conjecture

1. Start with M a simply-connected compact three-dimensional manifold.

2. Put an arbitrary Riemannian metric g_0 on M.

3. **Claim (Perelman II) :**
There **is** a well-defined **Ricci-flow-with-surgery**.

4. Run flow up to first singularity time (if there is one).

a. **First case :** Entire solution disappears.

Claim (Perelman II) :
If an entire solution disappears then the manifold is topologically equivalent to
(i) S^3/Γ (with Γ a finite subgroup of $SO(4)$ acting freely on S^3), or
(ii) $S^1 \times S^2$, or
(iii) $(S^1 \times S^2)/\mathbb{Z}_2$.

(Not assuming here that M is simply connected.)

b. **Next case :** If the entire solution doesn't disappear, say Ω is what's left at the singular time.

Form a **new manifold** M' by **surgering out the horns** in Ω.

Note : M' may be disconnected. Flow to next singularity time.

Remove high-curvature regions by **surgering out the horns** and by **throwing away isolated components**.

Continue.

5. **Claim (Perelman III, Colding-Minicozzi)**

After a finite time, there's nothing left. The entire solution went **extinct**.

Here we use the assumption that the original manifold was simply connected.

6. How to reconstruct the original manifold?

Pieces that went extinct, or were thrown away, were each diffeomorphic to

$$S^3/\Gamma,\ S^1 \times S^2 \text{ or } (S^1 \times S^2)/\mathbb{Z}_2.$$

Going from **after** a surgery to **before** a surgery amounts to piping components together, i.e. performing "connected sums".

(Possibly with some new $S^1 \times S^2$'s and $\mathbb{R}P^3$'s).

To reconstruct the original manifold

Let's go backwards in time.

We start with the empty set $\emptyset$.

We repeatedly see a new copy of S^3/Γ, $S^1 \times S^2$ or $(S^1 \times S^2)/\mathbb{Z}_2$ appearing and we pipe it in.

We end up with M.

7. Conclusion

The original manifold M is topologically equivalent to a connected sum :

$(S^3/\Gamma_1)\#\ldots\#(S^3/\Gamma_k)\#(S^1\times S^2)\#\ldots\#(S^1\times S^2)$.

By van Kampen's theorem,

$$\pi_1(M) = \Gamma_1 \star \ldots \star \Gamma_k \star \mathbb{Z} \star \ldots \star \mathbb{Z}.$$

But M is simply connected! So each Γ_i is trivial and there are no $S^1 \times S^2$ factors. Then

$$M \;=\; S^3\#\ldots\#S^3 \;=\; S^3.$$

This would prove the Poincaré Conjecture!

What if the starting manifold is not simply-connected?

The solution $g(t)$ could go on for all time t.

(Example : manifolds with constant negative sectional curvature.)

Shrink by a factor of t :

$$\widehat{g}(t) = \frac{g(t)}{t}.$$

(Motivation : if M has constant negative curvature then $g(t)$ increases linearly.)

Claim (Perelman II) : For large t, M decomposes into "thick" and "thin" pieces (either one possibly empty) :

$$M = M_{thick} \cup M_{thin}$$

where

a. The metric $\widehat{g}(t)$ on M_{thick} is close to constant sectional curvature $-\frac{1}{4}$. The interior of M_{thick} admits a **complete finite-volume Riemannian metric of constant negative sectional curvature**.

b. M_{thin} is a **graph manifold**. These are known to have a "geometric" decomposition.

c. The gluing of M_{thick} and M_{thin} is done along **incompressible 2-dimensional tori**
(i.e. $\pi_1(T^2) \to \pi_1(M)$ is 1-1).

This would prove the Geometrization Conjecture!

Important earlier case : when there are no singularities and $\sup_M |\mathrm{Riem}(g_t)| = O(t^{-1})$ (Hamilton, 1999).

Aside : What is a graph manifold?

Building blocks :

1. Solid doughnut

2. Solid doughnut with two wormholes

Take a bunch of these. Their boundaries are a lot of tori.

Pair up some of the tori. For each pair, glue the corresponding building blocks together.

By definition, the result is a *graph manifold*. These are understood.

Back to claims :

Claim (Perelman II) :
There **is** a well-defined Ricci-flow-with-surgery.

Two statements here :

1. We know how to do surgery if we encounter a singularity.
2. The surgery times do not accumulate.

How do we do surgery if we encounter a singularity?

Fact : Singularities are caused by curvature blowup.

If the solution exists on the time interval $[0, T)$, but no further, then

$$\lim_{t \to T^-} \sup_M |\mathrm{Riem}(g_t)| = \infty.$$

(Here $|\mathrm{Riem}|$ denotes the largest sectional curvature at a point, in absolute value.)

To do surgery, we need to know that singularities are caused by tiny necks collapsing.

How do we know that this is the case?

Blowup idea

Suppose that the solution goes singular.

Take a sequence of points and times approaching the singularity.

Blow up to bring those points up to a unit scale.

Try to take a limit of the geometry near these blowup points.

Hope that the limit geometry will satisfy some PDE and will be special.

Rescaling argument

Choose a sequence of times t_i and points x_i in M so that

1. $\lim_{i\to\infty} t_i = T$.
2. $\lim_{i\to\infty} |\mathrm{Riem}(x_i, t_i)| = \infty$.

Blowup

Zoom in to the spacetime point (x_i, t_i).

Define $r_i > 0$ by $r_i^{-2} = |\mathrm{Riem(x_i, t_i)}|$.

(The **intrinsic scale** at the point (x_i, t_i).)

Note $\lim_{i\to\infty} r_i = 0$.

Spatially expand M by a factor r_i^{-1} so that $|\mathrm{Riem(x_i, t_i)}|$ becomes 1.

Fact : If M is spatially expanded by a factor of r^{-1} then we still get a Ricci flow solution *provided* that we also expand time by r^{-2}.

We were looking at the original solution on a time interval of length approximately T, so the rescaled solution lives on a time interval of length approximately $T\,|\mathrm{Riem(x_i, t_i)}|$.

This goes to infinity as $i \to \infty$.

Get a **sequence** of Ricci flow solutions $g_i(t)$ on M, defined on larger and larger time intervals. (Shift time parameter so they all end at time 0.)

Try to take a "convergent subsequence" as $i \to \infty$ of the Ricci flow solutions $(M, g_i(t))$.

Call limit Ricci flow solution $(M_\infty, g_\infty(t))$ (if it exists).

It will live on an **infinite** time interval $(-\infty, 0]$.

"Ancient solution"

Very special type of Ricci flow solution!

If we can find a **near-cylinder** $S^2 \times [L, -L]$ in $(M_\infty, g_\infty(0))$ then there were tiny necks in the **original** solution near the (x_i, t_i)'s and we're in business.

With a bit more argument, get that **any** high-curvature region in the original solution is modeled by a rescaled chunk of an ancient solution.

Two problems :

1. Why can we take a convergent subsequence as $i \to \infty$ of the Ricci flow solutions $(M, g_i(t))$'s?

2. Even if we can, why is there a near-cylinder in $(M_\infty, g_\infty(0))$?

Why can we take a convergent subsequence as $i \to \infty$ of the Ricci flow solutions $(M, g_i(t))$?

Need two things (Cheeger-Gromov, Hamilton)

First thing :
By the blowup construction

$$|\mathrm{Riem}(x_i, 0)| = 1$$

for the solution $(M, g_i(t))$.

So the curvature is uniformly bounded *at* the spacetime points $(x_i, 0)$.

Still need to know that the curvature is uniformly bounded in *neighborhoods* of the points $(x_i, 0)$.

(Since the supposed limit will have bounded curvature in any compact spacetime region around $(x_\infty, 0)$.)

Second thing :

We want a *three-dimensional limit* $(M_\infty, g_\infty(t))$.

We need to know that the rescaled solutions $(M, g_i(t))$ look *uniformly* three-dimensional. (Enough to just show this for $t = 0$.)

More precisely, a unit ball in $(M_\infty, g_\infty(0))$ will (obviously) have positive three-dimensional volume.

So to have a limit, need to know that the unit balls around x_i in the rescaled metrics $(M, g_i(0))$ have *uniformly positive* volume.

Hamilton compactness theorem : If one has these two things then there is a blowup limit $(M_\infty, g_\infty(t))$ that is an ancient Ricci flow solution.

One gets the curvature bounds by choosing the blowup points (x_i, t_i) cleverly.

How to get lower volume bound?

Even if we can do this, why is there a near-cylinder in (M_∞, g_∞)?

Bad news possibility : $\mathbb{R} \times \textit{cigar soliton}$

A particular ancient solution, which has nothing like a cylinder in it.

If this appears in a blowup limit then we're in trouble.

This was the pre-Perelman status, as developed by Hamilton and others.

Perelman's first big innovation in Ricci flow :

No Local Collapsing Theorem

Say we have a Ricci flow solution on a finite time interval $[0, T)$.

The theorem says that at a spacetime point (x, t), the solution looks **noncollapsed at the intrinsic scale of the spacetime point**.

(Recall : intrinsic scale is $|\mathrm{Riem(x, t)}|^{-1/2}$.)

At a spacetime point (x, t), the solution looks **noncollapsed at the intrinsic scale of the spacetime point**.

More precisely, given the Ricci flow solution on the interval $[0, T)$ and a scale $\rho > 0$, we can find a number $\kappa > 0$ so that the following holds:

Suppose that a metric ball B in some time slice has radius r (less than ρ). If

$$|\mathrm{Riem}| \leq \mathrm{r}^{-2}$$

on B then

$$\mathrm{vol(B)} \geq \kappa\, \mathrm{r}^3.$$

We say that the Ricci flow solution is κ-*noncollapsed at scales less than* ρ.

Essentially scale-invariant, so it passes to a blowup limit!

In short,
Local curvature bound $\Longrightarrow$ local lower bound on volumes of balls

Of course, this is a statement about **Ricci flow solutions**.

With the **no local collapsing theorem**, we can take blowup limits. Furthermore, we won't get $\mathbb{R} \times cigar\ soliton$ as a limit.

Still need to show that the blowup limit actually has a near-cylinder in it (if it's non-compact).

I.e. have to understand "κ-noncollapsed ancient solutions".

Three-dimensional ancient solutions

(Perelman) : Any three-dimensional ancient solution that is κ-noncollapsed at all scales, for some $\kappa > 0$, falls into one of the following classes :

1. Isometric to S^3/Γ, where S^3 is the shrinking round 3-sphere.

2. Diffeomorphic to S^3 or $\mathbb{R}P^3$.

3. Shrinking round cylinder $\mathbb{R} \times S^2$ or its $\mathbb{Z}_2$-quotient.

4. Qualitatively similar to the Bryant soliton.

Using this, one can find the 2-dimensional spheres to do surgery.

Main problem : How do we know that the surgery times don't accumulate?

Technically difficult, must do surgery in a very precise way.

How to prove No Local Collapsing Theorem :

Find some "functional" $I(g)$ of metrics g with the following two properties :

1. If g is "locally collapsed" somewhere then $I(g)$ is very small.

2. If $g(t)$ is a Ricci flow solution then $I(g(t))$ is nondecreasing in t.

If we can find I then we're done!

Not at all clear that there is such a functional.

Perelman found **two** :

1. Entropy
2. Reduced volume.

<u>Conceptual framework</u> :

The Ricci flow is a *gradient flow* on

$$\{\text{Metrics modulo diffeomorphisms}\}.$$

Gradient flows

If F is a smooth function on a Riemannian manifold X then its gradient flow is

$$\frac{dx}{dt} = \nabla F\Big|_{x(t)}.$$

Easy fact : F is nondecreasing along a flow-line, i.e. $F(x(t))$ is nondecreasing in t.

Ricci flow as a gradient flow

Given a metric g on M, let $\lambda(g)$ be the smallest eigenvalue of $-4\triangle + R$.

Equivalently,

$$\lambda(g) = \inf_{f \in C^\infty(M), f \neq 0} \frac{\int_M \left(4|\nabla f|^2 + R f^2\right) d\text{vol}}{\int_M f^2 \, d\text{vol}}.$$

If $\phi : M \to M$ is a diffeomorphism then

$$\lambda(\phi^* g) = \lambda(g),$$

so λ descends to a function on $\frac{Metrics}{Diffeomorphisms}$.

(Perelman) The Ricci flow on $\frac{Metrics}{Diffeomorphisms}$ is the formal gradient flow of the function λ.

Consequence :

If $g(t)$ is a Ricci flow solution then $\lambda(g(t))$ is nondecreasing in t.