



Ricci measure for some singular Riemannian metrics

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Abstract We define the Ricci curvature, as a measure, for certain singular torsion-free connections on the tangent bundle of a manifold. The definition uses an integral formula and vector-valued half-densities. We give relevant examples in which the Ricci measure can be computed. In the time dependent setting, we give a weak notion of a Ricci flow solution on a manifold.

1 Introduction

There has been much recent work about metric measure spaces with lower Ricci bounds, particularly the Ricci limit spaces that arise as measured Gromov–Hausdorff limits of smooth manifolds with a uniform lower Ricci bound. In this paper we address the question of whether one can make sense of the Ricci curvature itself on singular spaces.

From one’s intuition about a two dimensional cone with total cone angle less than 2π , the Ricci curvature should exist at best as a measure. One natural approach toward a weak notion of Ricci curvature is to use an integral formula, such as the Bochner formula. The Bochner identity says that if ω_1 and ω_2 are smooth compactly supported 1-forms on a smooth Riemannian manifold M then

$$\langle \omega_1, \text{Ric}(\omega_2) \rangle = \int_M (\langle d\omega_1, d\omega_2 \rangle + \langle d^*\omega_1, d^*\omega_2 \rangle - \langle \nabla\omega_1, \nabla\omega_2 \rangle) \, \text{dvol}. \quad (1.1)$$

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Equivalently,

$$\langle \omega_1, \text{Ric}(\omega_2) \rangle = \int_M \sum_{i,j} \left(\nabla^i \omega_{1,i} \nabla^j \omega_{2,j} - \nabla^i \omega_{1,j} \nabla^j \omega_{2,i} \right) \text{dvol}. \tag{1.2}$$

Now consider a possibly singular Riemannian metric on M . In order to make sense of (1.2), one’s first attempt may be to require that $\nabla \omega_1$ and $\nabla \omega_2$ are square integrable. However, in the case of a two dimensional cone with total cone angle less than 2π , if one requires square integrability then one does not find any contribution from the vertex of the cone. That is, one would conclude that the cone is Ricci flat, which seems wrong. In order to see the curvature at the vertex, one needs to allow for more general test forms. It is not immediately evident what precise class of test forms should be allowed. A related fact is that the expression for the Ricci tensor, using local coordinates, makes distributional sense if the Christoffel symbols are square integrable. However, this is not the case for the cone.

Our resolution to this problem is by first passing from 1-forms to vector fields, and then passing to vector-valued half-densities. For V and W vector-valued half-densities, we consider the quadratic form

$$Q(V, W) = \int_M \sum_{i,j} \left[\left(\nabla_i V^i \right) \left(\nabla_j W^j \right) - \left(\nabla_i V^j \right) \left(\nabla_j W^i \right) \right]. \tag{1.3}$$

A compactly supported density on a manifold can be integrated, so two compactly supported half-densities can be multiplied and integrated. We require that V and W are compactly supported and Lipschitz regular on M . If one rewrote Q using 1-forms as in (1.2) then this would prescribe that the 1-forms should lie in certain weighted spaces.

One sees that (1.3) does not involve the Riemannian metric directly, but can be written entirely in terms of the connection. Hence we work in the generality of torsion-free connections on the tangent bundle. We also work with $C^{1,1}$ -manifolds M , with an eye toward limit spaces; it is known that Ricci limit spaces have a weak $C^{1,1}$ -structure [10]. Then we say that a possibly singular connection is *tame* if (1.3) makes sense for all compactly supported Lipschitz vector-valued half-densities V and W . We characterize tame connections in terms of integrability properties of their Christoffel symbols. We show that a tame connection, with Q bounded below, has a Ricci curvature that is well-defined as a measure with values in $S^2(T^*M) \otimes \mathcal{D}^*$, where \mathcal{D} is the density line bundle.

We prove stability results for this Ricci measure. We give examples to illustrate its meaning.

Proposition 1.1 *The Levi-Civita connection is tame and has a computable Ricci measure in the following cases : Alexandrov surfaces, Riemannian manifolds with boundary that are glued together, and families of cones.*

Passing to Ricci flow, one can use optimal transport to characterize supersolutions to the Ricci flow equation on a manifold [15, 20, 29]. There are also comparison principles

for Ricci flow supersolutions [2], [11, Sect. 2]. We give a weak notion of a Ricci flow solution (as opposed to supersolution), in the sense that the curvature tensor is not invoked, again on a fixed $C^{1,1}$ -manifold. The idea is that the Ricci tensor appearing in the Bochner integral formula can cancel the Ricci tensor appearing on the right-hand side of the Ricci flow equation. One could try to formulate such a time dependent integral identity just using the Bochner equality for 1-forms. However, one would get a term coming from the time derivative of the volume form, which unfortunately involves the scalar curvature. Using vector-valued half-densities instead, this term does not appear. We give examples of weak Ricci flow solutions, along with a convergence result and a compactness result.

To mention some earlier work, Lebedeva and Petrunin indicated the existence of a measure-valued curvature operator on an Alexandrov space that is a noncollapsed limit of Riemannian manifolds with a lower sectional curvature bound [14]. The Ricci form exists as a current on certain normal Kähler spaces and was used by Eyssidieux, Guedj and Zeriahi for Kähler–Einstein metrics [6]. Naber gave a notion of bounded Ricci curvature, in particular Ricci flatness, for metric measure spaces [21]. Gigli discussed Ricci curvature for certain metric measure spaces [8, Sect. 3.6].

The structure of this paper is as follows. In Sect. 2 we give some background information. Section 3 has the definitions of tame connection and Ricci measure, and proves some properties of these. Section 4 gives some relevant examples. Section 5 is about weak Ricci flow solutions.

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2 Background

Let M be an n -dimensional smooth manifold. Let FM denote the principal $GL(n, \mathbb{R})$ -frame bundle of M . For $c \in \mathbb{R}$, let $\rho_c : GL(n, \mathbb{R}) \rightarrow GL(1, \mathbb{R})$ be the homomorphism given by $\rho_c(M) = |\det M|^{-c}$. There is an associated real line bundle $\mathcal{D}^c = FM \times_{\rho_c} \mathbb{R}$, the c -density bundle. There is an isomorphism $\mathcal{D}^c \otimes \mathcal{D}^{c'} \rightarrow \mathcal{D}^{c+c'}$ of line bundles. A section of \mathcal{D}^c is called a c -density on M . A 1-density is just called a density. Compactly supported densities on M can be integrated, to give a linear functional $\int_M : C_c(M; \mathcal{D}) \rightarrow \mathbb{R}$. There is a canonical inner product on compactly supported half-densities, given by $\langle f_1, f_2 \rangle = \int_M f_1 f_2$.

Let ∇ be a torsion-free connection on TM . There is an induced connection on \mathcal{D} . Given a compactly supported vector-valued density V , i.e. a section $V \in C_c^\infty(M; TM \otimes \mathcal{D})$, the integral $\int_M \sum_i \nabla_i V^i$ of its divergence vanishes. With this fact, one can justify integration by parts.

The curvature of ∇ is a section of $\text{End}(TM) \otimes \Lambda^2(T^*M) = TM \otimes T^*M \otimes (T^*M \wedge T^*M)$. In terms of the latter description, the Ricci curvature of ∇ is the covariant 2-tensor field on M obtained by contracting the TM factor with the first T^*M factor in $(T^*M \wedge T^*M)$. In terms of indices, $R_{jl} = \sum_i R^i_{jil}$.

In this generality, the Ricci curvature need not be symmetric. As ∇ is torsion-free, the first Bianchi identity holds and one finds that $R_{jl} - R_{lj} = \sum_i R^i_{ijl}$. That is, the antisymmetric part of the Ricci tensor is the negative of the curvature of the induced connection on \mathcal{D} , and represents an obstruction to the local existence of a nonzero

parallel density. Of course, if ∇ is the Levi-Civita connection of a Riemannian metric then there is a nonzero parallel density, namely the Riemannian density.

Now let M be a $C^{1,1}$ -manifold. This means that there is an atlas $M = \bigcup_{\alpha} U_{\alpha}$ whose transition maps $\phi_{\alpha\beta}$ have a first derivative that is Lipschitz. We can take a maximal such atlas. The preceding discussion of c -densities still makes sense in this generality.

A $C^{1,1}$ -manifold admits an underlying smooth structure, in that we can find a subatlas with smooth transition maps. Furthermore, any two such smooth structures are diffeomorphic.

3 Tame connections and Ricci measure

In this section we give the notion of a tame connection and define its Ricci measure. In Sect. 3.1 we define tame connections, characterize them in terms of the Christoffel symbols, and prove stability under L^{∞} -perturbations of the connection. Section 3.2 has the definition of the Ricci measure. In Sect. 3.3 we extend the notion of Ricci measure to the case of weighted manifolds. Section 3.4 is about singular Riemannian metrics and Killing fields.

3.1 Tame connections

Let M be an n -dimensional $C^{1,1}$ -manifold. It makes sense to talk about the space $\mathcal{V}_{Lip}(M)$ of Lipschitz vector fields on M , meaning Lipschitz-regular sections of TM . Similarly, it makes sense to talk about Lipschitz vector-valued half-densities, i.e. Lipschitz-regular sections of $TM \otimes \mathcal{D}^{\frac{1}{2}}$.

Let $\mathcal{V}_{meas}(M)$ denote the measurable vector fields on M . Let ∇ be a measurable torsion-free connection on TM , i.e. an \mathbb{R} -bilinear map $\nabla : \mathcal{V}_{Lip}(M) \times \mathcal{V}_{Lip}(M) \rightarrow \mathcal{V}_{meas}(M)$ such that for $f \in Lip(M)$ and $X, Y \in \mathcal{V}_{Lip}(M)$, we have

- $\nabla_{fX}Y = f\nabla_XY$,
- $\nabla_X(fY) = (Xf)Y + f\nabla_XY$,
- $\nabla_XY - \nabla_YX = [X, Y]$.

Writing $\nabla_{\partial_i}\partial_j = \sum_k \Gamma^k_{ji}\partial_k$, the Christoffel symbols Γ^k_{ij} are measurable.

If V is a vector-valued half-density then we can locally write it as $V = \sum_j V^j\partial_j$, where V^i is a locally defined half-density. Then $\nabla_i V = \sum_j (\nabla_i V^j)\partial_j$, where $\nabla_i V^j$ is also a half-density. Further writing $V^j = v^j\sqrt{dx^1 \dots dx^n}$, we have

$$\nabla_i V = \sum_j (\nabla_i v^j)\partial_j \otimes \sqrt{dx^1 \dots dx^n}, \tag{3.1}$$

where

$$\nabla_i v^j = \partial_i v^j + \sum_k \Gamma^j_{ki}v^k - \frac{1}{2} \sum_k \Gamma^k_{ki}v^j \tag{3.2}$$

Given compactly supported Lipschitz vector-valued half-densities V and W on M , consider the formal expression

$$Q(V, W) = \int_M \sum_{i,j} \left[(\nabla_i V^i) (\nabla_j W^j) - (\nabla_i V^j) (\nabla_j W^i) \right]. \tag{3.3}$$

Note that the integrand of (3.3) is a density on M .

Definition 3.1 The connection ∇ is *tame* if the integrand in (3.3) is integrable for all V and W .

If $n = 1$ then Q vanishes identically.

Remark 3.2 Suppose that ∇ is the Levi-Civita connection of a Riemannian metric g . We can use the Riemannian half-density $\sqrt{\text{dvol}} = (\det g)^{\frac{1}{4}} \sqrt{dx^1 \dots dx^n}$ to trivialize $\mathcal{D}^{\frac{1}{2}}$. Using this trivialization, there is an isometric isomorphism between vector-valued half-densities and 1-forms, under which a 1-form $\omega = \sum_i \omega_i dx^i$ corresponds to a vector-valued half-density $V = \sum_i v^i \partial_i \otimes \sqrt{dx^1 \dots dx^n}$ with $v^i = \sum_j g^{ij} (\det g)^{\frac{1}{4}} \omega_j$. In this case, Q could be computed using (1.2), with the restriction on ω that each v^i in the local description of its isomorphic vector-valued half-density V should be Lipschitz.

We now characterize tameness of a connection in terms of its Christoffel symbols.

Proposition 3.3 *Suppose that $n > 1$. The connection ∇ is tame if and only if in any coordinate neighborhood, each Γ^i_{jk} is locally integrable and each $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj})$ is locally integrable.*

Proof Suppose that ∇ is tame. Let U be a coordinate neighborhood and choose $m \in U$. The point m has a neighborhood S with compact closure in U . Take V and W to have compact support in U . One finds that

$$Q(V, W) = Q_1(V, W) + Q_2(V, W), \tag{3.4}$$

where

$$Q_1(V, W) = \int_U \sum_{i,k,l} \left[\frac{1}{2} (\partial_k v^k) \Gamma^i_{il} w^l + \frac{1}{2} v^k \Gamma^i_{il} (\partial_k w^l) + \frac{1}{2} (\partial_l v^k) \Gamma^i_{ik} w^l + \frac{1}{2} v^k \Gamma^i_{ik} (\partial_l w^l) - (\partial_i v^k) \Gamma^i_{kl} w^l - v^k \Gamma^i_{kl} (\partial_i w^l) \right] dx^1 \dots dx^n \tag{3.5}$$

and

$$Q_2(V, W) = \int_U \sum_{i,j,k,l} v^k (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj}) w^l dx^1 \dots dx^n. \tag{3.6}$$

Given constant vectors $\{c^k\}_{k=1}^n$ and $\{d^l\}_{l=1}^n$, we can choose V and W so that $v^k = c^k$ and $w^l = d^l$ in S . Then the integrand of Q_1 vanishes in S . Hence the integrability of the integrand of $Q(V, W)$ implies the integrability of $\sum_{i,j,k,l} c^k (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj}) d^l$ in S , for any choice of $\{c^k\}_{k=1}^n$ and $\{d^l\}_{l=1}^n$. Letting m and S vary, this is equivalent to the local integrability of $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj})$ in U , for all k and l .

Returning to general V and W with compact support in U , we now know that the integrand of Q_2 is integrable. Hence the integrability of the integrand of Q implies the integrability of the integrand of Q_1 . Given a constant matrix $\{C_r^k\}_{r,k=1}^n$ and a constant vector $\{d^l\}_{l=1}^n$, we can choose V and W so that $v^k = \sum_r C_r^k x^r$ and $w^l = d^l$ in S . Then the integrand of Q_1 , over S , becomes

$$\sum_{i,k,l} \left[\frac{1}{2} C_k^k \Gamma^i_{il} d^l + \frac{1}{2} C_l^k \Gamma^i_{ik} d^l - C_i^k \Gamma^i_{kl} d^l \right]. \tag{3.7}$$

Taking first $C_k^l = \delta_k^l$, we see that $\frac{n-1}{2} \sum_i \Gamma^i_{il} d^l$ is integrable in S for any choice of $\{d^l\}_{l=1}^n$. Hence $\sum_i \Gamma^i_{il}$ is integrable in S for all l . It now follows from (3.7) that $\sum_{i,k,l} C_i^k \Gamma^i_{kl} d^l$ is integrable in S for any choices of $\{C_r^k\}_{r,k=1}^n$ and $\{d^l\}_{l=1}^n$. Hence Γ^i_{kl} is integrable in S for all i, k and l , so Γ^i_{kl} is locally integrable in U .

For the other direction of the proposition, suppose that in any coordinate neighborhood, each Γ^i_{jk} is locally integrable and each $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj})$ is locally integrable. Given V and W with compact support, we can cover $\text{supp}(V) \cup \text{supp}(W)$ by a finite number $\{U_r\}_{r=1}^N$ of open sets, each with compact closure in a coordinate neighborhood. Let $\{\phi_r\}_{r=1}^N$ be a subordinate Lipschitz partition of unity. Then

$$Q(V, W) = \sum_{r=1}^N Q(\phi_r V, W). \tag{3.8}$$

Looking at (3.5) and (3.6), we see that the integrand of $Q(\phi_r V, W)$ has support in U_r . Then from (3.5) and (3.6), we see that the integrand of $Q(\phi_r V, W)$ is integrable. The proposition follows. \square

Proposition 3.4 *Suppose that we have a fixed collection of coordinate neighborhoods that cover M , in each of which Γ^i_{jk} is locally integrable and $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj})$ is locally integrable.*

Then in any other coordinate neighborhood, Γ^i_{jk} is locally integrable and $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj})$ is locally integrable.

Proof This follows indirectly from the proof of Proposition 3.3, but can also be seen directly from the transformation formula

$$\tilde{\Gamma}^i_{jk} = \sum_{a,b,c} \frac{\partial y^i}{\partial x^a} \frac{\partial x^b}{\partial y^j} \frac{\partial x^c}{\partial y^k} \Gamma^a_{bc} + \sum_a \frac{\partial y^i}{\partial x^a} \cdot \frac{\partial^2 x^a}{\partial y^j \partial y^k} \tag{3.9}$$

for the Christoffel symbols under a change of coordinate from x to y , along with the fact that $\frac{\partial y}{\partial x}$ and $\frac{\partial x}{\partial y}$ are Lipschitz, and $\frac{\partial^2 x^a}{\partial y^j \partial y^k}$ is L^∞ . \square

We now show that tameness is preserved by bounded perturbations of the connection.

Proposition 3.5 *Suppose that ∇ is tame. Suppose that $T \cong T^i_{jk}$ is a measurable (1, 2)-tensor field, symmetric in the lower two indices, such that for all Lipschitz vector fields v , the (1, 1)-tensor field $T(v) = \sum_k T^i_{jk} v^k$ is a locally bounded section of $\text{End}(TM)$. Then $\nabla + T$ is tame.*

Proof Writing $T_i = T(\partial_i)$, we have

$$\begin{aligned} & \int_M \sum_{i,j} [(\nabla_i V^i + T_i V^i) (\nabla_j W^j + T_j W^j) - (\nabla_i V^j + T_i V^j) (\nabla_j W^i + T_j W^i)] \\ &= \int_M \sum_{i,j} [(\nabla_i V^i) (\nabla_j W^j) - (\nabla_i V^j) (\nabla_j W^i)] \\ &+ \int_M \sum_{i,j} [(T_i V^i) (\nabla_j W^j) + (\nabla_i V^i) (T_j W^j) - (T_i V^j) (\nabla_j W^i) \\ &- (\nabla_i V^j) (T_j W^i)] + \int_M \sum_{i,j} [(T_i V^i) (T_j W^j) - (T_i V^j) (T_j W^i)]. \end{aligned} \tag{3.10}$$

As before, we take V and W to have compact support. Since Proposition 3.3 tells us that in each coordinate neighborhood, the Christoffel symbols Γ^i_{jk} are locally L^1 , it follows that ∇V and ∇W are L^1 on M . Since TV and TW are L^∞ , the integrands in the second and third integrals on the right-hand side of (3.10) are integrable. The proposition follows. \square

3.2 Ricci measure

Suppose that ∇ is tame. We can rewrite the expression for $Q_1(V, W)$ in (3.5) as

$$Q_1(V, W) = \int_U \sum_{i,k,l} \left[\frac{1}{2} \Gamma^i_{il} \partial_k (v^k w^l) + \frac{1}{2} \Gamma^i_{ik} \partial_l (v^k w^l) - \Gamma^i_{kl} \partial_i (v^k w^l) \right] dx^1 \dots dx^n. \tag{3.11}$$

Using an underlying smooth structure for M , and taking V and W to be smooth with compact support in the coordinate neighborhood U for the moment, it follows that

$$Q(V, W) = \int_U \sum_{k,l} v^k R_{(kl)} w^l dx^1 \dots dx^n, \tag{3.12}$$

where the distribution

$$R_{(kl)} = \sum_i \left(\partial_i \Gamma^i_{kl} - \frac{1}{2} \partial_k \Gamma^i_{il} - \frac{1}{2} \partial_l \Gamma^i_{ik} \right) + \sum_{i,j} \left(\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj} \right) \tag{3.13}$$

is recognized as the symmetrized Ricci tensor.

Given a continuous vector bundle E on M , let $\mathcal{M}(M; E)$ denote the dual space to the topological vector space of compactly supported continuous sections of E^* . We can think of an element of $\mathcal{M}(M; E)$ as an E -valued measure on M .

In the rest of this subsection, we make the following assumption.

Assumption 3.6 There is some nonnegative $h \in \mathcal{M}(M; S^2(T^*M) \otimes \mathcal{D}^*)$ so that for all V and W ,

$$Q(V, W) \geq - \int_M \langle V, hW \rangle. \tag{3.14}$$

Assumption 3.6 implies that the distributional tensor field $R_{(kl)} + h_{kl}$ is nonnegative. It follows that it is a tensor-valued measure, and hence so is $R_{(kl)}$. The conclusion is that there is some $\mathcal{R} \in \mathcal{M}(M; S^2(T^*M) \otimes \mathcal{D}^*)$ so that for all compactly supported Lipschitz vector-valued half-densities V and W on M , we have

$$Q(V, W) = \int_M \langle V, \mathcal{R}W \rangle. \tag{3.15}$$

We call \mathcal{R} the Ricci measure of the connection ∇ .

We now prove a convergence result for the Ricci measure.

Proposition 3.7 *Let ∇ be a tame connection with Ricci measure \mathcal{R} . Let $\{T^{(r)}\}_{r=1}^\infty$ be a sequence of measurable (1, 2)-tensor fields as in Proposition 3.5. Suppose that the connections $\{\nabla + T^{(r)}\}_{r=1}^\infty$ satisfy Assumption 3.6 with a uniform choice of h . Let $\{\mathcal{R}^{(r)}\}_{r=1}^\infty$ be the Ricci measures of the connections $\{\nabla + T^{(r)}\}_{r=1}^\infty$. Suppose that for each compactly supported Lipschitz vector field v , we have $\lim_{r \rightarrow \infty} T^{(r)}(v) = 0$ in $L^\infty(M; \text{End}(TM))$. Then $\lim_{r \rightarrow \infty} \mathcal{R}^{(r)} = \mathcal{R}$ in the weak-* topology on $\mathcal{M}(M; S^2(T^*M) \otimes \mathcal{D}^*)$.*

Proof Let $Q^{(r)}$ be the quadratic form associated to the tame connection $\nabla + T^{(r)}$. From (3.10), for any V and W , we have $\lim_{r \rightarrow \infty} Q^{(r)}(V, W) = Q(V, W)$. It follows that $\lim_{r \rightarrow \infty} \mathcal{R}^{(r)} = \mathcal{R}$ distributionally. Since the relevant distributions are all measures, with $\mathcal{R}^{(r)} + h$ nonnegative, we have weak-* convergence. \square

Remark 3.8 If we further assume that ∇ has Christoffel symbols in L^q_{loc} , for $q > 1$, then we reach the same conclusion under the weaker assumption that $\lim_{r \rightarrow \infty} T^{(r)}(v) = 0$ in $L^{\max(2, q^*)}(M; \text{End}(TM))$, where $\frac{1}{q} + \frac{1}{q^*} = 1$.

3.3 Bakry-Emery-Ricci measure

Let ∇ be a measurable torsion-free connection on TM . We say that $f \in W^{1,1}_{loc}(M)$ if in any coordinate neighborhood U , we have $f \in L^1_{loc}(U)$ and there are $S_i \in L^1_{loc}(U)$ so that for any Lipschitz functions F^i with compact support in U , we have

$$\int_U f \sum_i \partial_i F^i dx^1 \dots dx^n = - \int_U \sum_i S_i F^i dx^1 \dots dx^n. \tag{3.16}$$

We let $\nabla_i f$ denote S_i . Given compactly supported Lipschitz vector-valued half-densities V and W on M , consider the formal expression

$$\begin{aligned} Q_f(V, W) &= \int_M \sum_{i,j} \left[e^f \left(\nabla_i \left(e^{-\frac{f}{2}} V^i \right) \right) \left(\nabla_j \left(e^{-\frac{f}{2}} W^j \right) \right) \right. \\ &\quad \left. - e^{-f} \left(\nabla_i \left(e^{\frac{f}{2}} V^j \right) \right) \left(\nabla_j \left(e^{\frac{f}{2}} W^i \right) \right) \right] \\ &= \int_M \sum_{i,j} \left[\left(\nabla_i V^i \right) \left(\nabla_j W^j \right) - \left(\nabla_i V^j \right) \left(\nabla_j W^i \right) \right. \\ &\quad \left. - \frac{1}{2} \left(\nabla_i f \right) \left(\nabla_j \left(V^i W^j + V^j W^i \right) \right) \right]. \end{aligned} \tag{3.17}$$

We say that the pair (∇, f) is tame if the integrand in (3.17) is integrable for all V and W .

Example 3.9 Suppose that f is semiconvex. Then $\text{Hess}(f) = \nabla \nabla f$ is well-defined as a measurable symmetric 2-tensor. Suppose that ∇ is tame in the sense of Definition 3.1 and Q satisfies Assumption 3.6. Then (∇, f) is tame and

$$Q_f(V, W) = \int_M \langle V, \mathcal{R}_f W \rangle, \tag{3.18}$$

where the $S^2(T^*M) \otimes \mathcal{D}^*$ -valued measure \mathcal{R}_f is given in a coordinate neighborhood U by

$$Q_f(V, W) = \int_U \sum_{k,l} v^k (R_{(kl)} + \text{Hess}(f)_{kl}) w^l dx^1 \dots dx^n \tag{3.19}$$

for V and W having compact support in U . In the Riemannian setting, we recognize $R_{(kl)} + \text{Hess}(f)_{kl}$ as the Bakry-Emery-Ricci tensor.

3.4 Riemannian metrics

Let g be a Riemannian metric on M , i.e. a measurable section of $S^2(T^*M)$ that is positive definite almost everywhere.

Definition 3.10 A Riemannian metric g lies in $W_{loc}^{1,1}$ if in any coordinate neighborhood U , we have $g_{ij} \in L^1_{loc}(U)$ and there are $S^l_{jk} \in L^1_{loc}(U)$ so that for any Lipschitz functions $\{f^{ijk}\}$ with compact support in U , we have

$$\int_U \sum_{i,j,k} g_{ij} \partial_k f^{ijk} dx^1 \dots dx^n = \int_U \sum_{i,j,k,l} g_{il} S^l_{jk} f^{ijk} dx^1 \dots dx^n. \tag{3.20}$$

If $g \in W_{loc}^{1,1}$ then it has a Levi-Civita connection with Christoffel symbols in L_{loc}^1 .

We recall the classical result that a smooth compact Riemannian manifold with negative Ricci curvature has no nonzero Killing vector fields. (In fact, this is the only result that we know for manifolds with negative Ricci curvature in dimension greater than two). We show that there is an analogous result in our setting.

Definition 3.11 A Lipschitz vector-valued half-density V is *Killing* if

$$\sum_k (g_{jk} \nabla_i V^k + g_{ik} \nabla_j V^k) = 0. \tag{3.21}$$

We note that if V is a Killing vector-valued half-density then writing $V = v \otimes \sqrt{d\text{vol}_g}$, the vector field v is a Killing vector field in the usual sense, at least where g is C^1 .

Proposition 3.12 *If M is compact, the Levi-Civita connection is tame and $Q(V, V) < 0$ for all nonzero V , then there is no nonzero Killing V .*

Proof If V is Killing then $\sum_i \nabla_i V^i = 0$ and $\nabla_j V^i = -\sum_{k,l} g_{jl} g^{ik} \nabla_k V^l$, so

$$Q(V, V) = \int_M \sum_{i,j,k,l} g_{jl} g^{ik} (\nabla_i V^j)(\nabla_k V^l) \geq 0. \tag{3.22}$$

The proposition follows. □

4 Examples

In this section we compute examples of the Ricci measure coming from Riemannian metrics in $W_{loc}^{1,1}$, in the sense of Definition 3.10. The examples are Alexandrov surfaces, Riemannian manifolds with boundary that are glued together, families of cones, Kähler manifolds and limit spaces of manifolds with lower bounds on Ricci curvature and injectivity radius. At the end of the section we make some remarks.

4.1 Alexandrov surfaces

We recall that there is a notion of a metric on a surface having bounded integral curvature [24]. This includes surfaces with Alexandrov curvature bounded below. (For us, the relevance of the latter is that they are the noncollapsed Gromov–Hausdorff limits of smooth Riemannian two-manifolds with Ricci curvature bounded below). Such a metric comes from a (possibly) singular Riemannian metric g . There exist local isothermal coordinates in which $g = e^{2\phi} ((dx^1)^2 + (dx^2)^2)$, where ϕ is the difference of two subharmonic functions (with respect to the Euclidean metric). The volume density $d\text{vol}_g$, given locally by $e^{2\phi} dx^1 dx^2$, lies in L_{loc}^1 .

Proposition 4.1 *The Levi-Civita connection ∇ is tame.*

Proof In the isothermal coordinates, we have

$$\Gamma^i_{jk} = \delta_{ij} \partial_k \phi + \delta_{i,k} \partial_j \phi - \delta_{jk} \partial_i \phi. \tag{4.1}$$

A subharmonic function f on a two-dimensional domain has $\partial_i f \in L^1_{loc}$ [13, Pf. of Lemma 1.6]. (The proof there is for functions defined on \mathbb{C} but can be localized). One finds that $\sum_{i,j} (\Gamma^i_{kl} \Gamma^j_{ji} - \Gamma^j_{ki} \Gamma^i_{lj}) = 0$. Proposition 3.3 implies that ∇ is tame. \square

The quadratic form Q can be described in a coordinate-free way as follows. Let K denote the curvature measure of g , given in local isothermal coordinates by $dK = -(\partial_1^2 + \partial_2^2) \phi dx^1 dx^2$. (That is, for any smooth function f with support in the coordinate chart, $\int_M f dK = -\int_{\mathbb{R}^2} (\partial_1^2 + \partial_2^2) f \phi dx^1 dx^2$). Given compactly supported Lipschitz vector-valued half-densities V and W , consider $\frac{g(V,W)}{dvol_g}$. We claim that this extends over the singularities of g to a Lipschitz function on M . To see this, in isothermal coordinates we can write $V = \sum_i v^i \partial_i \otimes \sqrt{dx^1 dx^2}$ and $W = \sum_i w^i \partial_i \otimes \sqrt{dx^1 dx^2}$, with $\{v^i\}_{i=1}^2$ and $\{w^i\}_{i=1}^2$ Lipschitz. Then $g(V, W) = e^{2\phi} (v^1 w^1 + v^2 w^2) dx^1 dx^2$ and $dvol_g = e^{2\phi} dx^1 dx^2$, from which the claim follows. One finds

Proposition 4.2

$$Q(V, W) = \int_M \frac{g(V, W)}{dvol_g} dK. \tag{4.2}$$

Example 4.3 If g is smooth and κ is the Gaussian curvature then $dK = \kappa dvol_g$, so

$$Q(V, W) = \int_M \kappa g(V, W). \tag{4.3}$$

Example 4.4 For $\alpha < 1$, put $g = ((x^1)^2 + (x^2)^2)^{-\alpha} ((dx^1)^2 + (dx^2)^2)$. Then (\mathbb{R}^2, g) is a cone with total cone angle $2\pi(1 - \alpha)$. One finds that

$$Q(V, W) = 2\pi\alpha \left(v^1(0, 0)w^1(0, 0) + v^2(0, 0)w^2(0, 0) \right). \tag{4.4}$$

In this case, $\partial_i \phi = -\alpha \frac{x^i}{(x^1)^2 + (x^2)^2}$, so Γ^i_{jk} lies in L^1_{loc} but not in L^2_{loc} .

Example 4.5 For $c > 0$, suppose that $g = e^{-2c|x^1|} ((dx^1)^2 + (dx^2)^2)$. Then

$$Q(V, W) = 2c \int_{-\infty}^{\infty} \left(v^1(0, x^2)w^1(0, x^2) + v^2(0, x^2)w^2(0, x^2) \right) dx^2. \tag{4.5}$$

In this example, Γ^i_{jk} lies in L^2_{loc} . The geometry can be described as follows. Take a two-dimensional cone with total cone angle $2\pi c$. Truncate the cone at distance $\frac{1}{c}$ from the vertex. Take two copies of such truncated cones and glue them along their circle boundaries. Remove the two vertex points and take the universal cover.

Note that all of the tangent cones are isometric to \mathbb{R}^2 , but the Ricci measure is not absolutely continuous.

4.2 Gluing manifolds along their boundaries

Let M_1 and M_2 be Riemannian manifolds with boundaries. Let $A_{ij}^{(1)}$ (resp. $A_{ij}^{(2)}$) denote the second fundamental form of ∂M_1 (resp. ∂M_2), taking values in normal vectors. Let $H^{(1)}$ (resp. $H^{(2)}$) denote the mean curvature of ∂M_1 (resp. ∂M_2), also taking value in normal vectors. Our conventions are such that for the unit ball in \mathbb{R}^n , if v is a nonzero tangent vector to the unit sphere then $A(v, v)$ is inward pointing.

Let $\phi : \partial M_1 \rightarrow \partial M_2$ be an isometric diffeomorphism. Using the local product structure near ∂M_1 (resp. ∂M_2) coming from the normal exponential map, the result $M = M_1 \cup_\phi M_2$ of gluing M_1 to M_2 acquires a smooth structure. It also acquires a C^0 -Riemannian metric. Let $X \subset M$ denote the gluing locus.

Given a compactly supported Lipschitz vector-valued half-density V on M , using the isomorphism $TM|_X = TX \oplus N_X M$, we can decompose V on X as $V = V^T + V^\perp$,

where V^T is a section of $TX \otimes \mathcal{D}_M^{\frac{1}{2}}|_X$ and V^\perp is a section of $N_X M \otimes \mathcal{D}_M^{\frac{1}{2}}|_X$.

Given $x \in X$, let n_x be the inward pointing unit normal vector to M_1 at x . Given V and W , decompose them along X as $V = V^T + V^\perp$ and $W = W^T + W^\perp$. Then $\langle A^{(1)}(V_x^T, W_x^T) - A^{(2)}(V_x^T, W_x^T), n_x \rangle$ lies in $\mathcal{D}_M|_{\{x\}}$. We would get the same result if we switched the roles of M_1 and M_2 .

Similarly, $\langle V_x^\perp, W_x^\perp \rangle$ lies in $\mathcal{D}_M|_{\{x\}}$. We can compute the number $\langle H_x^{(1)} - H_x^{(2)}, n_x \rangle$. We would get the same result if we switched the roles of M_1 and M_2 .

Let x^0 be a local coordinate at x so that $\frac{\partial}{\partial x^0}$ is a unit normal to X at x . There is a unique linear map $\mathcal{T}_x : \mathcal{D}_M|_{\{x\}} \rightarrow \mathcal{D}_X|_{\{x\}}$ so that $dx^0 \otimes \mathcal{T}_x(\omega_x) = \omega_x$. This extends to a map $\mathcal{T} : C^\infty(X; \mathcal{D}_M|_X) \rightarrow C^\infty(X; \mathcal{D}_X)$.

Proposition 4.6

$$\begin{aligned}
 Q(V, W) &= \int_{M_1} \langle V, \text{Ric}(W) \rangle + \int_{M_2} \langle V, \text{Ric}(W) \rangle \\
 &\quad + \int_X \mathcal{T} \left(\langle A^{(1)}(V^T, W^T) - A^{(2)}(V^T, W^T), n \rangle \right. \\
 &\quad \left. + \langle H^{(1)} - H^{(2)}, n \rangle \langle V^\perp, W^\perp \rangle \right). \tag{4.6}
 \end{aligned}$$

Proof The Levi-Civita connection ∇ on M has $\{\Gamma^i_{jk}\}$ in L^2_{loc} , so we can just compute the usual Ricci tensor (3.13) as a distribution. On the interior of M_1 (resp. M_2), we clearly get the usual Ricci tensor of M_1 (resp. M_2), so it suffices to look at what happens near X . Since V and W are compactly supported, we can effectively reduce to the case when X is compact. We can choose a local coordinate x^0 near X , with $n = \frac{\partial}{\partial x^0}|_X$ pointing into M_1 , so that the metric takes the form

$$g = (dx^0)^2 + h(x_0) + O\left((x^0)^2\right). \tag{4.7}$$

Here we have a metric $h(x^0)$ on X for $x^0 \in (-\epsilon, \epsilon)$. As a function of x^0 , the metric h is continuous on $(-\epsilon, \epsilon)$, smooth on $[0, \epsilon)$ and smooth on $(-\epsilon, 0]$. The second

fundamental form of ∂M_1 (resp. ∂M_2) is $A^{(1)} = -\frac{1}{2} \left(\lim_{x^0 \rightarrow 0^+} \frac{dh}{dx^0} \right) n$ (resp. $A^{(2)} = -\frac{1}{2} \left(\lim_{x^0 \rightarrow 0^-} \frac{dh}{dx^0} \Big|_{x^0=0} \right) n$). Using local coordinates $\{x^i\}$ on X , we have

$$\begin{aligned} \lim_{x^0 \rightarrow 0^+} \Gamma^0_{ij} &= \langle A^{(1)}_{ij}, n \rangle, \\ \lim_{x^0 \rightarrow 0^-} \Gamma^0_{ij} &= \langle A^{(2)}_{ij}, n \rangle, \\ \lim_{x^0 \rightarrow 0^+} \Gamma^i_{j0} &= -\langle A^{i,(1)}_j, n \rangle, \\ \lim_{x^0 \rightarrow 0^-} \Gamma^i_{j0} &= -\langle A^{i,(2)}_j, n \rangle, \end{aligned} \tag{4.8}$$

The relevant terms in (3.13) are

$$\begin{aligned} R_{(kl)} &= \partial_0 \Gamma^0_{kl} + \dots, \\ R_{(00)} &= -\partial_0 \Gamma^i_{i0} + \dots. \end{aligned} \tag{4.9}$$

Hence the singular part of the Ricci measure is

$$\begin{aligned} R_{(kl),sing} &= \langle A^{(1)}_{kl} - A^{(2)}_{kl}, n \rangle \delta_0(x^0), \\ R_{(00),sing} &= \langle H^{(1)} - H^{(2)}, n \rangle \delta_0(x^0). \end{aligned} \tag{4.10}$$

The proposition follows. □

Example 4.7 Let M_1 and M_2 each be the result of taking a two-dimensional cone with total cone angle $2\pi c$ and truncating it at a distance L from the vertex. Then the contribution to the Ricci measure of M from the circle gluing locus is $\frac{2}{L} \int_X \mathcal{T} \langle V, W \rangle$. This is consistent with Example 4.5.

Remark 4.8 Based on Proposition 4.6, if M is a Riemannian manifold with boundary then it would be natural to consider

$$Q(V, W) = \int_M \langle V, \text{Ric}(W) \rangle + \int_{\partial M} \mathcal{T} \left(\langle A(V^T, W^T), n \rangle + \langle H, n \rangle \langle V^\perp, W^\perp \rangle \right) \tag{4.11}$$

to define the Ricci measure of M .

4.3 Families of cones

We first consider the case of a single cone.

Proposition 4.9 *For $\alpha < 1$, put $g = |x|^{-2\alpha} \sum_{i=1}^n (dx^i)^2$ on \mathbb{R}^n . Then the Levi-Civita connection is tame. If $n = 2$ then $Q(V, W)$ is given by (4.4). If $n > 2$ then $Q(V, W) = \int_{\mathbb{R}^n} \langle V, \text{Ric}(W) \rangle$. That is, if $n > 2$ then there is no singular contribution to the Ricci measure from the vertex of the cone.*

Proof The case $n = 2$ was handled in Example 4.4. If $n > 2$ then we can use the formula for conformal transformations from (4.1), with $\phi = -\alpha \ln |x|$. In this case $\partial_i \phi \in L^2_{loc}$, so the formula (3.13) makes sense as a distribution. However, since $|\partial_i \phi| \sim |x|^{-1}$, there is no contribution to $Q(V, W)$ from the origin. (To have such a contribution, one would need to have $|\partial_i \phi| \sim |x|^{-(n-1)}$). \square

We now consider a family of cones. Let $\pi : M \rightarrow B$ be an n -dimensional real vector bundle over a Riemannian manifold B . Given $b \in B$, we write $M_b = \pi^{-1}(b)$. Let h be a Euclidean inner product on M and let D be an h -compatible connection. There is a natural Riemannian metric g_0 on M with $\pi : M \rightarrow B$ being a Riemannian submersion, so that the restrictions of g_0 to fibers are specified by h , and with horizontal subspaces coming from D . Let $s : B \rightarrow M$ denote the zero section and let Z denote its image. Given $\alpha < 1$, let g be the Riemannian metric on $M - Z$ obtained from g_0 , at $m \in M - Z$, by multiplying the fiberwise component of g_0 by $h(m, m)^{-\alpha}$.

Given $z \in Z$, let $\text{dvol}_{M_{\pi(z)}, z}$ denote the Riemannian density at z of the fiber $M_{\pi(z)}$, induced from h_z . There is a unique linear map $\mathcal{T}_z : \mathcal{D}_M|_{\{z\}} \rightarrow \mathcal{D}_Z|_{\{z\}}$ so that $\text{dvol}_{M_{\pi(z)}, z} \otimes \mathcal{T}_z(\omega_z) = \omega_z$ for all $\omega_z \in \mathcal{D}_M|_{\{z\}}$. This extends to a linear map $\mathcal{T} : C^\infty(Z; \mathcal{D}_M|_Z) \rightarrow C^\infty(Z; \mathcal{D}_Z)$.

Given a compactly supported Lipschitz vector-valued half-density V on M , we can decompose its restriction to Z orthogonally (with respect to g_0) as $V|_Z = V^{tan} + V^{nor}$, where V^{tan} is tangential to Z and V^{nor} is normal to Z , i.e. tangential to the fibers of the vector bundle.

Proposition 4.10 *The Levi-Civita connection ∇ of g is tame. If $n = 2$ then*

$$Q(V, W) = \int_M \langle V, \text{Ric}(W) \rangle + 2\pi\alpha \int_Z \mathcal{T}(\langle V^{nor}, W^{nor} \rangle_{g_0}). \tag{4.12}$$

If $n > 2$ then $Q(V, W) = \int_M \langle V, \text{Ric}(W) \rangle$.

Proof We can choose local coordinates $\{x^\beta, x^i\}$ for M so that the coordinates $\{x^\beta\}$ pullback from B and the coordinates $\{x^i\}$ restrict to the fibers as linear orthogonal coordinates with respect to h . In terms of such coordinates, we can write

$$g_0 = \sum_{\beta, \gamma} k_{\beta\gamma} dx^\beta dx^\gamma + \sum_i \left(dx^i + \sum_{\beta, j} C^i_{j\beta} x^j dx^\beta \right)^2, \tag{4.13}$$

where $\{C^i_{j\beta}\}$ is the local description of the connection D and $\{k_{\beta\gamma}\}$ is the local description of the Riemannian metric on B . Then

$$g = \sum_{\beta, \gamma} k_{\beta\gamma} dx^\beta dx^\gamma + \left(\sum_l (x^l)^2 \right)^{-\alpha} \sum_i \left(dx^i + \sum_{\beta, j} C^i_{j\beta} x^j dx^\beta \right)^2, \tag{4.14}$$

Let $\{\widehat{\Gamma}^\beta_{\gamma\delta}\}$ denote the Christoffel symbols of the Riemannian metric k on B . Put

$$F^i_{j\beta\gamma} = \partial_\beta C^i_{j\gamma} - \partial_\gamma C^i_{j\beta} + \sum_k C^i_{k\beta} C^k_{j\gamma} - \sum_k C^i_{k\gamma} C^k_{j\beta}, \tag{4.15}$$

the curvature of D .

Given $b \in B$, we can choose the coordinates $\{x^i\}$ near the fiber M_b so that $C^i_{j\beta}(b) = 0$. Then on M_b , we have

$$g|_{M_b} = \sum_{\beta,\gamma} k_{\beta\gamma} dx^\beta dx^\gamma + \left(\sum_l (x^l)^2 \right)^{-\alpha} \sum_i (dx^i)^2. \tag{4.16}$$

One finds that on M_b ,

$$\Gamma^i_{jk} = -\frac{\alpha}{|x|^2} (x_k \delta^i_j + x^j \delta^i_k - x^i \delta_{jk}), \tag{4.17}$$

$$\Gamma^i_{\beta\gamma} = \frac{1}{2} \sum_j x^j (\partial_\gamma C^i_{j\beta} + \partial_\beta C^i_{j\gamma}),$$

$$\Gamma^\beta_{\gamma i} = \Gamma^\beta_{i\gamma} = -\frac{1}{2} |x|^{-2\alpha} \sum_{\sigma,j} k^{\beta\sigma} x^j F^i_{j\sigma\gamma},$$

$$\Gamma^\beta_{\gamma\delta} = \widehat{\Gamma}^\beta_{\gamma\delta},$$

$$\Gamma^\beta_{ij} = \Gamma^i_{j\beta} = \Gamma^i_{\beta j} = 0.$$

Using Proposition 3.3, one can check that ∇ is tame.

The Ricci curvature of ∇ can be computed using the splitting of TM into its vertical and horizontal components relative to π . The corresponding O’Neill formulas still hold for the Ricci measure. In the present case, the fibers of π are totally geodesic with respect to the metric g . Relative to the vertical orthonormal coframe

$$\tau^i = |x|^{-\alpha} \left(dx^i + \sum_j C^i_{j\beta} x^j dx^\beta \right) \tag{4.18}$$

and a local orthonormal coframe $\{\tau^\beta\}$ for k , one finds that the curvature of the horizontal distribution is given by

$$A^i_{\beta\gamma} = \frac{1}{2} |x|^{-\alpha} \sum_j F^i_{j\beta\gamma} x^j. \tag{4.19}$$

Then using the O’Neill formulas, as given in [16, (4.7)], one finds that the only singular contribution to the Ricci measure is the fiberwise contribution coming from the singular points. Using Proposition 4.9, the proposition follows. \square

4.4 Kähler manifolds

Let M be a complex manifold of complex dimension n . Suppose that M admits a Kähler metric h which is $W_{loc}^{1,1}$ -regular, in the sense of Definition 3.10. Suppose that the Levi-Civita connection is tame. The Ricci measure of (M, h) can be described as follows. Let V be a compactly supported Lipschitz section of $T^{(1,0)}M \otimes \mathcal{D}^{\frac{1}{2}}$ and let W be a compactly supported Lipschitz section of $T^{(0,1)}M \otimes \mathcal{D}^{\frac{1}{2}}$. Then $Q(V, W) = \int_M q(V, W)$, where the measure $q(V, W)$ has the following description in local coordinates. Write $V = \sum_i v^i \partial_{z^i} \otimes \sqrt{dx^1 dy^1 \dots dx^n dy^n}$ and $W = \sum_j w^j \partial_{\bar{z}^j} \otimes \sqrt{dx^1 dy^1 \dots dx^n dy^n}$. Then

$$q(V, W) = - \sum_{i,j} (\partial_{z^i} \partial_{\bar{z}^j} \log \det h) v^i \bar{w}^j dx^1 dy^1 \dots dx^n dy^n. \tag{4.20}$$

4.5 Limit spaces of manifolds with lower bounds on Ricci curvature and injectivity radius

Given $n \in \mathbb{Z}^+$, $K \in \mathbb{R}$ and $i_0 > 0$, let (X, x) be a pointed Gromov–Hausdorff limit of a sequence $\{(M_i, m_i, g_i)\}_{i=1}^\infty$ of complete n -dimensional pointed Riemannian manifolds with $\text{Ric}(g_i) \geq K g_i$ and $\text{inj}_{m_i} \geq i_0$. From [1, Theorem 0.2 and p. 268], for any $p \in (n, \infty)$, the space X is a $L^{2,p}$ -manifold with a Riemannian metric g_X that is locally $L^{1,p}$ -regular. In particular, for any $\alpha \in (0, 1)$ the manifold X is also a $C^{1,\alpha}$ -manifold and hence has an underlying smooth structure, which is unique up to diffeomorphism [9, Theorem 2.10]. In order to apply the formalism of this paper, we extend the smooth structure to a $C^{1,1}$ -structure. Any two such $C^{1,1}$ -structures are related by a homeomorphism ϕ of X that is $L^{2,p}$ -regular for all $p \in (n, \infty)$.

As mentioned, with respect to the $L^{2,p}$ -manifold structure, g_X is locally $L^{1,p}$ -regular (and also locally C^α -regular). Hence the same will be true with respect to the smooth structure, and the ensuing $C^{1,1}$ -structure. By Hölder’s inequality, g_X is also locally $L^{1,1}$ -regular. Since g_X is nondegenerate and continuous in local coordinates, it has a continuous inverse. Putting $S^l_{jk} = -\sum_i g^{li} \partial_k g_{ij}$, the metric g_X lies in $W_{loc}^{1,1}$ in the sense of Definition 3.10. Its Christoffel symbols lie in L^2_{loc} (since g_X is locally $L^{1,2}$ -regular). Hence the Levi-Civita connection of g_X is tame. Also from [1], for large i there are pointed diffeomorphisms $\phi_i : (X, x) \rightarrow (M, m_i)$ so that $\lim_{i \rightarrow \infty} \phi_i^* g_i = g_X$ in $L^{1,p}_{loc}$. By Remark 3.8, it follows that $Q(V, W) \geq K \int_X g_{X,ij} V^i W^j$. Hence there is a Ricci measure \mathcal{R} .

Since the Christoffel symbols lie in L^2_{loc} , the quadratic form Q can be extended to compactly supported V and W in $\bigcap_{p \in (n, \infty)} L^{1,p}$, the latter of which also lies in $\bigcap_{\alpha \in (0,1)} C^\alpha$. It follows that Q is covariant with respect to diffeomorphisms ϕ of X that are $L^{2,p}$ -regular for all $p \in (n, \infty)$. Hence \mathcal{R} is independent of the choice of $C^{1,1}$ -structure.

4.6 Remarks

Remark 4.11 Let g be a Riemannian metric on M^n which lies in $W_{loc}^{1,1}$, in the sense of Definition 3.10. Suppose that the Levi-Civita connection is tame.

Suppose that the length metric gives a well-defined compact metric space X . If $Q \geq 0$ then a natural question is whether X has nonnegative n -Ricci curvature (with respect to the Hausdorff measure) in the sense of [17,28]. One way to answer this would be to show that X is the Gromov–Hausdorff limit of a sequence of smoothings $\{(M, g_i)\}_{i=1}^\infty$ of (M, g) , with the Ricci curvature of (M, g_i) bounded below by $-\frac{1}{i}g_i$. In the setting of Sect. 4.2, i.e. gluing Riemannian manifolds along boundaries, the argument for this appears in [22, Sect. 4].

Conversely, one can ask whether X having nonnegative n -Ricci curvature implies that $Q \geq 0$.

Remark 4.12 Suppose that the metric space X of Remark 4.11 has $Q \geq 0$ and non-negative Ricci curvature in the sense of [17,28]. One can ask if there is a relationship between the possible singularity of the Ricci measure and the existence of a singular stratum of X in the sense of [3]. Example 4.5 shows that there is no direct relationship, since the Ricci measure may be singular even if all of the tangent cones are Euclidean. However, one can ask whether the existence of a codimension-two singular stratum (i.e. $\mathcal{S}_{n-2} \neq \mathcal{S}_{n-3}$ in the notation of [3]) forces the Ricci measure to be singular.

Remark 4.13 If X has no singular strata of codimension less than three (i.e. $\mathcal{S} = \mathcal{S}_{n-3}$ in the notation of [3]) then one can ask whether a compactly supported Lipschitz vector-valued half-density V necessarily has ∇V square integrable. The relevance of this would be for nonmanifold spaces, where the square integrability of ∇V on the regular set would be a natural condition, whereas the requirement of V being Lipschitz may not make sense in a neighborhood of a singular point. See Example 5.12.

We note that the singular Kähler–Einstein metrics considered in [6] do not have any singular strata of real codimension two.

Remark 4.14 Suppose that X is a pointed Gromov–Hausdorff limit of a sequence of complete n -dimensional pointed Riemannian manifolds with a uniform lower bound on their Ricci curvature. For concreteness, we consider the case when the Hausdorff dimension of X is n , i.e. when the sequence is noncollapsing. As in [4, Sect. 3], the complement of a measure zero subset of X can be covered by a countable union of sets, each of which is biLipschitz to a Borel subset of \mathbb{R}^n . Furthermore, the transition maps between such sets can be taken to be $C^{1,1}$ -regular in a natural weak sense [10, Theorem 1.5]. Using this structure, one can define a Levi-Civita connection on X with measurable Christoffel symbols [10, Sect. 3]. The first question is whether the Levi-Civita connection is necessarily tame, in the sense of satisfying the equivalent condition of Proposition 3.3. This is true in the setting of Sect. 4.5.

If the Levi-Civita connection is tame then it should be possible to use this to construct the Ricci measure on the regular set of X (compare with Example 4.5 and Sect. 4.5). Based on Proposition 4.10, one would expect that any reasonable notion of the Ricci measure on the singular set should vanish on \mathcal{S}_{n-3} , and be given on \mathcal{S}_{n-2} by an analog of the last term in (4.12).

5 Weak Ricci flow

In this section we give notions of weak Ricci flow solutions. In Sect. 5.1 we prove an integral identity for smooth Ricci flow solutions. In Sect. 5.2 we define tame Ricci flow solutions, give a compactness result and discuss some examples. In Sect. 5.3 we define the broader class of cone-preserving Ricci flow solutions and give further examples.

5.1 An integral identity

Let M be a smooth manifold. Let $\{g(t)\}_{t \in [0, T]}$ be a smooth one-parameter family of Riemannian metrics on M .

Proposition 5.1 *Let $\{V(t)\}_{t \in [0, T]}$ and $\{W(t)\}_{t \in [0, T]}$ be one-parameter families of vector-valued half-densities on M . We assume that for each $T' \in [0, T)$, the family V has compact support in $M \times [0, T']$ and is Lipschitz there, and similarly for the family W . Then $(M, g(\cdot))$ satisfies the Ricci flow equation*

$$\frac{dg}{dt} = -2 \operatorname{Ric}_{g(t)} \tag{5.1}$$

if and only if for every such V and W , and every $t \in [0, T)$, we have

$$\begin{aligned} \int_M \sum_{ij} g_{ij}(t) V^i(t) W^j(t) &= \int_M \sum_{ij} g_{ij}(0) V^i(0) W^j(0) \\ &+ \int_0^t \int_M \sum_{i,j} \left[g_{ij}(\partial_s V^i) W^j + g_{ij} V^i (\partial_s W^j) \right. \\ &\left. - 2 (\nabla_i V^i) (\nabla_j W^j) + 2 (\nabla_i V^j) (\nabla_j W^i) \right] (s) ds. \end{aligned} \tag{5.2}$$

Proof Suppose that $(M, g(\cdot))$ is a Ricci flow solution. Then

$$\frac{d}{dt} \sum_{ij} g_{ij}(t) V^i(t) W^j(t) = -2 \sum_{ij} R_{ij} V^i W^j + \sum_{ij} g_{ij}(\partial_t V^i) W^j + \sum_{ij} g_{ij} V^i (\partial_t W^j). \tag{5.3}$$

Integrating gives (5.2). Conversely, if (5.2) holds then by taking V and W smooth and differentiating in t , we see that (5.3) holds for all smooth V and W . This implies that (5.1) holds. □

5.2 Tame Ricci flow

Now let M be a $C^{1,1}$ -manifold.

Definition 5.2 Let $\{g(t)\}_{t \in [0, T]}$ be a one-parameter family of $W_{loc}^{1,1}$ -Riemannian metrics on M , in the sense of Definition 3.10, which is locally- L^1 on $M \times [0, T)$. Suppose that for each $t \in [0, T)$, the Levi-Civita connection of $g(t)$ is tame in the sense of Definition 3.1. Suppose that there is an integrable function $c : (0, T) \rightarrow \mathbb{R}^+$ so that for all $t \in (0, T)$, the time- t Ricci measure satisfies $Q(V, W) \geq -c(t) \int_M g_{ij}(t) V^i W^j$. Let $\{V(t)\}_{t \in [0, T)}$ and $\{W(t)\}_{t \in [0, T)}$ be one-parameter families of vector-valued half-densities on M . We assume that for each $T' \in [0, T)$, the family V has compact support in $M \times [0, T')$ and is Lipschitz there, and similarly for the family W . We say that $\{g(t)\}_{t \in [0, T)}$ is a *tame Ricci flow solution* if (5.2) is satisfied for all such V and W , and all $t \in [0, T)$.

Example 5.3 Let $\{h(t)\}_{t \in [0, \infty)}$ be a smooth Ricci flow solution on M . Given a $C^{1,1}$ -diffeomorphism ϕ of M , put $g(t) = \phi^*h(t)$. Then $\{g(t)\}_{t \in [0, T)}$ is a tame Ricci flow solution. This is because equation (5.2) for g, V and W , is equivalent to equation (5.2) for h, ϕ_*V and ϕ_*W . Hence Proposition 5.1 applies.

Example 5.4 For all $t \geq 0$, let $g(t)$ be the metric of Example 4.4. We claim that if $\alpha \neq 0$ then $\{g(t)\}_{t \in [0, \infty)}$ is not a tame Ricci flow solution. This can be seen by taking V and W to be time-independent in (5.2).

We show that the property of being a tame Ricci flow solution passes to Lipschitz limits.

Proposition 5.5 *Let M be a $C^{1,1}$ -manifold. Let $\{g_i(\cdot)\}_{i=1}^\infty$ be a sequence of one-parameter families of Riemannian metrics on M , each defined for $t \in [0, T)$ and locally- L^1 on $M \times [0, T)$, with each $g_i(t)$ locally Lipschitz, satisfying (5.2). Let $\{g_\infty(t)\}_{t \in [0, T)}$ be a one-parameter family of locally Lipschitz Riemannian metrics on M . Suppose that for all $T' \in [0, T)$ and every coordinate neighborhood $U \subset M$ with compact closure,*

$$\lim_{i \rightarrow \infty} \sup_{t \in [0, T']} d_{\text{Lip}(U)}(g_i(t), g_\infty(t)) = 0. \tag{5.4}$$

Then $g_\infty(\cdot)$ is a tame Ricci flow solution.

Proof From the convergence assumption, g_∞ is locally- L^1 on $M \times [0, T)$. The Christoffel symbols of $\{g_i(t)\}_{i=1}^\infty$ and $g_\infty(t)$ are all locally- L^∞ , so the Levi-Civita connections are tame. Because the Levi-Civita connections of $\{g_i\}_{i=1}^\infty$ converge to that of g_∞ , in L_{loc}^∞ on $M \times [0, T)$, it follows that g_∞ satisfies (5.2). \square

Example 5.6 Let $\{g(t)\}_{t \in [0, T)}$ be a smooth Ricci flow solution on a smooth manifold M . Let $\{\phi_i\}_{i=1}^\infty$ be a sequence of smooth diffeomorphisms of M that $C^{1,1}$ -converge on compact subsets to a $C^{1,1}$ -diffeomorphism ϕ_∞ of M . Then $\lim_{i \rightarrow \infty} \phi_i^*g(\cdot) = \phi_\infty^*g(\cdot)$ with Lipschitz convergence on compact subsets of $M \times [0, T)$, and $\phi_\infty^*g(\cdot)$ is a tame Ricci flow solution.

We now give a compactness result for tame Ricci flow solutions.

Proposition 5.7 *Let M be a C^2 -manifold. Let $\{g_i(\cdot)\}_{i=1}^\infty$ be a sequence of tame Ricci flow solutions, defined for $t \in [0, T)$, consisting of C^1 -Riemannian metrics on M . Suppose that for all $T' \in [0, T)$ and every coordinate neighborhood $U \subset M$ with compact closure, the $g_i(t)$'s are uniformly bounded above and below on $U \times [0, T']$, and the $g_i(t)$'s and their first spatial partial derivatives are uniformly bounded and uniformly equicontinuous (in i) on $U \times [0, T']$. Then after passing to a subsequence, there is a tame Ricci flow solution $\{g_\infty(t)\}_{t \in [0, T)}$ on M consisting of C^1 -Riemannian metrics so that the $g_i(t)$'s C^1 -converge to $g_\infty(t)$, locally uniformly on $M \times [0, T)$.*

Proof By a diagonal argument, after passing to a subsequence we can assume that $\lim_{i \rightarrow \infty} g_i(\cdot) = g_\infty(\cdot)$ as stated in the proposition. Then Proposition 5.5 shows that $g_\infty(\cdot)$ is a tame Ricci flow solution. □

We now address when one can get a tame Ricci flow solution by appending a time-zero slice to a smooth Ricci flow solution defined for positive time.

Proposition 5.8 *Let $\{g(t)\}_{t \in (0, T)}$ be a smooth Ricci flow solution with $\text{Ric}(g(t)) \geq -c(t)g(t)$ for some positive integrable function c . Suppose that there is some $g(0) \in W_{loc}^{1,1}$ with tame Levi-Civita connection so that $\lim_{t \rightarrow 0^+} g(t) = g(0)$ in L_{loc}^1 . Then $\{g(t)\}_{t \in [0, T)}$ is a tame Ricci flow solution.*

Proof Let V and W be one-parameter families as in Definition 5.2. For any $t' \in [0, T)$ and $t \in [t', T)$, the analog of (5.2) holds with 0 replaced by t' . Taking $t' \rightarrow 0$ shows that (5.2) holds. □

Example 5.9 Let Σ be a compact two-dimensional metric space with curvature bounded in the Alexandrov sense. One can construct a Ricci flow solution starting from Σ , in an certain sense, which will be smooth for positive time [25]. Using [25, Lemma 3.3] and Proposition 5.8, we can extend the solution back to time zero to get a tame Ricci flow solution $g(\cdot)$ that exists on some time interval $[0, T)$, with Σ corresponding to $g(0)$.

Example 5.10 Let (\mathbb{R}^2, g_0) be a two-dimensional metric cone with total cone angle in $(0, 2\pi]$. There is a corresponding expanding soliton $\{g(t)\}_{t>0}$ with the property that at any positive time, the tangent cone at infinity is isometric to (\mathbb{R}^2, g_0) [5, Sect. 2.4]. Putting $g(0) = g_0$, one obtains a tame Ricci flow solution $\{g(t)\}_{t \geq 0}$.

Remark 5.11 Let g_0 be a Lipschitz-regular Riemannian metric on a compact $C^{1,1}$ -manifold M . Choose a compatible smooth structure on M . From [12, Theorem 5.3], there is a smooth solution $\{h(t)\}_{t \in (0, T)}$ to the DeTurck–Ricci flow with $\lim_{t \rightarrow 0^+} h(t) = g_0$. Fixing $t_0 \in (0, T)$ and using the estimate in [12, Theorem 5.3], we can integrate the vector field in the DeTurck trick backward in time from t_0 to 0, to obtain a homeomorphism ϕ of M . We can think of ϕ as giving a preferred smooth structure based on the Riemannian metric g_0 . We can also undo the vector field in the DeTurck trick, starting at time t_0 , to obtain a smooth Ricci flow solution $\{\widehat{g}(t)\}_{t \in (0, T)}$. Presumably ϕ is $C^{1,1}$ and there is a tame Ricci flow solution starting from g_0 , given by $g(t) = \phi^* \widehat{g}(t)$.

5.3 Cone-preserving Ricci flow

There has been recent work about Ricci flow on certain singular spaces with conical singularities along codimension-two strata, under the requirement that the flow preserve the conical singularities [18, 19, 23, 26, 27, 30, 31]. As seen in Example 5.4, such solutions may not be tame Ricci flow solutions. However, one can consider an alternative and less restrictive definition, which we call *cone-preserving Ricci flow solutions*, in which the V and W of Definition 5.2 are additionally required to have square-integrable covariant derivative.

Example 5.12 For all $t \geq 0$, let $g(t)$ be the metric of Example 4.4. We claim that $\{g(t)\}_{t \in [0, \infty)}$ is a cone-preserving Ricci flow solution. To see this, we can assume that $\alpha \neq 0$. Suppose that V is a compactly supported Lipschitz vector-valued half-density. Writing $V = \sum_i v^i \partial_i \otimes \sqrt{dx^1 dx^2}$, one finds $\nabla_i V = \sum_j \nabla_i v^j \otimes \sqrt{dx^1 dx^2}$, where

$$\begin{aligned} \nabla_1 v^1 &= \partial_1 v^1 + (\partial_2 \phi) v^2, \\ \nabla_1 v^2 &= \partial_1 v^2 - (\partial_2 \phi) v^1, \\ \nabla_2 v^1 &= \partial_2 v^1 - (\partial_1 \phi) v^2, \\ \nabla_2 v^2 &= \partial_2 v^2 + (\partial_1 \phi) v^1. \end{aligned} \tag{5.5}$$

Here $\phi(x^1, x^2) = -\frac{\alpha}{2} \log((x^1)^2 + (x^2)^2)$. Now the square norm of ∇V is

$$\int_M \sum_{i,j,k,l} g_{jl} g^{ik} (\nabla_i v^j)(\nabla_k v^l) dx^1 dx^2 = \int_M \sum_{i,j} (\nabla_i v^j)^2 dx^1 dx^2. \tag{5.6}$$

Suppose that this is finite. As each v^i is Lipschitz, and $\partial_i \phi = -\alpha \frac{x^i}{(x^1)^2 + (x^2)^2}$, it follows that $v^i(0) = 0$. Then (4.4) gives that $Q(V, W) = 0$. Looking at (5.2), the claim follows.

This example shows that there may be nonuniqueness among cone-preserving Ricci flow solutions with a given initial condition, in view of the expanding soliton solution mentioned in Example 5.10.

We see that in this example, the cone angle along the codimension-two stratum is unchanged. If one wants to give a notion of a weak Ricci flow solution along these lines on a nonmanifold space (which we do not address here), it is probably natural to impose the square integrability of ∇V as a requirement.

Example 5.13 For $k, n > 1$, consider $\mathbb{C}^n / \mathbb{Z}_k$, where the generator of \mathbb{Z}_k acts isometrically on the flat \mathbb{C}^n as multiplication by $e^{2\pi i/k}$. We expect that with any reasonable definition of a weak Ricci flow, this will give a static Ricci flow solution (since Proposition 4.9 indicates that the vertex of a cone with a smooth link, in real dimension greater than two, should not contribute to the Ricci measure).

On the other hand, there is an expanding soliton solution that exists for $t > 0$, and whose $t \rightarrow 0$ limit is $\mathbb{C}^n / \mathbb{Z}_k$ [7, Sect. 5]. This shows that one cannot expect a uniqueness result for weak Ricci flow solutions whose time-zero slice is a nonmanifold, without further restrictions.

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