The geometry of the space of measures and its applications

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Singer: 33 Children
Singer : 94 Grandchildren
Singer: 18 Great-grandchildren
Geometry of the space of probability measures

Motivation
- Optimal transport
- Formal Geometry of Wasserstein Space
- Metric geometry of Wasserstein space
- Ricci meets Wasserstein
- Some more metric geometry
- Generalized entropy functionals
- Abstract Ricci curvature

Applications
- Perelman’s reduced volume
- Formulas from Riemannian optimal transport
- Optimal transport for Ricci flow
- Monotonicity of the reduced volume
Infinite-dimensional spaces

Geometry and topology of infinite-dimensional spaces

Example: the space of connections modulo gauge transformations (Atiyah-Singer, ...)

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Today: the space of probability measures.
The motivation comes from questions about finite-dimensional spaces.

How can we understand Ricci curvature?

Does it make sense to talk about Ricci curvature for nonsmooth spaces?
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Claim: These questions can be answered in terms of optimal transport, or the geometry of the space of probability measures.
Partly joint work with Cedric Villani (ENS-Lyon).
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LV = Lott-Villani, S = Sturm

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Motivation

**Optimal transport**

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Dirtmoving

Given a *before* and an *after* dirtpile, what is the most efficient way to move the dirt from one place to the other?

Let’s say that the **cost** to move a gram of dirt from $x$ to $y$ is $d(x, y)^2$. 
Mémoire sur la théorie des déblais et des remblais (1781)

Memoir on the theory of excavations and fillings (1781)
MÉMOIRE
SUR LA
THÉORIE DES DÉBLAIS
ET DES REMBLAIS.
Par M. Monge.

Lorsqu'on doit transporter des terres d'un lieu dans un autre, on a coutume de donner le nom de Déblai au volume des terres que l'on doit transporter, et le nom de Remblai à l'espace qu'elles doivent occuper après le transport.

Le prix du transport d'une molécule étant, toutes choses égales, proportionnel à son poids et à l'espace qu'on lui fait parcourir, et par conséquent le prix du transport total devant être proportionnel à la somme des produits des molécules multipliées chacune par l'espace parcouru, il s'en suit que le déblai et le remblai étant donnés de figure et de position, il n'est pas indifférent que telle molécule du déblai soit transportée dans tel ou tel autre endroit du remblai, mais qu'il y a une certaine distribution à faire des molécules du premier dans le second, d'après laquelle la somme de ces produits fera la moindre possible, et le prix du transport total fera un minimum.

C'est la solution de cette question que je me propose de donner ici. Je diviserai ce Mémoire en deux parties, dans la première je supposerai que les déblais et les remblais ont des aires contenues dans un même plan ; dans le second, je supposerai que ce sont des volumes.

PREMIÈRE PARTIE.
Du transport des aires planes sur des aires comprises dans un même plan.

1.
Quelle que soit la route que doive suivre une molécule
Let \((X, d)\) be a compact metric space.

**Notation**

\(P(X)\) is the set of Borel probability measures on \(X\).

That is, \(\mu \in P(X)\) iff \(\mu\) is a nonnegative Borel measure on \(X\) with \(\mu(X) = 1\).

**Definition**

Given \(\mu_0, \mu_1 \in P(X)\), the Wasserstein distance \(W_2(\mu_0, \mu_1)\) is the square root of the minimal cost to transport \(\mu_0\) to \(\mu_1\).
Wasserstein space

\[ W_2(\mu_0, \mu_1)^2 = \inf \left\{ \int_{X \times X} d(x, y)^2 \, d\pi(x, y) \right\}, \]

where

\[ \pi \in P(X \times X), (p_0)_*\pi = \mu_0, (p_1)_*\pi = \mu_1. \]
Fact:

$(\mathcal{P}(X), W_2)$ is a metric space, called the Wasserstein space.

The metric topology is the weak-$\ast$ topology, i.e. $\lim_{i \to \infty} \mu_i = \mu$ if and only if for all $f \in C(X)$, $\lim_{i \to \infty} \int_X f \, d\mu_i = \int_X f \, d\mu$. 
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So to one compact metric space \((X, d)\), we’ve assigned another one \((P(X), W_2)\).
Fact:

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The metric topology is the weak-* topology, i.e. \(\lim_{i \to \infty} \mu_i = \mu\) if and only if for all \(f \in C(X)\), \(\lim_{i \to \infty} \int_X f \, d\mu_i = \int_X f \, d\mu\).

So to one compact metric space \((X, d)\), we’ve assigned another one \((P(X), W_2)\).

Note: There is an isometric embedding \(X \to P(X)\) by \(x \to \delta_x\).
Displacement interpolations

To move dirt in the real world, we would transport it along minimizing geodesics.
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\[ S = -\int \rho \log \rho \]

\[ \{ \mu_t \}_{t \in [0,1]} \]
Displacement interpolations

To move dirt in the real world, we would transport it along minimizing geodesics.

Take a snapshot at time $t$. We get a family of measures $\{\mu_t\}_{t \in [0,1]}$, called a displacement interpolation. We would like to say that this is a “geodesic” in $P(X)$. 
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If \((M, g)\) is a compact connected Riemannian manifold, let \(P^\infty(M) \subset P(M)\) be the smooth probability measures with positive density.

\[
P^\infty(M) = \{\rho \ d\text{vol}_M : \rho \in C^\infty(M), \rho > 0, \int_M \rho \ d\text{vol}_M = 1\}.
\]
Otto’s formal Riemannian metric on $P^\infty(M)$

Given $\mu = \rho \, \text{dvol}_M \in P^\infty(M)$, consider an infinitesimally nearby measure $\mu + \delta \mu$, i.e.

$$\delta \mu = (\delta \rho) \, \text{dvol}_M \in T_\mu P^\infty(M).$$

Solve $\delta \rho = - \sum_i \nabla^i (\rho \nabla_i \phi)$ for $\phi \in C^\infty(M)$, unique up to an additive constant.

Definition: $\langle \delta \mu, \delta \mu \rangle = \int_M |\nabla \phi|^2 \rho \, \text{dvol}_M$. This is the $H^{-1}$ Sobolev metric, in terms of $\rho$. 
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Say $c : [0, 1] \to P^\infty(M)$ is a smooth curve.

Write $c(t) = \rho(t) \, dvol_M$. 

Corresponding energy of a curve

Say \( c : [0, 1] \to P^\infty(M) \) is a smooth curve.

Write \( c(t) = \rho(t) \, d\text{vol}_M \).

Fact: We can solve

\[
\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi)
\]

for \( \phi \equiv \phi(t) \in C^\infty(M) \).
Benamou-Brenier variational problem

From $\{\rho(t)\}_{t \in [0,1]}$, we got $\{\phi(t)\}_{t \in [0,1]}$. 
Benamou-Brenier variational problem

From \( \{ \rho(t) \}_{t \in [0,1]} \), we got \( \{ \phi(t) \}_{t \in [0,1]} \).

**Definition**

\[
E(c) = \frac{1}{2} \int_{0}^{1} \int_{M} |\nabla \phi|^2 \rho \ d\text{vol}_M \ dt.
\]

This is the energy of the curve \( c \).
Theorem: (Otto-Westdickenberg 2005)

$$\frac{1}{2} W_2(\mu_0, \mu_1)^2 = \inf\{ E(c) : c(0) = \mu_0, c(1) = \mu_1 \}.$$
Theorem : (Otto-Westdickenberg 2005)

\[ \frac{1}{2} W_2(\mu_0, \mu_1)^2 = \inf \{ E(c) : c(0) = \mu_0, c(1) = \mu_1 \} . \]

That is, the geodesic distance coming from Otto’s metric is the Wasserstein distance \( W_2 \), at least on \( P^\infty(M) \).
Theorem: (Otto-Westdickenberg 2005)

\[ \frac{1}{2} W_2(\mu_0, \mu_1)^2 = \inf \{ E(c) : c(0) = \mu_0, c(1) = \mu_1 \}. \]

That is, the geodesic distance coming from Otto’s metric is the Wasserstein distance \( W_2 \), at least on \( P^\infty(M) \).

Note: the infimum may not be achieved. A minimizing \( c \) is a smooth displacement interpolation.
The Euler-Lagrange equation for the functional $E$ is

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\begin{align*}
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\end{align*}

We also had

\begin{align*}
\frac{\partial \rho}{\partial t} &= -\sum_i \nabla^i (\rho \nabla_i \phi).
\end{align*}

These are the equations for optimal transport and can be solved explicitly. (First worked out for Riemannian manifolds by Robert McCann 2001.)
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Can we extend these statements from formal results about $P^\infty(M)$ to rigorous results about $P(M)$?
Passing to metric geometry

Can we extend these statements from formal results about $P^\infty(M)$ to rigorous results about $P(M)$?

Or, more generally, about $P(X)$ for a nonsmooth space $X$?

Use ideas from metric geometry.
Review of length spaces

Say \((X, d)\) is a compact metric space and \(\gamma : [0, 1] \to X\) is a continuous map.
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The length of \(\gamma\) is

\[
L(\gamma) = \sup_J \sup_{0 = t_0 \leq t_1 \leq \ldots \leq t_J = 1} \sum_{j=1}^J d(\gamma(t_{j-1}), \gamma(t_j)).
\]
Definition

$(X, d)$ is a length space if the distance between two points $x_0, x_1 \in X$ equals the infimum of the lengths of curves joining them, i.e.

$$d(x_0, x_1) = \inf \{ L(\gamma) : \gamma(0) = x_0, \gamma(1) = x_1 \}.$$
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$$d(x_0, x_1) = \inf\{L(\gamma) : \gamma(0) = x_0, \gamma(1) = x_1\}.$$  

A length-minimizing curve is called a geodesic.
Length spaces

**Examples** of length spaces:
1. The underlying metric space of any Riemannian manifold.
2.
Length spaces

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1. The underlying metric space of any Riemannian manifold.
2.

Nonexamples:
1. A finite metric space with more than one point.
2. A circle with the chordal metric.
Proposition: (LV,S)

If $X$ is a length space then so is the Wasserstein space $P(X)$.

Hence we can talk about its (minimizing) geodesics $\{\mu_t\}_{t\in[0,1]}$, called \textit{Wasserstein geodesics}.
Proposition : (LV,S)

If $X$ is a length space then so is the Wasserstein space $P(X)$. Hence we can talk about its (minimizing) geodesics $\{\mu_t\}_{t\in[0,1]}$, called Wasserstein geodesics.

Proposition : (LV)

The Wasserstein geodesics are exactly the displacement interpolations $\{\mu_t\}_{t\in[0,1]}$. 
Curvature of Wasserstein space

**Formal calculation** : (Otto 2001)

$P(\mathbb{R}^n)$ has nonnegative sectional curvature.
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If a Riemannian manifold $M$ has nonnegative sectional curvature then the length space $P(M)$ has nonnegative curvature in the Alexandrov sense.
Curvature of Wasserstein space

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\( P(\mathbb{R}^n) \) has nonnegative sectional curvature.

Theorem: (LV,S)

If a Riemannian manifold \( M \) has nonnegative sectional curvature then the length space \( P(M) \) has nonnegative curvature in the Alexandrov sense.

Open question:

To what extent is \( P(M) \) an infinite-dimensional Riemannian manifold?
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Motivation
Optimal transport
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Review of Ricci curvature

Back to smooth manifolds.

Ricci curvature is an averaging of sectional curvature.

Fix a unit-length vector \( v \in T_mM \).

Definition

\[
\text{Ric}_M(v, v) = (n - 1) \cdot \text{(the average sectional curvature of the 2-planes } P \text{ containing } v).\
\]
Review of Ricci curvature

Back to smooth manifolds.

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Fix a unit-length vector \( v \in T_m M \).

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\[
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\]

Example: \( S^2 \times S^2 \) has *nonnegative* sectional curvatures but has *positive* Ricci curvatures.
Regularity Issue:

To define $\text{Ric}_M$, we need a Riemannian metric which is $C^2$-regular.

Can we make sense of Ricci curvature for nonsmooth spaces?
Can we make sense at least of “nonnegative Ricci curvature”?
Regularity Issue:

To define $\text{Ric}_M$, we need a Riemannian metric which is $C^2$-regular.

Can we make sense of Ricci curvature for nonsmooth spaces? Can we make sense at least of “nonnegative Ricci curvature”?

The analogous question for sectional curvature was solved by Alexandrov in the 1950’s.
Say $M$ is a compact Riemannian manifold.

**Definition:** The (negative) entropy functional $\mathcal{E} : P(M) \rightarrow \mathbb{R} \cup \infty$ is given by

$$\mathcal{E}(\mu) = \begin{cases} \int_M \rho \log \rho \ d\text{vol}_M & \text{if } \mu = \rho \ d\text{vol}_M, \\ \infty & \text{if } \mu \text{ is not a.c. w.r.t. } d\text{vol}_M. \end{cases}$$
Otto-Villani calculation

How does the entropy function behave along geodesics in $P(M)$?
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Suppose that $c(t) = \rho(t) \, d\text{vol}_M$ is a smooth Wasserstein geodesic.

We defined $\phi(t) \in \mathcal{C}^\infty(M)$ by

$$\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi).$$
How does the entropy function behave along geodesics in $P(M)$?

Suppose that $c(t) = \rho(t) \, d\text{vol}_M$ is a smooth Wasserstein geodesic.

We defined $\phi(t) \in C^\infty(M)$ by

$$\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi).$$

A local calculation: (Otto-Villani 2000)

Along the geodesic $c$,

$$\frac{d^2}{dt^2} \mathcal{E}(c(t)) = \int_M \left[ |\text{Hess}(\phi)|^2 + \text{Ric}_M(\nabla \phi, \nabla \phi) \right] \rho \, d\text{vol}_M.$$
Corollary:
If $\text{Ric}_M \geq 0$ then $\frac{d^2 \mathcal{E}}{dt^2} \geq 0$, i.e. $\mathcal{E}$ is convex along any Wasserstein geodesic $c$. 
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Rigorous proof on $P(M)$:
McCann-Erausquin-Cordero-Schmuckenschläger (2001)

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Rigorous proof on $P(M)$:
McCann-Erausquin-Cordero-Schmuckenschläger (2001)


A new way of thinking about Ricci curvature:
Nonnegative Ricci curvature is equivalent to convexity of $\mathcal{E}$ (on $P(M)$).

We will use this property to define the notion of “nonnegative Ricci curvature” for a nonsmooth space.
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Motivation
Optimal transport
Formal Geometry of Wasserstein Space
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Gromov-Hausdorff topology

A topology on the set of all compact metric spaces (modulo isometry).

\((X_1, d_1)\) and \((X_2, d_2)\) are close in the Gromov-Hausdorff topology if somebody with bad vision has trouble telling them apart.
Example: a cylinder with a small cross-section is Gromov-Hausdorff close to a line segment.
Theorem : (Gromov 1981)

Given $N \in \mathbb{Z}^+$ and $D > 0$,

\[ \{ (M, g) : \dim(M) = N, \text{diam}(M) \leq D, \text{Ric}_M \geq 0 \} \]

is precompact in the Gromov-Hausdorff topology on \{compact metric spaces\}/isometry.
Each point represents a compact metric space. Each interior point is a Riemannian manifold \((M, g)\) with \(\dim(M) = N\), \(\text{diam}(M) \leq D\) and \(\text{Ric}_M \geq 0\).
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The boundary points are compact metric spaces \((X, d)\) with \(\dim_H X \leq N\) and \(\text{diam}(X) \leq D\). They are generally not manifolds.
(Example : \(X = M/G\).)

In some moral sense, the boundary points are metric spaces with “nonnegative Ricci curvature”.
Question:

What can we say about the Gromov-Hausdorff limits of Riemannian manifolds with nonnegative Ricci curvature?
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To answer this, it turns out to be useful to consider instead metric-measure spaces.
Metric-measure spaces

Definition
A metric-measure space is a compact metric space \((X, d)\) equipped with a given probability measure \(\nu \in P(X)\).
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Canonical Example
If \(M\) is a compact Riemannian manifold then \(\left(M, d_M, \frac{d\text{vol}_M}{\text{vol}(M)}\right)\) is a metric-measure space.
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If \(M\) is a compact Riemannian manifold then \(\left(M, d_M, \frac{d\text{vol}_M}{\text{vol}(M)}\right)\) is a metric-measure space.

More generally, a smooth measured length space is a compact Riemannian manifold \((M, g)\) equipped with a smooth probability measure \(d\nu = e^{-\psi} \ d\text{vol}_M\).
An easy consequence of Gromov precompactness:

\[
\left\{ \left( M, g, \frac{d\text{vol}_M}{\text{vol}(M)} \right) : \dim(M) = N, \text{diam}(M) \leq D, \text{Ric}_M \geq 0 \right\}
\]

is precompact in the measured Gromov-Hausdorff topology on \{compact metric-measure spaces\}/isometry.
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is precompact in the measured Gromov-Hausdorff topology on \{compact metric-measure spaces\}/isometry.

What can we say about the limit points? (Work of Cheeger-Colding 1996-2000)

What are the smooth limit points?
Definition

\[ \lim_{i \to \infty} (X_i, d_i, \nu_i) = (X, d, \nu) \]

if there are Borel maps \( f_i : X_i \to X \) and a sequence \( \epsilon_i \to 0 \) such that

1. (Almost isometry) For all \( x, x' \in X_i \),

\[ |d_X(f_i(x), f_i(x')) - d_{X_i}(x, x')| \leq \epsilon_i. \]

2. (Almost surjective) For all \( x \in X \) and all \( i \), there is some \( x_i \in X_i \) such that

\[ d_X(f_i(x_i), x) \leq \epsilon_i. \]

3. \( \lim_{i \to \infty} (f_i)_* \nu_i = \nu \) in the weak-* topology.
Passage to $P(X)$

To one compact metric space we have assigned another.

$$(X, d) \longrightarrow (P(X), W_2)$$

**Proposition : (LV)**

If $X_i \to X$ in the Gromov-Hausdorff topology then $P(X_i) \to P(X)$ in the Gromov-Hausdorff topology.
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To one compact metric space we have assigned another.

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**Proposition : (LV)**

If $X_i \rightarrow X$ in the Gromov-Hausdorff topology then $P(X_i) \rightarrow P(X)$ in the Gromov-Hausdorff topology.

We will use the properties of the Wasserstein space $(P(X), W_2)$ to say something about the geometry of $(X, d)$. 
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**Proposition :** (LV)

If $X_i \rightarrow X$ in the Gromov-Hausdorff topology then $P(X_i) \rightarrow P(X)$ in the Gromov-Hausdorff topology.

We will use the properties of the Wasserstein space $(P(X), W_2)$ to say something about the geometry of $(X, d)$.

In particular, we will *define* what it means for $(X, \nu)$ to have “nonnegative Ricci curvature” in terms of $P(X)$. 
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**Generalized entropy functionals**

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**Notation**

- $X$ a compact Hausdorff space.

- $P(X) = \text{Borel probability measures on } X$, with weak-$*$ topology.

- Fix a background measure $\nu \in P(X)$. 
$N \in [1, \infty]$ a new parameter (possibly infinite).

It turns out that there’s not a single notion of “nonnegative Ricci curvature”, but rather a 1-parameter family. That is, for each $N$, there’s a notion of a space having “nonnegative $N$-Ricci curvature”.

Here $N$ is an effective dimension of the space, and must be inputted.
Definition of the “negative entropy” function

\[ E_N : P(X) \rightarrow \mathbb{R} \cup \infty \]
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\[ \mathcal{E}_N : \mathcal{P}(X) \rightarrow \mathbb{R} \cup \infty \]

Let

\[ \mu = \rho \nu + \mu_s \]

be the Lebesgue decomposition of \( \mu \) with respect to \( \nu \).
Definition of the “negative entropy” function

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Let

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be the Lebesgue decomposition of \( \mu \) with respect to \( \nu \).

For \( N \in [1, \infty) \), the “negative entropy” of \( \mu \) with respect to \( \nu \) is

\[ \mathcal{E}_N(\mu) = N - N \int_X \rho^{1-\frac{1}{N}} \, d\nu. \]
Definition of the “negative entropy” function

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Let

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For \( N \in [1, \infty) \), the “negative entropy” of \( \mu \) with respect to \( \nu \) is

\[ \mathcal{E}_N(\mu) = N - N \int_X \rho^{1 - \frac{1}{N}} \, d\nu. \]

For \( N = \infty \),

\[ \mathcal{E}_\infty(\mu) = \begin{cases} \int_X \rho \log \rho \, d\nu & \text{if } \mu \text{ is a.c. w.r.t. } \nu, \\ \infty & \text{otherwise.} \end{cases} \]
Geometry of the space of probability measures

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
Some more metric geometry
Generalized entropy functionals

Abstract Ricci curvature

Applications
Perelman’s reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume
Convexity on Wasserstein space

$(X, d)$ is a compact length space.

$\nu$ is a fixed probability measure on $X$. 
Convexity on Wasserstein space

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$\nu$ is a fixed probability measure on $X$.

We want to ask whether the negative entropy function $\mathcal{E}_N$ is a convex function on $P(X)$.

That is, given $\mu_0, \mu_1 \in P(X)$, whether $\mathcal{E}_N$ restricts to a convex function along a Wasserstein geodesic $\{\mu_t\}_{t\in[0,1]}$ from $\mu_0$ to $\mu_1$. 
Nonnegative $N$-Ricci curvature

**Definition**
Given $N \in [1, \infty]$, we say that a compact measured length space $(X, d, \nu)$ has nonnegative $N$-Ricci curvature if:

For all $\mu_0, \mu_1 \in \mathcal{P}(X)$ with $\text{supp} (\mu_0) \subset \text{supp} (\nu)$ and $\text{supp} (\mu_1) \subset \text{supp} (\nu)$, there is some Wasserstein geodesic $\{\mu_t\}_{t \in [0,1]}$ from $\mu_0$ to $\mu_1$ so that for all $t \in [0,1]$, $E_N(\mu_t) \leq t E_N(\mu_1) + (1 - t) E_N(\mu_0)$. 
Definition
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$$\mathcal{E}_N(\mu_t) \leq t \mathcal{E}_N(\mu_1) + (1 - t) \mathcal{E}_N(\mu_0).$$
Nonnegative $N$-Ricci curvature

Note: We only require convexity along some geodesic from $\mu_0$ to $\mu_1$, not all geodesics. There's also a notion of "N-Ricci curvature bounded below by $K$."
Nonnegative $N$-Ricci curvature

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Nonnegative $N$-Ricci curvature

Note: We only require convexity along *some* geodesic from $\mu_0$ to $\mu_1$, not all geodesics.

There’s also a notion of “$N$-Ricci curvature bounded below by $K$”.
**Main result**

**Theorem : (LV,S)**

Let \( \{(X_i, d_i, \nu_i)\}_{i=1}^{\infty} \) be a sequence of compact measured length spaces with

\[
\lim_{i \to \infty} (X_i, d_i, \nu_i) = (X, d, \nu)
\]

in the measured Gromov-Hausdorff topology.

For any \( N \in [1, \infty] \), if each \((X_i, d_i, \nu_i)\) has nonnegative \( N \)-Ricci curvature then \((X, d, \nu)\) has nonnegative \( N \)-Ricci curvature.
What does all this have to do with Ricci curvature?

Let \((M, g)\) be a compact connected \(n\)-dimensional Riemannian manifold.

We could take the Riemannian measure, but let’s be more general and consider any smooth measured length space.
What does all this have to do with \textbf{Ricci} curvature?

Let \((M, g)\) be a compact connected \(n\)-dimensional Riemannian manifold.

We could take the Riemannian measure, but let’s be more general and consider any smooth measured length space.

Say \(\psi \in C^\infty(M)\) has

\[
\int_M e^{-\psi} \, d\text{vol}_M = 1.
\]

Put \(\nu = e^{-\psi} \, d\text{vol}_M\).
For $N \in [1, \infty]$, define the $N$-Ricci tensor $\text{Ric}_N$ of $(M^n, g, \nu)$ by

\[
\begin{cases}
\text{Ric} + \text{Hess}(\psi) & \text{if } N = \infty, \\
\text{Ric} + \text{Hess}(\psi) - \frac{1}{N-n} d\psi \otimes d\psi & \text{if } n < N < \infty, \\
\text{Ric} + \text{Hess}(\psi) - \infty (d\psi \otimes d\psi) & \text{if } N = n, \\
-\infty & \text{if } N < n,
\end{cases}
\]

where by convention $\infty \cdot 0 = 0$.

$\text{Ric}_N$ is a symmetric covariant 2-tensor field on $M$ that depends on $g$ and $\psi$. 

The $N$-Ricci tensor

For $N \in [1, \infty]$, define the $N$-Ricci tensor $\text{Ric}_N$ of $(M^n, g, \nu)$ by

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\text{Ric}_N &= \text{Ric} + \text{Hess}(\Psi) - \infty (d\psi \otimes d\psi) \\
-\infty &= \infty \\
\end{align*}
$$

where by convention $\infty \cdot 0 = 0$.

$\text{Ric}_N$ is a symmetric covariant 2-tensor field on $M$ that depends on $g$ and $\Psi$.

(If $N = n$ then $\text{Ric}_N$ is $-\infty$ except where $d\psi = 0$. There, $\text{Ric}_N = \text{Ric}$.)

$\text{Ric}_\infty = \text{Bakry-Emery tensor} = \text{right-hand side of Perelman’s modified Ricci flow equation.}$
Abstract Ricci recovers classical Ricci

Recall that $\nu = e^{-\psi} \, d\text{vol}_M$.

**Theorem :** (LV, S)
For $N \in [1, \infty]$, the measured length space $(M, g, \nu)$ has nonnegative $N$-Ricci curvature if and only if $\text{Ric}_N \geq 0$. 
Abstract Ricci recovers classical Ricci

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**Theorem : (LV, S)**
For $N \in [1, \infty]$, the measured length space $(M, g, \nu)$ has nonnegative $N$-Ricci curvature if and only if $\text{Ric}_N \geq 0$.

Classical case : $\Psi$ constant, so $\nu = \frac{d\text{vol}}{\text{vol}(M)}$.

Then $(M^n, g, \nu)$ has abstract nonnegative $N$-Ricci curvature if and only if it has classical nonnegative Ricci curvature, as soon as $N \geq n$. 
Geometry of the space of probability measures

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
Some more metric geometry
Generalized entropy functionals
Abstract Ricci curvature

Applications
Perelman’s reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume
Smooth limit spaces

Had Gromov precompactness theorem. What are the limit spaces \((X, d, \nu)\)? Suppose that the limit space is a smooth measured length space, i.e.

\[
(X, d, \nu) = (B, g_B, e^{-\psi} \, d\text{vol}_B)
\]

for some \(n\)-dimensional smooth Riemannian manifold \((B, g_B)\) and some \(\psi \in C^\infty(B)\).
Smooth limit spaces

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**Theorem :** (LV)

If \((B, g_B, e^{-\psi} \text{dvol}_B)\) is a measured Gromov-Hausdorff limit of Riemannian manifolds with nonnegative Ricci curvature and dimension at most \(N\) then \(\text{Ric}_N(B) \geq 0\).
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**Theorem : (LV)**

If \((B, g_B, e^{-\psi} \text{dvol}_B)\) is a measured Gromov-Hausdorff limit of Riemannian manifolds with nonnegative Ricci curvature and dimension at most \(N\) then \(\text{Ric}_N(B) \geq 0\).

Note : the dimension can drop on taking limits.
Theorem: (LV, S)

If \((X, d, \nu)\) has nonnegative \(N\)-Ricci curvature and \(x \in \text{supp}(\nu)\) then \(r^{-N} \nu(B_r(x))\) is nonincreasing in \(r\).
If \((M, g)\) is a compact Riemannian manifold, let \(\lambda_1\) be the smallest positive eigenvalue of the Laplacian \(-\nabla^2\).

**Theorem : (Lichnerowicz 1964)**

If \(\text{dim}(M) = n\) and \(M\) has Ricci curvatures bounded below by \(K > 0\) then

\[
\lambda_1 \geq \frac{n}{n - 1} K.
\]
Theorem : (LV)

If \((X, d, \nu)\) has \(N\)-Ricci curvature bounded below by \(K > 0\) and \(f\) is a Lipschitz function on \(X\) with \(\int_X f \, d\nu = 0\) then

\[
\int_X f^2 \, d\nu \leq \frac{N - 1}{N} \frac{1}{K} \int_X |\nabla f|^2 \, d\nu.
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Theorem: (LV)

If \((X, d, \nu)\) has \(N\)-Ricci curvature bounded below by \(K > 0\) and \(f\) is a Lipschitz function on \(X\) with \(\int_X f \, d\nu = 0\) then

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Here

\[
|\nabla f|(x) = \limsup_{y \to x} \frac{|f(y) - f(x)|}{d(y, x)}.
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(There are also log Sobolev inequalities and Sobolev inequalities.)
Theorem (O’Neill 1966) Sectional curvature is nondecreasing under a Riemannian submersion.
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Some open questions

Do measured length spaces with nonnegative $N$-Ricci curvature admit isoperimetric inequalities?
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Do measured length spaces with nonnegative $N$-Ricci curvature admit isoperimetric inequalities?

To what extent does the Cheeger-Gromoll splitting principle hold for measured length spaces with nonnegative $N$-Ricci curvature?
Geometry of the space of probability measures

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
Some more metric geometry
Generalized entropy functionals
Abstract Ricci curvature
Applications

Perelman’s reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume
$M$ a compact, connected $n$-dimensional manifold.

Say $(M, g(t))$ is a Ricci flow solution, i.e. $\frac{dg}{dt} = -2 \text{ Ric.}$
$M$ a compact, connected $n$-dimensional manifold.

Say $(M, g(t))$ is a Ricci flow solution, i.e. $\frac{dg}{dt} = -2 \text{ Ric.}$

Fix $t_0$ and put $\tau = t_0 - t$. Then $\frac{dg}{d\tau} = 2 \text{ Ric.}$

An important tool: monotonic quantities.
Reduced volume

Fix $p \in M$. Say $\gamma : [0, \bar{\tau}] \to M$ is a smooth curve with $\gamma(0) = p$. (The graph of $\gamma$ goes “backward in time”.)
Reduced volume

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(The graph of \( \gamma \) goes “backward in time”.)

**Definition**

\( L \)-length \( L(\gamma) = \int_0^\tau \sqrt{\tau} \left( |\dot{\gamma}|^2_{g(\tau)} + R(\gamma(\tau), \tau) \right) \, d\tau \).
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$L$-length $\mathcal{L}(\gamma) = \int_0^\tau \sqrt{\tau} \left( |\dot{\gamma}|^2_{g(\tau)} + R(\gamma(\tau), \tau) \right) \, d\tau$.

**Definition**

reduced distance Given $q \in M$, put

$$\bar{L}(q, \tau) = \inf \{ \mathcal{L}(\gamma) : \gamma(0) = p, \gamma(\tau) = q \}.$$
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$\overline{L}(q, \tau) = \inf \{ L(\gamma) : \gamma(0) = p, \gamma(\tau) = q \}$.

Put $l(q, \tau) = \frac{\overline{L}(q, \tau)}{2\sqrt{\tau}}$. 

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Put $l(q, \bar{\tau}) = \frac{L(q, \bar{\tau})}{2\sqrt{\tau}}$.

**Definition**

reduced volume $\widetilde{V}(\bar{\tau}) = \tau^{-\frac{n}{2}} \int_M e^{-l(q, \bar{\tau})} \text{dvol}(q)$. 
Theorem: (Perelman 2002)
\( \tilde{V} \) is nonincreasing in \( \tau \), i.e. nondecreasing in \( t \).
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\( \tilde{V} \) is nonincreasing in \( \tau \), i.e. nondecreasing in \( t \).

An “entropy” functional for Ricci flow.

The only assumption: \( g(t) \) satisfies the Ricci flow equation.

Main application: Perelman’s “no local collapsing” theorem.
Perelman’s heuristic derivation

Put $\overline{M} = M \times S^N \times \mathbb{R}^+$. 

Here $N$ is a free parameter and $\tau$ is the coordinate on $\mathbb{R}^+$. 

Fact: As $N \to \infty$, $\text{Ric}(\overline{M}) = O(N^{-1})$. 

Bishop-Gromov: $r - \text{dim vol}(B_r(p))$ is nonincreasing in $r$ if $\text{Ric} \geq 0$. 

Apply formally to $M$ and take $N \to \infty$. Get monotonicity of $\overline{V}$. 
Perelman’s heuristic derivation

Put $\overline{M} = M \times S^N \times \mathbb{R}^+$. Here $N$ is a free parameter and $\tau$ is the coordinate on $\mathbb{R}^+$. Put

$$
\overline{g} = g(\tau) + 2N\tau g_{S^N} + \left( \frac{N}{2\tau} + R \right) d\tau^2.
$$

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Apply formally to $\overline{M}$ and take $N \to \infty$. Get monotonicity of $\tilde{V}$. 

We know how to characterize nonnegative Ricci curvature using optimal transport. Apply to $\overline{M}$ and translate down to $M$. 
Heuristic relation to optimal transport

We know how to characterize nonnegative Ricci curvature using optimal transport. Apply to \( \overline{M} \) and translate down to \( M \).

This should give an optimal transport problem on \( M \) with which we can derive the monotonicity of \( \tilde{V} \).
We know how to characterize nonnegative Ricci curvature using optimal transport. Apply to $\widetilde{M}$ and translate down to $M$.

This should give an optimal transport problem on $M$ with which we can derive the monotonicity of $\widetilde{V}$.

We’ll describe a (re)proof of the monotonicity of $\widetilde{V}$, using optimal transport methods.
Say $c : [0, 1] \rightarrow P^\infty(M)$ is a smooth curve.

Write $c(t) = \rho(t) \, \text{dvol}_M$.

Solve
\[ \frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi) \]
for $\phi \equiv \phi(t) \in C^\infty(M)$.

From $\{\rho(t)\}_{t \in [0, 1]}$, we got $\{\phi(t)\}_{t \in [0, 1]}$. Put
\[ E(c) = \frac{1}{2} \int_0^1 \int_M |\nabla \phi|^2 \rho \, \text{dvol}_M \, dt. \]
The Euler-Lagrange equation for the functional $E$ is

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2} |\nabla \phi|^2.$$
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Then

$$\frac{d^2}{dt^2} \int_M \rho \log \rho \, d\text{vol}_M = \int_M \left[ |\text{Hess} \phi|^2 + \text{Ric}_M(\nabla \phi, \nabla \phi) \right] \rho \, d\text{vol}_M.$$
Geometry of the space of probability measures

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
Some more metric geometry
Generalized entropy functionals
Abstract Ricci curvature
Applications
Perelman’s reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume
Question

Can we do something similar for the Ricci flow?

Principle: Satisfying $\text{Ric} = 0$ in the Riemannian case is like satisfying the Ricci flow equation in the spacetime case.

Optimal transport in a Ricci flow background was first considered by Peter Topping, with application to another monotonic quantity ($\mathcal{W}$-functional).
Can we do something similar for the Ricci flow?

Principle: Satisfying $\text{Ric} = 0$ in the Riemannian case is like satisfying the Ricci flow equation in the spacetime case.

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Note: The Ricci flow equation

$$\frac{dg}{dt} = -2 \, \text{Ric}$$

implies

$$\frac{d\text{vol}_M}{dt} = -R \, d\text{vol}_M.$$
Assume hereafter that \((M, g(t))\) satisfies the Ricci flow equation.

Given \(c : [t_0, t_1] \to P^\infty(M)\), write \(c(t) = \rho(t) \, \text{dvol}_M\). Solve

\[
\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i (\rho \nabla_i \phi) + R \rho
\]

for \(\phi \equiv \phi(t) \in C^\infty(M)\).
Assume hereafter that \((M, g(t))\) satisfies the Ricci flow equation.

Given \(c : [t_0, t_1] \rightarrow P^\infty(M)\), write \(c(t) = \rho(t) \text{ dvol}_M\). Solve

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\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi) + R \rho
\]

for \(\phi \equiv \phi(t) \in C^\infty(M)\).

**Definition**

\[
E_0(c) = \frac{1}{2} \int_{t_0}^{t_1} \int_M (|\nabla \phi|^2 + R) \rho \text{ dvol}_M \, dt
\]
**$E_0$ functional**

Assume hereafter that $(M, g(t))$ satisfies the Ricci flow equation.

Given $c : [t_0, t_1] \rightarrow P^\infty(M)$, write $c(t) = \rho(t)\,\text{dvol}_M$. Solve

$$\frac{\partial \rho}{\partial t} = - \sum_i \nabla^i (\rho \nabla_i \phi) + R\,\rho$$

for $\phi \equiv \phi(t) \in C^\infty(M)$.

**Definition**

$$E_0(c) = \frac{1}{2} \int_{t_0}^{t_1} \int_M \left( |\nabla \phi|^2 + R \right) \rho \, \text{dvol}_M \, dt$$

Euler-Lagrange equation for $E_0$:

$$\frac{\partial \phi}{\partial t} = - \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} R.$$
Convexity statement

Proposition

If $c$ satisfies the Euler-Lagrange equation then

$$\frac{d^2}{dt^2} \int_M \left( \rho \ln \rho - \phi \rho \right) d\text{vol}_M = \int_M | \text{Ric} - \text{Hess } \phi |^2 \rho \ d\text{vol}_M .$$
Convexity statement

**Proposition**

If $c$ satisfies the Euler-Lagrange equation then

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\frac{d^2}{dt^2} \int_M (\rho \ln \rho - \phi \rho) \ dvol_M = \int_M | \text{Ric} - \text{Hess} \phi |^2 \rho \ dvol_M .
$$

**Corollary**

If $c$ satisfies the Euler-Lagrange equation then

$$
\int_M (\rho \ln \rho - \phi \rho) \ dvol_M \text{ is convex in } t .
$$
Say we want to transport a measure $\mu_0$ (at time $t_0$) to a measure $\mu_1$ (at time $t_1$).

Take the cost to transport a unit of mass from $p$ to $q$ to be

$$\min\{L_0(\gamma) : \gamma(t_0) = p, \gamma(t_1) = q\},$$

where

$$L_0(\gamma) = \frac{1}{2} \int_{t_0}^{t_1} \left( |\dot{\gamma}|^2 g(t) + R(\gamma(t), t) \right) dt.$$  

There is a corresponding notion of optimal transport, displacement interpolation, etc.
$E_-$ functional

Fix $t_0$ and put $\tau = t_0 - t$. The Ricci flow equation is

$$\frac{dg}{d\tau} = 2 \text{ Ric}.$$
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\]

Given \( c : [\tau_0, \tau_1] \rightarrow P^\infty(M) \), write \( c(\tau) = \rho(\tau) \text{ dvol}_M \). Solve

\[
\frac{\partial \rho}{\partial \tau} = - \sum_i \nabla^i (\rho \nabla_i \phi) - R \rho
\]

for \( \phi = \phi(\tau) \in C^\infty(M) \).
Fix $t_0$ and put $\tau = t_0 - t$. The Ricci flow equation is
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for $\phi = \phi(\tau) \in C^\infty(M)$.

**Definition**

$E_-(c) = \int_{\tau_0}^{\tau_1} \int_M \sqrt{\tau} \left( |\nabla \phi|^2 + R \right) \rho \text{ dvol}_M \ d\tau$
Fix $t_0$ and put $\tau = t_0 - t$. The Ricci flow equation is

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Given $c : [\tau_0, \tau_1] \rightarrow P^\infty(M)$, write $c(\tau) = \rho(\tau) \text{dvol}_M$. Solve

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**Definition**

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**Euler-Lagrange equation for $E_-$**:

$$\frac{\partial \phi}{\partial \tau} = -\frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} R - \frac{1}{2\tau} \phi.$$
Proposition

If \( c \) satisfies the Euler-Lagrange equation then

\[
\left( \tau^{\frac{3}{2}} \frac{d}{d\tau} \right)^2 \left( \int_M \left( \rho \ln \rho + \phi \rho \right) \, d\text{vol}_M + \frac{n}{2} \ln \tau \right) = \tau^3 \int_M \left| \text{Ric} + \text{Hess} \phi - \frac{g}{2\tau} \right|^2 \rho \, d\text{vol}_M.
\]
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Proposition

If $c$ satisfies the Euler-Lagrange equation then

$$\left( \frac{3}{2} \frac{d}{d\tau} \right)^2 \left( \int_M (\rho \ln \rho + \phi \rho) \ d\text{vol}_M + \frac{n}{2} \ln \tau \right) =$$

$$\tau^3 \int_M \left| \text{Ric} + \text{Hess} \phi - \frac{g}{2\tau} \right|^2 \rho \ d\text{vol}_M.$$ 

Corollary

If $c$ satisfies the Euler-Lagrange equation then

$$\int_M (\rho \ln \rho + \phi \rho) \ d\text{vol}_M + \frac{n}{2} \ln \tau \text{ is convex in the variable } s = \tau^{-\frac{1}{2}}.$$
Geometry of the space of probability measures

Motivation
Optimal transport
Formal Geometry of Wasserstein Space
Metric geometry of Wasserstein space
Ricci meets Wasserstein
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Generalized entropy functionals
Abstract Ricci curvature
Applications
Perelman's reduced volume
Formulas from Riemannian optimal transport
Optimal transport for Ricci flow
Monotonicity of the reduced volume
Take $\tau_0 \to 0$, $\mu_0 = \delta_p$ and $\mu_1$ an absolutely continuous measure.

The displacement interpolation is along $\mathcal{L}$-geodesics emanating from $p$.

In this case, $\phi = l$. 
Proposition

In this case, \( \int_M (\rho \ln \rho + \phi \rho) \, d\text{vol}_M + \frac{n}{2} \ln \tau \) is nondecreasing in \( \tau \).
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Proof.

We know that it is convex in \( s = \tau^{-\frac{1}{2}} \). As \( s \to \infty \), i.e. as \( \tau \to 0 \), it approaches a constant. (Almost Euclidean situation.) So it is nonincreasing in \( s \), i.e. nondecreasing in \( \tau \). \( \square \)
Trivial fact: The minimizer of

$$\int_M \left( \rho \ln \rho + \phi \rho \right) \, d\text{vol}_M + \frac{n}{2} \ln \tau,$$

as $\rho \, d\text{vol}_M$ ranges over absolutely continuous probability measures, is

$$- \ln \left( \tau^{-\frac{n}{2}} \int_M e^{-\phi} \, d\text{vol}_M \right).$$

The minimizing measure is given by

$$\rho = \frac{e^{-\phi}}{\int_M e^{-\phi} \, d\text{vol}_M}.$$
Monotonicity of reduced volume

Proposition

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Proof: Say \( \tau' < \tau'' \). Recall that \( \phi = l \). Take \( \mu'(\tau'') = \rho(\tau'') \, d\text{vol}_M \) with

\[ \rho(\tau'') = \frac{e^{-\phi(\tau'')}}{\int_M e^{-\phi(\tau'')} \, d\text{vol}_M}. \]
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Transport it to \( \delta_\rho \) (at time zero). At the intermediate time \( \tau' \) we see a measure \( \mu(\tau') = \rho(\tau') \text{dvol}_M \).
Proof

Then

$$- \ln \left( (\tau')^{-\frac{n}{2}} \int_M e^{-\phi(\tau')} \, d\text{vol}_M \right)$$

$$\leq \int_M \left[ \rho(\tau') \ln \rho(\tau') + \phi(\tau') \rho(\tau') \right] \, d\text{vol}_M + \frac{n}{2} \ln \tau'$$

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End of proof
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J.L., McCann-Topping Suppose that $((M, g(t)))$ is a Ricci flow solution. Suppose that $\mu_0(t)$ and $\mu_1(t)$ are two solutions of the backward heat flow on measures $d\mu/dt = -\nabla^2_g \mu$. Then $W_2(\mu_0(t), \mu_1(t))$ is nondecreasing in $t$. 

Topping Extension to a statement about the $L$-transport distance between $\mu_0$ and $\mu_1$ at distinct but related times.
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