DEGREES OF FREEDOM AND THE QUANTIZATION OF ANOMALOUS GAUGE THEORIES

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We show, for the two-dimensional case, that the anomalous gauge theory of chiral fermions yields degrees of freedom whose number depends on the regularization procedure. For a particular regularization, the gauge fields have dim[$\mathscr{G}$]−rank[$\mathscr{G}$] surviving degrees of freedom, while for others this number changes to 2 dim[$\mathscr{G}$]. Our procedure and results are compared with Faddeev's recent suggestions on how to quantize anomalous gauge theories. We conclude with some remarks on the four-dimensional case.

In gauge theories of chiral fermions, generators of gauge transformations acquire anomalous terms in their commutators which prevent physical states from being defined as those which are gauge invariant. Faddeev [1] has suggested that these systems may nevertheless be quantized consistently if their constraints are recognized as being of the second class, and treated appropriately.

Subsequently it has been shown that in two dimensions, both the abelian [2,3] and the non-abelian [4] gauge theories of chiral fermions, although anomalous, lead to consistent quantum theories, if regularized appropriately. Consistency and unitarity were demonstrated for $\alpha > 1$, where $\alpha/2$ represents the *a priori* undetermined coefficient of the $A_\mu A^\mu$ counterterm in the effective action. For $\alpha < 1$, these theories were shown to be non-unitary.

More recently, Faddeev [5] has suggested that a proper way to quantize chiral gauge theories may be to introduce an extra gauge-group valued field, with a gauged Wess–Zumino [WZ] term in the action, designed to cancel the anomaly in the original system. Such WZ terms also appear [6] when one attempts to decouple a chiral fermion from Salam–Weinberg type models. Faddeev also mentions that in his theory of the chiral system the vector field acquires, loosely speaking, an extra "half-degree of polarization", as compared to the corresponding gauge invariant non-chiral system.

Faddeev's proposal [5] is couched in the more important four-dimensional context. But it is useful to compare within the simpler two-dimensional context, the physical content of his proposal with that of the original system studied in refs. [2–4] where no extra fields are added, nor the anomaly cancelled away by hand. In particular, it is interesting to see whether, and in what sense the "half degree of polarization" appears in the original two-dimensional model. These are the aims of our paper. We will employ bosonization techniques [7] to obtain our results.

It will be seen that Faddeev's enlarged system, which is gauge invariant, reduces upon gauge fixing, to the original chiral system. For the latter, it has already been shown [2–4] that with the $\alpha > 1$ regularization, the space components A_1^a, of the vector-field multiplet (and their canonical momenta E^a) survive as dynamical variables, in addition to the matter fields. Thus, as compared to the corresponding anomaly-free vector gauge theory in two dimensions, where no components of A_μ^a are dynamical, the $\alpha > 1$ regularized chiral theory permits one full polarization component for the entire vector field multiplet. This corresponds to 2 dim [$\mathscr{G}$] degrees of freedom in phase

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space arising from the gauge fields. ($\mathcal{G}$ is the gauge group and we are referring to the number of phase-space degrees of freedom at each space point x)

However, it will be shown below that with the $\alpha = 1$ regularization, the chiral theory, while still consistent, permits only $N(\mathcal{G}) = \dim[\mathcal{G}] - \mathrm{rank}[\mathcal{G}]$ degrees of freedom for the gauge field. $N(\mathcal{G})$ is an even number for any compact group $\mathcal{G}$, so that one still has an integral number of canonical pairs at each x. But $N(\mathcal{G})$ is in general not an even multiple of $\dim[\mathcal{G}]$. Thus, in configuration space the number of dynamical fields surviving from the gauge field multiplet is $\frac{1}{2}N(\mathcal{G})$, which roughly equals $\frac{1}{2}\dim[\mathcal{G}]$ when $\dim[\mathcal{G}] \gg \mathrm{rank}[\mathcal{G}]$. All this is in addition to matter field degrees of freedom whose number remains the same as in the non-interacting theory.

Let us begin by recalling the $\alpha = 1$ case for the chiral U(1) problem, which has already been analyzed [3,8]. Its bosonized action is

$$S[\phi, A_\mu] = \int \mathrm{d}x\, \mathrm{d}t [\tfrac{1}{2}(\partial_\mu\phi + A_\mu)(\partial^\mu\phi + A^\mu) - \epsilon^{\mu\nu}\partial_\mu\phi A_\nu - (1/4e^2)F_{\mu\nu}F^{\mu\nu}]. \tag{1}$$

The associated hamiltonian, in terms of canonical pairs (ϕ, π), (A_0, π_0) and $(A_1, E/e^2)$ is

$$H = \int \mathrm{d}x [\tfrac{1}{2}\pi^2 + \tfrac{1}{2}(\partial_1\phi)^2 + (1/2e^2)E^2 - (1/e^2)\partial_1 E A_0 + (\pi + \partial_1\phi + A_1)(A_1 - A_0) + v\pi_0], \tag{2}$$

with

$$\pi_0(x) = 0 \tag{3}$$

as a constraint. As permitted, a term $v\pi_0$, where $v(x)$ is a Lagrange multiplier field, has been added. The preservation of the constraint (3) under time evolution leads to three further constraints.

$$\partial_0\pi_0 = \{\pi_0, H\}_{\mathrm{P.B.}} \equiv G(x) = (1/e^2)\partial_1 E + \pi + \partial_1\phi + A_1 = 0, \qquad \partial_0 G = E = 0, \tag{4,5}$$

and

$$\partial_0 E = -e^2(\pi + \partial_1\phi + 2A_1 - A_0) = 0, \qquad \text{implying } A_0 - A_1 = 0. \tag{6}$$

The requirement $\{A_0 - A_1, H\}_{\mathrm{P.B.}} = 0$ can be fulfilled by choosing $v = E + \partial_1 A_0$. (In ref. [3], the presence of constraint (6) was erroneously overlooked.) Eqs. (3)–(6) form a set of second-class constraints in Dirac's terminology [9]. Before quantizing the theory, Dirac brackets must be employed. Then A_0, π_0, A_1 and E can be eliminated using eqs. (3)–(6). Only the matter degrees of freedom ϕ and π remain, with $\{\phi(x), \pi(y)\}_{\mathrm{Dirac}} = \delta(x-y)$ and a reduced hamiltonian

$$H = \int \mathrm{d}x\, \tfrac{1}{2}[\pi^2 + (\partial_1\phi)^2]. \tag{7}$$

The number of non-matter degrees of freedom, namely zero, agrees with $\dim[\mathcal{G}] - \mathrm{rank}[\mathcal{G}]$ since $\dim[\mathcal{G}] = \mathrm{rank}[\mathcal{G}] = 1$ for this U(1) theory. It is worth noting that, unlike the anomaly-free Schwinger model, in this $\alpha = 1$ regularized chiral model there are no interaction effects due to A_μ. Only the free matter field is left. This feature will persist even if A_μ were further coupled vectorially to other matter fields.

Next, let us turn to the non-abelian chiral gauge theory in two dimensions. Consider n massless Dirac fermions whose right-current is coupled to a U(n) Lie-algebra valued gauge field A_μ. The bosonized action for this system, regularized at $\alpha = 1$ is [4]

$$S[U, A_\mu] = \int \mathrm{d}x\, \mathrm{d}t\, \mathrm{Tr}[(1/8\pi)\partial_\mu U \partial^\mu U^{-1} - J_\mu A^\mu + (1/8\pi)A_\mu A^\mu - (1/4e^2)F_{\mu\nu}F^{\mu\nu}] + \Gamma_{\mathrm{WZ}}, \tag{8}$$

where $U(x,t)$ is a U(n) group valued field and Γ_{WZ} is the Wess–Zumino term [7]

$$\Gamma_{\mathrm{WZ}}(U) \equiv \frac{1}{12\pi}\int \mathrm{d}^3y\, \epsilon^{ijk}\, \mathrm{Tr}(U^{-1}\partial_i U\, U^{-1}\partial_j U\, U^{-1}\partial_k U), \tag{9}$$

and

$$J^\mu \equiv (\mathrm{i}/4\pi)(g^{\mu\nu} + \epsilon^{\mu\nu})\, U^{-1}\partial_\nu U. \tag{10}$$

The field equations, in light-cone components ($x_\pm \equiv (x_0 \pm x_1)/\sqrt{2}$, etc.) are

$$D_- J_+ = (1/2\pi)\,\partial_- A_+, \qquad D_- F_{-+} = (e^2/4\pi) A_-, \qquad D_+ F_{-+} = e^2 J_+ - (e^2/4\pi) A_+. \tag{11, 12, 13}$$

These equations imply

$$\partial_- A_+ - \partial_+ A_- = 0, \tag{14}$$

where the left-hand side gives the anomaly with the $\alpha = 1$ regularization.

We will obtain the number of degrees of freedom of this system in two ways. First let us perturb the system around a class of solutions of (11)–(13), linearize these equations and count the number of perturbed fields for which Cauchy data can be independently specified. One class of solutions of the field equations is

$$A_-^{(0)} = 0, \qquad F_{-+}^{(0)} = \partial_- A_+^{(0)} = 0, \qquad J_+^{(0)} = (1/4\pi) A_+^{(0)}. \tag{15}$$

Here $A_+^{(0)}$ is an arbitrary Lie-algebra valued matrix function of x_+. Let us choose it to be diagonal. Let

$$A_- = a_-, \qquad A_+ = A_+^{(0)} + a_+, \qquad J_+ = (1/4\pi) A_+^{(0)} + j_+. \tag{16}$$

On linearizing eqs. (11)–(13) around the solution (15), the perturbations $a_\pm$ and j_+ obey, to first order,

$$\partial_- j_+ = (1/2\pi)\,\partial_+ a_- + \mathrm{i}[a_-, j_+ + (1/4\pi) A_+^{(0)}], \qquad \partial_+ [a_-, A_+^{(0)}] = \mathrm{i}[A_+^{(0)}, [a_-, A_+^{(0)}]] + e^2 (j_+ - a_+/4\pi), \tag{17, 18}$$

and

$$[\partial_- a_-, A_+^{(0)}] = (\mathrm{i} e^2/4\pi)\, a_-, \tag{19}$$

Since $A_+^{(0)}$ has been chosen diagonal, eq. (19) implies

$$\mathrm{Diag}(a_-) = 0. \tag{20}$$

This eliminates rank $[\mathcal{G}]$ degrees of freedom from a_-. The remaining dim $[\mathcal{G}]$ – rank $[\mathcal{G}]$ components of a_- can be specified arbitrarily as initial data and their time evolution obtained using eq. (19). The field a_+ is fully constrained by (18), while (17) places no restriction on the initial data of j_+. Thus, the number of matter degrees of freedom remains unchanged, while the perturbed gauge field (a_+, a_-) has altogether dim $[\mathcal{G}]$ – rank $[\mathcal{G}]$ degrees of freedom.

Alternately, let us perform a canonical constraint analysis of the system specified by the action (8). The hamiltonian associated with this action is [4]

$$H = \int \mathrm{d}x\ \mathrm{Tr}\{\pi(\rho_R \rho_R + \rho_L \rho_L) - [\rho_R - (1/4\pi) A_1](A_1 - A_0) + (1/e^2)(\tfrac{1}{2} E^2 - D_1 E \cdot A_0) + v\pi_0\}. \tag{21}$$

Here all fields (including the Lagrange multiplier field v) are Lie-algebra valued hermitian matrices. (A_0, π_0) and $(A_1, E/e^2)$ are canonical pairs while $\rho_{R,L}$ are the right (left) charge densities of the free Fermi field, obeying the familiar Kac–Moody current algebra [4]. In component notation,

$$\{\rho^a_{R,L}(x), \rho^b_{R,L}(y)\}_{\mathrm{P.B.}} = -f^{abc} \rho^c_{R,L}\, \delta(x-y) \pm (\delta^{ab}/2\pi)\, \delta'(x-y), \qquad \{\rho^a_R(x), \rho^b_L(y)\}_{\mathrm{P.B.}} = 0.$$

The constraint

$$\pi_0(x) = 0 \tag{22}$$

requires for its preservation further constraints. We have

$$\partial_0 \pi_0 = (1/e^2)\, D_1 E + (1/4\pi) A_1 - \rho_R \equiv G = 0. \tag{23}$$

Then

$$\partial_0 G = (1/4\pi) E + \mathrm{i}[A_0, G + (1/4\pi) A_1] = 0,$$

which implies, given (23) that

$$B \equiv E + \mathrm{i}[A_0, A_1] = 0, \tag{24}$$

and

$$\partial_0 B = e^2 \rho_R + (e^2/4\pi)(A_0 - 2A_1) + \mathrm{i}[A_0, 2E + D_1 A_0] + \mathrm{i}[v, A_1] \equiv C + \mathrm{i}[v, A_1] = 0. \tag{25}$$

Eq. (25) cannot be inverted for the Lagrange multiplier field v. Upon choosing a basis in which A_1 is diagonal, (25) implies a further constraint

$$\mathrm{Diag}[C] = 0. \tag{26}$$

For generic A_1, eq. (26) gives rank $[\mathcal{G}]$ constraints. The non-diagonal components of (25) can be satisfied by choosing v suitably. This still leaves the diagonal components of v unspecified, but these will be fixed by requiring that the constraint (26) be preserved in time. Altogether, eqs. (22)–(24), and (26) yield 3 dim$[\mathcal{G}]$ + rank$[\mathcal{G}]$ constraints which can be imposed on the 4 dim$[\mathcal{G}]$ gauge field variables π_0, A_0, E and A_1. This leaves us with $N(\mathcal{G})$ = dim$[\mathcal{G}]$ − rank$[\mathcal{G}]$ gauge field degrees of freedom (in addition to the usual matter degrees of freedom contained in $\rho_{R,L}$) in agreement with the earlier linearized analysis.

Constraints (22)–(24) and (26) are of the second class. Before quantizing the theory, one must employ Dirac brackets to convert the constraints into strong equations. They can then be used to eliminate dependent variables. We will not complete this exercise here. Suffice it to note that when constraints (22) and (23) are used, the hamiltonian (21) can be written as

$$H = \int \mathrm{d}x\ \mathrm{Tr}\{\pi[\rho_R - (1/2\pi)A_1]^2 + \pi\rho_L\rho_L + (1/2e^2)E^2\}. \tag{27}$$

The hamiltonian is thus positive on the constrained subspace. When Dirac brackets are converted to commutators, one has formally a consistent unitary quantum theory for the $\alpha = 1$ regularized non-abelian chiral model (8), with dim$[\mathcal{G}]$ − rank$[\mathcal{G}]$ degrees of freedom for the gauge field.

Now let us follow Faddeev's suggestion and enlarge these systems by adding an extra field h with a gauged WZ action. For the U(1) case, the action in (1) is enlarged to

$$S[\phi, A_\mu, h] = S[\phi, A_\mu] + \int \mathrm{d}x\ \mathrm{d}t(\epsilon^{\mu\nu}\partial_\mu h A_\nu). \tag{28}$$

This action is, by design, gauge invariant under $\phi \to \phi - \chi, A_\mu \to A_\mu + \partial_\mu \chi$ and $h \to h - \chi$. Its canonical analysis yields two first-class constraints, $\pi_0 = 0$ and $\widetilde{G} \equiv (1/e^2)\partial_1 E + \pi_\phi + \partial_1\phi + \pi_h + \partial_1 h = 0$, along with two second-class constraints $E = 0$ and $\pi_h - A_1 = 0$. In quantizing this theory, one has to fix a gauge, through a subsidiary condition. Consider the gauge $h = 0$. Preservation of this condition forces another constraint $A_0 - A_1 = 0$. Altogether these constraints reduce the system (28) back to the original system described by eqs. (2)–(7).

In the path integral formalism, when the gauge fixing factor $\delta(h(x,t))$ is inserted into the path integral of $\exp[\mathrm{i}S(\phi, A_\mu, h)]$ over the fields ϕ, A_μ and h, it reduces to

$$\widetilde{P} = \int \mathrm{D}\phi\ \mathrm{D}A_\mu\ \exp[\mathrm{i}S(\phi, A_\mu)]. \tag{29}$$

This is just the naive path integral of the original system (1) over all its fields. Since $S[\phi, A_\mu]$ is non-singular (recall that it is not gauge-invariant), $\widetilde{P}$ is well defined. To see if $\widetilde{P}$ correctly reflects the constrained quantum theory of the system (1), compare it with

$$P = \int \mathrm{D}\phi\ \exp\left(\mathrm{i}\int \mathrm{d}x\ \mathrm{d}t\, \tfrac{1}{2}\, \partial_\mu \phi \partial^\mu \phi\right). \tag{30}$$

In P, only the independent field ϕ appears, with a free action obtained from the free hamiltonian (7). The fields constrained to vanish by virtue of eqs. (2)–(6) do not appear at all in P, whereas they do appear in $\widetilde{P}$. However, for on-shell S-matrix purposes $\widetilde{P}$ is equivalent to P. This is easy to check. Since $S[\phi, A_\mu]$ is quadratic, the associated propagator in terms of (ϕ, A_+, A_-) is given by the matrix

$$\frac{\mathrm{i}}{k^2}\begin{bmatrix} 1-k^2/e^2 & \mathrm{i}k_+(k^2/e^2-2) & (\mathrm{i}k_-)(k^2/e^2) \\ -\mathrm{i}k_+(k^2/e^2-2) & k_+k_+(k^2/e^2-4) & k^2(k_+k_+/e^2-1) \\ (-\mathrm{i}k_-)k^2/e^2 & k^2(k_+k_+/e^2-1) & (k^2/e^2)k_-k_- \end{bmatrix}. \tag{31}$$

The only pole in (31) is at $k^2 = 0$, where the residue matrix has only one non-zero eigenvalue, corresponding to the ϕ–ϕ propagator. Propagators involving fields which are constrained to vanish in the canonical analysis are non-zero in (31), but they are all local (i.e. polynomials in k_+, k_-) and will not affect the S-matrix. For instance the ϕ–A_- propagator is just $(-k_-/e^2)$. In short, on-shell, the naive path integral (29) reflects the second-class constraints and is equivalent to (30).

Note that this Faddeev enlargement procedure can also be done for the $\alpha > 1$ regularization. The original bosonized action for the abelian case is [2]

$$S_{\alpha>1}[\phi, A_\mu] = \int \mathrm{d}x\,\mathrm{d}t\,[\tfrac{1}{2}\,\partial_\mu\phi\partial^\mu\phi + (g^{\mu\nu} - \epsilon^{\mu\nu})\,\partial_\mu\phi A_\nu + \tfrac{1}{2}\,\alpha A_\mu A^\mu - (1/4e^2)\,F_{\mu\nu}F^{\mu\nu}]\,. \tag{32}$$

Upon introducing the field h, the enlarged gauge invariant action is

$$S_{\alpha>1}[\phi, A_\mu, h] = S_{\alpha>1}[\phi, A_\mu] + \int \mathrm{d}x\,\mathrm{d}t[\epsilon^{\mu\nu}\partial_\mu h A_\nu + \tfrac{1}{2}(\alpha-1)(\partial_\mu h\partial^\mu h + 2A_\mu\partial^\mu h)]. \tag{33}$$

Although $(\partial_\mu h)^2$ occurs in (33), no new degrees of freedom are introduced as compared to the system (32). Gauge must be fixed for the system (33), and the gauge choice $h = 0$ reduces it to the original system (32).

For the nonabelian chiral theory, the Faddeev enlargement of the $\alpha = 1$ action (8) yields the new action

$$S[U, A_\mu, h] = S[U, A_\mu] - \Gamma_{\mathrm{WZ}}(h) - \frac{\mathrm{i}}{4\pi}\int \mathrm{d}x\,\mathrm{d}t\,\mathrm{Tr}[\epsilon^{\mu\nu}h^{-1}\partial_\nu h A_\mu]. \tag{34}$$

Here h is a U(n) group valued matrix field. The gauged WZ terms involving h that have been added in (34) are designed to make $S[U, A_\mu, h]$ gauge invariant under $U \to Ug$, $h \to hg$, $A_\mu \to g^{-1}(A_\mu + \mathrm{i}\partial_\mu)g$. Notice that $S[U, A_\mu, h]$ is just the gauge transform of $S[U, A_\mu]$ under $U \to Uh^{-1}$ and $A_\mu \to h(A_\mu + \mathrm{i}\partial_\mu)h^{-1}$. If the field h were integrated over, without gauge fixing, that would formally amount to averaging correlation functions of the original theory over the gauge group. However, gauge must be fixed. Upon using the ghost-free gauge $h = 1$, one again restores the original system in eq. (8). Our earlier result that the original system contains $\dim[\mathcal{G}] - \mathrm{rank}[\mathcal{G}]$ degrees of freedom for the gauge field, clearly holds for the enlarged system (34) as well.

We conclude with some comments about the more important four-dimensional case. We have seen that in two dimensions, the $\alpha = 1$ regularization yields $\dim[\mathcal{G}] - \mathrm{rank}[\mathcal{G}]$ degrees of freedom for the gauge field. For the $\alpha > 1$ regularization, this number changes [4] to $2\dim[\mathcal{G}]$, while for $\alpha < 1$ the theory is non-unitary. We expect that this dependence of the consistency and the number of degrees of freedom on the regularization would also be a property of the four-dimensional chiral gauge theory.

In four dimensions, bosonization techniques are not avaiable and one must work with the action in fermionic form. Then Faddeev's method of introducing an extra field h and adding gauged WZ terms to this classical fermionic action provides a natural way of treating the constraints of the chiral gauge theory in the path integral formalism. For instance, consider the abelian chiral theory with a regularization such that the anomaly is proportional to $F^{\mu\nu}\tilde{F}_{\mu\nu}$. The corresponding gauged WZ term to be added to the action is $h\,F_{\mu\nu}\tilde{F}^{\mu\nu}$, where h is a scalar field which gauge transforms as $h \to h - \chi$. This leads to a gauge invariant quantum theory. The $h = 0$ gauge reproduces the original chiral theory. Alternately, integration over the field h forces the constraint $\delta(F_{\mu\nu}\tilde{F}^{\mu\nu})$ in the path integral (compare this with eq. (2.19) of ref. [10]). We believe that in order to test the unitarity of this chiral theory one must use a perturbation expansion which respects this constraint at each order. (This is also suggested by ref. [5] when applied to the abelian case.) One way to obtain such an expansion may be to parametrize the surface $F_{\mu\nu}\tilde{F}^{\mu\nu} = 0$ and rewrite the action using these coordinates, as is done with σ models [11]. Another approach may be to use the perturbation expansion for a composite-field effective action [12], which for our problem would break gauge invariance at the lowest order. In any case, the question of whether anomalous gauge theories are consistent in four dimensions needs to be further examined.

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