Positive scalar curvature on noncompact manifolds

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Scalar curvature

Given a Riemannian n-manifold (M, g), at a point $m \in M$, the scalar curvature $R(m)$ is $n(n-1)$ times the average sectional curvature at m.

Basic question: Given a smooth compact manifold M, does it admit a Riemannian metric with positive scalar curvature (psc)?

The answer is known when M is simply connected (Gromov-Lawson 1980, Stolz 1992).

Open conjecture: If M is aspherical, i.e. has vanishing higher homotopy groups, then M does not admit a psc metric.

Known to be true if $n \leq 5$ (Schoen-Yau 1979, Gromov-Lawson 1983, Chodosh-Li 2024, Gromov).

Classifying spaces

If M is a connected manifold with fundamental group Γ then there is an aspherical classifying space $B\Gamma$ along with a classifying map $\nu : M \to B\Gamma$ that is an isomorphism on π_1.

Here $B\Gamma$ is a CW-complex that only depends on Γ. Both $B\Gamma$ and ν are uniquely defined up to homotopy.

How to get $B\Gamma$? Start with M. Ask if $\pi_2(M)$ is nonzero. If not, attach 3-disks to kill it. Then ask if π_3 of the result is nonzero. If not, attach 4-disks to kill it. Continue to get $B\Gamma$. There's an inclusion map $\nu : M \to B\Gamma$.

Generalized conjecture

We have the map $\nu : M \to B\Gamma$.

Generalized open conjecture: If M is compact and oriented, let $[M] \in \mathrm{H}_n(M; \mathbb{Q})$ be its fundamental class. If $\nu_*[M]$ is nonzero in $\mathrm{H}_n(B\Gamma; \mathbb{Q})$ then M does not admit a psc metric.

If M is already aspherical then we can take $B\Gamma = M$ and $\nu = \mathrm{Id}$, so we recover the previous conjecture.

The generalized conjecture is known to be true if M is spin and $\pi_1(M)$ satisfies the Strong Novikov Conjecture. (Conceivably, all discrete groups do.)

Noncompact manifolds

What about complete metrics on noncompact manifolds? We can ask about obstructions for uniformly positive scalar curvature, i.e. $R \geq r_0 > 0$, or just positive scalar curvature, i.e $R > 0$. The answers are not the same, e.g. if $M = \mathbb{R}^2$.

Test question: Suppose that Y is a connected oriented compact manifold-with-boundary, with connected boundary ∂Y. Put $\Gamma = \pi_1(Y, y_0)$ and $\Gamma' = \pi_1(\partial Y, y_0)$. There is a classifying map $\nu : (Y, \partial Y) \to (B\Gamma, B\Gamma')$. Is nonvanishing of $\nu_*[Y, \partial Y]$ in $\mathrm{H}_n(B\Gamma, B\Gamma'; \mathbb{Q})$ an obstruction to the existence of a complete psc metric on the interior of Y, provided that

1. $\Gamma' \to \Gamma$ is injective, or
2. The Riemannian metric has finite volume?

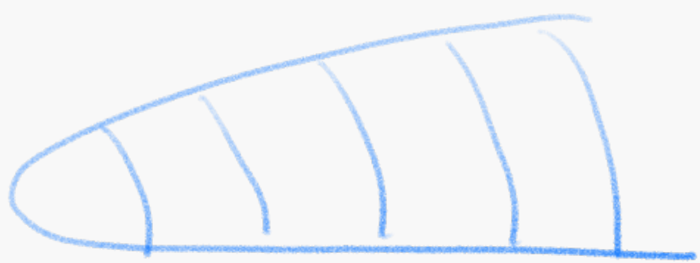
Example

Had assumptions

1. $\Gamma' \to \Gamma$ is injective, or
2. The complete Riemannian metric on the interior has finite volume.

A condition like 1 or 2 is necessary.

Example: Suppose that Y is D^2, so $\partial Y = S^1$. Then $\nu_*[Y, \partial Y]$ is nonzero. Nevertheless, there is a psc metric on the interior of Y, i.e. $\mathbb{R}^2$.



In this case, the map $\pi_1(S^1) \to \pi_1(D^2)$ is not injective, and also the psc metric has infinite volume.

Tools

The main tools that I use:

1. Almost flat vector bundles (Connes-Gromov-Moscovici 1990)
2. Almost flat relative vector bundles (Kubota 2022)
3. Callias-type Dirac operators (Callias 1978, ..., Cecchini-Zeidler 2024, ...)

Using these tools, one can give *localized* obstructions to positive scalar curvature.

Another set of tools comes from μ-bubbles. It would be interesting if one could derive analogous results using them.

Almost flat vector bundles

If X is a compact manifold then elements of $\mathrm{K}^0(X)$ can be represented as formal differences $E^+ - E^-$ of vector bundles on X.

Give X a Riemannian metric.

Definition: A class $\beta \in \mathrm{K}^0(X)$ is almost flat if for each $\epsilon > 0$, we can find

1. A $\mathbb{Z}_2$-graded Hermitian vector bundle $E^{\pm}$ representing β, and
2. A Hermitian connection $\nabla^{\pm}$ on $E^{\pm}$ whose curvature satisfies $\| F^{\pm} \| < \epsilon$.

Note: As ϵ decreases, the rank of E will generally go to infinity.

Index theorem

Theorem: (Connes-Gromov-Moscovici 1990) If M is a compact even dimensional spin manifold with a psc metric then for any almost flat $\beta \in \mathrm{K}^0(M)$, we have

$$\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0.$$

Proof: If D is the Dirac operator on spinors coupled to E then Lichnerowicz says that

$$D^2 = \nabla^*\nabla + \frac{R}{4} - \frac{1}{4}\sum_{\mu,\nu}[\gamma^\mu, \gamma^\nu]F^E_{\mu\nu}.$$

So for sufficiently small ϵ, the kernel of D vanishes. But Atiyah-Singer says that the index of D is $\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta)$.

The odd case

Elements of $\mathrm{K}^{-1}(X)$ can be represented by pairs (V, σ) where V is a vector bundle on X and σ is an automorphism of V.

Definition: A class $\beta \in \mathrm{K}^{-1}(X)$ is almost flat if for each $\epsilon > 0$, we can find

1. A Hermitian vector bundle V equipped with an isometric automorphism σ that together represent β, and

2. A Hermitian connection ∇^V on V so that $\| F^V \| < \epsilon$ and $\| \nabla^V \sigma \| < \epsilon$.

Relevance of almost flat bundles

If M has psc and $\beta \in \mathrm{K}^*(M)$ is a.f. then $\int_M \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0$.

Where do almost flat K-theory classes come from? They pullback from classifying spaces.

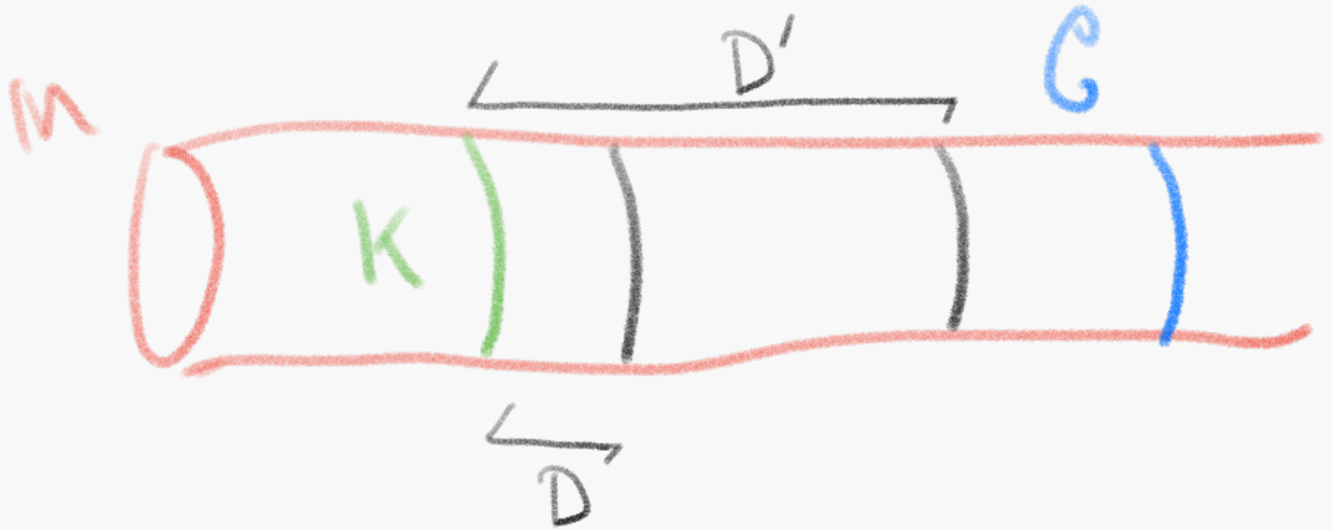
Definition: If Γ is a discrete group, let $\mathrm{K}^*_{af}(B\Gamma)$ be the elements $\eta \in \mathrm{K}^*(B\Gamma)$ so that for any compact manifold X and any $\nu : X \to B\Gamma$, the pullback $\nu^*\eta$ is almost flat on X.

Conceivably, $\mathrm{K}^*_{af}(B\Gamma)$ is all of $\mathrm{K}^*(B\Gamma)$, at least rationally. This is known for many Γ, such as word hyperbolic groups.

If so, we conclude that if M has a psc metric then for any $\nu : M \to B\Gamma$, the pushforward $\nu_*(\star\widehat{A}(TM))$ vanishes in $\mathrm{H}_*(B\Gamma; \mathbb{Q})$.

In particular, $\nu_*(\star 1) = \nu_*[M]$ vanishes in $\mathrm{H}_n(B\Gamma; \mathbb{Q})$.

The geometric setup



Given $r_0, D > 0$, put $r_0' = \frac{1}{256} r_0^2 D^2$ and $D' = D + \frac{32}{r_0 D}$.

- M is a Riemannian spin manifold-with-boundary.
- K is a compact submanifold of M containing ∂M.
- $R > 0$ on K.
- $R \geq r_0$ on $N_D(K) - K$.
- $R \geq -r_0'$ on $N_{D'}(K) - N_D(K)$.
- $N_{D'}(K)$ lies in a compact submanifold $\mathcal{C}$.

First result

Terminology: The boundary of a Riemannian manifold-with-boundary is mean convex if it has nonnegative mean curvature. For example, ∂B^n is mean convex.

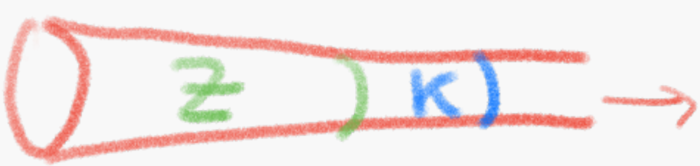
Theorem 1: Suppose that ∂M is mean convex. If $\beta \in \mathrm{K}^*(\mathcal{C})$ is almost flat then

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\Big|_{\partial M}\right) = 0.$$

Exhaustion of finite volume manifolds

Proposition: Let M be a complete finite volume oriented Riemannian manifold, of dimension at most seven, with compact boundary. Then there is an exhaustion of M by compact submanifolds-with-boundary Z so that ∂Z (away from ∂M) is mean convex as seen from $M - Z$.

Corollary: There is no complete finite volume psc metric on $[0, \infty) \times T^{n-1}$, provided that $n \leq 7$.



Relative K-theory

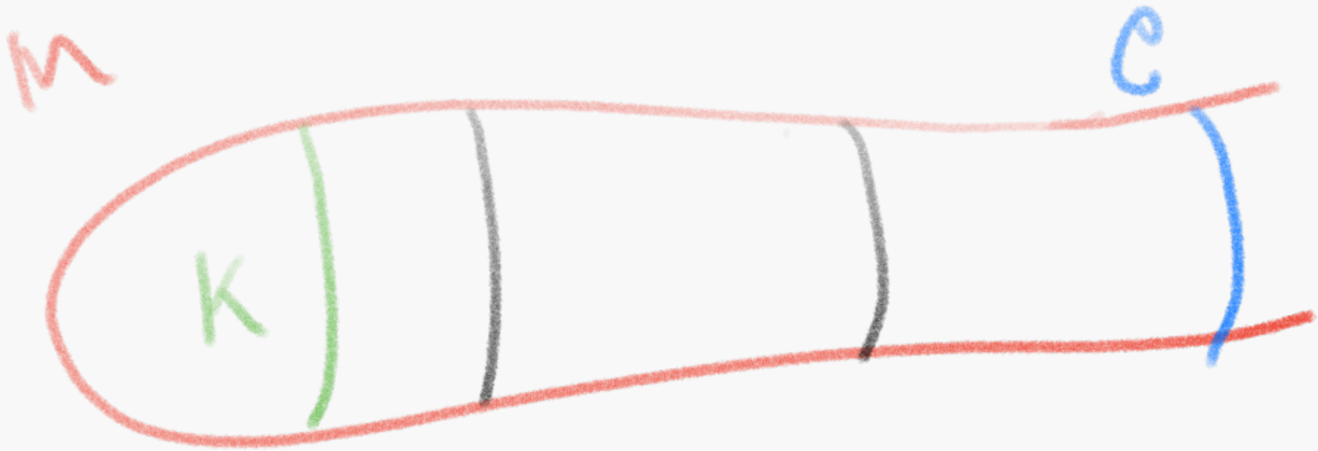
If X is a compact manifold and $Y \subset X$ is a submanifold then a generator of the relative K-group $\mathrm{K}^0(X, Y)$ is a formal difference $E^+ - E^-$ of vector bundles on X, along with an isomorphism $\sigma : E^+\big|_Y \to E^-\big|_Y$.

Give X a Riemannian metric.

Definition: (Kubota) A class $\beta \in \mathrm{K}^0(X, Y)$ is almost flat if for each $\epsilon > 0$, we can find

1. A $\mathbb{Z}_2$-graded Hermitian vector bundle $E^\pm$ on X and an isometric isomorphism $\sigma : E^+\big|_Y \to E^-\big|_Y$ so that (E, σ) represents β, and
2. A Hermitian connection $\nabla^\pm$ on $E^\pm$ so that $\| F^\pm \| < \epsilon$ and $\| \nabla\sigma \| < \epsilon$.

Second result



Theorem 2: If $\beta \in \mathrm{K}^*(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$ is almost flat then

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0.$$

Remark: If M is compact and $K = \mathcal{C} = M$ then this becomes the Connes-Gromov-Moscovici result.

Application to test question

Where do almost flat elements of $\mathrm{K}^0(X, Y)$ come from? From pullbacks under maps of pairs $\nu : (X, Y) \to (B\Gamma, B\Gamma')$, where $h : \Gamma' \to \Gamma$ is an *injective* homomorphism.

If h is not injective then we cannot expect that elements of $\mathrm{K}^0(B\Gamma, B\Gamma')$ pullback to almost flat elements of $\mathrm{K}^0(X, Y)$. Example: $\Gamma' = \mathbb{Z}$ and $\Gamma = \{e\}$.

Corollary: Suppose that Y^n is a connected compact spin manifold-with-boundary, with connected boundary. Put $\Gamma = \pi_1(Y, y_0)$ and $\Gamma' = \pi_1(\partial Y, y_0)$. If the interior of Y has a complete psc metric, and $\mathrm{K}^*_{af}(B\Gamma, B\Gamma')$ equals $\mathrm{K}^*(B\Gamma, B\Gamma')$ rationally, then $\nu_*[Y, \partial Y]$ vanishes in $\mathrm{H}_n(B\Gamma, B\Gamma'; \mathbb{Q})$.

Note: If $\Gamma' \to \Gamma$ is not injective, replace Γ' by its image.

Almost flat stably

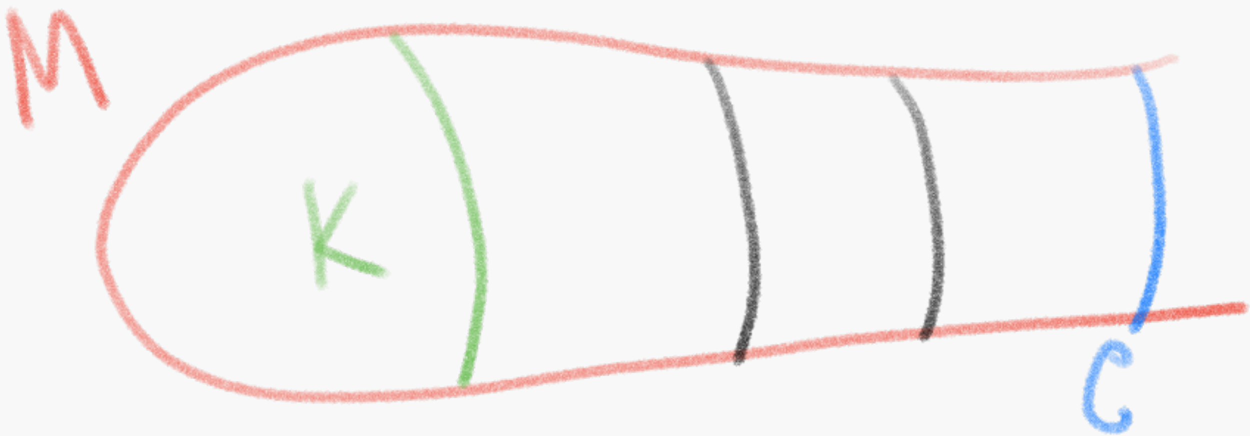
Suppose that X is a compact manifold and $Y \subset X$ is a submanifold. Give X a Riemannian metric.

Definition: (Kubota) A class $\beta \in \mathrm{K}^0(X, Y)$ is almost flat stably if for each $\epsilon > 0$, we can find

1. A $\mathbb{Z}_2$-graded Hermitian vector bundle E on X, a Hermitian vector bundle V on Y, and an isometric isomorphism $\sigma : E^+\big|_Y \oplus V \to E^-\big|_Y \oplus V$ so that (E, V, σ) represents β, and

2. Hermitian connections $\nabla^{E^\pm}$ on E and ∇^V on V so that $\| F^{E^\pm} \| < \epsilon$, $\| F^V \| < \epsilon$ and $\| \nabla\sigma \| < \epsilon$.

Third result



Theorem 3: If ∂K is mean convex as seen from $M - K$, and $\beta \in \mathrm{K}^*(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$ is almost flat stably, then

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0.$$

Application to test question

Corollary: Suppose that Y is a connected compact spin manifold-with-boundary, with connected boundary, of dimension $n \leq 7$. Put $\Gamma = \pi_1(Y, y_0)$ and $\Gamma' = \pi_1(\partial Y, y_0)$.

If the interior of Y has a complete finite volume psc metric, and $K^*_{af,st}(B\Gamma, B\Gamma')$ equals $K^*(B\Gamma, B\Gamma')$ rationally, then $\nu_*[Y, \partial Y]$ vanishes in $H_n(B\Gamma, B\Gamma'; \mathbb{Q})$.

Scalar curvature

Simplicial volume

The simplicial volume $\| M \|$ of a closed oriented manifold M roughly measures how many simplices it takes to triangulate the manifold.

Definition:

$$\| M \| = \inf \left\{ \sum_i |a_i| : \left[\sum_i a_i c_i \right] = [M] \right\}$$

Here

- $a_i \in \mathbb{R}$,
- $c_i : \triangle^n \to M$ is a singular n-simplex,
- $[M] \in \mathrm{H}_n(M; \mathbb{R})$ is the fundamental class, and
- The sum is finite.

Simplicial volume conjecture

A manifold has almost nonnegative scalar curvature if for each $\epsilon > 0$ it admits an appropriately normalized Riemannian metric with $R \geq -\epsilon$.

What are possible topological obstructions for a compact manifold to have almost nonnegative scalar curvature?

Conjecture: (Gromov 1986) For each $n \in \mathbb{Z}^+$, there is some $c_n > 0$ so that if M is a compact connected oriented n-dimensional Riemannian manifold with $R \geq -\sigma^2$ then $\| M \| \leq c_n \, \sigma^n \, \mathrm{vol}(M)$.

Remark: This is open even if $\sigma = 0$.

Almost nonnegative scalar curvature

Conjecture: (Gromov 1986) For each $n \in \mathbb{Z}^+$, there is some $c_n > 0$ so that if M is a compact connected oriented n-dimensional Riemannian manifold with $R \geq -\sigma^2$ then $\| M \| \leq c_n \, \sigma^n \, \mathrm{vol}(M)$.

Two ways to think of this conjecture:

1. If we normalize $\sigma = 1$, it says that simplicial volume is an obstruction to volume-collapsing with a lower scalar curvature bound.

2. If we normalize $\mathrm{vol} = 1$, it says that simplicial volume is an obstruction to having almost nonnegative scalar curvature, relative to the volume.

We will think about it the second way.

Known results

Conjecture: (Gromov 1986) For each $n \in \mathbb{Z}^+$, there is some $c_n > 0$ so that if M is a compact connected oriented n-dimensional Riemannian manifold with $\mathrm{vol} = 1$ and $R \geq -\sigma^2$ then $\| M \| \leq c_n \, \sigma^n$.

It's true if scalar curvature is replaced by Ricci curvature (Gromov 1982). In fact, there's a gap theorem: there is some $\epsilon_n > 0$ so that if $\mathrm{vol} = 1$ and $\mathrm{Ric} \geq -\epsilon_n$ then $\| M \| = 0$.

Also, the conjecture is true if scalar curvature is replaced by "macroscopic scalar curvature", along with a gap result (Braun-Sauer 2021).

Diameter bound

Can we verify the conjecture in the "easy" case when we impose some additional curvature bound?

Theorem 4: Given $n \in \mathbb{Z}^+$ and $D, \Lambda < \infty$, there is some $\epsilon = \epsilon(n, D, \Lambda) > 0$ with the following property. Let M^n be a compact connected spin manifold so that $\pi_1(M)$ satisfies the Strong Novikov Conjecture. Suppose that g is a Riemannian metric on M so that (M, g) has

- Diameter bounded above by D,
- Curvature operator bounded below by $-\Lambda$, and
- Scalar curvature bounded below by $-\epsilon$.

Then $\| M \| = 0$.

Remark: One cannot remove the lower bound on the curvature operator (Lohkamp 1999).

Volume bound

Going to back to the original conjecture, let's add a double sided bound on the sectional curvatures.

Question: Given $n \in \mathbb{Z}^+$ and $\Lambda < \infty$, is there some $\epsilon = \epsilon(n, \Lambda) > 0$ with the following property? Let M^n be a compact connected manifold. Suppose that g is a Riemannian metric on M so that (M, g) has

- Volume equal to one,
- Sectional curvatures bounded in magnitude by Λ, and
- Scalar curvature bounded below by $-\epsilon$.

Then $\| M \| = 0$.

Suppose not. Let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence that gives a counterexample.

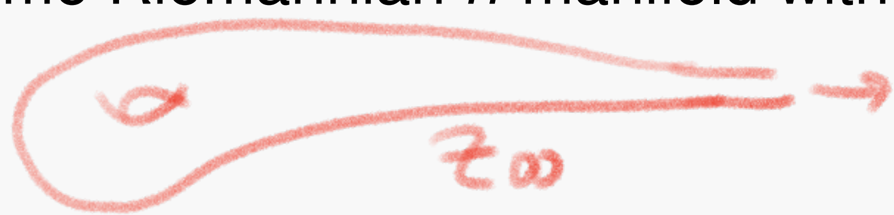
Contradiction argument

Each (M_i, g_i) is a compact connected Riemannian n-manifold with

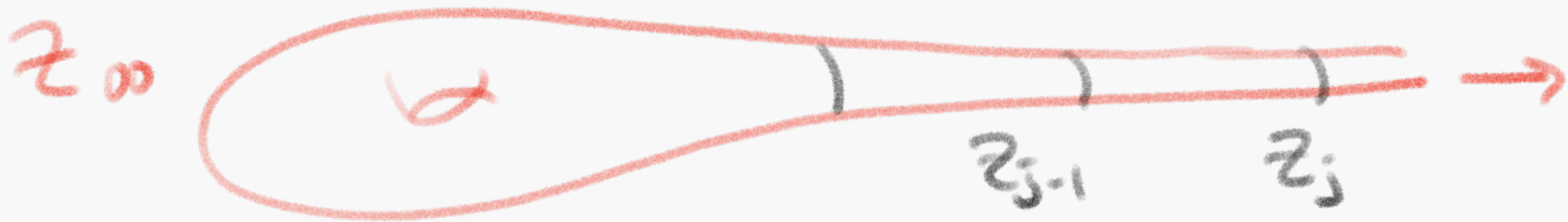
- Volume equal to one,
- Sectional curvatures bounded in magnitude by Λ, and
- Scalar curvature bounded below by $-\frac{1}{i}$, but
- $\| M_i \| \neq 0$.

From the previous result, we can assume that $\mathrm{diam}(M_i, g_i)$ goes to infinity.

After passing to a subsequence, we get a multipointed limit $\lim_{i \to \infty}(M_i, g_i) = (Z_\infty, g_\infty)$, where (Z_∞, g_∞) is a complete noncompact finite volume Riemannian n-manifold with *positive* scalar curvature.



Thick-thin decomposition



Based on Theorem 3, it is reasonable to assume that Z_∞ has an exhaustion by compact submanifolds $\{Z_j\}_{j=1}^{\infty}$ so that the image of $[Z_j, Z_j - \mathrm{int}(Z_{j-1})]$ vanishes in $\mathrm{H}_n(B\pi_1(Z_j), B\pi_1(Z_j - \mathrm{int}(Z_{j-1})); \mathbb{Q})$.

This implies that the relative simplicial volume $\| Z_j, Z_j - \mathrm{int}(Z_{j-1}) \|$ vanishes.

Lifting Z_j to M_i for large i, we get a decomposition $M_i = M_i^{thick} \cup M_i^{thin}$.

Gluing problem

We have a decomposition $M_i = M_i^{thick} \cup M_i^{thin}$ where

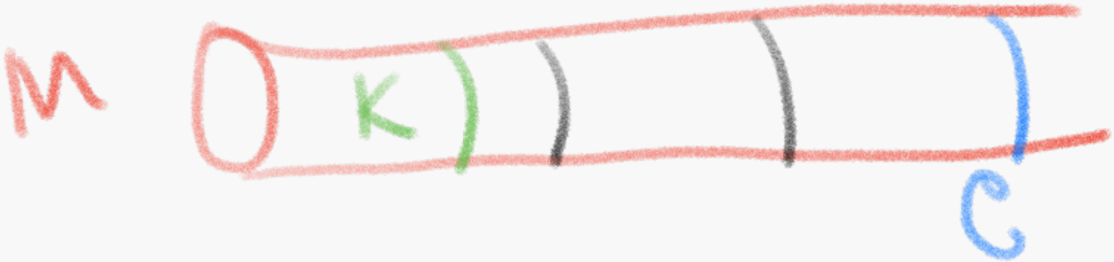
- $\partial M_i^{thick} \subset \mathrm{int}(M_i^{thin})$ and $\partial M_i^{thin} \subset \mathrm{int}(M_i^{thick})$,
- $\| M_i^{thick}, M_i^{thick} \cap M_i^{thin} \| = 0$, and
- M_i^{thin} is locally volume collapsed.

Since M_i^{thin} is locally volume collapsed relative to a sectional curvature bound, it has an "amenable open cover" of multiplicity at most n. In particular, $\| M_i^{thin}, M_i^{thick} \cap M_i^{thin} \| = 0$ (Gromov, Ivanov, Löh-Sauer).

We would now like to say that $\| M_i \| = 0$, which would give a contradiction. Is this true? A gluing problem for simplicial volume!

Scalar curvature

Statement of Theorem 1



Theorem 1: Suppose that ∂M is mean convex. If $\beta \in \mathrm{K}^*(\mathcal{C})$ is almost flat then

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\Big|_{\partial M}\right) = 0.$$

Sketch of proof: Suppose that $\dim(M)$ is even. Given $\epsilon > 0$, let (V, σ) represent $\beta \in \mathrm{K}^{-1}(\mathcal{C})$, where

- V is a Hermitian vector bundle on $\mathcal{C}$,
- σ is an isometric automorphism of V,
- ∇^V is a Hermitian connection on V, and
- $\| F^V \| < \epsilon$ and $\| \nabla^V \sigma \| < \epsilon$.

Callias operator

Put $\widehat{D} = D + \frac{16}{r_0 D} < D'$.

Suppose for simplicity that d_K is smooth away from K, and $\mathcal{N} = \overline{N_{\widehat{D}}(K)}$ is a smooth manifold-with-boundary.

Let S be the spinor bundle on $\mathcal{N}$, with $\mathbb{Z}_2$-grading operator ϵ_S.

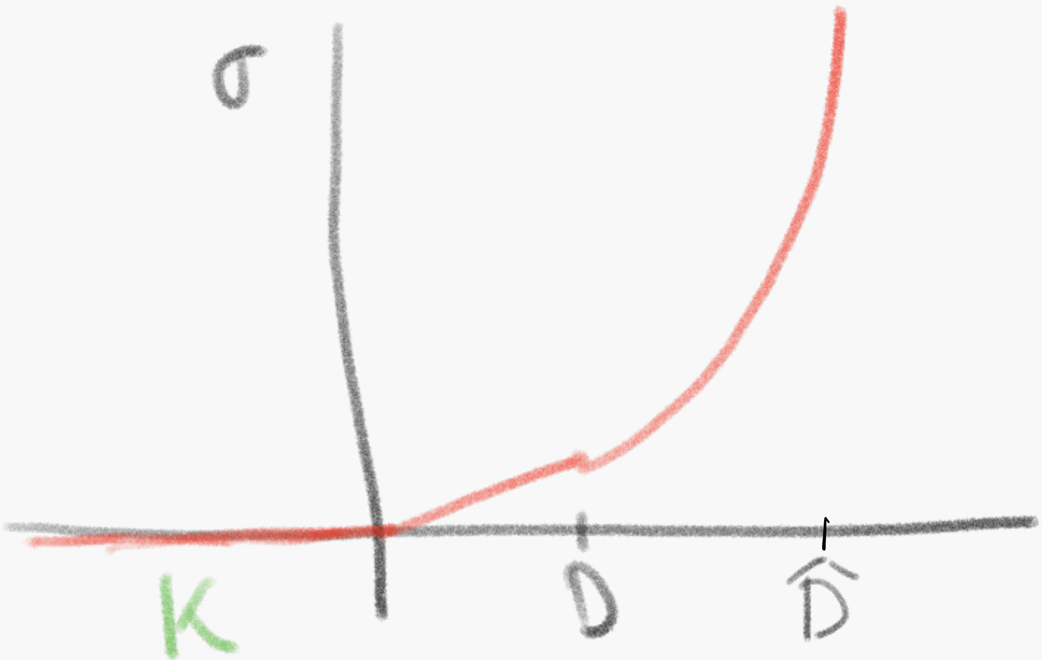
Let E be the restriction of $V \oplus V$ to $\mathcal{N}$.

Let $\mathcal{D}^V$ be the Dirac operator on $C^\infty(\mathcal{N}; S \otimes V)$.

For an appropriate function f, define $\mathcal{D}^E$ on $C^\infty(\mathcal{N}; S \otimes E)$ by

$$\mathcal{D}^E = \begin{pmatrix} \mathcal{D}^V & \epsilon_S f \sigma^{-1} \\ \epsilon_S f \sigma & \mathcal{D}^V \end{pmatrix}.$$

Warping function



Take $f = \sigma \circ d_K$, where σ is a slight smoothing of

$$\begin{cases} 0 & \text{if } t \leq 0, \\ \frac{r_0}{8} t & \text{if } 0 \leq t \leq D, \\ \frac{2}{\widehat{D} - t} & \text{if } D \leq t \leq \widehat{D}. \end{cases}$$

Local boundary condition

Define a self-adjoint operator Π on $C^\infty\left(\partial\mathcal{N}, (S \otimes E)\big|_{\partial\mathcal{N}}\right)$ by

$$\Pi = \begin{cases} \begin{pmatrix} 0 & \sqrt{-1}\epsilon_S\gamma^n \\ \sqrt{-1}\epsilon_S\gamma^n & 0 \end{pmatrix} & \text{on } \partial M, \\ \begin{pmatrix} 0 & \sqrt{-1}\epsilon_S\gamma^n\sigma^{-1} \\ \sqrt{-1}\epsilon_S\gamma^n\sigma & 0 \end{pmatrix} & \text{on } \partial\mathcal{N} - \partial M. \end{cases}.$$

Boundary condition: $\Pi\left(\psi\big|_{\partial\mathcal{N}}\right) = \psi\big|_{\partial\mathcal{N}}$.

Claim: With this boundary condition, $\mathcal{D}^E$ is invertible on $\mathcal{N}$.

Idea: Show that with this choice of f, for any nonzero ψ we have $\int_{\mathcal{N}} \langle \mathcal{D}^E\psi, \mathcal{D}^E\psi\rangle \,\mathrm{dvol} > 0$.

Index computation

Claim: The index of $\mathcal{D}^E$, going from $C^\infty(\mathcal{N}; (S \otimes E)^+)$ to $C^\infty(\mathcal{N}; (S \otimes E)^-)$ is

$$\int_{\partial M} \widehat{A}(T\partial M) \wedge \mathrm{ch}\left(\beta\Big|_{\partial M}\right).$$

Idea: Without changing the index, we can

- Deform f to zero.
- Make the Riemannian metric a product near $\partial\mathcal{N}$.
- Make the connection ∇^V a product near $\partial\mathcal{N}$.

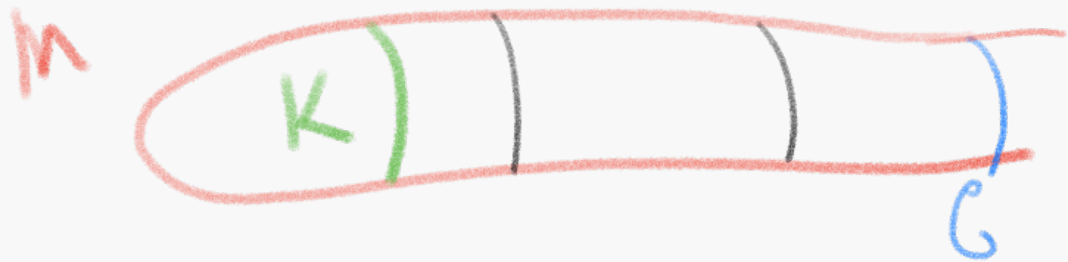
The boundary conditions are such that $\mathcal{D}^E$ is the same as the corresponding operator on the $\mathbb{Z}_2$-invariant spinors on the double $D\mathcal{N}$.

Index computation II

Then the index of $\mathcal{D}^E$ is the same as the $\mathbb{Z}_2$-invariant index on $D\mathcal{N}$. To compute it, make the connection ∇^{E^-} on $E^- \cong V$ equal to $\nabla^{E^+} = \nabla^V$ very close to ∂M, and equal to $\sigma \circ \nabla^{E^+} \circ \sigma^{-1}$ away from a δ-neighborhood of ∂M. Then the index is

$$\begin{aligned}
\int_{\mathcal{N}} \widehat{A}(T\mathcal{N}) \wedge \operatorname{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) &= \int_{\partial M \times [0,\delta]} \widehat{A}(T\mathcal{N}) \wedge \operatorname{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \int_{[0,\delta]} \operatorname{tr}_s \left(e^{\frac{i}{2\pi}(\nabla^E)^2} \right) \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \operatorname{ch}(V, \sigma)\big|_{\partial M} \\
&= \int_{\partial M} \widehat{A}(T\partial M) \wedge \operatorname{ch}\left(\beta\big|_{\partial M} \right).
\end{aligned}$$

Statement of Theorem 2



Theorem 2: If $\beta \in \mathrm{K}^*(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$ is almost flat then

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0.$$

Sketch of proof: Suppose that $\dim(M)$ is even. Given $\epsilon > 0$, let (E, σ) represent $\beta \in \mathrm{K}^0(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$, where

- $E^{\pm}$ is a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{C}$,
- $\sigma : E^+\big|_{\mathcal{C} - \mathrm{int}(K))} \to E^-\big|_{\mathcal{C} - \mathrm{int}(K))}$ is an isometric isomorphism,
- $\nabla^{\pm}$ is a Hermitian connection on $E^{\pm}$, and
- $\| F^{\pm} \| < \epsilon$ and $\| \nabla\sigma \| < \epsilon$.

Callias operator

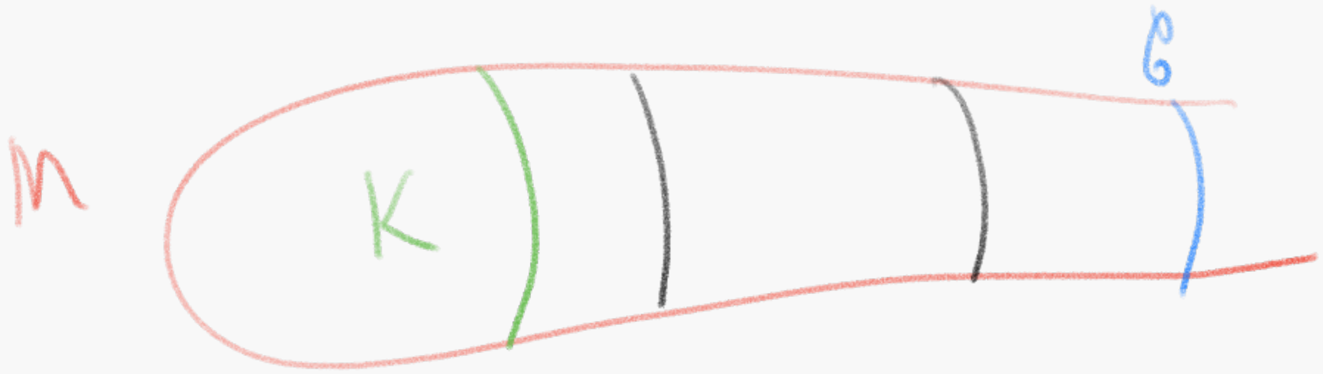
Define the warping function f as in the proof of Theorem 1.

Define $\mathcal{D}$ on $C^\infty(\mathcal{N}; S \otimes E)$ by

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}^{E^+} & \epsilon_S f \sigma^{-1} \\ \epsilon_S f \sigma & \mathcal{D}^{E^-} \end{pmatrix}.$$

Impose local boundary conditions on $\partial\mathcal{N}$ and proceed as in the proof of Theorem 1.

Statement of Theorem 3

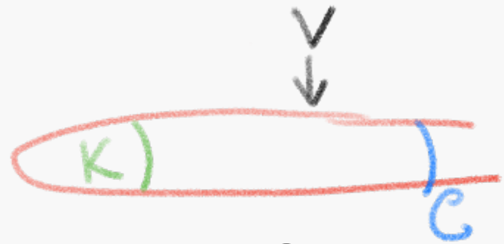


Theorem 3: If ∂K is mean convex as seen from $M - K$, and $\beta \in \mathrm{K}^*(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$ is almost flat stably, then

$$\int_{\mathcal{C}} \widehat{A}(TM) \wedge \mathrm{ch}(\beta) = 0.$$

Sketch of proof: Combine the setups of Theorems 1 and 2.

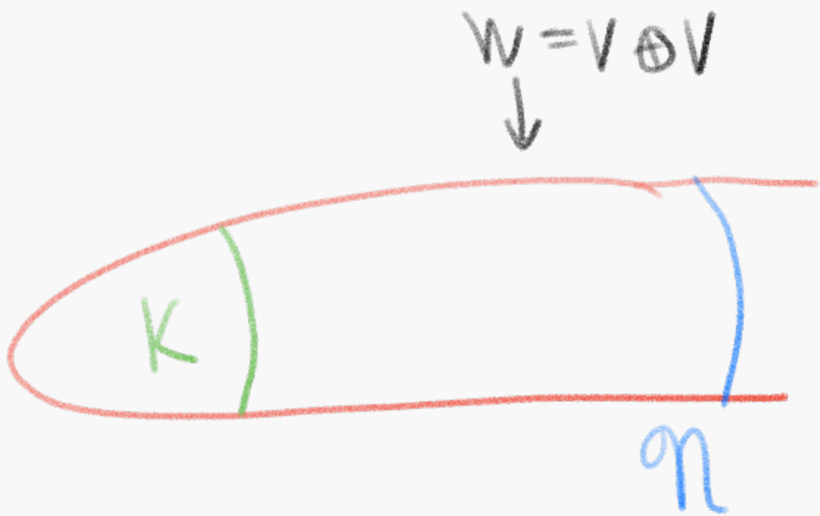
Setup



Suppose that $\dim(M)$ is even. Given $\epsilon > 0$, let (E, V, σ) represent $\beta \in \mathrm{K}^0(\mathcal{C}, \mathcal{C} - \mathrm{int}(K))$, where

- $E^{\pm}$ is a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{C}$,
- V is a Hermitian vector bundle on $\mathcal{C} - \mathrm{int}(K)$,
- $\sigma : E^+\big|_{\mathcal{C}-\mathrm{int}(K)} \oplus V \to E^-\big|_{\mathcal{C}-\mathrm{int}(K)} \oplus V$ is an isometric isomorphism,
- $\nabla^{E^{\pm}}$ is a Hermitian connection on $E^{\pm}$,
- ∇^V is a Hermitian connection on V, and
- $\| F^{E^{\pm}} \| < \epsilon$, $\| F^V \| < \epsilon$ and $\| \nabla\sigma \| < \epsilon$.

Domain of the operator



Put $W = V \oplus V$, a $\mathbb{Z}_2$-graded Hermitian vector bundle on $\mathcal{C} - \mathrm{int}(K)$, with Hermitian connection $\nabla^W = \nabla^V \oplus \nabla^V$.

The Callias operator $\mathcal{D}$ will act on the subspace of

$$C^\infty(\mathcal{N}; S \otimes E) \oplus C^\infty(\mathcal{N} - \mathrm{int}(K); S \otimes W)$$

that satisfies certain local boundary conditions.

Callias operator

On K, the operator $\mathcal{D}$ is the usual Dirac-type operator on $C^\infty(K; S \otimes E)$.

On $\mathcal{N} - \mathrm{int}(K)$, put $Z = E\Big|_{\mathcal{N}-\mathrm{int}(K)} \oplus W$. Then $\mathcal{D}$ acts on $C^\infty(\mathcal{N} - \mathrm{int}(K); S \otimes Z)$ by

$$\mathcal{D} = \begin{pmatrix} \mathcal{D}^{Z^+} & \epsilon_S f \sigma^{-1} \\ \epsilon_S f \sigma & \mathcal{D}^{Z^-} \end{pmatrix}.$$

Proceed as in the proofs of Theorems 1 and 2.

Statement of Theorem 4

Theorem 4: Given $n \in \mathbb{Z}^+$ and $D, \Lambda < \infty$, there is some $\epsilon = \epsilon(n, D, \Lambda) > 0$ with the following property. Let M^n be a compact connected spin manifold so that $\pi_1(M)$ satisfies the Strong Novikov Conjecture. Suppose that g is a Riemannian metric on M so that (M, g) has

- Diameter bounded above by D,
- Curvature operator bounded below by $-\Lambda$, and
- Scalar curvature bounded below by $-\epsilon$.

Then $\| M \| = 0$.

Suppose not. Let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence that gives a counterexample.

Contradiction argument

Each (M_i, g_i) is a compact connected Riemannian n-manifold with

- Diameter at most D,
- Curvature operator bounded below by $-\Lambda$, and
- Scalar curvature bounded below by $-\frac{1}{i}$, but
- $\| M_i \| \neq 0$.

From Gromov's gap theorem, there is a uniform lower bound $\mathrm{vol}(M_i, g_i) \geq v_0 > 0$ (coming from the lower Ricci curvature bound).

We can run the Ricci flow for a uniform amount of time to obtain a new metric g_i' with $R(M_i, g_i') \geq -\frac{1}{i}$ and a uniform bound $|\,\mathrm{Rm}(M_i, g_i')| \leq \Lambda'$ (Bamler-Cabezas-Rivas-Wilking 2019).

Limit space

From distortion estimates under Ricci flow, there is a uniform upper bound $\operatorname{diam}(M_i, g_i') \leq D'$.

From Gromov's gap theorem, there is again a uniform lower bound $\operatorname{vol}(M_i, g_i') \geq v_0' > 0$.

After passing to a subsequence, there is a smooth limit $\lim_{i \to \infty}(M_i, g_i') = (M_\infty, g_\infty)$.

Necessarily, $R(M_\infty, g_\infty) \geq 0$.

Since (M_∞, g_∞) is also the result of running Ricci flow, either $\operatorname{Ric}(M_\infty, g_\infty) = 0$ or $R(M_\infty, g_\infty) > 0$.

End of proof

If $\mathrm{Ric}(M_\infty, g_\infty) = 0$ then $\pi_1(M_\infty)$ is virtually abelian and so $\| M_\infty \| = 0$.

If $R(M_\infty, g_\infty) > 0$ then SNC implies that $\nu_*[M_\infty]$ vanishes in $\mathrm{H}_n(B\pi_1(M); \mathbb{Q})$. This implies that $\| M_\infty \| = 0$.

In either case, $\| M_\infty \| = 0$. Since M_∞ is diffeomorphic to M_i, for large i, this is a contradiction.