

What did Perelman prove, how did he do it, and what's left?

Berkeley MUSA, October 12, 2026

How a changing geometry reveals the shape of a space.

Perelman's theorems, 2002–2003

Poincaré conjecture

A closed, connected 3-manifold in which every loop shrinks to a point is a 3-sphere.

The formulation follows [KL, Appendix I]. The precise statement comes shortly.

Perelman's theorems, 2002–2003

Poincaré conjecture

A closed, connected 3-manifold in which every loop shrinks to a point is a 3-sphere.

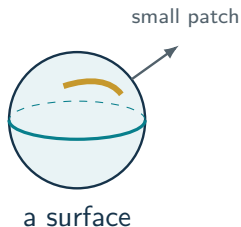
Geometrization, in outline

A closed orientable 3-manifold is a connected sum of pieces of **graph-manifold**, **hyperbolic**, or **mixed** type.

The formulation follows [KL, Appendix I]. The precise statement comes shortly.

Local dimension, global shape

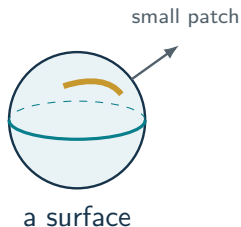
A **2-manifold** looks locally like an open piece of $\mathbb{R}^2$.



Local dimension, global shape

A **2-manifold** looks locally like an open piece of $\mathbb{R}^2$.

A **3-manifold** looks locally like an open piece of $\mathbb{R}^3$.



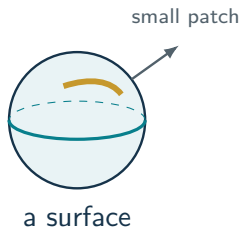
Local dimension, global shape

A **2-manifold** looks locally like an open piece of $\mathbb{R}^2$.

A **3-manifold** looks locally like an open piece of $\mathbb{R}^3$.

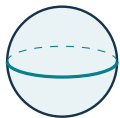
The interesting information is how these little pieces fit together.

Closed means compact and without boundary.



Classification of closed orientable surfaces

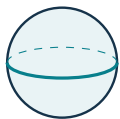
Every closed, connected, orientable surface is a sphere with g handles, for a unique integer $g \geq 0$.



S^2 : genus 0

Classification of closed orientable surfaces

Every closed, connected, orientable surface is a sphere with g handles, for a unique integer $g \geq 0$.



S^2 : genus 0



T^2 : genus 1

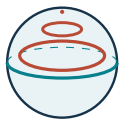


Σ_2 : genus 2

...

Classification of closed orientable surfaces

Every closed, connected, orientable surface is a sphere with g handles, for a unique integer $g \geq 0$.



S^2 : genus 0



T^2 : genus 1



Σ_2 : genus 2

...

A handle supplies a loop that cannot contract.

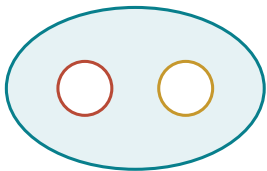
$$\Sigma \cong S^2 \iff \text{every loop in } \Sigma \text{ contracts.}$$

Here Σ is a closed, connected, orientable surface.

What did Perelman prove?

A pair of pants

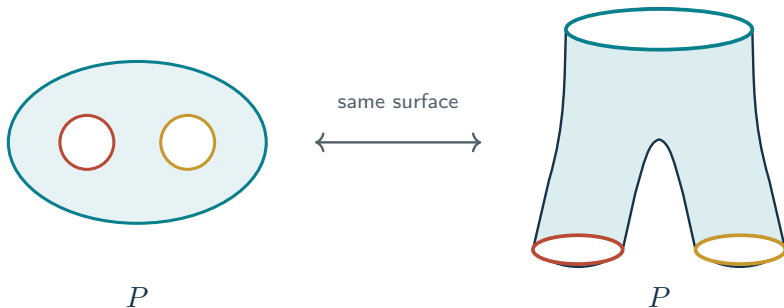
A disk with two holes removed is the same surface as a pair of trousers.



P

A pair of pants

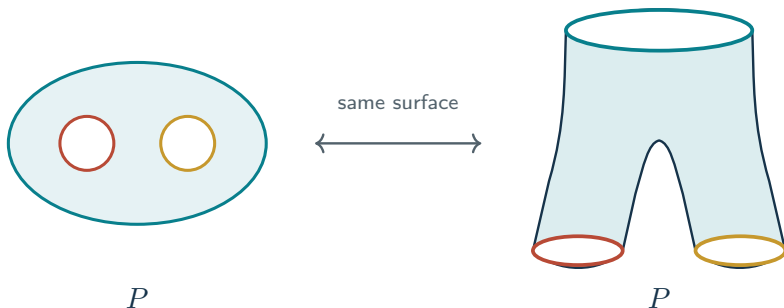
A disk with two holes removed is the same surface as a pair of trousers.



The waist and two cuffs give **three boundary circles**.

A pair of pants

A disk with two holes removed is the same surface as a pair of trousers.



The waist and two cuffs give **three boundary circles**.

Gluing these circles is a useful way to build surfaces.

What did Perelman prove?

Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



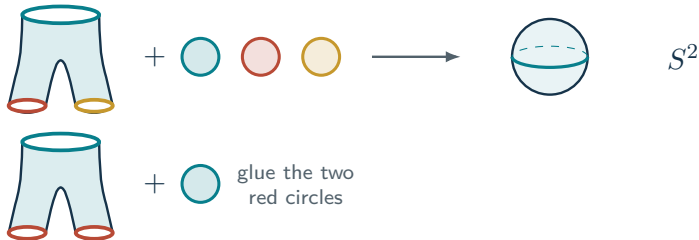
Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



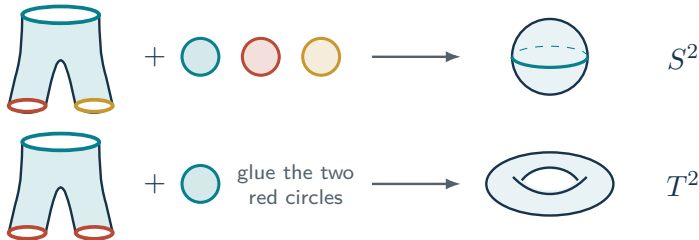
Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



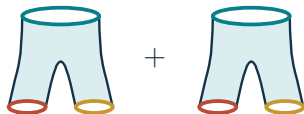
Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



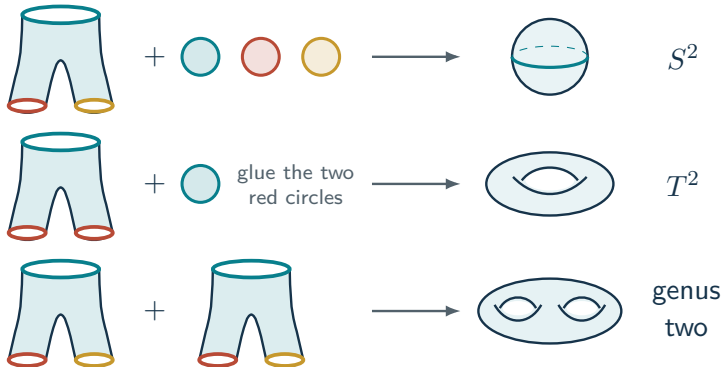
Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



Making surfaces from pants and disks

Match boundary circles of the same color and glue them together.



Two closed 3-manifolds

The 3-sphere

$$S^3 = \{x \in \mathbb{R}^4 : x_1^2 + \cdots + x_4^2 = 1\}.$$

One equation in four variables gives
three local coordinates.

Two closed 3-manifolds

The 3-sphere

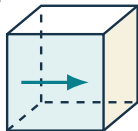
$$S^3 = \{x \in \mathbb{R}^4 : x_1^2 + \cdots + x_4^2 = 1\}.$$

One equation in four variables gives three local coordinates.

The 3-torus

$$T^3 = \mathbb{R}^3 / \mathbb{Z}^3.$$

Identify opposite faces of a cube.



The Poincaré Conjecture

Let M be a **closed, connected 3-manifold**.

The Poincaré Conjecture

Let M be a **closed, connected 3-manifold**.

If **every loop in M contracts to a point**, then

$$M \cong S^3.$$

That is, M is homeomorphic to the 3-sphere.

The Poincaré Conjecture

Let M be a **closed, connected 3-manifold**.

If **every loop in M contracts to a point**, then

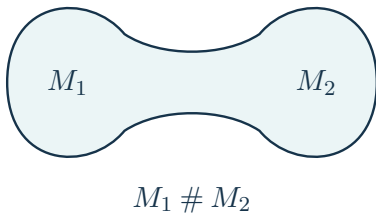
$$M \cong S^3.$$

That is, M is homeomorphic to the 3-sphere.

This is the 3D analogue of the characterization of S^2 by contracting loops.

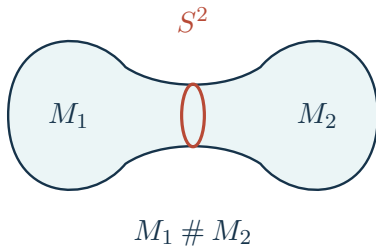
Now a theorem, proved by Perelman.

Cutting along a sphere



Schematic cross-section of a three-dimensional neck.

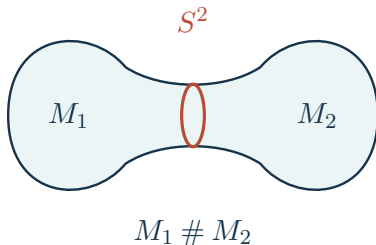
Cutting along a sphere



Cut the neck, then fill the two boundary spheres with 3-balls.

Schematic cross-section of a three-dimensional neck.

Cutting along a sphere



Cut the neck, then fill the two boundary spheres with 3-balls.

Connected sum is the reverse operation.

Schematic cross-section of a three-dimensional neck.

Take the product with a circle

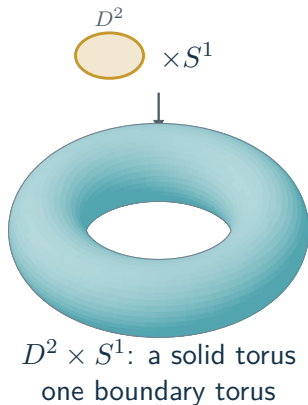
Take the products of the **2D building blocks** D^2 and P with S^1 .



Right: the outer wall is shown transparent to reveal the two tunnels. Both figures are complete, uncut rings.
What did Perelman prove?

Take the product with a circle

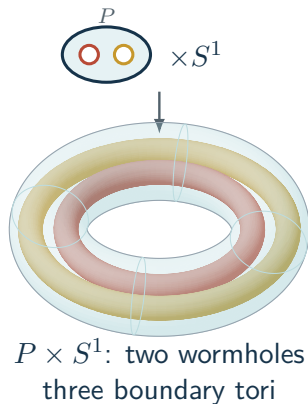
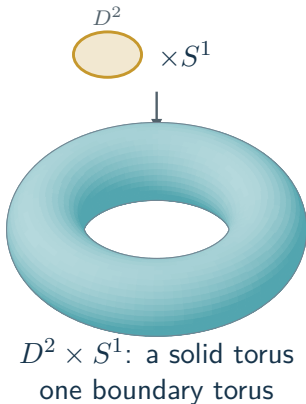
Take the products of the **2D building blocks** D^2 and P with S^1 .



Right: the outer wall is shown transparent to reveal the two tunnels. Both figures are complete, uncut rings.
What did Perelman prove?

Take the product with a circle

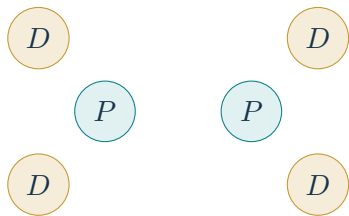
Take the products of the **2D building blocks** D^2 and P with S^1 .



Right: the outer wall is shown transparent to reveal the two tunnels. Both figures are complete, uncut rings.
What did Perelman prove?

Gluing the blocks into a graph manifold

The building blocks are solid tori $D^2 \times S^1$ and $P \times S^1$.



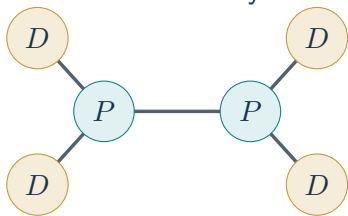
A D -vertex is a solid torus. A P -vertex is a $P \times S^1$ block.

[KL, Appendix I]. Choose gluings giving an orientable manifold. Unpaired tori remain as boundary.

Gluing the blocks into a graph manifold

The building blocks are solid tori $D^2 \times S^1$ and $P \times S^1$.

Pair their boundary tori and glue each pair by a homeomorphism.



A D -vertex is a solid torus. A

P -vertex is a $P \times S^1$ block.

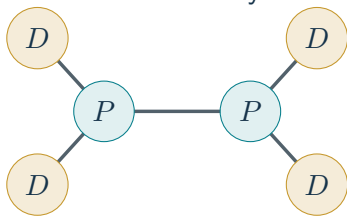
An edge records a glued pair of tori.

[KL, Appendix I]. Choose gluings giving an orientable manifold. Unpaired tori remain as boundary.

Gluing the blocks into a graph manifold

The building blocks are solid tori $D^2 \times S^1$ and $P \times S^1$.

Pair their boundary tori and glue each pair by a homeomorphism.



A D -vertex is a solid torus. A

P -vertex is a $P \times S^1$ block.

An edge records a glued pair of tori.

The edges also need **gluing maps**.

The maps may mix the two circle directions on a torus.

[KL, Appendix I]. Choose gluings giving an orientable manifold. Unpaired tori remain as boundary.

Some familiar graph manifolds

Manifold	How to recognize it
----------	---------------------

S^3	Two solid tori, glued suitably
-------	--------------------------------

For S^3 , glue so that a meridian on one side becomes a longitude on the other.

Here $g \geq 2$. We use the broad graph-manifold convention of [KL, Appendix I].

Some familiar graph manifolds

Manifold	How to recognize it
----------	---------------------

S^3	Two solid tori, glued suitably
-------	--------------------------------

T^3	$S^1 \times T^2$
-------	------------------

For S^3 , glue so that a meridian on one side becomes a longitude on the other.

Here $g \geq 2$. We use the broad graph-manifold convention of [KL, Appendix I].

Some familiar graph manifolds

Manifold	How to recognize it
----------	---------------------

S^3	Two solid tori, glued suitably
-------	--------------------------------

T^3	$S^1 \times T^2$
-------	------------------

$S^1 \times \Sigma_g$	Cut the surface Σ_g into pants
-----------------------	---------------------------------------

For S^3 , glue so that a meridian on one side becomes a longitude on the other.

Different gluing maps can give different 3-manifolds.

Here $g \geq 2$. We use the broad graph-manifold convention of [KL, Appendix I].

What does hyperbolic geometry look like?



the Poincaré disk model of $\mathbb{H}^2$

“Straight lines” are shortest paths, called **geodesics**.

The model preserves angles, but distorts distances. See [C, §1].

What does hyperbolic geometry look like?



the Poincaré disk model of $\mathbb{H}^2$

“Straight lines” are shortest paths, called **geodesics**.

A geodesic triangle has angle sum **less than** 180° .

The model preserves angles, but distorts distances. See [C, §1].

What does hyperbolic geometry look like?



the Poincaré disk model of $\mathbb{H}^2$

“Straight lines” are shortest paths, called **geodesics**.

A geodesic triangle has angle sum **less than** 180° .

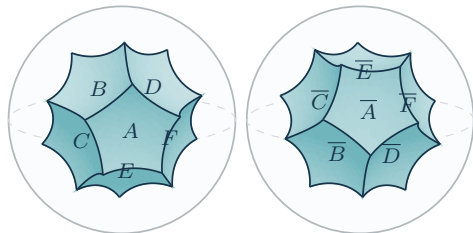
The boundary circle is infinitely far away.

In dimension three, the analogous ball model is $\mathbb{H}^3$: constant curvature -1 in every two-dimensional direction.

The model preserves angles, but distorts distances. See [C, §1].

The Seifert–Weber manifold

Glue each face of a regular **hyperbolic dodecahedron** to its opposite, with a $3/10$ turn: $108^\circ = 3\pi/5$.



front view

back view

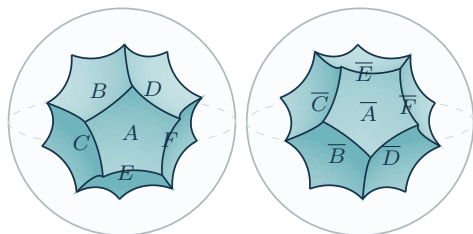
$$A \sim \bar{A}, B \sim \bar{B}, \dots, F \sim \bar{F}$$

Left: the Poincaré ball model of $\mathbb{H}^3$. Its boundary sphere is at infinity. [C1]; [T].

What did Perelman prove?

The Seifert–Weber manifold

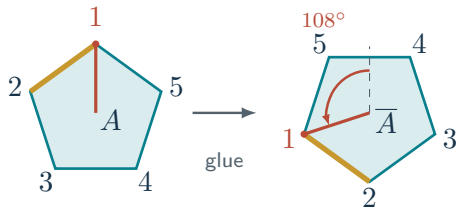
Glue each face of a regular **hyperbolic dodecahedron** to its opposite, with a $3/10$ turn: $108^\circ = 3\pi/5$.



front view

back view

$$A \sim \bar{A}, B \sim \bar{B}, \dots, F \sim \bar{F}$$



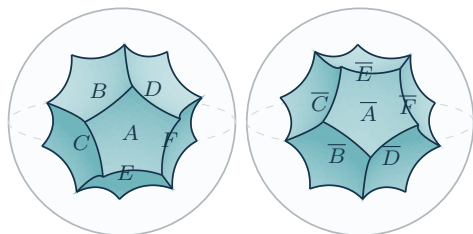
One opposite face pair, viewed in the same direction.
Match the numbered vertices. Repeat for all six pairs.

Left: the Poincaré ball model of $\mathbb{H}^3$. Its boundary sphere is at infinity. [C1]; [T].

What did Perelman prove?

The Seifert–Weber manifold

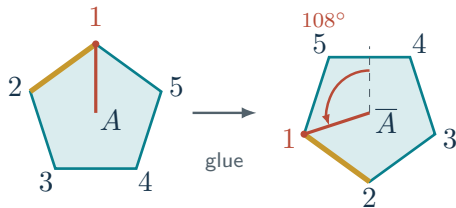
Glue each face of a regular **hyperbolic dodecahedron** to its opposite, with a $3/10$ turn: $108^\circ = 3\pi/5$.



front view

back view

$$A \sim \bar{A}, B \sim \bar{B}, \dots, F \sim \bar{F}$$



One opposite face pair, viewed in the same direction. Match the numbered vertices. Repeat for all six pairs.

Dihedral angle 72° : five wedges meet along each edge, so $5 \times 72^\circ = 360^\circ$.

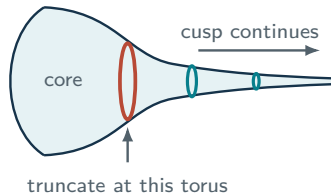
The quotient is closed: no boundary, finite volume, finite diameter.

Left: the Poincaré ball model of $\mathbb{H}^3$. Its boundary sphere is at infinity. [C1]; [T].

What did Perelman prove?

Finite volume can include infinite cusps

A noncompact, complete, finite-volume orientable hyperbolic 3-manifold has a compact core and finitely many **torus cusps**.



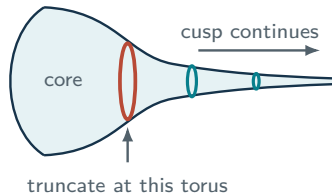
The full manifold has no boundary. Truncating its cusps creates the boundary tori used in geometrization.

Finite volume can include infinite cusps

A noncompact, complete, finite-volume orientable hyperbolic 3-manifold has a compact core and finitely many **torus cusps**.

Each cusp is $T^2 \times [0, \infty)$.

Complete: escaping takes infinite distance.



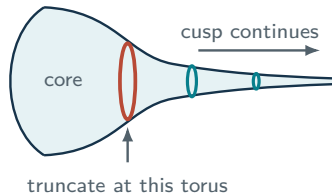
The full manifold has no boundary. Truncating its cusps creates the boundary tori used in geometrization.

Finite volume can include infinite cusps

A noncompact, complete, finite-volume orientable hyperbolic 3-manifold has a compact core and finitely many **torus cusps**.

Each cusp is $T^2 \times [0, \infty)$.

Complete: escaping takes infinite distance.



At distance r , torus area is $A_0 e^{-2r}$, so the cusp has volume

$$\int_0^\infty A_0 e^{-2r} dr = \frac{A_0}{2}.$$

The full manifold has no boundary. Truncating its cusps creates the boundary tori used in geometrization.

Thurston's Geometrization Conjecture: three cases

Every closed connected orientable 3-manifold is a connected sum of closed pieces. Each piece has one of these forms:

1. **Graph case.** The whole piece is a graph manifold.

Perelman's theorem, in the formulation of [KL, Appendix I, pp. 2854–2855].

What did Perelman prove?

Thurston's Geometrization Conjecture: three cases

Every closed connected orientable 3-manifold is a connected sum of closed pieces. Each piece has one of these forms:

1. **Graph case.** The whole piece is a graph manifold.
2. **Hyperbolic case.** The whole piece is a closed hyperbolic 3-manifold.

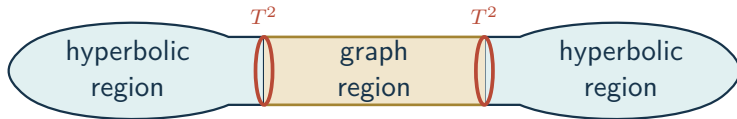
Perelman's theorem, in the formulation of [KL, Appendix I, pp. 2854–2855].

What did Perelman prove?

Thurston's Geometrization Conjecture: three cases

Every closed connected orientable 3-manifold is a connected sum of closed pieces. Each piece has one of these forms:

1. **Graph case.** The whole piece is a graph manifold.
2. **Hyperbolic case.** The whole piece is a closed hyperbolic 3-manifold.
3. **Mixed case.** A nonempty compact graph-manifold region has a nonempty complete finite-volume hyperbolic complement. Its boundary tori are incompressible in the closed piece.



Mixed case: a schematic after truncating the hyperbolic cusps.

Perelman's theorem, in the formulation of [KL, Appendix I, pp. 2854–2855].

What did Perelman prove?

Why the tori must be incompressible

A loop that cannot shrink on the torus must still be unable to shrink in the surrounding piece.



this meridian bounds
a disk inside

The boundary of a solid torus fails this test.

The reason for the incompressibility condition in [KL, Appendix I].

Why the tori must be incompressible

A loop that cannot shrink on the torus must still be unable to shrink in the surrounding piece.



this meridian bounds
a disk inside

The boundary of a solid torus fails this test.
For example, S^3 can be written as a solid torus
glued to the exterior of a hyperbolic knot.

The reason for the incompressibility condition in [KL, Appendix I].

Why the tori must be incompressible

A loop that cannot shrink on the torus must still be unable to shrink in the surrounding piece.



this meridian bounds
a disk inside

The boundary of a solid torus fails this test.
For example, S^3 can be written as a solid torus
glued to the exterior of a hyperbolic knot.

Incompressibility excludes this misleading “hyperbolic piece” of S^3 .

The reason for the incompressibility condition in [KL, Appendix I].

Hamilton's idea: let the geometry change

A metric is a rule for measuring lengths and angles.

Hamilton, 1982 [H]. The manifold stays fixed while the solution is smooth.

How did he do it?

Hamilton's idea: let the geometry change

A metric is a rule for measuring lengths and angles.

Hamilton's **Ricci flow** changes that rule according to curvature:

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}(g).$$

Hamilton, 1982 [H]. The manifold stays fixed while the solution is smooth.

Hamilton's idea: let the geometry change

A metric is a rule for measuring lengths and angles.

Hamilton's **Ricci flow** changes that rule according to curvature:

$$\frac{\partial g}{\partial t} = -2 \operatorname{Ric}(g).$$

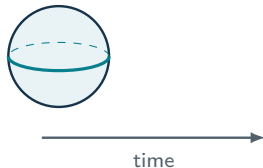
Positive Ricci curvature contracts lengths in the corresponding directions.

Could this evolution reveal the pieces in geometrization?

Hamilton, 1982 [H]. The manifold stays fixed while the solution is smooth.

Three easy examples

Round sphere: shrinks.

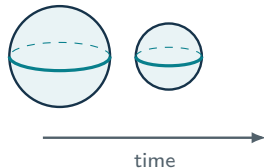


The picture represents a round 3-sphere by a 2-sphere.

How did he do it?

Three easy examples

Round sphere: shrinks.

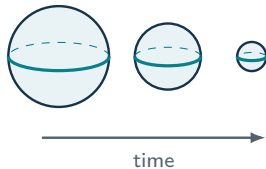


The picture represents a round 3-sphere by a 2-sphere.

How did he do it?

Three easy examples

Round sphere: shrinks.



A round sphere disappears after a finite amount of time.

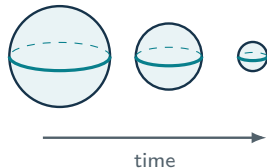
The picture represents a round 3-sphere by a 2-sphere.

Three easy examples

Round sphere: shrinks.

Flat torus: stays fixed.

Hyperbolic manifold: expands.

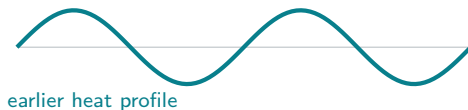


A round sphere disappears after a finite amount of time.

The picture represents a round 3-sphere by a 2-sphere.

Smoothing can coexist with a singularity

Heat diffusion smooths temperature differences.



Smoothing can coexist with a singularity

Heat diffusion smooths temperature differences.



Smoothing can coexist with a singularity

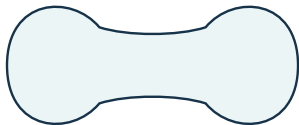
Heat diffusion smooths temperature differences.

Ricci flow also has a smoothing effect, but its nonlinear curvature terms can amplify large curvature.



For geometry, a narrow neck can pinch even while other regions smooth out.

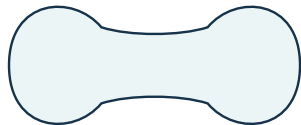
A neck can pinch



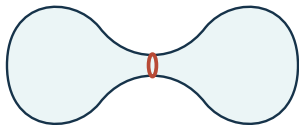
earlier

Schematic neckpinch. No claim that every initial metric follows this picture.

A neck can pinch



earlier

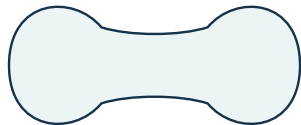


near a singular time

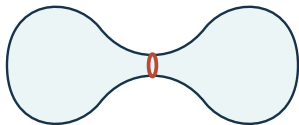
A small cross-sectional S^2 can shrink faster than the large regions beside it.

Schematic neckpinch. No claim that every initial metric follows this picture.

A neck can pinch



earlier



near a singular time

A small cross-sectional S^2 can shrink faster than the large regions beside it.

The smooth differential equation then stops making sense.

Schematic neckpinch. No claim that every initial metric follows this picture.

Looking closely at a singularity

Magnify a region where curvature is becoming very large.

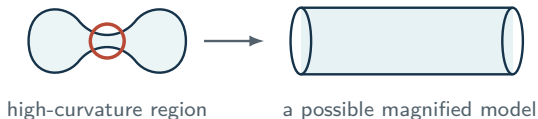


high-curvature region

Hamilton's compactness theory and Perelman's estimates [P1], [KL].

Looking closely at a singularity

Magnify a region where curvature is becoming very large.

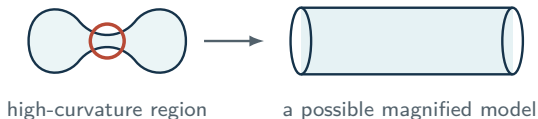


The aim is to find a recognizable local shape.

Hamilton's compactness theory and Perelman's estimates [P1], [KL].

Looking closely at a singularity

Magnify a region where curvature is becoming very large.



The aim is to find a recognizable local shape.

The difficulty is proving that the magnified geometry does not degenerate.

Hamilton's compactness theory and Perelman's estimates [P1], [KL].

Perelman's key control

A monotone quantity

Perelman found an entropy that can only increase as the flow evolves.

[P1–P2]. Constants depend on the initial metric and a finite time interval. Collapse at infinite time remains possible.

Perelman's key control

A monotone quantity

Perelman found an entropy that can only increase as the flow evolves.

No local collapsing at finite time

At the scale set by curvature, a forming singularity cannot become arbitrarily thin.

[P1–P2]. Constants depend on the initial metric and a finite time interval. Collapse at infinite time remains possible.

Perelman's key control

A monotone quantity

Perelman found an entropy that can only increase as the flow evolves.

No local collapsing at finite time

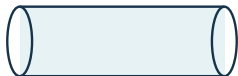
At the scale set by curvature, a forming singularity cannot become arbitrarily thin.

Together with Hamilton's estimates, this makes the local models manageable.

[P1–P2]. Constants depend on the initial metric and a finite time interval. Collapse at infinite time remains possible.

Recognizable high-curvature regions

In dimension three, Perelman's estimates give **canonical neighborhoods**.

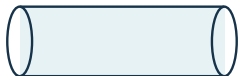


neck: $S^2 \times (-L, L)$

[P1, §§11–12], [P2]. Diagrams suppress one dimension.

Recognizable high-curvature regions

In dimension three, Perelman's estimates give **canonical neighborhoods**.



neck: $S^2 \times (-L, L)$

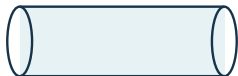


cap

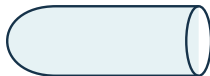
[P1, §§11–12], [P2]. Diagrams suppress one dimension.

Recognizable high-curvature regions

In dimension three, Perelman's estimates give **canonical neighborhoods**.



neck: $S^2 \times (-L, L)$



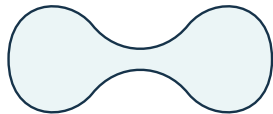
cap

There are also compact positively curved components.

The geometry is controlled after rescaling to the curvature scale.

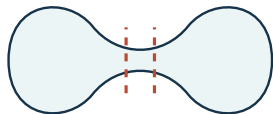
[P1, §§11–12], [P2]. Diagrams suppress one dimension.

Surgery and continued evolution



[P2]. Certain components of known topology are discarded.

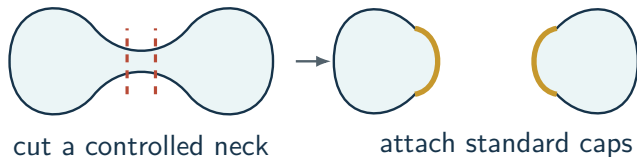
Surgery and continued evolution



cut a controlled neck

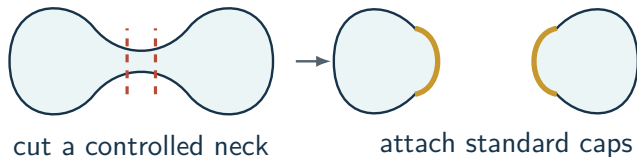
[P2]. Certain components of known topology are discarded.

Surgery and continued evolution



[P2]. Certain components of known topology are discarded.

Surgery and continued evolution



Restart Ricci flow and keep a record of each topological change.

Perelman's estimates keep this procedure under control, surgery after surgery.

[P2]. Certain components of known topology are discarded.

Why this proves Poincaré

If every loop in M contracts, Perelman proves **finite extinction**: eventually the flow has no components left.

[P3], with surgery topology from [P2].

Why this proves Poincaré

If every loop in M contracts, Perelman proves **finite extinction**: eventually the flow has no components left.

Reversing the surgeries reconstructs M as a connected sum of quotients of S^3 and copies of $S^2 \times S^1$.

[P3], with surgery topology from [P2].

Why this proves Poincaré

If every loop in M contracts, Perelman proves **finite extinction**: eventually the flow has no components left.

Reversing the surgeries reconstructs M as a connected sum of quotients of S^3 and copies of $S^2 \times S^1$.

The loop condition eliminates $S^2 \times S^1$ and every nontrivial quotient.

[P3], with surgery topology from [P2].

Why this proves Poincaré

If every loop in M contracts, Perelman proves **finite extinction**: eventually the flow has no components left.

Reversing the surgeries reconstructs M as a connected sum of quotients of S^3 and copies of $S^2 \times S^1$.

The loop condition eliminates $S^2 \times S^1$ and every nontrivial quotient.

What remains is a connected sum of 3-spheres, hence a 3-sphere.

[P3], with surgery topology from [P2].

If the flow survives: thick and thin

At large time, view the manifold using a steadily larger measuring scale.

Thick regions

They retain three-dimensional volume
at that scale.

Hyperbolic pieces emerge.

[P1–P2], [KL, §§90–92 and Appendix I], and the collapsing theorem [KL2].

If the flow survives: thick and thin

At large time, view the manifold using a steadily larger measuring scale.

Thick regions

They retain three-dimensional volume at that scale.

Hyperbolic pieces emerge.

Thin regions

Some directions become very small.
Collapsing theory identifies graph manifolds.

[P1–P2], [KL, §§90–92 and Appendix I], and the collapsing theorem [KL2].

If the flow survives: thick and thin

At large time, view the manifold using a steadily larger measuring scale.

Thick regions

They retain three-dimensional volume at that scale.

Hyperbolic pieces emerge.

Thin regions

Some directions become very small.

Collapsing theory identifies graph manifolds.

The boundary tori are incompressible. We have recovered exactly the picture from the beginning.

[P1–P2], [KL, §§90–92 and Appendix I], and the collapsing theorem [KL2].

Progress after Perelman

A canonical flow through singularities

Kleiner–Lott and Bamler–Kleiner developed a flow that no longer depends on surgery choices.

[KL3], [BK], [B]. These results go beyond Perelman's original proof.

Progress after Perelman

A canonical flow through singularities

Kleiner–Lott and Bamler–Kleiner developed a flow that no longer depends on surgery choices.

Eventually the singularities stop

Bamler proved eventual smoothness and quantitative decay of curvature.

[KL3], [BK], [B]. These results go beyond Perelman's original proof.

Progress after Perelman

A canonical flow through singularities

Kleiner–Lott and Bamler–Kleiner developed a flow that no longer depends on surgery choices.

Eventually the singularities stop

Bamler proved eventual smoothness and quantitative decay of curvature.

We can then ask: what shape does a surviving component approach?

[KL3], [BK], [B]. These results go beyond Perelman's original proof.

Even the 3-torus leaves a question

The torus has a flat metric. But start Ricci flow with an arbitrary metric.

[L26, Theorem 1.1]. Hamilton proved the corresponding result for the 2-torus in 1988.

Even the 3-torus leaves a question

The torus has a flat metric. But start Ricci flow with an arbitrary metric.

Must the flow eventually go flat?

[L26, Theorem 1.1]. Hamilton proved the corresponding result for the 2-torus in 1988.

Even the 3-torus leaves a question

The torus has a flat metric. But start Ricci flow with an arbitrary metric.

Must the flow eventually go flat?

Theorem (John Lott, 2026 preprint)

Yes. Its eventual smooth flow converges to a flat metric.

The convergence is smooth and exponentially fast. The limiting flat torus depends on the initial metric.

[L26, Theorem 1.1]. Hamilton proved the corresponding result for the 2-torus in 1988.

Why small curvature is not enough

A flat torus can be very long and very thin.



a flat 2-torus: identify opposite sides

A geometric example, not a family evolving by Ricci flow. Taking a product with a circle gives 3-tori.

Why small curvature is not enough

A flat torus can be very long and very thin.



a flat 2-torus: identify opposite sides

All curvatures are zero, yet the diameter can be huge.

A geometric example, not a family evolving by Ricci flow. Taking a product with a circle gives 3-tori.

Why small curvature is not enough

A flat torus can be very long and very thin.



a flat 2-torus: identify opposite sides

All curvatures are zero, yet the diameter can be huge.

To understand convergence, one also needs to control the **overall size**.

A geometric example, not a family evolving by Ricci flow. Taking a product with a circle gives 3-tori.

A small direction can disappear

Keep one circumference of a flat 2-torus fixed and let the other shrink to zero.



A small direction can disappear

Keep one circumference of a flat 2-torus fixed and let the other shrink to zero.



A small direction can disappear

Keep one circumference of a flat 2-torus fixed and let the other shrink to zero.

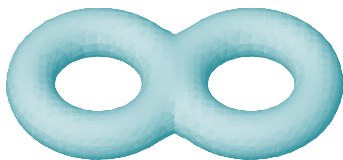


Unwrap the short loops by passing to a covering space.

The geometry upstairs can retain the dimensions that disappeared downstairs.

More graph manifolds to understand

Start with a **closed surface of genus at least two**, with a hyperbolic metric.

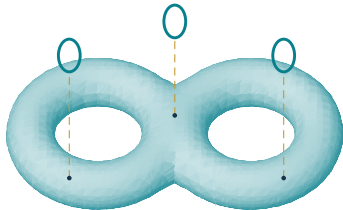


a closed genus-two surface

[L26, §1.4]. The remaining homogeneous types are named in the optional slides.

More graph manifolds to understand

Start with a **closed surface of genus at least two**, with a hyperbolic metric.



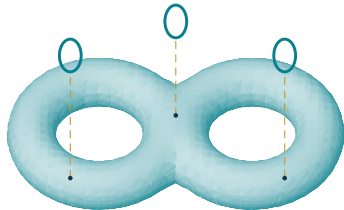
a closed genus-two surface

Place a circle over every point.
The circle direction may twist as one
moves around the surface.

[L26, §1.4]. The remaining homogeneous types are named in the optional slides.

More graph manifolds to understand

Start with a **closed surface of genus at least two**, with a hyperbolic metric.



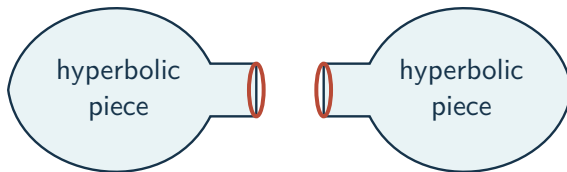
a closed genus-two surface

Place a circle over every point.
The circle direction may twist as one moves around the surface.
These circle bundles are **graph manifolds**.

For general initial metrics, what is the precise long-time geometry?

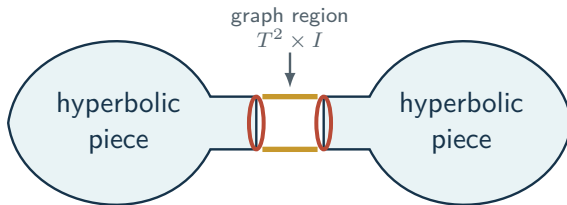
[L26, §1.4]. The remaining homogeneous types are named in the optional slides.

Where hyperbolic and graph regions meet



Schematic topology, not a proposed Ricci-flow metric. General perspective: [L10], [L26, §1.4].

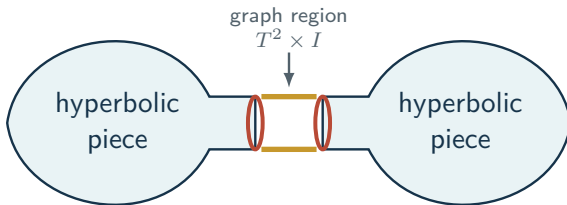
Where hyperbolic and graph regions meet



Join two truncated hyperbolic pieces through a torus collar. The collar is a graph manifold.

Schematic topology, not a proposed Ricci-flow metric. General perspective: [L10], [L26, §1.4].

Where hyperbolic and graph regions meet



Join two truncated hyperbolic pieces through a torus collar. The collar is a graph manifold.

How does the connecting region change under Ricci flow?

Schematic topology, not a proposed Ricci-flow metric. General perspective: [L10], [L26, §1.4].

The three answers

What did Perelman prove?

Poincaré, and the decomposition into hyperbolic and graph-manifold regions.

The three answers

What did Perelman prove?

Poincaré, and the decomposition into hyperbolic and graph-manifold regions.

How did he do it?

Let geometry evolve, control the singularities, and read off the topology.

The three answers

What did Perelman prove?

Poincaré, and the decomposition into hyperbolic and graph-manifold regions.

How did he do it?

Let geometry evolve, control the singularities, and read off the topology.

What's left?

Understand the detailed geometry of the flow where directions collapse and where the pieces meet.

Questions

Sources for Perelman's proof

- [H] R. Hamilton, *Three-manifolds with positive Ricci curvature*, J. Differential Geom. 17 (1982), 255–306.
- [P1] G. Perelman, *The entropy formula for the Ricci flow and its geometric applications* (2002). [arXiv:math/0211159](#)
- [P2] G. Perelman, *Ricci flow with surgery on three-manifolds* (2003). [arXiv:math/0303109](#)
- [P3] G. Perelman, *Finite extinction time for the solutions to the Ricci flow on certain three-manifolds* (2003). [arXiv:math/0307245](#)
- [KL] B. Kleiner and J. Lott, *Notes on Perelman's papers*, Geom. Topol. 12 (2008), 2587–2858, especially Appendix I, pp. 2854–2855. [Linked notes](#), [Appendix I](#)
- [Scott] P. Scott, *The geometries of 3-manifolds*, Bull. London Math. Soc. 15 (1983), 401–487.
- [KL2] B. Kleiner and J. Lott, *Locally collapsed 3-manifolds*, Astérisque 365 (2014), 7–99.

Sources for hyperbolic geometry

[C] D. Calegari, *Chapter 2: Hyperbolic Geometry*, notes for a book on 3-manifolds (January 2025). Models and constructions.

[Author's chapter](#)

[C1] D. Calegari, *Chapter 1: Combinatorial Foundations*, Example 4.23: the Seifert–Weber construction and its hyperbolic metric.

[Author's chapter, Example 4.23](#)

[T] S. Trettel, *Regular Polytopes in the Poincaré Ball* (2026). Explicit sphere equations for the hyperbolic dodecahedron.

[Construction in the ball model](#)

Sources for the long-time questions

- [KL3] B. Kleiner and J. Lott, *Singular Ricci flows I*, Acta Math. 219 (2017), 65–134.
- [BK] R. Bamler and B. Kleiner, *Uniqueness and stability of Ricci flow through singularities*, Acta Math. 228 (2022), 1–215. [arXiv:1709.04122](#)
- [B] R. Bamler, *Long-time behavior of 3-dimensional Ricci flow*, Introduction and Parts A–D, Geom. Topol. 22 (2018). [Introduction: arXiv:1411.6658](#)
- [L10] J. Lott, *Dimensional reduction and the long-time behavior of Ricci flow*, Comment. Math. Helv. 85 (2010), 485–534. [arXiv:0711.4063](#)
- [LS] J. Lott and N. Šešum, *Ricci flow on three-dimensional manifolds with symmetry*, Comment. Math. Helv. 89 (2014), 1–32. [arXiv:1102.4384](#)
- [L26] J. Lott, *Under Ricci flow, a 3-torus goes flat*, preprint dated September 13, 2026, especially Theorem 1.1 and §§1.2–1.4. [math.berkeley.edu/~lott/riccitorus.pdf](#)

Optional: the Appendix I formulation

For every closed connected orientable 3-manifold M ,

$$M \cong M_1 \# \cdots \# M_n.$$

Each M_i has a compact codimension-zero submanifold G_i such that

- G_i is a graph manifold.

[KL, Appendix I]. The manifolds G_i and the hyperbolic complements may be disconnected.

References and optional details

Optional: the Appendix I formulation

For every closed connected orientable 3-manifold M ,

$$M \cong M_1 \# \cdots \# M_n.$$

Each M_i has a compact codimension-zero submanifold G_i such that

- G_i is a graph manifold.
- $M_i \setminus G_i$ is complete hyperbolic with finite volume.

[KL, Appendix I]. The manifolds G_i and the hyperbolic complements may be disconnected.

References and optional details

Optional: the Appendix I formulation

For every closed connected orientable 3-manifold M ,

$$M \cong M_1 \# \cdots \# M_n.$$

Each M_i has a compact codimension-zero submanifold G_i such that

- G_i is a graph manifold.
- $M_i \setminus G_i$ is complete hyperbolic with finite volume.
- Every boundary torus $T \subset \partial G_i$ is incompressible: $\pi_1(T) \longrightarrow \pi_1(M_i)$ is injective.

[KL, Appendix I]. The manifolds G_i and the hyperbolic complements may be disconnected.

References and optional details

Optional: the Appendix I formulation

For every closed connected orientable 3-manifold M ,

$$M \cong M_1 \# \cdots \# M_n.$$

Each M_i has a compact codimension-zero submanifold G_i such that

- G_i is a graph manifold.
- $M_i \setminus G_i$ is complete hyperbolic with finite volume.
- Every boundary torus $T \subset \partial G_i$ is incompressible: $\pi_1(T) \longrightarrow \pi_1(M_i)$ is injective.

Graph case: $G_i = M_i$. Hyperbolic case: $G_i = \emptyset$. Mixed case: both regions are nonempty.

[KL, Appendix I]. The manifolds G_i and the hyperbolic complements may be disconnected.

References and optional details

Optional: the eight geometries

Family	Simply connected model spaces
Constant curvature	S^3 , $\mathbb{R}^3$, $\mathbb{H}^3$

Homogeneous: every point has the same geometry.

Thurston's geometries. See [Scott]. You do not need to memorize this list.

Optional: the eight geometries

Family	Simply connected model spaces
Constant curvature	$S^3, \mathbb{R}^3, \mathbb{H}^3$
Products	$S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$

Homogeneous: every point has the same geometry.

Thurston's geometries. See [Scott]. You do not need to memorize this list.

Optional: the eight geometries

Family	Simply connected model spaces
Constant curvature	$S^3, \mathbb{R}^3, \mathbb{H}^3$
Products	$S^2 \times \mathbb{R}, \mathbb{H}^2 \times \mathbb{R}$
Other homogeneous models	$\text{Nil}, \text{Sol}, \widetilde{\text{SL}(2, \mathbb{R})}$

Homogeneous: every point has the same geometry.

The last three models can have different geometry in different directions.

Thurston's geometries. See [Scott]. You do not need to memorize this list.

Optional: three exact solutions

If g_0 has constant sectional curvature K_0 in dimension three,

$$\text{Ric}(g_0) = 2K_0g_0, \quad g(t) = (1 - 4K_0t)g_0.$$

K_0	Evolution	What happens to lengths?
-------	-----------	--------------------------

Optional: three exact solutions

If g_0 has constant sectional curvature K_0 in dimension three,

$$\text{Ric}(g_0) = 2K_0g_0, \quad g(t) = (1 - 4K_0t)g_0.$$

K_0	Evolution	What happens to lengths?
-------	-----------	--------------------------

+1	$(1 - 4t)g_0$	Shrink to zero at $t = \frac{1}{4}$
----	---------------	-------------------------------------

Optional: three exact solutions

If g_0 has constant sectional curvature K_0 in dimension three,

$$\text{Ric}(g_0) = 2K_0g_0, \quad g(t) = (1 - 4K_0t)g_0.$$

K_0	Evolution	What happens to lengths?
-------	-----------	--------------------------

+1	$(1 - 4t)g_0$	Shrink to zero at $t = \frac{1}{4}$
----	---------------	-------------------------------------

0	g_0	Stay fixed
---	-------	------------

Optional: three exact solutions

If g_0 has constant sectional curvature K_0 in dimension three,

$$\text{Ric}(g_0) = 2K_0g_0, \quad g(t) = (1 - 4K_0t)g_0.$$

K_0	Evolution	What happens to lengths?
+1	$(1 - 4t)g_0$	Shrink to zero at $t = \frac{1}{4}$
0	g_0	Stay fixed
-1	$(1 + 4t)g_0$	Grow like $\sqrt{t}$

Optional: the entropy formula

On a closed n -manifold, let

$$u = (4\pi\tau)^{-n/2}e^{-f}, \quad \int_M u \, dV = 1.$$

[P1, equation (3.4)]. R denotes scalar curvature and $\nabla^2 f$ the Hessian.

References and optional details

Optional: the entropy formula

On a closed n -manifold, let

$$u = (4\pi\tau)^{-n/2}e^{-f}, \quad \int_M u \, dV = 1.$$

Set

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n]u \, dV.$$

[P1, equation (3.4)]. R denotes scalar curvature and $\nabla^2 f$ the Hessian.

References and optional details

Optional: the entropy formula

On a closed n -manifold, let

$$u = (4\pi\tau)^{-n/2}e^{-f}, \quad \int_M u \, dV = 1.$$

Set

$$\mathcal{W}(g, f, \tau) = \int_M [\tau(R + |\nabla f|^2) + f - n]u \, dV.$$

Along $g_t = -2 \operatorname{Ric}$, $\tau' = -1$, and $u_t = -\Delta u + Ru$,

$$\frac{d\mathcal{W}}{dt} = 2\tau \int_M \left| \operatorname{Ric} + \nabla^2 f - \frac{g}{2\tau} \right|^2 u \, dV \geq 0.$$

[P1, equation (3.4)]. R denotes scalar curvature and $\nabla^2 f$ the Hessian.

References and optional details

Optional: torus bundles and gluing maps

Glue the ends of $T^2 \times [0, 1]$ by a map $A \in \mathrm{SL}(2, \mathbb{Z})$:

$$(x, 1) \sim (Ax, 0).$$

Gluing	Example of A	Geometry
No twist	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Euclidean

These are the working models after passing to finite covers in [L26, §3].

Optional: torus bundles and gluing maps

Glue the ends of $T^2 \times [0, 1]$ by a map $A \in \mathrm{SL}(2, \mathbb{Z})$:

$$(x, 1) \sim (Ax, 0).$$

Gluing	Example of A	Geometry
No twist	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Euclidean
Shear	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	Nil

These are the working models after passing to finite covers in [L26, §3].

Optional: torus bundles and gluing maps

Glue the ends of $T^2 \times [0, 1]$ by a map $A \in \mathrm{SL}(2, \mathbb{Z})$:

$$(x, 1) \sim (Ax, 0).$$

Gluing	Example of A	Geometry
No twist	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	Euclidean
Shear	$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	Nil
Stretch and contract	$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$	Sol

These are the working models after passing to finite covers in [L26, §3].

Optional: a natural scale for the size

In the expanding hyperbolic example, lengths grow roughly like $\sqrt{t}$.

[L26, §§1.2–1.3]. This bound complements Bamler's curvature estimates.

Optional: a natural scale for the size

In the expanding hyperbolic example, lengths grow roughly like $\sqrt{t}$.
So compare later geometries after dividing all distances by $\sqrt{t}$.

[L26, §§1.2–1.3]. This bound complements Bamler's curvature estimates.

Optional: a natural scale for the size

In the expanding hyperbolic example, lengths grow roughly like $\sqrt{t}$.
So compare later geometries after dividing all distances by $\sqrt{t}$.

A key step in the 3-torus theorem is the bound

$$\text{diam}(M, g(t)) \leq C\sqrt{t}.$$

The rescaled manifold cannot stretch indefinitely in length.

[L26, §§1.2–1.3]. This bound complements Bamler's curvature estimates.

Optional: why short loops help

Temperature differences around a short circle even out quickly.



earlier

Heat-equation analogy for [L26, §4] and [LS]. Radial variation depicts temperature.

Optional: why short loops help

Temperature differences around a short circle even out quickly.



Heat-equation analogy for [L26, §4] and [LS]. Radial variation depicts temperature.

Optional: why short loops help

Temperature differences around a short circle even out quickly.



earlier



later



almost uniform

The paper uses analogous smoothing along very small torus fibers.

This helps compare a general flow with a simpler, symmetric one.

Heat-equation analogy for [L26, §4] and [LS]. Radial variation depicts temperature.

Optional: heat on a short circle

A heat equation on a circle of circumference $2\pi\varepsilon$ is

$$u_t = \varepsilon^{-2} u_{\theta\theta}.$$

Exact scalar toy model for the symmetry-decay idea in [L26, §4].

Optional: heat on a short circle

A heat equation on a circle of circumference $2\pi\varepsilon$ is

$$u_t = \varepsilon^{-2} u_{\theta\theta}.$$

For a Fourier mode,

$$u(\theta, t) = e^{-k^2 t / \varepsilon^2} \cos(k\theta), \quad k \neq 0.$$

Exact scalar toy model for the symmetry-decay idea in [L26, §4].

Optional: heat on a short circle

A heat equation on a circle of circumference $2\pi\varepsilon$ is

$$u_t = \varepsilon^{-2} u_{\theta\theta}.$$

For a Fourier mode,

$$u(\theta, t) = e^{-k^2 t / \varepsilon^2} \cos(k\theta), \quad k \neq 0.$$

Small ε makes nonconstant modes decay rapidly.

The constant mode remains: **averaging removes the short-scale oscillations.**

Exact scalar toy model for the symmetry-decay idea in [L26, §4].

Optional: compact and lifted limits

Theorem (John Lott, 2026 preprint)

For immortal flows on closed manifolds of these types:

Type	Compact limit of $t^{-1}g(t)$	Lifted limit
Euclidean	Point	Euclidean space

An expanding soliton evolves by scaling and changes of coordinates. Lifted limits use space and time rescaling.

[L26, Theorem 1.1]. The Sol circle or interval has positive length.

Optional: compact and lifted limits

Theorem (John Lott, 2026 preprint)

For immortal flows on closed manifolds of these types:

Type	Compact limit of $t^{-1}g(t)$	Lifted limit
Euclidean	Point	Euclidean space
Nil	Point	Nil expanding soliton

An expanding soliton evolves by scaling and changes of coordinates. Lifted limits use space and time rescaling.

[L26, Theorem 1.1]. The Sol circle or interval has positive length.

Optional: compact and lifted limits

Theorem (John Lott, 2026 preprint)

For immortal flows on closed manifolds of these types:

Type	Compact limit of $t^{-1}g(t)$	Lifted limit
Euclidean	Point	Euclidean space
Nil	Point	Nil expanding soliton
Sol	Circle or interval	Sol expanding soliton

An expanding soliton evolves by scaling and changes of coordinates. Lifted limits use space and time rescaling.

[L26, Theorem 1.1]. The Sol circle or interval has positive length.

Optional: a diameter conjecture

For an immortal Ricci flow on a closed 3-manifold, consider

$$\text{diam}(M, g(t)) = O(\sqrt{t}) \iff M \text{ admits a locally homogeneous metric.}$$

Status described in the preprint

Conjectural equivalence: [L26, §1.4], referring to [L10, Remarks 1.4–1.5 and Proposition 3.5].

Optional: a diameter conjecture

For an immortal Ricci flow on a closed 3-manifold, consider

$$\text{diam}(M, g(t)) = O(\sqrt{t}) \iff M \text{ admits a locally homogeneous metric.}$$

Status described in the preprint

The forward implication follows from earlier work together with Bamler's curvature bound.

Conjectural equivalence: [L26, §1.4], referring to [L10, Remarks 1.4–1.5 and Proposition 3.5].

Optional: a diameter conjecture

For an immortal Ricci flow on a closed 3-manifold, consider

$$\text{diam}(M, g(t)) = O(\sqrt{t}) \iff M \text{ admits a locally homogeneous metric.}$$

Status described in the preprint

The forward implication follows from earlier work together with Bamler's curvature bound.

The reverse implication remains open for the $\mathbb{H}^2 \times \mathbb{R}$ and $\widetilde{\text{SL}}(2, \mathbb{R})$ types.

Conjectural equivalence: [L26, §1.4], referring to [L10, Remarks 1.4–1.5 and Proposition 3.5].