Chern-Simons, differential K-theory and operator theory

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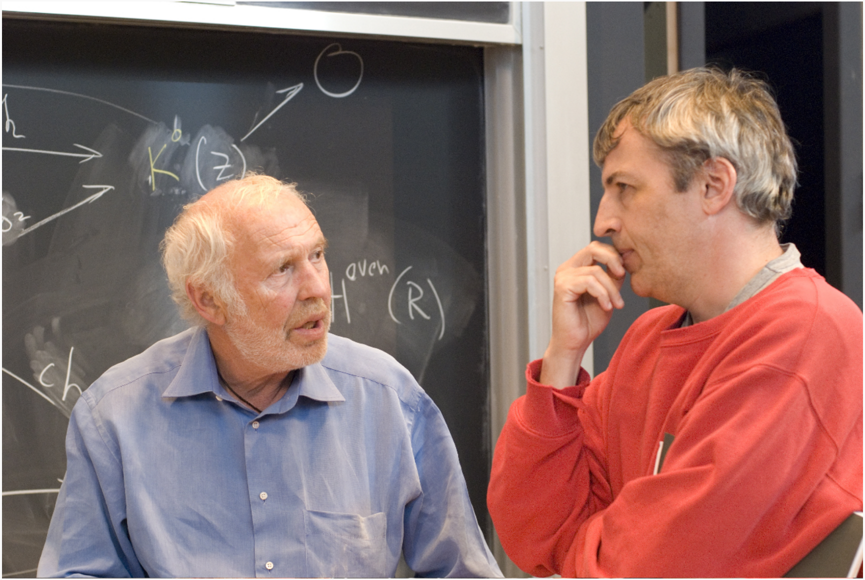
Chern

Simons



Attribution

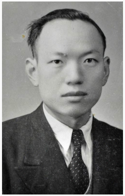
Joint work with Alexander Gorokhovsky

Structure of the talk

- Chern-Simons forms
- Differential K-theory
- Hilbert bundles
- Infinite dimensional cycles
- (Differential) twisted K-theory

Chern-Simons, differential K-theory and operator theory

Chern character form



Suppose that M is a smooth manifold, E is a finite dimensional Hermitian vector bundle over M and ∇ is a Hermitian connection on E. The Chern character form of ∇ is

$$\mathrm{ch}(\nabla) = \mathrm{Tr}\left(e^{-\nabla^2}\right) \in \Omega^{\mathrm{even}}(M).$$

It is a closed form whose de Rham cohomology class is independent of ∇.

Chern-Simons form

Suppose that ∇_0 and ∇_1 are two Hermitian connections on E. Putting $\nabla_s = s\nabla_1 + (1-s)\nabla_0$, the Chern-Simons form is

$$CS(\nabla_0, \nabla_1) = \int_0^1 \mathrm{Tr}\left(\frac{d\nabla_s}{ds} e^{-\nabla_s^2}\right) ds \in \Omega^{\mathrm{odd}}(M)/\operatorname{Im}(d).$$

Then

$$dCS(\nabla_0, \nabla_1) = \mathrm{ch}(\nabla_0) - \mathrm{ch}(\nabla_1).$$

Quillen's Chern character I

Another approach to the Chern-Simons form: On $E \oplus E$, put

$$A_s = s\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \nabla_0 & 0 \\ 0 & \nabla_1 \end{pmatrix} = sV + \nabla.$$

(Adding a mass term.) Then

$$A_s^2 = s^2 V^2 + s(\nabla V + V\nabla) + \nabla^2.$$

We think of V as being an odd variable, so

$$\begin{aligned} \nabla V + V\nabla &= \left(\sum_\alpha dx^\alpha \nabla_\alpha\right) V + V \sum_\alpha dx^\alpha \nabla_\alpha \\ &= \sum_\alpha dx^\alpha (\nabla_\alpha V - V\nabla_\alpha) \\ &= \sum_\alpha dx^\alpha [\nabla_\alpha, V]. \end{aligned}$$

Quillen's Chern character II

Then

$$A_s^2 = s^2 V^2 + s(\nabla V + V\nabla) + \nabla^2$$
$$= s^2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + s \begin{pmatrix} 0 & \nabla_0 - \nabla_1 \\ \nabla_1 - \nabla_0 & 0 \end{pmatrix} + \begin{pmatrix} \nabla_0^2 & 0 \\ 0 & \nabla_1^2 \end{pmatrix}.$$

Define

$$\mathrm{ch}(A_s) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-A_s^2} \right)$$
$$= \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-\begin{pmatrix} s^2 + \nabla_0^2 & s(\nabla_0 - \nabla_1) \\ s(\nabla_1 - \nabla_0) & s^2 + \nabla_1^2 \end{pmatrix}} \right).$$

Quillen's Chern character III

$$\mathrm{ch}(A_s) = \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} e^{-\begin{pmatrix} s^2 + \nabla_0^2 & s(\nabla_0 - \nabla_1) \\ s(\nabla_1 - \nabla_0) & s^2 + \nabla_1^2 \end{pmatrix}} \right).$$

Then $\mathrm{ch}(A_s)$ is closed and its de Rham cohomology class is independent of s.

When $s = 0$, we get $\mathrm{ch}(A_s) = \mathrm{ch}(\nabla_0) - \mathrm{ch}(\nabla_1)$.

Also, $\lim_{s\to\infty} \mathrm{ch}(A_s) = 0$, because of the $-s^2$ in the exponent.

Quillen's Chern character IV

We can construct the Chern-Simons form as

$$CS(\nabla_0, \nabla_1) = \int_0^\infty \mathrm{Tr}\left(\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \frac{dA_s}{ds} e^{-A_s^2} \right) ds.$$

Instead of interpolating between ∇_0 and ∇_1, we are now interpolating between $\nabla = \begin{pmatrix} \nabla_0 & 0 \\ 0 & \nabla_1 \end{pmatrix}$ and ∞.

More conceptually,

$E \oplus E$ is a $\mathbb{Z}_2$-graded vector bundle,

$V = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is an operator on $E \oplus E$, of odd degree, and

$A_s = sV + \nabla$ is a superconnection in the sense of Quillen.

K-theory

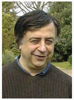
M is a smooth manifold.

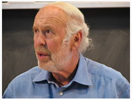
$K^0(M)$ is the free abelian group generated by isomorphism classes of finite dimensional complex vector bundles on M, quotiented by the relations $[E_2] = [E_1] + [E_3]$ if there is a short exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0.$$

Generators of differential K-theory






Differential K-theory combines vector bundles and differential forms. There are various models for the differential K-group $\check{K}^0(M)$. Here is a "standard" model.

A generator for $\check{K}^0(M)$ is a quadruple $\mathcal{E} = (E, h^E, \nabla^E, \omega)$, where

- E is a finite dimensional complex vector bundle on M.
- h^E is a Hermitian metric on E.
- ∇^E is a Hermitian connection on E.
- $\omega \in \Omega^{\mathrm{odd}}(M)/\operatorname{Im}(d)$.

(There's a model due to Simons and Sullivan where ω gets absorbed into the connection.)

Relations for $\check{K}^0(M)$

Given three such quadruples, we impose the relation

$$\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$$

if there is a short exact sequence of Hermitian vector bundles

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0,$$

and

$$\omega_2 = \omega_1 + \omega_3 - CS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) \in \Omega^{\mathrm{odd}}(M)/\operatorname{Im}(d).$$

Here the Chern-Simons form CS satisfies

$$dCS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) = \mathrm{ch}\left(\nabla^{E_2}\right) - \mathrm{ch}\left(\nabla^{E_1}\right) - \mathrm{ch}\left(\nabla^{E_3}\right).$$

Exact sequences

Quotienting by the relations defines $\check{K}^0(M)$. There are a forgetful map

$$f : \check{K}^0(M) \to K^0(M),$$

and a Chern character map

$$\mathrm{Ch} : \check{K}^0(M) \to \Omega^{\mathrm{even}}_K(M)$$

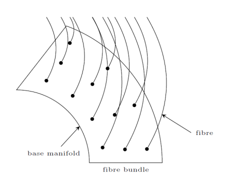
coming from

$$\mathrm{Ch}(E, h^E, \nabla^E, \omega) = \mathrm{ch}(\nabla^E) + d\omega.$$

$$0 \longrightarrow K^{-1}(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^0(M) \xrightarrow{\mathrm{Ch}} \Omega^{\mathrm{even}}_K(M) \longrightarrow 0$$

$$0 \longrightarrow \frac{\Omega^{\mathrm{odd}}(M)}{\Omega^{\mathrm{odd}}_K(M)} \longrightarrow \check{K}^0(M) \xrightarrow{f} K^0(M) \longrightarrow 0$$

Atiyah-Singer families index theorem



Suppose that $\pi : M \to B$ is a fiber bundle.

Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin*^{c}.

Geometric assumptions: Riemannian metrics on the fibers, Hermitian connection on the associated *spin*^{c} line bundle.

There are index maps

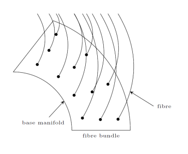
$$\mathrm{ind}_{an}, \mathrm{ind}_{top} : K^0(M) \to K^0(B).$$

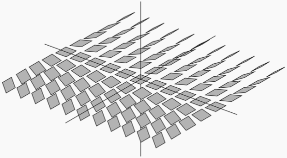
Atiyah-Singer families index theorem

$$\mathrm{ind}_{an} = \mathrm{ind}_{top}\,.$$

Index theorem in differential K-theory

Suppose in addition that there is a horizontal distribution on the fiber bundle.





(Freed-L.) There are index maps

$$\mathrm{ind}_{an}, \mathrm{ind}_{top} : \check{K}^0(M) \to \check{K}^0(B).$$

Their construction uses local index theory methods.

(Simons and Sullivan gave an alternative construction of ind_{top} in terms of "$\hat{K}$-characters".)

Index theorem in differential K-theory

Theorem

(Freed-L.)

$$\mathrm{ind}_{an} = \mathrm{ind}_{top}$$

as maps from $\check{K}^0(M)$ to $\check{K}^0(B)$.

Applying f, one recovers the Atiyah-Singer families index theorem. Applying Ch, one recovers Bismut's local version of the families index theorem.

Consequences

The index theorem in differential K-theory packages many of the results of local index theory into a semitopological setting. Some consequences:

- $\mathbb{R}/\mathbb{Z}$-index theorem
- Computation of $\mathbb{R}/\mathbb{Z}$-valued eta invariants.
- Computation of the determinant line bundle, along with its Quillen metric and compatible connection (up to isomorphism).

Chern-Simons, differential K-theory and operator theory

Slogans

Differential K-theory is a "K-theory of finite dimensional vector bundles with connections".

It is closely linked to local index theory.

Today: Differential K-theory as a "K-theory of infinite dimensional vector bundles with (super)connections".

Some motivation:

1. It unifies various earlier models for differential K-theory.
2. The analytic index becomes almost tautological.
3. The even and odd cases can be treated similarly.
4. Extension to twisting by H^3.

Motivation

From the viewpoint of analytic index theory, it is natural to use infinite dimensional vector bundles.

Unbounded Kasparov KK-theory: $K^0(M) \cong KK^0(\mathbb{C}, C(M))$, the latter being given in terms of unbounded Fredholm operators on $\mathbb{Z}_2$-graded Hilbert $C(M)$-modules. Can we give a model for differential K-theory along these lines?

If E is a finite dimensional $\mathbb{Z}_2$-graded vector bundle then

$$\mathrm{ch}(\nabla) = \mathrm{Tr}_s\, e^{-\nabla^2} = \mathrm{Tr}\left(\epsilon e^{-\nabla^2}\right),$$

where ϵ is the $\mathbb{Z}_2$-grading operator.

Problem: This doesn't make sense if E is infinite dimensional.

Solution: Replace the connection ∇ by a superconnection.

Superconnections

E is a finite dimensional $\mathbb{Z}_2$-graded vector bundle on M.

(Quillen) A superconnection on E is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathrm{End}_{\mathrm{odd}}(E))$
- $A_{[1]}$ is a connection on E
- $A_{[i]} \in \Omega^i(M; \mathrm{End}(E))$ for $i \geq 2$, with odd total parity.

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-A^2} \in \Omega^{\mathrm{even}}(M).$$

In the previous description of $\check{K}^0(M)$, you can replace connections by superconnections.

Supertraces

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where $A_{[0]}$ is a section of $\mathrm{End}_{\mathrm{odd}}(E)$.

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-A^2} \in \Omega^{\mathrm{even}}(M).$$

If we expand $\mathrm{ch}(A)$ in the form degree,

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-A_{[0]}^2} + \dots = \mathrm{Index}(A_{[0]}) + \dots,$$

where $\mathrm{Index}(A_{[0]}) \in C^\infty(M)$ is the fiberwise index.

On an infinite dimensional vector bundle, if $\mathrm{Tr}_s\, e^{-A_{[0]}^2}$ has a chance of making sense then $\mathrm{Tr}_s\, e^{-A^2}$ has a chance of making sense.

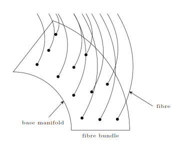
Hilbert bundles

Suppose that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle. We want to be able to talk about superconnections on $\mathcal{H}$.

What is the right structure group for the bundle? Say that H is a fiber of the bundle. The structure group should be a subgroup of $U_{\text{even}}(H)$.

All of $U_{\text{even}}(H)$ is too big. (Any infinite-dimensional Hilbert bundle with structure group given by the unitary operators, in the norm or strong topology, is topologically trivial.)

A special case



Suppose that $X \to M$ is a fiber bundle with compact fiber Z. Its structure group is contained in $\mathrm{Diff}(Z)$.

The functions on the fibers form a vector bundle on the base. More formally, $\mathrm{Diff}(Z)$ acts on the Hilbert space H of square-integrable half-densities $L^2(Z)$. That is, there's an (injective) homomorphism $\rho : \mathrm{Diff}(Z) \to U(H)$, and an associated Hilbert bundle $\mathcal{H} \to M$ with fiber H.

We should be able to include this case, i.e. deal with structure groups $\rho(\mathrm{Diff}(Z)) \subset U(H)$.

Goal

For a $\mathbb{Z}_2$-graded Hilbert space H, the goal is to find the right notion of a structure group $G \subset U_{\text{even}}(H)$, so that

1. It is general enough to include the preceding example coming from a fiber bundle.

2. It is restrictive enough that we can make sense of the Chern character of a superconnection of a Hilbert bundle with structure group G.

We will construct G using a pseudodifferential calculus based on a “Dirac operator” D.

"Analysis on manifolds" without manifolds

Say H is a $\mathbb{Z}_2$-graded Hilbert space,

$D = \begin{pmatrix} 0 & \partial_+^* \\ \partial_+ & 0 \end{pmatrix}$ is a self-adjoint operator.

Assume that $\mathrm{Tr}\, e^{-\theta D^2} < \infty$ for all $\theta > 0$.

For $s \in \mathbb{Z}^{\geq 0}$, put $H^s = \mathrm{Dom}(|D|^s)$, a "Sobolev space".

For $s \in \mathbb{Z}^{<0}$, put $H^s = (H^{-s})^*$.

Put $H^\infty = \bigcap_{s \geq 0} H^s$.

Abstract pseudodifferential operators

Definition

op^k consists of the closed operators F on H so that $F(H^\infty) \subset H^\infty$ and for all $s \in \mathbb{Z}$, F extends to a bounded operator from H^s to H^{s-k}.

The space of "Dirac-type operators":

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & P_+^* \\ P_+ & 0 \end{pmatrix} \in op^1 : \frac{1}{\sqrt{P^2+1}} \in op^{-1} \right\}.$$

Clearly $D \in \mathcal{P}$.

Lemma

$\mathcal{P}$ is closed under order-zero perturbations.

The structure group

As a group,

$$G = U_{\text{even}}(H) \cap op^0.$$

What is the smooth structure? Since we only care about Hilbert bundles over *finite dimensional* manifolds, it's enough to know what a smooth map $\mathbb{R}^k \to G$ is. (Diffeology)

A map $\mathbb{R}^k \to G$ is declared to be "smooth" if it preserves the smooth maps $\mathbb{R}^k \to H^s$ and $\mathbb{R}^k \to op^l$.

Here H^s and op^l have Fréchet topologies.

Superconnection

Suppose that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group G. It now makes sense to say that a superconnection on $\mathcal{H}$ is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathcal{P})$
- $A_{[1]} = d + A_\alpha$ locally, with $A_\alpha \in \Omega^1(U_\alpha; op^{k_1})$
- $A_{[i]} \in \Omega^i(M; op^{k_i})$ for $i \geq 2$, with odd total parity.

Then

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-A^2} \in \Omega^{\mathrm{even}}(M)$$

makes sense, using a Duhamel expansion of e^{-A^2}.

Chern-Simons forms

Suppose that A and A' are two superconnections on the Hilbert bundle. Then $\mathrm{ch}(A)$ and $\mathrm{ch}(A')$ are closed forms on M.

When can we say that their difference is exact?

It turns out to be enough for their 0-th terms to differ by a pseudodifferential operator of order zero.

If $A_{[0]} - A'_{[0]} \in \Omega^0(M; op^0)$, put

$$\eta(A, A') = \int_0^1 \mathrm{Tr}_s \left(\frac{dB}{dt} e^{-B^2(t)} \right) dt,$$

where $B(t) = (1-t)A + tA'$.

Then

$$d\eta(A, A') = \mathrm{ch}(A) - \mathrm{ch}(A').$$

So $\eta(A, A')$ is the Chern-Simons form in this setting.

Interpolating to infinity

Suppose that $A_{[0]}$ is fiberwise invertible. Put

$$\eta(A, \infty) = \int_1^\infty \mathrm{Tr}_s \left(\frac{dA_t}{dt} e^{-A_t^2} \right) dt,$$

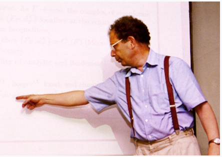
where

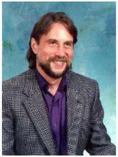
$$A_t = tA_{[0]} + A_{[1]} + t^{-1} A_{[2]} + \dots$$

Then

$$d\eta(A, \infty) = \mathrm{ch}(A).$$

Here $\eta(A, \infty)$ is the analog of the Bismut-Cheeger eta form.





Chern-Simons, differential K-theory and operator theory

Generators for $\check{K}^0(M)$

Generators are triples $(\mathcal{H}, A, \omega)$, where

- $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group G.
- A is a superconnection on $\mathcal{H}$.
- $\omega \in \Omega^{\text{odd}}(M)/\operatorname{Im}(d)$.

Relations for $\check{K}^0(M)$

1. $$[\mathcal{H}, A, \omega] + [\mathcal{H}', A', \omega'] = [\mathcal{H} \oplus \mathcal{H}', A \oplus A', \omega + \omega']$$
2. If $A_{[0]}$ is fiberwise invertible then
$$[\mathcal{H}, A, \omega] = [0, 0, \omega + \eta(A, \infty)].$$
3. If $A_{[0]} - A'_{[0]} \in \Omega^0(M; op^0)$ then
$$[\mathcal{H}, A, \omega] = [\mathcal{H}', A', \omega' + \eta(A, A')].$$

Theorem (Gorokhovsky-L.) The natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism, where $\check{K}^0_{stan}(M)$ is the "standard" differential K-group defined using finite dimensional vector bundles and connections.

Comparison map

The inverse map $q : \check{K}^0(M) \to \check{K}^0_{stan}(M)$ in a special case:

Suppose that $\mathrm{Ker}(A_{[0]})$ forms a $\mathbb{Z}_2$-graded finite dimensional vector bundle on M.

Let Q be fiberwise orthogonal projection on $\mathrm{Ker}(A_{[0]})$.

Then

$$q(\mathcal{H}, A, \omega) = \\ \left[\mathrm{Ker}(A_{[0]}), QA_{[1]}Q, \omega + \eta(A, B) + \eta((I - Q)A(I - Q), \infty)\right],$$

where $B = (I - Q)A(I - Q) + QA_{[1]}Q$.

Unification

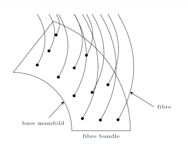
The Hilbert bundle version $\check{K}^0(M)$ of differential K-theory unifies some other models. First, the natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism.

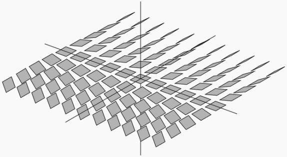
Bunke and Schick have a "geometric families" model of differential K-theory.


There is a natural map $\check{K}^0_{geom.fam.}(M) \to \check{K}^0(M)$ that is an isomorphism.

On the other hand, there are no obvious comparison maps with the Hopkins-Singer model for differential K-theory.

Pushforward





Suppose that $\pi : M \to B$ is a fiber bundle.

Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin*^{c}.

Geometric assumptions: Riemannian metrics on the fibers, Hermitian connection on the associated *spin*^{c} line bundle, horizontal distribution

There was an analytic index map (Freed-L.)

$$\mathrm{ind}_{an} : \check{K}^0_{stan}(M) \to \check{K}^0_{stan}(B).$$

An easier description

Say $[E, A, \omega]$ is a finite dimensional cycle for $\check{K}^0(M)$. Let $\mathcal{H}$ be the bundle on B of fiberwise spinor fields with values in E, i.e.

$$C^\infty(B; \mathcal{H}^\infty) = C^\infty(M; E \otimes S^V M)$$
$$= C^\infty(M; E) \otimes_{C^\infty(M)} C^\infty(M; S^V M).$$

Define the pushforward superconnection, acting on $C^\infty(B; \mathcal{H}^\infty)$, by

$$\pi_* A = m(A \otimes Id) + Id \otimes \mathcal{B},$$

where m is the Clifford action of $T^* M$ on $\pi^* \Lambda^* TB \otimes S^V M$, and $\mathcal{B}$ is the Bismut superconnection for the bundle $\pi : M \to B$. Put

$$\omega' = \int_{M/B} \mathrm{Td}\left(\nabla^{T^V M}\right) \wedge \omega + \lim_{u \to 0} \eta((\pi_* A)_u, \pi_* A) \in \Omega^{\mathrm{odd}}(B) / \operatorname{Im}(d).$$

Pushforward theorem

Theorem (Gorokhovsky-L.)

If (E, A, ω) is a finite dimensional generator of $\check{K}^0(M)$ then

$$\mathrm{ind}_{an}([E, A, \omega]) = [\mathcal{H}, \pi_* A, \omega']$$

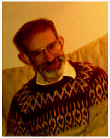
in $\check{K}^0(B) \cong \check{K}^0_{stan}(B)$.

This gives an almost tautological pushforward of *finite dimensional* cycles in differential K-theory.

Can one also push forward infinite dimensional cycles? Formally yes, but there are some technical questions.

Twisted K-theory





There's a notion of *twisted* K-theory, where one twists by an element of $\mathrm{H}^3(M; \mathbb{Z})$.

Using finite dimensional vector bundles, one can only handle *torsion* elements of $\mathrm{H}^3(M; \mathbb{Z})$. To deal with all of $\mathrm{H}^3(M; \mathbb{Z})$, one needs to use infinite dimensional vector bundles.

Can one extend the previous model from differential K-theory to differential twisted K-theory?

$U(1)$-bundles

$\mathrm{H}^2(M;\mathbb{Z})$ classifies principal $U(1)$-bundles on M. We can twist a vector bundle on M by the associated complex line bundle.

Data for a $U(1)$-bundle:

- An open cover $\{U_\alpha\}$ of M.
- A smooth map $\phi_{\alpha\beta} : U_\alpha \cap U_\beta \to U(1)$ on each nonempty intersection, so that
- $\phi_{\alpha\beta}\phi_{\beta\gamma} = \phi_{\alpha\gamma}$ on each nonempty $U_\alpha \cap U_\beta \cap U_\gamma$.

Gerbes

$H^3(M;\mathbb{Z})$ classifies $U(1)$-gerbes on M. We'll twist by coupling to a gerbe.

Data for a gerbe:

- An open cover $\{U_\alpha\}$ of M.
- A complex line bundle $\mathcal{L}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.
- Isomorphisms $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \to \mathcal{L}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$ so that
- $\mu_{\alpha\gamma\delta} \circ (\mu_{\alpha\beta\gamma} \otimes \mathrm{Id}) = \mu_{\alpha\beta\delta} \circ (\mathrm{Id} \otimes \mu_{\beta\gamma\delta})$ on $\mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\delta}$.

$U(1)$-connection on a gerbe

We have line bundles $\mathcal{L}_{\alpha\beta}$ on overlaps. A $U(1)$-connection on the gerbe consists of

- A Hermitian metric on $\mathcal{L}_{\alpha\beta}$.
- Connective structure: A Hermitian connection $\nabla_{\alpha\beta}$ on each $\mathcal{L}_{\alpha\beta}$ so
$$\mu^*_{\alpha\beta\gamma}\nabla_{\alpha\gamma} = (\nabla_{\alpha\beta} \otimes I) + (I \otimes \nabla_{\beta\gamma}).$$
- Curving: $\kappa_\alpha \in \Omega^2(U_\alpha)$ so
$$\nabla^2_{\alpha\beta} = \kappa_\alpha - \kappa_\beta.$$

Then $H = d\kappa_\alpha$ is a globally defined closed 3-form on M, the de Rham representative of the gerbe's class in $\mathrm{H}^3(M; \mathbb{Z})$.

Chern character in twisted K-theory

A twisted Hilbert bundle $\mathcal{H}$ is given by Hilbert bundles $\mathcal{H}_\alpha$ over the U_α's, with isomorphisms $\phi_{\alpha\beta} : \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{H}_\beta$ on $U_\alpha \cap U_\beta$.

A superconnection on $\mathcal{H}$ is given by superconnections A_α on the $\mathcal{H}_\alpha$'s so $\phi^*_{\alpha\beta} A_\beta = (A_\alpha \otimes I) + (I \otimes \nabla_{\alpha\beta})$ on $U_\alpha \cap U_\beta$.

Put

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-(A_\alpha^2 + \kappa_\alpha)} \in \Omega^{\mathrm{even}}(M).$$

Then

$$(d + H\wedge)\, \mathrm{ch}(A) = 0.$$

Model for differential twisted K-theory

The generators for differential twisted K-theory are now triples $(\mathcal{H}, A, \omega)$ as before. Quotienting by the relations, one gets the differential twisted K-theory group.

Theorem (Gorokhovsky-L.): Up to isomorphism, the differential twisted K-group only depends on the gerbe through its isomorphism class. It is independent of the choices of connective structure and curving.

This gives an explicit model for differential twisted K-theory. It remains to show that it agrees with other models (Bunke-Nikolaus).