Some names

Optimal transport in Riemannian geometry

Otto, Otto-Villani, McCann, J.L.-Villani, Sturm

Optimal transport in Ricci flow

J.L., McCann-Topping, Topping

Ricci flow

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Fix t_0 and put $\tau = t_0 - t$. Then $\frac{dg}{d\tau} = 2 \text{ Ric}$.

An important tool : monotonic quantities.

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Definition

reduced volume $\widetilde{V}(\overline{\tau}) = \overline{\tau}^{-\frac{n}{2}} \int_M e^{-l(q, \overline{\tau})} \, \mathrm{dvol}(q)$.

Monotonicity of the reduced volume

Theorem

(Perelman) $\widetilde{V}$ is nonincreasing in $\overline{\tau}$, i.e. nondecreasing in t.

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An "entropy" functional for Ricci flow.

The only assumption : $g(t)$ satisfies the Ricci flow equation.

Main application : Perelman's "no local collapsing" theorem.

Perelman's heuristic derivation

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$$\overline{g} = g(\tau) + \tau g_{S^N} + \left(\frac{N}{2\tau} + R \right) d\tau^2,$$

where g_{S^N} has constant sectional curvature $\frac{1}{2N}$.

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Bishop-Gromov : $r^{-\dim} \operatorname{vol}(B_r(p))$ is nonincreasing in r if $\mathrm{Ric} \geq 0$.

Apply formally to $\overline{M}$ and take $N \to \infty$. Get monotonicity of $\widetilde{V}$.

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This should give an optimal transport problem on M with which we can derive the monotonicity of $\widetilde{V}$.

Optimal transport

(M, g) a compact Riemannian manifold

$P(M)$ = Borel probability measures on M

$P^\infty(M) = \{\rho \, \mathrm{dvol}_M \, : \, \rho \in C^\infty(M), \, \rho > 0, \, \int_M \rho \, \mathrm{dvol}_M \, = \, 1\}.$

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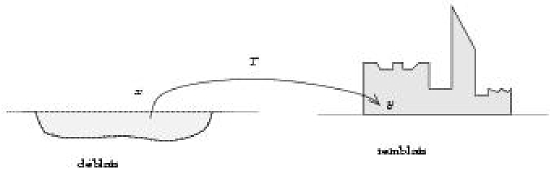
Transport problem Given $\mu_0, \mu_1 \in P(M)$, we want to move μ_0 to μ_1 most efficiently.



Wasserstein metric

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Wasserstein metric W_2 on $P(M)$

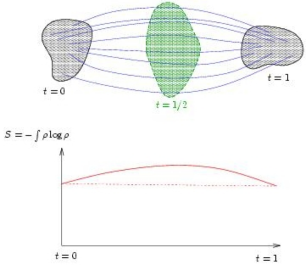
$$W_2(\mu_0,\mu_1)^2 = \inf_\Pi \int_{M\times M} d(x,y)^2 \, d\Pi(x,y),$$

where $\Pi \in P(M \times M)$, $(p_0)_*\Pi = \mu_0$, $(p_1)_*\Pi = \mu_1$.

(A minimizer always exists.)

Displacement interpolation

The transport is done along geodesics in M.



Take a snapshot of the mass at each time $t \in [0, 1]$, get a 1-parameter family of measures $\{\mu_t\}_{t\in[0,1]}$.

Eulerian formulation

Variational problem for $\{\mu_t\}_{t\in[0,1]}$
(Benamou-Brenier)

Say $c : [0,1] \to P^\infty(M)$ is a smooth curve.

Write $c(t) = \rho(t)\ \mathrm{dvol}_M$.

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Fact : We can solve

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi)$$

for $\phi \equiv \phi(t) \in C^\infty(M)$.

Here ϕ is unique up to an additive constant.

Energy of a curve of measures

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Theorem

Otto-Westdickenberg

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Note : the infimum may not be achieved. A minimizing c is a smooth displacement interpolation.

Euler-Lagrange equations

The Euler-Lagrange equation for the functional E is

Hamilton-Jacobi equation

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We also had

Conservation equation

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i (\rho \nabla_i \phi).$$

These are the equations for optimal tranport and can be solved explicitly.

Entropy

Definition

$\mathcal{E} \ : \ P^{\infty}(M) \to \mathbb{R}$ is given by

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Proposition

Otto-Villani Along a smooth displacement interpolation c,

$$\frac{d^2}{dt^2}\mathcal{E}(c(t)) = \int_M \Big[|\operatorname{Hess} \phi|^2 + \operatorname{Ric}(\nabla\phi, \nabla\phi) \Big] \rho \, \mathrm{dvol}_M .$$

Synthetic Ricci curvature

Corollary

If $\mathrm{Ric}_M \geq 0$ *then* $\mathcal{E}$ *is convex along smooth displacement interpolations in* $P^\infty(M)$.

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J.L.-Villani, Sturm One can take convexity of $\mathcal{E}$ (along displacement interpolations) as a definition of "nonnegative Ricci curvature" for a metric-measure space.

Question

Can we do something similar for the Ricci flow?

Motivation : Satisfying $\mathrm{Ric} = 0$ in the Riemannian case is like satisfying the Ricci flow equation in the spacetime case.

Optimal transport in a Ricci flow spacetime was first considered by Topping.

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Note : The Ricci flow equation

$$\frac{dg}{dt} = -2\ \mathrm{Ric}$$

implies

$$\frac{\mathrm{dvol}_M}{dt} = -R\ \mathrm{dvol}_M\,.$$

E_0 functional

Assume hereafter that $(M, g(t))$ satisfies the Ricci flow equation.

Given $c : [t_0, t_1] \to P^\infty(M)$, write $c(t) = \rho(t)$ dvol$_M$. Solve

$$\frac{\partial \rho}{\partial t} = -\sum_i \nabla^i(\rho \nabla_i \phi) + R\,\rho$$

for $\phi \equiv \phi(t) \in C^\infty(M)$.

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Euler-Lagrange equation for E_0 :

$$\frac{\partial \phi}{\partial t} = -\frac{1}{2}\,|\nabla \phi|^2 + \frac{1}{2}\,R.$$

Convexity statement

Proposition

If c satisfies the Euler-Lagrange equation then

$$\frac{d^2}{dt^2}\int_M (\rho \ln \rho - \phi\,\rho)\,\mathrm{dvol}_M = \int_M |\,\mathrm{Ric} - \mathrm{Hess}\,\phi|^2\,\rho\,\mathrm{dvol}_M\,.$$

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Corollary

If c satisfies the Euler-Lagrange equation then $\int_M (\rho \ln \rho \; - \; \phi\, \rho) \text{ dvol}_M$ *is convex in* t.

Corresponding optimal transport problem

Say we want to transport a measure μ_0 (at time t_0) to a measure μ_1 (at time t_1).

Take the cost to transport a unit of mass from p to q to be

$$\min\{\mathcal{L}_0(\gamma) \,:\, \gamma(t_0) = p, \gamma(t_1) = q\},$$

where

$$\mathcal{L}_0(\gamma) \;=\; \frac{1}{2} \int_{t_0}^{t_1} \left(|\dot{\gamma}|^2_{g(t)} \;+\; R(\gamma(t), t) \right) \, dt.$$

There is a corresponding notion of optimal transport, displacement interpolation, etc.

Convexity result

Theorem

$\int_M (\rho \ln \rho - \phi \rho) \, \mathrm{dvol}_M$ *is convex along a displacement interpolation between absolutely continuous measures* $\mu_0, \mu_1 \in P(M)$.

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The proof uses results of Bernard-Buffoni/Topping for optimal transport problems with a time-dependent cost.

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Euler-Lagrange equation for E_- :

$$\frac{\partial \phi}{\partial \tau} = - \frac{1}{2} |\nabla \phi|^2 + \frac{1}{2} R - \frac{1}{2\tau} \phi.$$

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Proposition

If c satisfies the Euler-Lagrange equation then

$$\left(\tau^{\frac{3}{2}}\frac{d}{d\tau}\right)^2 \left(\int_M (\rho\,\ln\rho\ +\ \phi\,\rho)\ \mathrm{dvol}_M + \frac{n}{2}\,\ln\tau\right)\ =$$
$$\tau^3\ \int_M \left|\mathrm{Ric} + \mathrm{Hess}\,\phi - \frac{g}{2\tau}\right|^2\ \rho\ \mathrm{dvol}_M\,.$$

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$$\left(\tau^{\frac{3}{2}} \frac{d}{d\tau}\right)^2 \left(\int_M (\rho \ln \rho + \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2} \ln \tau\right) = \tau^3 \int_M \left|\mathrm{Ric} + \mathrm{Hess}\, \phi - \frac{g}{2\tau}\right|^2 \rho \, \mathrm{dvol}_M .$$

Corollary

If c satisfies the Euler-Lagrange equation then $\int_M (\rho \ln \rho + \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2} \ln \tau$ *is convex in* $\tau^{-\frac{1}{2}}$*.*

Perelman's reduced volume

Riemannian optimal transport

Optimal transport for Ricci flow

Monotonicity of the reduced volume

The transport problem

Take $\tau_0 \to 0$, $\mu_0 = \delta_p$ and μ_1 an absolutely continuous measure.

The displacement interpolation is along $\mathcal{L}$-geodesics emanating from p.

In this case, $\phi = l$.

From convexity to monotonicity

Proposition

In this case, $\int_M (\rho \ln \rho + \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2} \ln \tau$ *is nondecreasing in* τ.

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Proof.

We know that it is convex in $s = \tau^{-\frac{1}{2}}$. As $s \to \infty$, i.e. as $\tau \to 0$, it approaches a constant. (Almost Euclidean situation.) So it is nonincreasing in s, i.e. nondecreasing in τ. □

Free energy

Trivial fact : The minimizer of

$$\int_M (\rho \ln \rho + \phi \rho) \, \mathrm{dvol}_M + \frac{n}{2} \ln \tau,$$

as $\rho \, \mathrm{dvol}_M$ ranges over absolutely continuous probability measures, is

$$- \ln \left(\tau^{-\frac{n}{2}} \int_M e^{-\phi} \, \mathrm{dvol}_M \right).$$

The minimizing measure is given by

$$\rho = \frac{e^{-\phi}}{\int_M e^{-\phi} \, \mathrm{dvol}_M}.$$

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Proof : Say $\tau' < \tau''$. Recall $\phi = l$. Take $\mu(\tau'') = \rho(\tau'') \, \mathrm{dvol}_M$ with

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Transport it to δ_p (at time zero). At the intermediate time τ' we see a measure $\mu(\tau') \;=\rho(\tau') \; \mathrm{dvol}_M$.

Proof

Then

$$- \ln \left((\tau')^{-\frac{n}{2}} \int_M e^{-\phi(\tau')} \, \mathrm{dvol}_M \right)$$

$$\leq \int_M \left[\rho(\tau') \ln \rho(\tau') + \phi(\tau') \rho(\tau') \right] \mathrm{dvol}_M + \frac{n}{2} \ln \tau'$$

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=\ & - \ln\left((\tau'')^{-\frac{n}{2}} \int_M e^{-\phi(\tau'')} \, \mathrm{dvol}_M \right).
\end{aligned}
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End of proof

Optimal transport and heat flow

Otto, Sturm-von Renesse, Otto-Westdickenberg Suppose that a compact Riemannian manifold has Ric ≥ 0. If $\mu_0(t)$ and $\mu_1(t)$ are two solutions of the heat flow on measures then $W_2(\mu_0(t), \mu_1(t))$ is nonincreasing in t.

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J.L., McCann-Topping Suppose that $(M, g(t))$ is a Ricci flow solution. Suppose that $\mu_0(t)$ and $\mu_1(t)$ are two solutions of the backward heat flow on measures

$$\frac{d\mu}{dt} = - \nabla^2_{g(t)} \mu.$$

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Topping Extension to a statement about the $\mathcal{L}$-transport distance between μ_0 and μ_1 at distinct but related times.