A Hilbert bundle description of differential $K$-theory

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Introduction

Summary of differential $K$-theory

Superconnections on Hilbert bundles

Infinite dimensional cycles

Twisted differential $K$-theory
Joint work with Alexander Gorokhovsky
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It is closely linked to local index theory, as will be described.
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Today: Differential $K$-theory as a “$K$-theory of infinite dimensional vector bundles with (super)connections”.

Some motivation:

1. It unifies various earlier models for differential $K$-theory.
2. The analytic index becomes almost tautological.
3. The even and odd cases can be treated similarly.
4. Extension to twisting by $H^3$. 
Structure of the talk

- Summary of differential $K$-theory
- Superconnections on Hilbert bundles
- Infinite dimensional cycles
- Twisted differential $K$-theory
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Summary of differential $K$-theory

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Twisted differential $K$-theory
$K$-theory

$M$ is a smooth manifold.

$K^0(M)$ is the free abelian group generated by isomorphism classes of finite dimensional complex vector bundles on $M$, quotiented by the relations $[E_2] = [E_1] + [E_3]$ if there is a short exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0.$$
Generators of differential $K$-theory

Differential $K$-theory combines vector bundles and differential forms. There are various models for the differential $K$-group $\hat{K}_0(M)$. Here is a "standard" model.

A generator for $\hat{K}_0(M)$ is a quadruple $E = (E, h_E, \nabla_E, \omega)$, where

- $E$ is a finite dimensional complex vector bundle on $M$.
- $h_E$ is a Hermitian metric on $E$.
- $\nabla_E$ is a Hermitian connection on $E$.
- $\omega \in \Omega^{\text{odd}}(M)/\text{Im}(d)$. 
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Given three such quadruples, we impose the relation

\[ \mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3 \]

if there is a short exact sequence of Hermitian vector bundles

\[ 0 \rightarrow E_1 \rightarrow E_2 \rightarrow E_3 \rightarrow 0, \]

and

\[ \omega_2 = \omega_1 + \omega_3 - CS \left( \nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3} \right) \in \Omega^{odd}(M)/\text{Im}(d). \]
Relations for $\tilde{K}^0(M)$

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$$\omega_2 = \omega_1 + \omega_3 - CS \left( \nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3} \right) \in \Omega^{odd}(M)/\operatorname{Im}(d).$$

Here the Chern-Simons form $CS$ satisfies

$$dCS \left( \nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3} \right) = \operatorname{ch} \left( \nabla^{E_2} \right) - \operatorname{ch} \left( \nabla^{E_1} \right) - \operatorname{ch} \left( \nabla^{E_3} \right).$$
Quotienting by the relations defines $\tilde{K}^0(M)$. There are a forgetful map

$$f : \tilde{K}^0(M) \to K^0(M),$$

and a Chern character map

$$Ch : \tilde{K}^0(M) \to \Omega_K^{even}(M)$$

coming from

$$Ch(E, h^E, \nabla^E, \omega) = ch(\nabla^E) + d\omega.$$
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\[
\begin{align*}
0 & \longrightarrow K^{-1}(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \tilde{K}^0(M) \overset{Ch}{\longrightarrow} \Omega_K^{even}(M) \longrightarrow 0 \\
0 & \longrightarrow \frac{\Omega^{odd}(M)}{\Omega_K^{odd}(M)} \longrightarrow \tilde{K}^0(M) \overset{f}{\longrightarrow} K^0(M) \longrightarrow 0
\end{align*}
\]
Suppose that $\pi : M \rightarrow B$ is a fiber bundle.

**Topological assumptions:** The fibers are compact and even dimensional. The fiberwise tangent bundle is $spin^c$.

**Geometric assumptions:** Riemannian metrics on the fibers, Hermitian connection on the associated $spin^c$ line bundle.
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There are index maps

$$\text{ind}_{an}, \text{ind}_{top} : K^0(M) \to K^0(B).$$
Atiyah-Singer families index theorem

$$\text{ind}_{\text{an}} = \text{ind}_{\text{top}}.$$
Index theorem in differential $K$-theory

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Their construction uses local index theory methods.
Index theorem in differential $K$-theory

Theorem
(Freed-L.)

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as maps from $\tilde{K}^0(M)$ to $\tilde{K}^0(B)$. 
Index theorem in differential $K$-theory

**Theorem**
*(Freed-L.)*

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\text{ind}_{an} = \text{ind}_{top}
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*as maps from $\tilde{K}^0(M)$ to $\tilde{K}^0(B)$.*

Applying $f$, one recovers the Atiyah-Singer families index theorem. Applying $\text{Ch}$, one recovers Bismut’s local version of the families index theorem.
The index theorem in differential $K$-theory packages many of the results of local index theory into a semitopological setting. Some consequences:

- $\mathbb{R}/\mathbb{Z}$-index theorem
- Computation of $\mathbb{R}/\mathbb{Z}$-valued eta invariants.
- Computation of the determinant line bundle, along with its Quillen metric and compatible connection (up to isomorphism).
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Twisted differential $K$-theory
From the viewpoint of analytic index theory, it is more natural to use infinite dimensional vector bundles.

Unbounded Kasparov KK-theory: $K^0(M) \cong KK^0(\mathbb{C}, C(M))$, the latter being given in terms of unbounded Fredholm operators on $\mathbb{Z}_2$-graded Hilbert $C(M)$-modules. Can we give a model for differential $K$-theory along these lines?

If $E$ is a finite dimensional $\mathbb{Z}_2$-graded vector bundle then $\text{ch}(\nabla) = \text{Tr} - \nabla^2$.

Problem: This doesn’t make sense if $E$ is infinite dimensional.

Solution: Replace the connection $\nabla$ by a superconnection.
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Motivation

From the viewpoint of analytic index theory, it is more natural to use infinite dimensional vector bundles.

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Superconnections

$E$ is a finite dimensional $\mathbb{Z}_2$-graded vector bundle on $M$.

(Quillen) A superconnection on $E$ is a sum


where

- $A[0] \in \Omega^0(M; \text{End}_{odd}(E))$
- $A[1]$ is a connection on $E$
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In the previous description of $\tilde{K}^0(M)$, you can replace connections by superconnections.
Say that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle. We want to be able to talk about superconnections on $\mathcal{H}$.
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What is the right structure group for the bundle? It should be general enough to include the case of Bismut superconnections, but restrictive enough so that one can still define the Chern character of a superconnection.
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What is the right structure group for the bundle? It should be general enough to include the case of Bismut superconnections, but restrictive enough so that one can still define the Chern character of a superconnection.

Say that $H$ is a fiber of the bundle. $U(H)$ is too big. We will restrict this using a pseudodifferential calculus based on a “Dirac operator” $D$. 
Abstract Sobolev spaces

Say $H$ is a $\mathbb{Z}_2$-graded Hilbert space,

$$D = \begin{pmatrix} 0 & \partial^* \\ \partial_+ & 0 \end{pmatrix}$$

is a self-adjoint operator.

Assume that $\text{Tr} e^{-\theta D^2} < \infty$ for all $\theta > 0$. 
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For $s \in \mathbb{Z}^{\geq 0}$, put $H^s = \text{Dom}(|D|^s)$.

For $s \in \mathbb{Z}^{< 0}$, put $H^s = (H^{-s})^*$.
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Put $H^\infty = \bigcap_{s \geq 0} H^s$. 
Definition

$op^k$ consists of the closed operators $F$ on $H$ so that $F(H^\infty) \subset H^\infty$ and for all $s \in \mathbb{Z}$, $F$ extends to a bounded operator from $H^s$ to $H^{s-k}$.
Abstract pseudodifferential operators

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The space of “Dirac-type operators”:

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & P^\ast_+ \\ P_+ & 0 \end{pmatrix} \in op^1 : \frac{1}{\sqrt{P^2 + 1}} \in op^{-1} \right\}.$$  

Clearly $D \in \mathcal{P}$. 
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Clearly $D \in \mathcal{P}$.

Lemma
$\mathcal{P}$ is closed under order-zero perturbations.
The structure group

As a set,

\[ G = U(H) \cap \text{op}^0. \]
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What is the smooth structure? Since we only care about Hilbert bundles over \textit{finite dimensional} manifolds, it’s enough to know what a smooth map \( \mathbb{R}^k \to G \) is. (Diffeology)
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What is the smooth structure? Since we only care about Hilbert bundles over finite dimensional manifolds, it’s enough to know what a smooth map \( \mathbb{R}^k \to G \) is. (Diffeology)

A map \( \mathbb{R}^k \to G \) is declared to be “smooth” if it preserves the smooth maps \( \mathbb{R}^k \to H^s \) and \( \mathbb{R}^k \to \text{op}^k \).

Here \( H^s \) and \( \text{op}^k \) have Fréchet topologies.
Suppose that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group $G$. It now makes sense to say that a superconnection on $\mathcal{H}$ is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \ldots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathcal{P})$
- $A_{[1]} = d + A_\alpha$ locally, with $A_\alpha \in \Omega^1(U_\alpha; op^{k_1})$
- $A_{[i]} \in \Omega^i(M; op^{k_i})$ for $i \geq 2$, with odd total parity.
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Then

$$\text{ch}(A) = \text{Tr}_s e^{-A^2} \in \Omega^{even}(M)$$

makes sense, using a Duhamel expansion of $e^{-A^2}$. 
If $A[0] - A'[0] \in \Omega^0(M; op^0)$, put

$$\eta(A, A') = \int_0^1 \text{Tr}_S \left( \frac{dB}{dt} e^{-B^2(t)} \right) dt,$$

where $B(t) = (1 - t)A + tA'$. 
Relative Chern character

If $A[0] - A'[0] \in \Omega^0(M; op^0)$, put

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$$d\eta(A, A') = \text{ch}(A) - \text{ch}(A').$$
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$$d\eta(A, A') = \text{ch}(A) - \text{ch}(A').$$

Suppose that $A[0]$ is fiberwise invertible. Put

$$\eta(A, \infty) = \int_1^\infty \text{Tr}_s \left( \frac{dA_t}{dt} e^{-A_t^2} \right) dt,$$

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Generators for $\tilde{K}^0(M)$

Generators are triples $(\mathcal{H}, A, \omega)$, where

- $\mathcal{H} \rightarrow M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group $G$.
- $A$ is a superconnection on $\mathcal{H}$.
- $\omega \in \Omega^{odd}(M)/\text{Im}(d)$. 
Relations for $\tilde{K}^0(M)$

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\[
[H, A, \omega] + [H', A', \omega'] = [H \oplus H', A \oplus A', \omega + \omega']
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\[ [\mathcal{H}, A, \omega] + [\mathcal{H}', A', \omega'] = [\mathcal{H} \oplus \mathcal{H}', A \oplus A', \omega + \omega'] \]

2. If $A_0$ is fiberwise invertible then
\[ [\mathcal{H}, A, \omega] = [0, 0, \omega + \eta(A, \infty)]. \]
Relations for $\tilde{K}^0(M)$

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$$[\mathcal{H}, A, \omega] + [\mathcal{H}', A', \omega'] = [\mathcal{H} \oplus \mathcal{H}', A \oplus A', \omega + \omega']$$

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$$[\mathcal{H}, A, \omega] = [0, 0, \omega + \eta(A, \infty)].$$

3. If $A[0] - A'[0] \in \Omega^0(M; op^0)$ then
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Relations for $\mathcal{K}^0(M)$

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3. If $A_0 - A_0' \in \Omega^0(M; op^0)$ then
   $$[\mathcal{H}, A, \omega] = [\mathcal{H}', A', \omega' + \eta(A, A')].$$

**Theorem** (Gorokhovsky-L.) The natural map $\mathcal{K}^0_{stan}(M) \to \mathcal{K}^0(M)$ is an isomorphism, where $\mathcal{K}^0_{stan}(M)$ is the “standard” differential $K$-group defined using finite dimensional vector bundles and connections.
The inverse map $q : \check{K}^0(M) \to \check{K}^0_{stan}(M)$ in a special case:

Suppose that $\text{Ker}(A_{[0]})$ forms a $\mathbb{Z}_2$-graded finite dimensional vector bundle on $M$. 

Let $Q$ be fiberwise orthogonal projection on $\text{Ker}(A_{[0]})$. Then

$$q(H, A, \omega) = [\text{Ker}(A_{[0]}), QA_{[1]}Q, \omega + \eta(A, B) + \eta((I - Q)A(I - Q), \infty)]$$

where $B = (I - Q)A(I - Q) + QA_{[1]}Q$. 

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where \( B = (I - Q)A(I - Q) + QA_{[1]} Q \).
The Hilbert bundle version $\tilde{K}^0(M)$ of differential $K$-theory unifies some other models. First, the natural map $\tilde{K}^0_{stan}(M) \to \tilde{K}^0(M)$ is an isomorphism.
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Bunke and Schick have a “geometric families” model of differential $K$-theory.

There is a natural map $\tilde{K}_\text{geom.fam.}^0(M) \to \tilde{K}^0(M)$ that is an isomorphism.
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On the other hand, there are no obvious comparison maps with the Hopkins-Singer model for differential $K$-theory.
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There was an analytic index map (Freed-L.)

$$\text{ind}_an : \tilde{K}^0_{stan}(M) \to \tilde{K}^0_{stan}(B).$$
An easier description

Say \([E, A, \omega]\) is a finite dimensional cycle for \(\tilde{K}^0(M)\).
An easier description

Say $[E, A, \omega]$ is a finite dimensional cycle for $K^0(M)$.

Let $\mathcal{H}$ be the bundle of $L^2$ vertical spinors with values in $E$, i.e.

$$C^\infty(B, \mathcal{H}^\infty) = C^\infty(M; E \otimes S^V M)$$

$$= C^\infty(M; E) \otimes_{C^\infty(M)} C^\infty(M; S^V M).$$
Say \([E, A, \omega]\) is a finite dimensional cycle for \(\tilde{K}^0(M)\).

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\]

Define the pushforward superconnection, acting on \(C^\infty(B, \mathcal{H}^\infty)\), by

\[
\pi_* A = m(A \otimes \text{Id}) + \text{Id} \otimes B,
\]

where \(m\) is the Clifford action of \(T^*M\) on \(\pi^* \Lambda^* TB \otimes S^V M\), and \(B\) is the Bismut superconnection for the bundle \(\pi : M \to B\). Put
Say $[E, A, \omega]$ is a finite dimensional cycle for $\tilde{K}^0(M)$.

Let $\mathcal{H}$ be the bundle of $L^2$ vertical spinors with values in $E$, i.e.

$$C^\infty(B, \mathcal{H}^\infty) = C^\infty(M; E \otimes S^VM) = C^\infty(M; E) \otimes_{C^\infty(M)} C^\infty(M; S^VM).$$

Define the pushforward superconnection, acting on $C^\infty(B, \mathcal{H}^\infty)$, by

$$\pi_* A = m(A \otimes Id) + Id \otimes B,$$

where $m$ is the Clifford action of $T^*M$ on $\pi^* \Lambda^* TB \otimes S^VM$, and $B$ is the Bismut superconnection for the bundle $\pi : M \to B$. Put

$$\omega' = \int_{M/B} \text{Td} \left( \nabla^{T^VM} \right) \wedge \omega + \lim_{u \to 0} \eta((\pi_* A)_u, \pi_* A) \in \Omega^{\text{odd}}(B)/\text{Im}(d).$$
Theorem (Gorokhovsky-L.)

If \((E, \nabla^E, \omega)\) is a generator of \(\check{K}^0_{stan}(M)\) then

\[
\text{ind}_{an}([E, \nabla^E, \omega]) = [\mathcal{H}, \pi_* A, \omega']
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This gives an almost tautological pushforward of \textit{finite dimensional} cycles in differential \(K\)-theory.

Can one also push forward infinite dimensional cycles? Formally yes, but there are some technical questions.
A Hilbert bundle description of differential $K$-theory

Introduction

Summary of differential $K$-theory

Superconnections on Hilbert bundles

Infinite dimensional cycles

Twisted differential $K$-theory
Twisted $K$-theory

There’s a notion of twisted $K$-theory, where one twists by an element of $H^3(M; \mathbb{Z})$. Can one extend the previous model from differential $K$-theory to twisted differential $K$-theory?
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Using finite dimensional vector bundles, one can only handle torsion elements of $H^3(M; \mathbb{Z})$. To deal with all of $H^3(M; \mathbb{Z})$, one needs to use infinite dimensional vector bundles.
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Can one extend the previous model from differential $K$-theory to twisted differential $K$-theory?
$H^3(\mathcal{M}; \mathbb{Z})$ classifies $U(1)$-gerbes on $M$. We’ll twist by coupling to a gerbe.
Gerbes

$H^3(M; \mathbb{Z})$ classifies $U(1)$-gerbes on $M$. We’ll twist by coupling to a gerbe.

Data for a gerbe:

- An open cover $\{U_\alpha\}$ of $M$.
- A complex line bundle $\mathcal{L}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.
- An isomorphism $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \to \mathcal{L}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$. 
We have line bundles $L_{\alpha\beta}$ on overlaps. A $U(1)$-connection on the gerbe consists of

- A Hermitian metric on $L_{\alpha\beta}$.
- Connective structure: A Hermitian connection $\nabla_{\alpha\beta}$ on $L_{\alpha\beta}$ so
  \[ \mu^*_{\alpha\beta\gamma} \nabla_{\alpha\gamma} = (\nabla_{\alpha\beta} \otimes I) + (I \otimes \nabla_{\beta\gamma}). \]
- Curving: $\kappa_{\alpha} \in \Omega^2(U_\alpha)$ so
  \[ \nabla^2_{\alpha\beta} = \kappa_{\alpha} - \kappa_{\beta}. \]
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Then $H = d\kappa_\alpha$ is a globally defined closed 3-form on $M$, the de Rham representative of the gerbe’s class in $H^3(M; \mathbb{Z})$. 
A twisted Hilbert bundle $\mathcal{H}$ is given by Hilbert bundles $\mathcal{H}_\alpha$ over the $U_\alpha$’s, with isomorphisms $\phi_{\alpha\beta} : \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{H}_\beta$ on $U_\alpha \cap U_\beta$. 

The generators for twisted differential $K$-theory are now triples $(\mathcal{H}, A, \omega)$ as before. Quotienting by the relations, one gets the twisted differential $K$-theory group.
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A superconnection on $\mathcal{H}$ is given by superconnections $A_\alpha$ on the $\mathcal{H}_\alpha$’s so $\phi_{\alpha\beta}^* A_\beta = (A_\alpha \otimes I) + (I \otimes \nabla_{\alpha\beta})$ on $U_\alpha \cap U_\beta$. 

Put $\text{ch}(A) = \text{Tr} \left( - \left( A_\alpha^2 + \kappa_\alpha \right) \right) \in \Omega_{\text{even}}(M)$.

Then $(d + H \wedge) \text{ch}(A) = 0$.

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The generators for twisted differential $K$-theory are now triples $(\mathcal{H}, A, \omega)$ as before. Quotienting by the relations, one gets the twisted differential $K$-theory group.
Theorem (Gorokhovsky-L.): Up to isomorphism, the twisted differential $K$-group only depends on the gerbe through its isomorphism class. It is independent of the choices of connective structure and curving.
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This gives an explicit model for twisted differential $K$-theory. It remains to show that it agrees with other models (Bunke-Nikolaus).
Happy Birthday!