A Hilbert bundle description of differential K-theory

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Introduction

Summary of differential K-theory

Superconnections on Hilbert bundles

Infinite dimensional cycles

Twisted differential K-theory

Attribution

Joint work with Alexander Gorokhovsky

Slogans

Differential K-theory is a "K-theory of finite dimensional vector bundles with connections".

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It is closely linked to local index theory, as will be described.

Today: Differential K-theory as a "K-theory of infinite dimensional vector bundles with (super)connections".

Some motivation:

1. It unifies various earlier models for differential K-theory.
2. The analytic index becomes almost tautological.
3. The even and odd cases can be treated similarly.
4. Extension to twisting by H^3.

Structure of the talk

- Summary of differential K-theory
- Superconnections on Hilbert bundles
- Infinite dimensional cycles
- Twisted differential K-theory

Introduction

Summary of differential K-theory

Superconnections on Hilbert bundles

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K-theory

M is a smooth manifold.

$K^0(M)$ is the free abelian group generated by isomorphism classes of finite dimensional complex vector bundles on M, quotiented by the relations $[E_2] = [E_1] + [E_3]$ if there is a short exact sequence

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0.$$

Generators of differential K-theory

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A generator for $\check{K}^0(M)$ is a quadruple $\mathcal{E} = (E, h^E, \nabla^E, \omega)$, where

- E is a finite dimensional complex vector bundle on M.
- h^E is a Hermitian metric on E.
- ∇^E is a Hermitian connection on E.
- $\omega \in \Omega^{odd}(M)/\operatorname{Im}(d)$.

Relations for $\check{K}^0(M)$

Given three such quadruples, we impose the relation

$$\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$$

if there is a short exact sequence of Hermitian vector bundles

$$0 \longrightarrow E_1 \longrightarrow E_2 \longrightarrow E_3 \longrightarrow 0,$$

and

$$\omega_2 = \omega_1 + \omega_3 - CS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) \in \Omega^{odd}(M)/\operatorname{Im}(d).$$

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$$\omega_2 = \omega_1 + \omega_3 - CS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) \in \Omega^{odd}(M)/\operatorname{Im}(d).$$

Here the Chern-Simons form CS satisfies

$$dCS\left(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}\right) = \operatorname{ch}\left(\nabla^{E_2}\right) - \operatorname{ch}\left(\nabla^{E_1}\right) - \operatorname{ch}\left(\nabla^{E_3}\right).$$

Exact sequences

Quotienting by the relations defines $\check{K}^0(M)$. There are a forgetful map

$$f : \check{K}^0(M) \to K^0(M),$$

and a Chern character map

$$\mathrm{Ch} : \check{K}^0(M) \to \Omega_K^{even}(M)$$

coming from

$$\mathrm{Ch}(E, h^E, \nabla^E, \omega) = \mathrm{ch}(\nabla^E) + d\omega.$$

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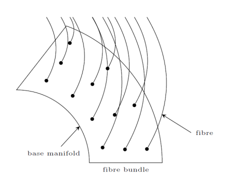
$$0 \longrightarrow K^{-1}(M; \mathbb{R}/\mathbb{Z}) \longrightarrow \check{K}^0(M) \xrightarrow{\mathrm{Ch}} \Omega_K^{even}(M) \longrightarrow 0$$

$$0 \longrightarrow \frac{\Omega^{odd}(M)}{\Omega_K^{odd}(M)} \longrightarrow \check{K}^0(M) \xrightarrow{f} K^0(M) \longrightarrow 0$$

Atiyah-Singer families index theorem



Suppose that $\pi : M \to B$ is a fiber bundle.

Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin*^{c}.

Geometric assumptions: Riemannian metrics on the fibers, Hermitian connection on the associated *spin*^{c} line bundle.

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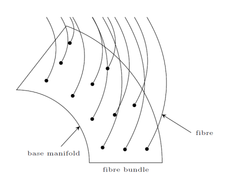

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There are index maps

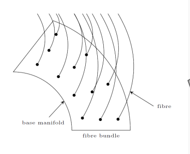
$$\mathrm{ind}_{an}, \mathrm{ind}_{top} : K^0(M) \to K^0(B).$$

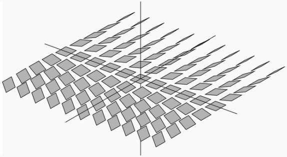
Atiyah-Singer families index theorem

$$\mathrm{ind}_{an} = \mathrm{ind}_{top}\,.$$

Index theorem in differential K-theory

Suppose in addition that there is a horizontal distribution on the fiber bundle.





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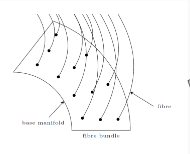

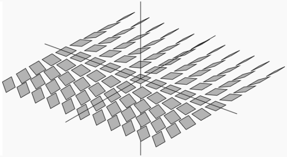

(Freed-L.) There are index maps

$$\mathrm{ind}_{an}, \mathrm{ind}_{top} : \check{K}^0(M) \to \check{K}^0(B).$$

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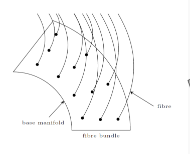

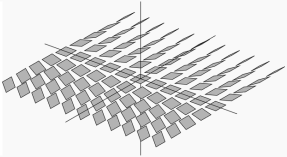

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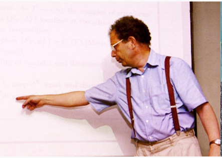
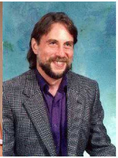
Their construction uses local index theory methods.

Index theorem in differential K-theory

Theorem
(Freed-L.)

$$\mathrm{ind}_{an} = \mathrm{ind}_{top}$$

as maps from $\check{K}^0(M)$ *to* $\check{K}^0(B)$.

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as maps from $\check{K}^0(M)$ to $\check{K}^0(B)$.

Applying f, one recovers the Atiyah-Singer families index theorem. Applying Ch, one recovers Bismut's local version of the families index theorem.

Consequences

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The index theorem in differential K-theory packages many of the results of local index theory into a semitopological setting. Some consequences:

- $\mathbb{R}/\mathbb{Z}$-index theorem
- Computation of $\mathbb{R}/\mathbb{Z}$-valued eta invariants.
- Computation of the determinant line bundle, along with its Quillen metric and compatible connection (up to isomorphism).

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Motivation

From the viewpoint of analytic index theory, it is more natural to use infinite dimensional vector bundles.

Unbounded Kasparov KK-theory: $K^0(M) \cong KK^0(\mathbb{C}, C(M))$, the latter being given in terms of unbounded Fredholm operators on $\mathbb{Z}_2$-graded Hilbert $C(M)$-modules. Can we give a model for differential K-theory along these lines?

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If E is a finite dimensional $\mathbb{Z}_2$-graded vector bundle then

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Problem: This doesn't make sense if E is infinite dimensional.

Solution: Replace the connection ∇ by a superconnection.

Superconnections

E is a finite dimensional $\mathbb{Z}_2$-graded vector bundle on M.

(Quillen) A superconnection on E is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathrm{End}_{odd}(E))$
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In the previous description of $\check{K}^0(M)$, you can replace connections by superconnections.

Hilbert bundles

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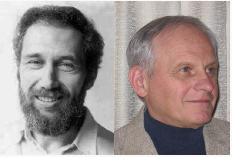
Say that H is a fiber of the bundle. $U(H)$ is too big. We will restrict this using a pseudodifferential calculus based on a "Dirac operator" D.

Abstract Sobolev spaces



Say H is a $\mathbb{Z}_2$-graded Hilbert space,

$D = \begin{pmatrix} 0 & \partial_+^* \\ \partial_+ & 0 \end{pmatrix}$ is a self-adjoint operator.

Assume that $\operatorname{Tr} e^{-\theta D^2} < \infty$ for all $\theta > 0$.

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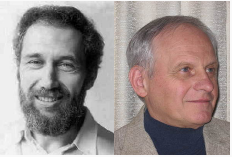

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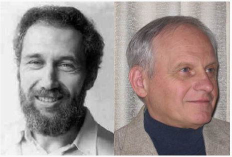

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For $s \in \mathbb{Z}^{<0}$, put $H^s = (H^{-s})^*$.

Put $H^\infty = \bigcap_{s \geq 0} H^s$.

Abstract pseudodifferential operators

Definition

op^k consists of the closed operators F on H so that $F(H^\infty) \subset H^\infty$ and for all $s \in \mathbb{Z}$, F extends to a bounded operator from H^s to H^{s-k}.

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The space of "Dirac-type operators":

$$\mathcal{P} = \left\{ \begin{pmatrix} 0 & P_+^* \\ P_+ & 0 \end{pmatrix} \in op^1 : \frac{1}{\sqrt{P^2+1}} \in op^{-1} \right\}.$$

Clearly $D \in \mathcal{P}$.

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Clearly $D \in \mathcal{P}$.

Lemma

$\mathcal{P}$ is closed under order-zero perturbations.

The structure group

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A map $\mathbb{R}^k \to G$ is declared to be "smooth" if it preserves the smooth maps $\mathbb{R}^k \to H^s$ and $\mathbb{R}^k \to op^k$.

Here H^s and op^k have Fréchet topologies.

Superconnection

Suppose that $\mathcal{H} \to M$ is a $\mathbb{Z}_2$-graded Hilbert bundle with structure group G. It now makes sense to say that a superconnection on $\mathcal{H}$ is a sum

$$A = A_{[0]} + A_{[1]} + A_{[2]} + \dots,$$

where

- $A_{[0]} \in \Omega^0(M; \mathcal{P})$
- $A_{[1]} = d + A_\alpha$ locally, with $A_\alpha \in \Omega^1(U_\alpha; op^{k_1})$
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Then

$$\operatorname{ch}(A) = \operatorname{Tr}_s e^{-A^2} \in \Omega^{even}(M)$$

makes sense, using a Duhamel expansion of e^{-A^2}.

Relative Chern character

If $A_{[0]} - A'_{[0]} \in \Omega^0(M; op^0)$, put

$$\eta(A, A') = \int_0^1 \mathrm{Tr}_s \left(\frac{dB}{dt} e^{-B^2(t)} \right) dt,$$

where $B(t) = (1 - t)A + tA'$.

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Suppose that $A_{[0]}$ is fiberwise invertible. Put

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Generators for $\check{K}^0(M)$

Generators are triples $(\mathcal{H}, A, \omega)$, where

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- A is a superconnection on $\mathcal{H}$.
- $\omega \in \Omega^{odd}(M)/\operatorname{Im}(d)$.

Relations for $\check{K}^0(M)$

1. $$[\mathcal{H}, A, \omega] + [\mathcal{H}', A', \omega'] = [\mathcal{H} \oplus \mathcal{H}', A \oplus A', \omega + \omega']$$

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Theorem (Gorokhovsky-L.) The natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism, where $\check{K}^0_{stan}(M)$ is the "standard" differential K-group defined using finite dimensional vector bundles and connections.

Comparison map

The inverse map $q : \check{K}^0(M) \to \check{K}^0_{stan}(M)$ in a special case:

Suppose that $\mathrm{Ker}(A_{[0]})$ forms a $\mathbb{Z}_2$-graded finite dimensional vector bundle on M.

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Then

$$q(\mathcal{H}, A, \omega) = \left[\mathrm{Ker}(A_{[0]}), QA_{[1]}Q, \omega + \eta(A, B) + \eta((I - Q)A(I - Q), \infty)\right],$$

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where $B = (I - Q)A(I - Q) + QA_{[1]}Q$.

Unification

The Hilbert bundle version $\check{K}^0(M)$ of differential K-theory unifies some other models. First, the natural map $\check{K}^0_{stan}(M) \to \check{K}^0(M)$ is an isomorphism.

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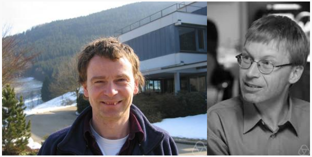

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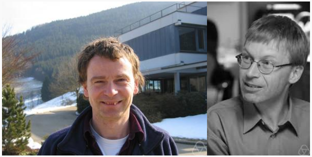

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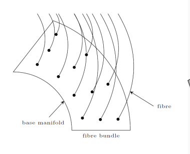
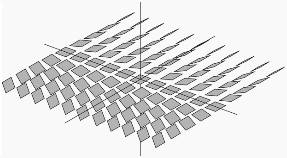
On the other hand, there are no obvious comparison maps with the Hopkins-Singer model for differential K-theory.

Pushforward





Suppose that $\pi : M \to B$ is a fiber bundle.

Topological assumptions: The fibers are compact and even dimensional. The fiberwise tangent bundle is *spin^c*.

Geometric assumptions: Riemannian metrics on the fibers, Hermitian connection on the associated *spin^c* line bundle, horizontal distribution

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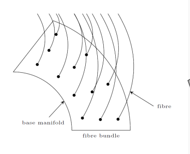

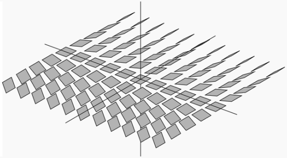

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There was an analytic index map (Freed-L.)

$$\mathrm{ind}_{an} : \check{K}^0_{stan}(M) \to \check{K}^0_{stan}(B).$$

An easier description

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$$\begin{aligned} C^\infty(B, \mathcal{H}^\infty) &= C^\infty(M; E \otimes S^V M) \\ &= C^\infty(M; E) \otimes_{C^\infty(M)} C^\infty(M; S^V M). \end{aligned}$$

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$$\omega' = \int_{M/B} \mathrm{Td}\left(\nabla^{T^V M}\right) \wedge \omega + \lim_{u \to 0} \eta((\pi_* A)_u, \pi_* A) \in \Omega^{odd}(B) / \operatorname{Im}(d).$$

Pushforward theorem

Theorem (Gorokhovsky-L.)

If (E, ∇^E, ω) is a generator of $\check{K}^0_{stan}(M)$ then

$$\mathrm{ind}_{an}([E, \nabla^E, \omega]) = [\mathcal{H}, \pi_* A, \omega']$$

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Can one also push forward infinite dimensional cycles? Formally yes, but there are some technical questions.

Introduction

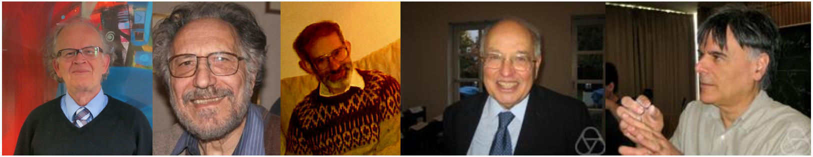
Summary of differential K-theory

Superconnections on Hilbert bundles

Infinite dimensional cycles

Twisted differential K-theory

Twisted K-theory



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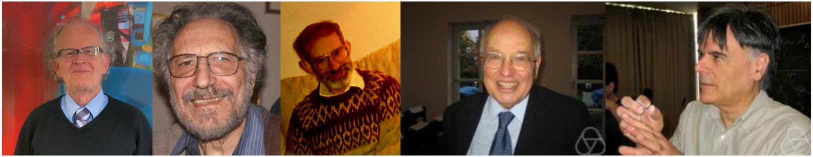
Twisted K-theory



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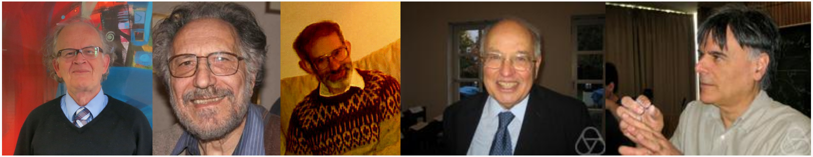

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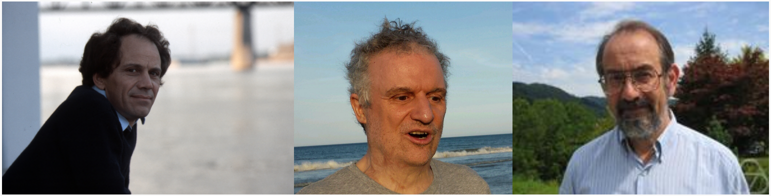
Can one extend the previous model from differential K-theory to twisted differential K-theory?

Gerbes



$H^3(M; \mathbb{Z})$ classifies $U(1)$-gerbes on M. We'll twist by coupling to a gerbe.

Gerbes



$\mathrm{H}^3(M;\mathbb{Z})$ classifies $U(1)$-gerbes on M. We'll twist by coupling to a gerbe.

Data for a gerbe:

- An open cover $\{U_\alpha\}$ of M.
- A complex line bundle $\mathcal{L}_{\alpha\beta}$ on $U_\alpha \cap U_\beta$.
- An isomophism $\mu_{\alpha\beta\gamma} : \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \to \mathcal{L}_{\alpha\gamma}$ on $U_\alpha \cap U_\beta \cap U_\gamma$.

$U(1)$-connection on a gerbe

We have line bundles $\mathcal{L}_{\alpha\beta}$ on overlaps. A $U(1)$-connection on the gerbe consists of

- A Hermitian metric on $\mathcal{L}_{\alpha\beta}$.
- Connective structure: A Hermitan connection $\nabla_{\alpha\beta}$ on $\mathcal{L}_{\alpha\beta}$ so
$$\mu^*_{\alpha\beta\gamma}\nabla_{\alpha\gamma} = (\nabla_{\alpha\beta} \otimes I) + (I \otimes \nabla_{\beta\gamma}).$$
- Curving: $\kappa_\alpha \in \Omega^2(U_\alpha)$ so
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Then $H = d\kappa_\alpha$ is a globally defined closed 3-form on M, the de Rham representative of the gerbe's class in $\mathrm{H}^3(M; \mathbb{Z})$.

Twisted differential K-theory

A twisted Hilbert bundle $\mathcal{H}$ is given by Hilbert bundles $\mathcal{H}_\alpha$ over the U_α's, with isomorphisms $\phi_{\alpha\beta} : \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \to \mathcal{H}_\beta$ on $U_\alpha \cap U_\beta$.

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A superconnection on $\mathcal{H}$ is given by superconnections A_α on the $\mathcal{H}_\alpha$'s so $\phi^*_{\alpha\beta} A_\beta = (A_\alpha \otimes I) + (I \otimes \nabla_{\alpha\beta})$ on $U_\alpha \cap U_\beta$.

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Put

$$\mathrm{ch}(A) = \mathrm{Tr}_s\, e^{-(A_\alpha^2 + \kappa_\alpha)} \in \Omega^{even}(M).$$

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The generators for twisted differential K-theory are now triples $(\mathcal{H}, A, \omega)$ as before. Quotienting by the relations, one gets the twisted differential K-theory group.

Model for twisted differential K-theory

Theorem (Gorokhovsky-L.): Up to isomorphism, the twisted differential K-group only depends on the gerbe through its isomorphism class. It is independent of the choices of connective structure and curving.

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This gives an explicit model for twisted differential K-theory. It remains to show that it agrees with other models (Bunke-Nikolaus).

Happy Birthday!