The transverse index problem for Riemannian foliations

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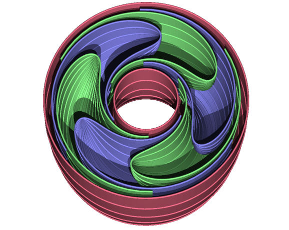
Attribution

Joint work with Alexander Gorokhovsky

Foliation index theory



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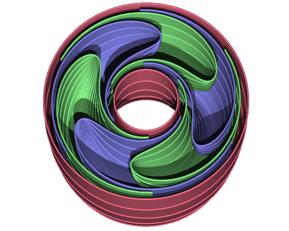

Let $(M, \mathcal{F})$ be a compact foliated manifold.

There are two types of foliated index theories,

1. Longitudinal index theory, and
2. Transverse index theory.

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More precisely, $\mathrm{Index}(D) \in K_*(C_r^*(M;\mathcal{F}))$, where $C_r^*(M;\mathcal{F})$ is the reduced C^*-algebra of the foliation.

The longitudinal index theorem gives a topological equivalent of $\mathrm{Index}(D)$.

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In general, we would like a formula for $\text{Index}(D)$ in terms of the local geometry of the foliated manifold.

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where

1. $\{\Sigma_i\}$ are the strata of the orbifold,
2. m_i is the multiplicity of Σ_i and
3. The characteristic class $\mathcal{N}_i$ is computed from the normal data of Σ_i and the auxiliary vector bundle E.

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3. Usual local index theory methods (McKean-Singer technique) don't work.

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- *$W_{\max}$ is the deepest stratum in the space of leaf closures.*
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2. If $\mathcal{F}$ is transversely oriented then the basic signature of $(M, \mathcal{F})$ equals the signature of W_{max}.

3. If $\mathcal{F}$ has a transverse spin structure, D is the basic Dirac operator and the leaves are noncompact then Index $(D) = 0$.

The transverse index problem for Riemannian foliations

What is a foliation?

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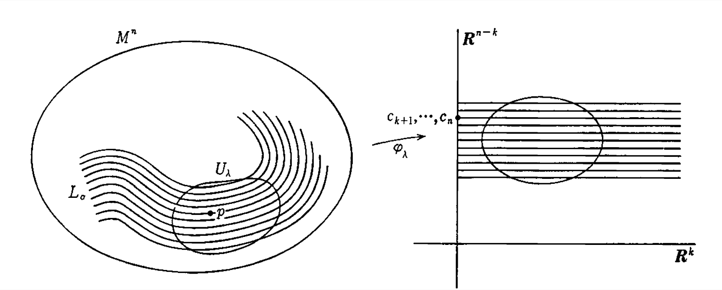
A k-dimensional foliation of a manifold M is a covering of M by foliation charts

$$\phi_i : U_i \to \mathbb{R}^k \times \mathbb{R}^{n-k}$$



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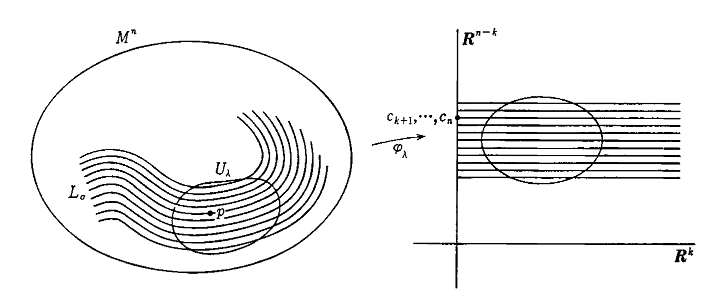

whose transition maps are of the form

$$\phi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$$

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A Riemannian foliation is a foliation $(M, \mathcal{F})$ along with an inner product on $N\mathcal{F}$ that locally pulls back, under the submersion $U \to \mathbb{R}^{n-k}$, from a Riemannian metric on $\mathbb{R}^{n-k}$.

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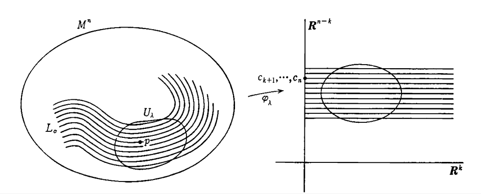
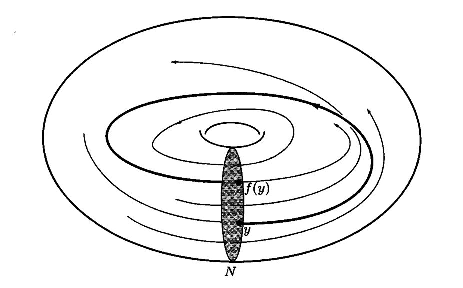


Complete transversal



A complete transversal $\mathcal{T}$ is a (possibly disconnected) submanifold, with $\dim(\mathcal{T}) = \text{codim}(\mathcal{F})$, that is transverse to $\mathcal{F}$ and hits every leaf of $(M, \mathcal{F})$. It always exists.

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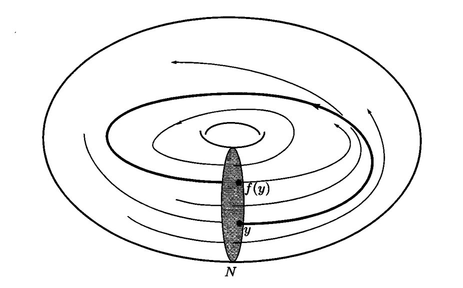

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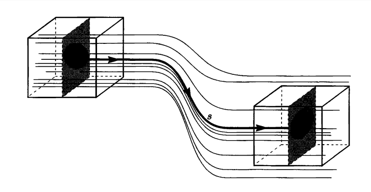
$\mathcal{T}$ acquires a Riemannian metric, pulled back from the local diffeomorphisms $\mathcal{T} \cap U \to \mathbb{R}^{n-k}$.

Holonomy



Start at a point $p \in \mathcal{T}$. Slide along a path in the leaf through p, until you hit a point $q \in \mathcal{T}$. This gives a germ of a diffeomorphism sending $p \in \mathcal{T}$ to $q \in \mathcal{T}$, the holonomy element.

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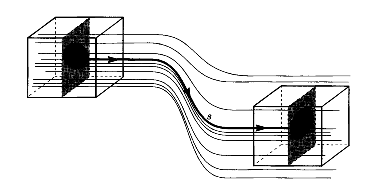

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"Basic" means holonomy-invariant on $\mathcal{T}$. For example, a Riemannian foliation has a basic Riemannian metric.

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Basic operators

If $(M, \mathcal{F})$ is a Riemannian foliation, it makes sense to talk about a basic $\mathbb{Z}_2$-graded Clifford module $\mathcal{E}$ on $\mathcal{T}$ and a basic Dirac-type operator D, acting on the basic sections of $\mathcal{E}$.

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One approach

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One can lift D to an operator on $F_{\mathrm{Spin}(q)}M$, which then descends to a Spin(q)-transversally elliptic operator $\widehat{D}$ on $\widehat{W}$.

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Index theorem for an operator which is transversally elliptic with respect to a compact Lie group action (Atiyah, Berline-Vergne, Vergne, Paradan, . . .)

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Approach of Brüning-Kamber-Richardson : do equivariant modifications of $\widehat{W}$ to simplify the isotropy group structure. Keep track of how the index changes under the modifications. Get a semilocal formula for Index(D).

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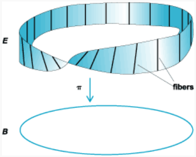
Put $M = (\widetilde{Y} \times Z)/\Gamma$, a fiber bundle over Y.



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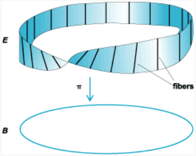

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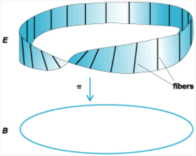

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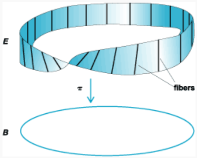

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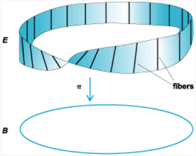

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A transverse Dirac-type operator on $(M, \mathcal{F})$ amounts to a Γ-invariant Dirac-type operator D on Z,

or, equivalently, a G-invariant Dirac-type operator D on Z.

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Problem : $\int_{T^k} \mathcal{L}(g,z)\, d\mu_{T^k}(g)$ generally diverges!

Index computation

Suppose that $G = T^k$.

$$\mathrm{Index}\,(D_{\mathrm{inv}}) = \int_{T^k} \mathrm{Index}(g)\, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left(\int_{Z^g} \mathcal{L}(g) \right) \, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left(\int_{Z^{T^k}} \mathcal{L}(g,z)\, dz \right) \, d\mu_{T^k}(g)$$

$$\stackrel{?}{=} \int_{Z^{T^k}} \left(\int_{T^k} \mathcal{L}(g,z)\, d\mu_{T^k}(g) \right) \, dz$$

Problem : $\int_{T^k} \mathcal{L}(g,z)\, d\mu_{T^k}(g)$ generally diverges!

Get cancellations from various components of Z^{T^k}.

Divergences

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But what to do for more general Riemannian foliations, which may not reduce to compact Lie group actions?

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A Riemannian foliation may not come from a Lie group action, but there is always a local action by a Lie algebra $\mathfrak{g}$.

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Idea : prove a Kirillov-type delocalized index formula, in terms of $X \in \mathfrak{g}$.

If $\mathfrak{g}$ is abelian, we can replace the nonexistent "integral over G" by an averaging over $X \in \mathfrak{g}$.

The transverse index problem for Riemannian foliations

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The transverse index problem becomes a question about the invariant index of an operator on a Riemannian groupoid.

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Local structure of a Riemannian groupoid (Haefliger, Molino).

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Let $\phi \in C_c^\infty(\mathcal{T})$ be a cutoff function so that

$$\int_{\overline{\mathcal{G}}^p} \phi^2(s(g))\, d\mu^p(g) \;=\; 1.$$

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Hence $P = \alpha \circ \beta$ is an orthogonal projection on $L^2(\mathcal{T};\mathcal{E})$, with image isomorphic to $L^2(\mathcal{T};\mathcal{E})^{\overline{\mathcal{G}}}$. Get inner product on $L^2(\mathcal{T};\mathcal{E})^{\overline{\mathcal{G}}}$.

Spectral triple

Let D_0 be a $\mathcal{G}$-invariant Dirac-type operator on $L^2(\mathcal{T};\mathcal{E})$. (Give it APS boundary conditions).

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Put $D_{\text{inv}} = \beta \circ D_0 \circ \alpha$, an operator on $L^2(\mathcal{T};\mathcal{E})^{\overline{\mathcal{G}}}$. It is unitarily equivalent to $P \circ D_0 \circ P$ on $\mathrm{Im}(P)$.

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Corollary

D_{inv} is Fredholm. In fact, $e^{-tD_{\mathrm{inv}}^2}$ is trace-class for all $t > 0$.

The transverse index problem for Riemannian foliations

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Put $W_{\max} = \{w \in W : K_w \simeq T^k\}$.

Index theorem

Theorem

Under the assumptions of the previous slide, let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded basic Clifford module on $\mathcal{T}$. Let D_{inv} be the basic Dirac-type operator on $L^2(\mathcal{T};\mathcal{E})^{\overline{\mathcal{G}}}$. Then

$$\mathrm{Index}(D_{\mathrm{inv}}) = \int_{W_{\max}} \widehat{A}\,(TW_{\max})\;\mathcal{N}_{\mathcal{E}}.$$

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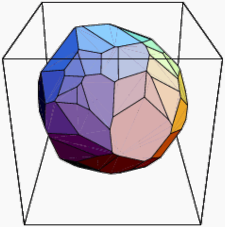
Here $\mathcal{N}_{\mathcal{E}}$ is a characteristic class on W_{max}, which is computed from the normal data of W_{max}, along with $\mathcal{E}$.

The transverse index problem for Riemannian foliations

Step 1



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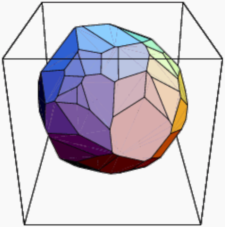

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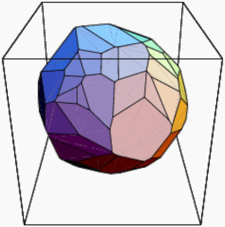

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- $B(V_\alpha)$ extends to a closed Riemannian manifold Y_α with a T^{k_α}-action.

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Step 2

Fix a Haar measure $d\mu_{\mathfrak{g}}$ on $\mathfrak{g} \simeq \mathbb{R}^k$. If $F \in C^\infty(\mathbb{R}^k)$ is a finite sum of periodic functions, put

$$AV_X F = \lim_{R \to \infty} \frac{\int_{B(0,R)} F(X)\, d\mu_{\mathfrak{g}}(X)}{\int_{B(0,R)} 1\, d\mu_{\mathfrak{g}}(X)}.$$

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Fix a Haar measure $d\mu_{\mathfrak{g}}$ on $\mathfrak{g} \simeq \mathbb{R}^k$. If $F \in C^\infty(\mathbb{R}^k)$ is a finite sum of periodic functions, put

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Proposition

$$\mathrm{Index}(D_{\mathrm{inv}}) = AV_X \left(\sum_\alpha \mathrm{Tr}_s \left(e^{-(tD_\alpha^2 + \mathcal{L}_X)} \eta_\alpha \right) \right.$$

$$\left. + \frac{1}{2} \sum_\alpha \int_0^t \mathrm{Tr}_s \left(e^{-(sD_\alpha^2 + \mathcal{L}_X)} D_\alpha [D_\alpha, \eta_\alpha]\, ds \right) \right).$$

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For $t > 0$ and $\epsilon \in \mathbb{C}$, put

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Proposition

$F_{t,2}(X)$ has a holomorphic extension to $X \in \mathbb{C}^k$.

Step 4

Proposition

If $X \in \mathbb{R}^k$ *then*

$$\lim_{t \to 0} F_{t,1}(iX) = \sum_{\alpha} \int_{Y_\alpha} \widehat{A}(iX, Y_\alpha)\ \mathrm{ch}(iX, \mathcal{E}_\alpha / S)\, \eta_\alpha .$$

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Proposition

For any $Q \in \mathbb{C}^k$,

$$AV_X \sum_\alpha \int_{Y_\alpha} \widehat{A}(X, Y_\alpha)\ \mathrm{ch}(X, \mathcal{E}_\alpha/S)\, \eta_\alpha =$$

$$AV_X \int_{W_{\max}} \widehat{A}(TW_{\max}) \frac{\mathrm{ch}_{\mathcal{W}}(e^{-X+Q})}{\mathrm{ch}_{S_N}(e^{-X+Q})}.$$

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For generic $Q \in \mathbb{C}^k$, define $\mathcal{N}_{\mathcal{E},Q} \in \Omega^*(W_{\max})$ by

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Corollary

$$\mathrm{Index}\,(D_{\mathrm{inv}}) = \int_{W_{\max}} \widehat{A}(TW_{\max})\,\mathcal{N}_{\mathcal{E},Q}$$