The transverse index problem for Riemannian foliations

John Lott
UC-Berkeley
http://math.berkeley.edu/~lott

May 27, 2013
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
Joint work with Alexander Gorokhovsky
Foliation index theory

Let \((M, \mathcal{F})\) be a compact foliated manifold.
Let $(M, \mathcal{F})$ be a compact foliated manifold.

There are two types of foliated index theories,

1. Longitudinal index theory, and
2. Transverse index theory.
Longitudinal index theorem (Connes-Skandalis)
What I won’t be talking about

Longitudinal index theorem (Connes-Skandalis)

Suppose that there is a differential operator $D$ on $M$ that is Dirac-type, except that it only differentiates in the leaf directions.
Longitudinal index theorem (Connes-Skandalis)

Suppose that there is a differential operator $D$ on $M$ that is Dirac-type, except that it only differentiates in the leaf directions.

We can think $D$ as a “family” of leaf-wise operators, parametrized by the “leaf space” of the foliation.
Longitudinal index theorem (Connes-Skandalis)

Suppose that there is a differential operator $D$ on $M$ that is Dirac-type, except that it only differentiates in the leaf directions.

We can think $D$ as a “family” of leaf-wise operators, parametrized by the “leaf space” of the foliation.

One can define its “families index” in the “K-theory of the leaf space of the foliation”.
Longitudinal index theorem (Connes-Skandalis)

Suppose that there is a differential operator $D$ on $M$ that is Dirac-type, except that it only differentiates in the leaf directions.

We can think $D$ as a “family” of leaf-wise operators, parametrized by the “leaf space” of the foliation.

One can define its “families index” in the “K-theory of the leaf space of the foliation”.

More precisely, $\text{Index}(D) \in K_*(C^*_r(M; \mathcal{F}))$, where $C^*_r(M; \mathcal{F})$ is the reduced $C^*$-algebra of the foliation.
What I won’t be talking about

Longitudinal index theorem (Connes-Skandalis)

Suppose that there is a differential operator $D$ on $M$ that is Dirac-type, except that it only differentiates in the leaf directions.

We can think $D$ as a “family” of leaf-wise operators, parametrized by the “leaf space” of the foliation.

One can define its “families index” in the “K-theory of the leaf space of the foliation”.

More precisely, $\text{Index}(D) \in K_*(C^*_r(M; \mathcal{F}))$, where $C^*_r(M; \mathcal{F})$ is the reduced $C^*$-algebra of the foliation.

The longitudinal index theorem gives a topological equivalent of $\text{Index}(D)$. 
Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the \textit{transverse} direction to the foliation, and it’s invariant under sliding along leaves.
Transverse index problem

Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the transverse direction to the foliation, and it’s invariant under sliding along leaves.

Then $D$ is like an operator on the leaf space.
Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the \textit{transverse} direction to the foliation, and it’s invariant under sliding along leaves.

Then $D$ is like an operator \textit{on} the leaf space.

To make sense of $D$, we must assume that $(M, \mathcal{F})$ has a transverse Riemannian metric.
Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the transverse direction to the foliation, and it’s invariant under sliding along leaves.

Then $D$ is like an operator on the leaf space.

To make sense of $D$, we must assume that $(M, \mathcal{F})$ has a transverse Riemannian metric.

**Fact:** (El-Kacimi, Glazebrook-Kamber) $D$ is Fredholm.
Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the transverse direction to the foliation, and it’s invariant under sliding along leaves.

Then $D$ is like an operator on the leaf space.

To make sense of $D$, we must assume that $(M, \mathcal{F})$ has a transverse Riemannian metric.

**Fact:** (El-Kacimi, Glazebrook-Kamber) $D$ is Fredholm.

Hence $\text{Index}(D) \in \mathbb{Z}$ is well-defined.
Suppose instead that $D$ is a Dirac-type operator, except that it only differentiates in the transverse direction to the foliation, and it’s invariant under sliding along leaves.

Then $D$ is like an operator on the leaf space.

To make sense of $D$, we must assume that $(M, \mathcal{F})$ has a transverse Riemannian metric.

Fact : (El-Kacimi, Glazebrook-Kamber) $D$ is Fredholm.

Hence $\text{Index}(D) \in \mathbb{Z}$ is well-defined.

What is it?
Transverse index problem

Index($D$) $\in \mathbb{Z}$. We want a local formula for it.
Index($D$) $\in \mathbb{Z}$. We want a local formula for it.

Prototypical example: the Atiyah-Singer index theorem.
Index($D$) $\in \mathbb{Z}$. We want a local formula for it.

Prototypical example: the Atiyah-Singer index theorem.

Suppose that $M$ is foliated by points.
Index($D$) $\in \mathbb{Z}$. We want a local formula for it.

Prototypical example: the Atiyah-Singer index theorem.

Suppose that $M$ is foliated by points. Then $D$ is just a Dirac-type operator on $M$, and

\[
\text{Index}(D) = \int_M \hat{A}(TM) \text{ch}(E).
\]
Index($D$) $\in \mathbb{Z}$. We want a local formula for it.

**Prototypical example:** the Atiyah-Singer index theorem.

Suppose that $M$ is foliated by points. Then $D$ is just a Dirac-type operator on $M$, and

$$\text{Index}(D) = \int_M \hat{A}(TM) \text{ch}(E).$$

In general, we would like a formula for Index($D$) in terms of the local geometry of the foliated manifold.
Another known case

Suppose that the leaves of \((M, \mathcal{F})\) are all compact.
Another known case

Suppose that the leaves of \((M, \mathcal{F})\) are all compact.

Then the leaf space of \((M, \mathcal{F})\) is an orbifold. The operator \(D\) becomes an orbifold Dirac-type operator.

Orbifold index theorem (Kawasaki 1981):

\[
\text{Index}(D) = \sum_i \int_{\Sigma_i} m_i \hat{A}(T\Sigma_i) N_i,
\]

1. \(\{\Sigma_i\}\) are the strata of the orbifold,
2. \(m_i\) is the multiplicity of \(\Sigma_i\) and
3. The characteristic class \(N_i\) is computed from the normal data of \(\Sigma_i\) and the auxiliary vector bundle \(E\).
Another known case

Suppose that the leaves of \((M, \mathcal{F})\) are all compact.

Then the leaf space of \((M, \mathcal{F})\) is an orbifold. The operator \(D\) becomes an orbifold Dirac-type operator.

Orbifold index theorem (Kawasaki 1981):

\[
\text{Index}(D) = \sum_i \int_{\Sigma_i} \frac{1}{m_i} \hat{A}(T\Sigma_i) \hat{N}_i,
\]
Another known case

Suppose that the leaves of \((M, F)\) are all compact.

Then the leaf space of \((M, F)\) is an orbifold. The operator \(D\) becomes an orbifold Dirac-type operator.

Orbifold index theorem (Kawasaki 1981):

\[
\text{Index}(D) = \sum \int_{\Sigma_i} \frac{1}{m_i} \hat{A}(T\Sigma_i) \mathcal{N}_i,
\]

where

1. \(\{\Sigma_i\}\) are the strata of the orbifold,
2. \(m_i\) is the multiplicity of \(\Sigma_i\) and
3. The characteristic class \(\mathcal{N}_i\) is computed from the normal data of \(\Sigma_i\) and the auxiliary vector bundle \(E\).
Why is this problem interesting?

1. It has been open for twenty years.
2. It leads to questions about analysis on Riemannian groupoids.
3. Usual local index theory methods (McKean-Singer technique) don't work.
Why is this problem interesting?

1. It has been open for twenty years.
1. It has been open for twenty years.

2. It leads to questions about analysis on Riemannian groupoids.
Why is this problem interesting?

1. It has been open for twenty years.

2. It leads to questions about analysis on Riemannian groupoids.

3. Usual local index theory methods (McKean-Singer technique) don’t work.
Main result

Theorem

Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Then

$$\text{Index} \left( D \right) = \int_{W_{\text{max}}} \hat{A} \left( T W_{\text{max}} \right) N_{E}.$$ 

Here $W_{\text{max}}$ is the deepest stratum in the space of leaf closures, and $N_{E}$ is a "renormalized" characteristic class, which is computed from the normal data of $W_{\text{max}}$ along with $E$. 

Main result

Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $E$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $E$. Then $\text{Index}(D) = \int_{\tilde{W}^\text{max}} \hat{A}(T\tilde{W}^\text{max}) N_E$. Here $\tilde{W}^\text{max}$ is the deepest stratum in the space of leaf closures. $N_E$ is a "renormalized" characteristic class, which is computed from the normal data of $\tilde{W}^\text{max}$, along with $E$. 
Main result

Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $E$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $E$. Then

$$\text{Index}(D) = \int_{W_{\text{max}}} \hat{A}(T W_{\text{max}}) N_E.$$ 

Here $W_{\text{max}}$ is the deepest stratum in the space of leaf closures. $N_E$ is a “renormalized” characteristic class, which is computed from the normal data of $W_{\text{max}}$, along with $E$. 
Main result

Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $\mathcal{E}$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $\mathcal{E}$. Then

$$\text{Index}(D) = \int_{W_{\text{max}}^\hat{A}(TW_{\text{max}})} N_E.$$ 

Here $W_{\text{max}}$ is the deepest stratum in the space of leaf closures. $N_E$ is a "renormalized" characteristic class, which is computed from the normal data of $W_{\text{max}}$, along with $E$. 
Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $\mathcal{E}$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $\mathcal{E}$. Then

$$\text{Index}(D) = \int_{W_{\text{max}}} \hat{A}(TW_{\text{max}}) \ \mathcal{N}_\mathcal{E}.$$
Main result

Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $\mathcal{E}$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $\mathcal{E}$. Then

$$\text{Index} (D) = \int_{W_{\text{max}}} \hat{A}(TW_{\text{max}}) \mathcal{N}_\mathcal{E}.$$

Here

- $W_{\text{max}}$ is the deepest stratum in the space of leaf closures.
Theorem
Let $M$ be a compact connected manifold equipped with a Riemannian foliation $\mathcal{F}$. Suppose that the Molino Lie algebra is abelian and the isotropy groups are connected. Let $\mathcal{E}$ be a holonomy-invariant $\mathbb{Z}_2$-graded normal Clifford module on $M$. Let $D$ be the basic Dirac-type operator acting on holonomy-invariant sections of $\mathcal{E}$. Then

$$\text{Index}(D) = \int_{W_{\text{max}}} \widehat{A}(TW_{\text{max}}) \mathcal{N}_\mathcal{E}.$$ 

Here

- $W_{\text{max}}$ is the deepest stratum in the space of leaf closures.
- $\mathcal{N}_\mathcal{E}$ is a “renormalized” characteristic class, which is computed from the normal data of $W_{\text{max}}$, along with $\mathcal{E}$. 
Corollaries

Under the hypotheses of the preceding theorem,

1. The basic Euler characteristic of \((M, F)\) equals the Euler characteristic of \(W_{\text{max}}\).

2. If \(F\) is transversely oriented then the basic signature of \((M, F)\) equals the signature of \(W_{\text{max}}\).

3. If \(F\) has a transverse spin structure, \(D\) is the basic Dirac operator and the leaves are noncompact then \(\text{Index}(D) = 0\).
Under the hypotheses of the preceding theorem,

1. The basic Euler characteristic of \((M, \mathcal{F})\) equals the Euler characteristic of \(W_{\text{max}}\).
Corollaries

Under the hypotheses of the preceding theorem,

1. The basic Euler characteristic of \((M, \mathcal{F})\) equals the Euler characteristic of \(W_{\text{max}}\).

2. If \(\mathcal{F}\) is transversely oriented then the basic signature of \((M, \mathcal{F})\) equals the signature of \(W_{\text{max}}\).
Corollaries

Under the hypotheses of the preceding theorem,

1. The basic Euler characteristic of \((M, \mathcal{F})\) equals the Euler characteristic of \(W_{\text{max}}\).

2. If \(\mathcal{F}\) is transversely oriented then the basic signature of \((M, \mathcal{F})\) equals the signature of \(W_{\text{max}}\).

3. If \(\mathcal{F}\) has a transverse spin structure, \(D\) is the basic Dirac operator and the leaves are noncompact then \(\text{Index}(D) = 0\).
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
What is a foliation?

A $k$-dimensional foliation of a manifold $M$ is a covering of $M$ by foliation charts $\varphi_i : U_i \to \mathbb{R}^k \times \mathbb{R}^{n-k}$ whose transition maps are of the form $\varphi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y))$. 
A $k$-dimensional foliation of a manifold $M$ is a covering of $M$ by foliation charts

$$\phi_i : U_i \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$
What is a foliation?

A $k$-dimensional foliation of a manifold $M$ is a covering of $M$ by foliation charts

$$\phi_i : U_i \rightarrow \mathbb{R}^k \times \mathbb{R}^{n-k}$$

whose transition maps are of the form

$$\phi_{ij}(x, y) = (g_{ij}(x, y), h_{ij}(y)).$$
What is a Riemannian foliation?

If $(M, F)$ is a foliation, its normal bundle is $N_F = TM/T_F$. A Riemannian foliation is a foliation $(M, F)$ along with an inner product on $N_F$ that locally pulls back, under the submersion $U \to \mathbb{R}^{n-k}$, from a Riemannian metric on $\mathbb{R}^{n-k}$. 
What is a Riemannian foliation?

If \((M, \mathcal{F})\) is a foliation, its normal bundle is

\[ \mathcal{N}\mathcal{F} = TM / T\mathcal{F}. \]
What is a Riemannian foliation?

If \((M, \mathcal{F})\) is a foliation, its normal bundle is

\[
N\mathcal{F} = TM/T\mathcal{F}.
\]

A **Riemannian foliation** is a foliation \((M, \mathcal{F})\) along with an inner product on \(N\mathcal{F}\) that locally pulls back, under the submersion \(U \to \mathbb{R}^{n-k}\), from a Riemannian metric on \(\mathbb{R}^{n-k}\).
What is a Riemannian foliation?

If \((M, \mathcal{F})\) is a foliation, its normal bundle is

\[
N\mathcal{F} = TM / TF.
\]

A **Riemannian foliation** is a foliation \((M, \mathcal{F})\) along with an inner product on \(N\mathcal{F}\) that locally pulls back, under the submersion \(U \rightarrow \mathbb{R}^{n-k}\), from a Riemannian metric on \(\mathbb{R}^{n-k}\).
A complete transversal $\mathcal{T}$ is a (possibly disconnected) submanifold, with $\dim(\mathcal{T}) = \text{codim}(\mathcal{F})$, that is transverse to $\mathcal{F}$ and hits every leaf of $\mathcal{F}$. It always exists.
A complete transversal $\mathcal{T}$ is a (possibly disconnected) submanifold, with $\dim(\mathcal{T}) = \text{codim}(\mathcal{F})$, that is transverse to $\mathcal{F}$ and hits every leaf of $(M, \mathcal{F})$. It always exists.

$\mathcal{T}$ acquires a Riemannian metric, pulled back from the local diffeomorphisms $\mathcal{T} \cap U \to \mathbb{R}^{n-k}$. 
Holonomy

Start at a point \( p \in \mathcal{T} \). Slide along a path in the leaf through \( p \), until you hit a point \( q \in \mathcal{T} \). This gives a germ of a diffeomorphism sending \( p \in \mathcal{T} \) to \( q \in \mathcal{T} \), the holonomy element.
Holonomy

Start at a point \( p \in \mathcal{T} \). Slide along a path in the leaf through \( p \), until you hit a point \( q \in \mathcal{T} \). This gives a germ of a diffeomorphism sending \( p \in \mathcal{T} \) to \( q \in \mathcal{T} \), the holonomy element.

“Basic” means holonomy-invariant on \( \mathcal{T} \). For example, a Riemannian foliation has a basic Riemannian metric.
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
If \((M, \mathcal{F})\) is a Riemannian foliation, it makes sense to talk about a basic \(\mathbb{Z}_2\)-graded Clifford module \(\mathcal{E}\) on \(\mathcal{T}\) and a basic Dirac-type operator \(D\), acting on the basic sections of \(\mathcal{E}\).
If \((M, \mathcal{F})\) is a Riemannian foliation, it makes sense to talk about a basic \(\mathbb{Z}_2\)-graded Clifford module \(\mathcal{E}\) on \(\mathcal{T}\) and a basic Dirac-type operator \(D\), acting on the basic sections of \(\mathcal{E}\).

\[
\text{Index}(D) = \dim \text{Ker}(D_+) - \dim \text{Ker}(D_-).
\]
If \((M, \mathcal{F})\) is a Riemannian foliation, it makes sense to talk about a basic \(\mathbb{Z}_2\)-graded Clifford module \(\mathcal{E}\) on \(\mathcal{T}\) and a basic Dirac-type operator \(D\), acting on the basic sections of \(\mathcal{E}\).

\[
\text{Index}(D) = \dim \ker (D_+) - \dim \ker (D_-).
\]

What is it?
One approach

Suppose (for simplicity) that $\mathcal{T}$ has a basic spin structure. Putting $q = \text{codim}(\mathcal{F})$, there is a principal Spin($q$)-bundle

$$F_{\text{Spin}(q)}M \rightarrow M.$$
One approach

Suppose (for simplicity) that $\mathcal{T}$ has a basic spin structure. Putting $q = \text{codim}(\mathcal{F})$, there is a principal Spin$(q)$-bundle

$$F_{\text{Spin}(q)} M \to M.$$ 

There is a lifted foliation $\widehat{\mathcal{F}}$ of $F_{\text{Spin}(q)} M$, with $\dim(\widehat{\mathcal{F}}) = \dim(\mathcal{F})$. The closures of its leaves form the fibers of a fiber bundle $F_{\text{Spin}(q)} M \to \widehat{\mathcal{W}}$, which is Spin$(q)$-equivariant.
One approach

Suppose (for simplicity) that $\mathcal{T}$ has a basic spin structure. Putting $q = \text{codim}(\mathcal{F})$, there is a principal Spin($q$)-bundle

$$F_{\text{Spin}(q)} M \rightarrow M.$$ 

There is a lifted foliation $\hat{\mathcal{F}}$ of $F_{\text{Spin}(q)} M$, with $\text{dim}(\hat{\mathcal{F}}) = \text{dim}(\mathcal{F})$. The closures of its leaves form the fibers of a fiber bundle $F_{\text{Spin}(q)} M \rightarrow \hat{\mathcal{W}}$, which is Spin($q$)-equivariant.

$$F_{\text{Spin}(q)} M \quad \rightarrow \quad \hat{\mathcal{W}}$$
$$\downarrow \quad \downarrow$$
$$M \quad \rightarrow \quad \mathcal{W}$$
One approach

Suppose (for simplicity) that $\mathcal{T}$ has a basic spin structure. Putting $q = \text{codim}(\mathcal{F})$, there is a principal Spin$(q)$-bundle

$$F_{\text{Spin}(q)}M \rightarrow M.$$ 

There is a lifted foliation $\hat{\mathcal{F}}$ of $F_{\text{Spin}(q)}M$, with $\dim(\hat{\mathcal{F}}) = \dim(\mathcal{F})$. The closures of its leaves form the fibers of a fiber bundle $F_{\text{Spin}(q)}M \rightarrow \hat{W}$, which is Spin$(q)$-equivariant.

$$F_{\text{Spin}(q)}M \rightarrow \hat{W}$$

$$\downarrow \quad \downarrow$$

$$M \rightarrow W$$

One can lift $D$ to an operator on $F_{\text{Spin}(q)}M$, 

$$F_{\text{Spin}(q)}M \rightarrow F_{\text{Spin}(q)}M.$$
Suppose (for simplicity) that $\mathcal{T}$ has a basic spin structure. Putting $q = \text{codim}(\mathcal{F})$, there is a principal Spin($q$)-bundle

$$F_{\text{Spin}(q)} M \rightarrow M.$$ 

There is a lifted foliation $\widehat{\mathcal{F}}$ of $F_{\text{Spin}(q)} M$, with $\dim(\widehat{\mathcal{F}}) = \dim(\mathcal{F})$. The closures of its leaves form the fibers of a fiber bundle $F_{\text{Spin}(q)} M \rightarrow \widehat{\mathcal{W}}$, which is Spin($q$)-equivariant.

$$F_{\text{Spin}(q)} M \rightarrow \widehat{\mathcal{W}}$$

$$M \rightarrow \mathcal{W}$$

One can lift $D$ to an operator on $F_{\text{Spin}(q)} M$, which then descends to a Spin($q$)-transversally elliptic operator $\widehat{D}$ on $\widehat{\mathcal{W}}$. 
One approach

Index theorem for an operator which is transversally elliptic with respect to a compact Lie group action (Atiyah, Berline-Vergne, Vergne, Paradan, ...)

Does not seem to be explicit enough (so far) to compute $\text{Index}(D)$. Approach of Br"{u}ning-Kamber-Richardson: do equivariant modifications of $\hat{W}$ to simplify the isotropy group structure. Keep track of how the index changes under the modifications. Get a semilocal formula for $\text{Index}(D)$. 
One approach

Index theorem for an operator which is transversally elliptic with respect to a compact Lie group action (Atiyah, Berline-Vergne, Vergne, Paradan, ...)

Does not seem to be explicit enough (so far) to compute $\text{Index}(D)$. 
Index theorem for an operator which is transversally elliptic with respect to a compact Lie group action (Atiyah, Berline-Vergne, Vergne, Paradan, . . . )

Does not seem to be explicit enough (so far) to compute $\text{Index}(D)$.

Approach of Brüning-Kamber-Richardson: do equivariant modifications of $\hat{W}$ to simplify the isotropy group structure. Keep track of how the index changes under the modifications. Get a semilocal formula for $\text{Index}(D)$. 
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
Suppose that we are given

1. A discrete group $\Gamma$,
Suppose that we are given

1. A discrete group $\Gamma$,
2. A compact Lie group $G$,
3. An injective homomorphism $\Gamma \to G$ with dense image,
4. A closed manifold $Y$ with fundamental group $\Gamma$,
5. A closed Riemannian manifold $Z$ on which $G$ acts isometrically.

Put $\tilde{M} = (\tilde{Y} \times Z) / \Gamma$, a fiber bundle over $Y$. 

Suspension foliations
Suppose that we are given
1. A discrete group $\Gamma$,
2. A compact Lie group $G$,
3. An injective homomorphism $\Gamma \to G$ with dense image,

Put $M = (\tilde{Y} \times Z)/\Gamma$, a fiber bundle over $Y$. 
Suppose that we are given

1. A discrete group $\Gamma$,
2. A compact Lie group $G$,
3. An injective homomorphism $\Gamma \to G$ with dense image,
4. A closed manifold $Y$ with fundamental group $\Gamma$, and
Suppose that we are given

1. A discrete group $\Gamma$,
2. A compact Lie group $G$,
3. An injective homomorphism $\Gamma \rightarrow G$ with dense image,
4. A closed manifold $Y$ with fundamental group $\Gamma$, and
5. A closed Riemannian manifold $Z$ on which $G$ acts isometrically.
Suppose that we are given

1. A discrete group $\Gamma$,
2. A compact Lie group $G$,
3. An injective homomorphism $\Gamma \to G$ with dense image,
4. A closed manifold $Y$ with fundamental group $\Gamma$, and
5. A closed Riemannian manifold $Z$ on which $G$ acts isometrically.

Put $M = (\tilde{Y} \times Z)/\Gamma$, a fiber bundle over $Y$. 
There is a horizontal Riemannian foliation $\mathcal{F}$ on $M$. 
Suspension foliations

\[ M = (\tilde{Y} \times Z)/\Gamma \]

There is a horizontal Riemannian foliation \( \mathcal{F} \) on \( M \).

A fiber \( Z \) of the fiber bundle is a complete transversal.
Suspension foliations

\[ M = (\tilde{Y} \times Z)/\Gamma \]

There is a horizontal Riemannian foliation \( \mathcal{F} \) on \( M \).

A fiber \( Z \) of the fiber bundle is a complete transversal.

A transverse Dirac-type operator on \((M, \mathcal{F})\) amounts to a \( \Gamma \)-invariant Dirac-type operator \( D \) on \( Z \),
Suspension foliations

\[ M = (\tilde{Y} \times Z)/\Gamma \]

There is a horizontal Riemannian foliation \( \mathcal{F} \) on \( M \).

A fiber \( Z \) of the fiber bundle is a complete transversal.

A transverse Dirac-type operator on \( (M, \mathcal{F}) \) amounts to a \( \Gamma \)-invariant Dirac-type operator \( D \) on \( Z \),

or, equivalently, a \( G \)-invariant Dirac-type operator \( D \) on \( Z \).
Suppose that $G = T^k$. 

Problem: $\int T^k L(g, z) d\mu_{T^k}(g)$ generally diverges!

Get cancellations from various components of $Z_{T^k}$. 

Index computation
Index computation

Suppose that $G = T^k$.

$$\text{Index} \left( D_{\text{inv}} \right) = \int_{T^k} \text{Index}(g) \, d\mu_{T^k}(g)$$
Index computation

Suppose that $G = T^k$.

$$\text{Index} (D_{\text{inv}}) = \int_{T^k} \text{Index} (g) \, d\mu_T(g)$$

$$= \int_{T^k} \left( \int_{Zg} \mathcal{L}(g) \right) \, d\mu_T(g)$$

Problem: $\int_{T^k} \mathcal{L}(g) \, d\mu_T(g)$ generally diverges!

Get cancellations from various components of $Z_{T^k}$. 
Index computation

Suppose that $G = T^k$.

\[
\text{Index} \left( D_{\text{inv}} \right) = \int_{T^k} \text{Index}(g) \, d\mu_{T^k}(g)
\]

\[
= \int_{T^k} \left( \int_{Zg} \mathcal{L}(g) \right) \, d\mu_{T^k}(g)
\]

\[
= \int_{T^k} \left( \int_{ZT^k} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g)
\]
Index computation

Suppose that $G = T^k$.

$$\text{Index}(D_{\text{inv}}) = \int_{T^k} \text{Index}(g) \, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left( \int_{Z^g} \mathcal{L}(g) \right) \, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left( \int_{Z^{T^k}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g)$$

$$? = \int_{Z^{T^k}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz$$

Problem: $\int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g)$ generally diverges!

Get cancellations from various components of $Z^{T^k}$. 

Note: The equality symbol ($) is used to denote the equality of expressions.
Suppose that $G = T^k$.

$$\text{Index} \left( D_{\text{inv}} \right) = \int_{T^k} \text{Index}(g) \, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left( \int_{Z^g} \mathcal{L}(g) \right) \, d\mu_{T^k}(g)$$

$$= \int_{T^k} \left( \int_{Z^{T^k}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g)$$

$$\overset{?}{=} \int_{Z^{T^k}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz$$

**Problem**: $\int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g)$ generally diverges!
Index computation

Suppose that $G = T^k$.

\[
\text{Index} (D_{\text{inv}}) = \int_{T^k} \text{Index}(g) \, d\mu_{T^k}(g)
\]

\[
= \int_{T^k} \left( \int_{Z^g} \mathcal{L}(g) \right) \, d\mu_{T^k}(g)
\]

\[
= \int_{T^k} \left( \int_{Z^{T^k}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g)
\]

\[
? = \int_{Z^{T^k}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz
\]

Problem: $\int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g)$ generally diverges!

Get cancellations from various components of $Z^{T^k}$.
Divergences

\[
\int_{T^k} \left( \int_{Z^k} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g) = \int_{Z^k} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz
\]
Divergences

\[ \int_{T^k} \left( \int_{Z^{T^k}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g) \overset{?}{=} \int_{Z^{T^k}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz \]

McKean-Singer-type localization, to compute Index \((D_{\text{inv}})\), doesn’t work.
Divergences

\[
\int_{T^k} \left( \int_{Z^{Tk}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g) \overset{?}{=} \int_{Z^{Tk}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz
\]

McKean-Singer-type localization, to compute Index \((D_{\text{inv}})\), doesn’t work.

In this case, we can subtract off the divergent terms of 
\(\int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g)\) by hand. We know that they will cancel out in the end.
Divergences

\[ \int_{T^k} \left( \int_{Z^{T^k}} \mathcal{L}(g, z) \, dz \right) \, d\mu_{T^k}(g) \overset{?}{=} \int_{Z^{T^k}} \left( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \right) \, dz \]

McKean-Singer-type localization, to compute Index \((D_{\text{inv}})\), doesn’t work.

In this case, we can subtract off the divergent terms of \( \int_{T^k} \mathcal{L}(g, z) \, d\mu_{T^k}(g) \) by hand. We know that they will cancel out in the end.

But what to do for more general Riemannian foliations, which may not reduce to compact Lie group actions?
A way out

A Riemannian foliation may not come from a Lie group action, but there is always a local action by a Lie algebra $g$. 
A Riemannian foliation may not come from a Lie group action, but there is always a local action by a Lie algebra $\mathfrak{g}$.

Idea: prove a Kirillov-type delocalized index formula, in terms of $X \in \mathfrak{g}$. 
A Riemannian foliation may not come from a Lie group action, but there is always a local action by a Lie algebra \( g \).

Idea: prove a Kirillov-type delocalized index formula, in terms of \( X \in g \).

If \( g \) is abelian, we can replace the nonexistent “integral over \( G \)” by an averaging over \( X \in g \).
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
Let $T$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. 

Furthermore, $G$ is a Riemannian groupoid, meaning that $\mathcal{T}$ has a Riemannian metric, and if $g \in G$ then $dg : T_{s(g)} \to T_{r(g)}$ is an isometric isomorphism. 

The transverse index problem becomes a question about the invariant index of an operator on a Riemannian groupoid.
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$. 
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. 
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \to \mathcal{T}.$$
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \to \mathcal{T}.$$ 

$$\mathcal{G}_p = s^{-1}(p), \quad \mathcal{G}^p = r^{-1}(p), \quad \mathcal{G}_p^p = s^{-1}(p) \cap r^{-1}(p)$$
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \to \mathcal{T}.$$ 

$$\mathcal{G}_p = s^{-1}(p), \quad \mathcal{G}^p = r^{-1}(p), \quad \mathcal{G}_p^p = s^{-1}(p) \cap r^{-1}(p)$$

Furthermore, $\mathcal{G}$ is a Riemannian groupoid,
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \rightarrow \mathcal{T}.$$ 

$$\mathcal{G}_p = s^{-1}(p), \quad \mathcal{G}^p = r^{-1}(p), \quad \mathcal{G}_p^p = s^{-1}(p) \cap r^{-1}(p)$$

Furthermore, $\mathcal{G}$ is a Riemannian groupoid, meaning that

- $\mathcal{T}$ has a Riemannian metric, and
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \to \mathcal{T}.$$ 

$$\mathcal{G}_p = s^{-1}(p), \quad \mathcal{G}^p = r^{-1}(p), \quad \mathcal{G}_p^p = s^{-1}(p) \cap r^{-1}(p)$$

Furthermore, $\mathcal{G}$ is a Riemannian groupoid, meaning that

- $\mathcal{T}$ has a Riemannian metric, and
- If $g \in \mathcal{G}$ then $dg : T_{s(g)}\mathcal{T} \to T_{r(g)}\mathcal{T}$ is an isometric isomorphism.
Let $\mathcal{T}$ be a complete transversal for the Riemannian foliation $(M, \mathcal{F})$. It is the unit space of an étale groupoid $\mathcal{G}$.

The elements of $\mathcal{G}$ are the holonomy elements of $(M, \mathcal{F})$. Source and range maps

$$s, r : \mathcal{G} \to \mathcal{T}.$$ 

$$\mathcal{G}_p = s^{-1}(p), \quad \mathcal{G}^p = r^{-1}(p), \quad \mathcal{G}^p_p = s^{-1}(p) \cap r^{-1}(p)$$

Furthermore, $\mathcal{G}$ is a Riemannian groupoid, meaning that

- $\mathcal{T}$ has a Riemannian metric, and
- If $g \in \mathcal{G}$ then $dg : T_{s(g)} \mathcal{T} \to T_{r(g)} \mathcal{T}$ is an isometric isomorphism.

The transverse index problem becomes a question about the invariant index of an operator on a Riemannian groupoid.
One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $\mathcal{T}$. However, it also has a topology in which it becomes an étale groupoid $G_\delta$. Example: suspension foliation $G = \mathbb{Z} \rtimes \Gamma$.
One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $\mathcal{T}$. It is a Lie groupoid but is generally not étale.
Groupoid closure

One can take the closure $\overline{\mathcal{G}}$ of $\mathcal{G}$ in the 1-jet topology on $\mathcal{T}$. It is a Lie groupoid but is generally not étale.

However, it also has a topology in which it becomes an étale groupoid $\overline{\mathcal{G}}_\delta$. 

Example: suspension foliation $\mathcal{G} = \mathbb{Z} \rtimes \Gamma$ 

$\mathcal{G} = \mathbb{Z} \rtimes \Gamma$ 

$\mathcal{G}_\delta = \mathbb{Z} \rtimes \Gamma$
One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $T$. It is a Lie groupoid but is generally not étale.

However, it also has a topology in which it becomes an étale groupoid $\overline{G}_\delta$.

Example : suspension foliation
Groupoid closure

One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $\mathcal{T}$. It is a Lie groupoid but is generally not étale.

However, it also has a topology in which it becomes an étale groupoid $\overline{G}_\delta$.

Example: suspension foliation

$$\mathcal{G} = \mathbb{Z} \rtimes \Gamma$$
Groupoid closure

One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $\mathcal{T}$. It is a Lie groupoid but is generally not étale. However, it also has a topology in which it becomes an étale groupoid $\overline{G}_\delta$.

Example: suspension foliation

$$G = \mathbb{Z} \rtimes \Gamma$$

$$\overline{G} = \mathbb{Z} \rtimes G$$
One can take the closure $\overline{G}$ of $G$ in the 1-jet topology on $\mathcal{T}$. It is a Lie groupoid but is generally not étale.

However, it also has a topology in which it becomes an étale groupoid $\overline{G}_\delta$.

Example: suspension foliation

$$G = \mathbb{Z} \rtimes \Gamma$$

$$\overline{G} = \mathbb{Z} \rtimes G$$

$$\overline{G}_\delta = \mathbb{Z} \rtimes G_\delta$$
The Lie algebroid of $\mathcal{G}$ is a locally constant sheaf of Lie algebras.
The Lie algebroid of $\mathcal{G}$ is a locally constant sheaf of Lie algebras.

Its stalk $\mathfrak{g}$ could be any finite-dimensional Lie algebra.
The Lie algebroid of $\mathcal{G}$ is a locally constant sheaf of Lie algebras.

Its stalk $\mathfrak{g}$ could be any finite-dimensional Lie algebra.

Local representation of $\mathfrak{g}$ by Killing vector fields on $\mathcal{T}$. 
The **Lie algebroid** of \( \bar{\mathcal{G}} \) is a locally constant sheaf of Lie algebras.

Its stalk \( g \) could be any finite-dimensional Lie algebra.

Local representation of \( g \) by Killing vector fields on \( \mathcal{T} \).

Parallel to Cheeger-Fukaya-Gromov theory of Nil-structures in bounded curvature collapse.
The Lie algebroid of $\mathfrak{G}$ is a locally constant sheaf of Lie algebras.

Its stalk $\mathfrak{g}$ could be any finite-dimensional Lie algebra.

Local representation of $\mathfrak{g}$ by Killing vector fields on $\mathcal{T}$.

Parallel to Cheeger-Fukaya-Gromov theory of Nil-structures in bounded curvature collapse.

Local structure of a Riemannian groupoid (Haefliger, Molino).
$\mathcal{G}$ is a proper Lie groupoid, so we can integrate over it.
$ar{G}$ is a proper Lie groupoid, so we can integrate over it.

Let $\{d\mu^p\}_{p \in T}$ be a Haar system for $\bar{G}$, with $\mu^p$ a measure on $G^p$. 
\( \bar{G} \) is a proper Lie groupoid, so we can integrate over it.

Let \( \{ d\mu^p \}_{p \in T} \) be a Haar system for \( \bar{G} \), with \( \mu^p \) a measure on \( \mathcal{G}^p \).

Let \( \phi \in C^\infty_c(\mathcal{T}) \) be a cutoff function so that

\[
\int_{\mathcal{G}^p} \phi^2(s(g)) \, d\mu^p(g) = 1.
\]
Let $E$ be a $G$-equivariant vector bundle on $\mathcal{T}$. 
Invariant subspace

Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant vector bundle on $\mathcal{T}$.

Define an injection $\alpha : L^2(\mathcal{T}; \mathcal{E})^\mathcal{G} \to L^2(\mathcal{T}; \mathcal{E})$ by

$$\alpha(\xi) = \phi \xi,$$
Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant vector bundle on $\mathcal{T}$.

Define an injection $\alpha : L^2(\mathcal{T}; \mathcal{E}) \rightarrow L^2(\mathcal{T}; \mathcal{E})$ by

$$\alpha(\xi) = \phi \xi,$$

and a surjection $\beta : L^2(\mathcal{T}; \mathcal{E}) \rightarrow L^2(\mathcal{T}; \mathcal{E})$ by

$$(\beta(\eta))_p = \int_{\mathcal{G}^p} \left( \eta_{s(g)} \cdot g^{-1} \right) \phi(s(g)) \, d\mu^p(g).$$
Let $\mathcal{E}$ be a $G$-equivariant vector bundle on $\mathcal{T}$.

Define an injection $\alpha : L^2(\mathcal{T}; \mathcal{E})^\mathcal{G} \rightarrow L^2(\mathcal{T}; \mathcal{E})$ by

$$\alpha(\xi) = \phi \xi,$$

and a surjection $\beta : L^2(\mathcal{T}; \mathcal{E}) \rightarrow L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$ by

$$(\beta(\eta))_p = \int_{\mathcal{G}^p} \left( \eta_{s(g)} \cdot g^{-1} \right) \phi(s(g)) \, d\mu^p(g).$$

Then

$\beta \circ \alpha = \text{Id}$
Invariant subspace

Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant vector bundle on $\mathcal{T}$.

Define an injection $\alpha : L^2(\mathcal{T}; \mathcal{E})^\mathcal{G} \to L^2(\mathcal{T}; \mathcal{E})$ by

$$\alpha(\xi) = \phi \xi,$$

and a surjection $\beta : L^2(\mathcal{T}; \mathcal{E}) \to L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$ by

$$(\beta(\eta))_p = \int_{\mathcal{G}^p} \left( \eta_{s(g)} \cdot g^{-1} \right) \phi(s(g)) \, d\mu^p(g).$$

Then

- $\beta \circ \alpha = \text{Id}$
- $\beta = \alpha^*$
Invariant subspace

Let $\mathcal{E}$ be a $\mathcal{G}$-equivariant vector bundle on $\mathcal{T}$.

Define an injection $\alpha : L^2(\mathcal{T}; \mathcal{E})^\mathcal{G} \to L^2(\mathcal{T}; \mathcal{E})$ by

$$\alpha(\xi) = \phi \xi,$$

and a surjection $\beta : L^2(\mathcal{T}; \mathcal{E}) \to L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$ by

$$(\beta(\eta))_p = \int_{\mathcal{G}^p} \left( \eta_{s(g)} \cdot g^{-1} \right) \phi(s(g)) \ d\mu^p(g).$$

Then

- $\beta \circ \alpha = \text{Id}$
- $\beta = \alpha^*$

Hence $P = \alpha \circ \beta$ is an orthogonal projection on $L^2(\mathcal{T}; \mathcal{E})$, with image isomorphic to $L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$. Get inner product on $L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$. 
Let $D_0$ be a $\mathcal{G}$-invariant Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})$. (Give it APS boundary conditions).
Spectral triple

Let $D_0$ be a $\mathcal{G}$-invariant Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})$. (Give it APS boundary conditions).

**Proposition**

$(C^\infty_c(\overline{\mathcal{G}}), L^2(\mathcal{T}; \mathcal{E}), D_0)$ is a spectral triple of dimension $q$. 
Let $D_0$ be a $\mathcal{G}$-invariant Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})$. (Give it APS boundary conditions).

**Proposition**

$(C_\infty^\infty(\mathcal{G}), L^2(\mathcal{T}; \mathcal{E}), D_0)$ is a spectral triple of dimension $q$.

Put $D_{\text{inv}} = \beta \circ D_0 \circ \alpha$, an operator on $L^2(\mathcal{T}; \mathcal{E})^{\overline{\mathcal{G}}}$. It is unitarily equivalent to $P \circ D_0 \circ P$ on $\text{Im}(P)$.
Let $D_0$ be a $G$-invariant Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})$. (Give it APS boundary conditions).

**Proposition**

$(C^\infty_c(G), L^2(\mathcal{T}; \mathcal{E}), D_0)$ is a spectral triple of dimension $q$.

Put $D_{\text{inv}} = \beta \circ D_0 \circ \alpha$, an operator on $L^2(\mathcal{T}; \mathcal{E})^G$. It is unitarily equivalent to $P \circ D_0 \circ P$ on $\text{Im}(P)$. One finds

$$D_{\text{inv}} = D_0 - \frac{1}{2} c(\tau),$$

where $\tau$ is a certain closed $G$-invariant 1-form on $\mathcal{T}$. 
Let $D_0$ be a $\mathcal{G}$-invariant Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})$. (Give it APS boundary conditions).

**Proposition**

$(C^\infty_c(\mathcal{G}), L^2(\mathcal{T}; \mathcal{E}), D_0)$ is a spectral triple of dimension $q$.

Put $D_{\text{inv}} = \beta \circ D_0 \circ \alpha$, an operator on $L^2(\mathcal{T}; \mathcal{E})\overline{\mathcal{G}}$. It is unitarily equivalent to $P \circ D_0 \circ P$ on $\text{Im}(P)$. One finds

$$D_{\text{inv}} = D_0 - \frac{1}{2} c(\tau),$$

where $\tau$ is a certain closed $\mathcal{G}$-invariant 1-form on $\mathcal{T}$.

**Corollary**

$D_{\text{inv}}$ is Fredholm. In fact, $e^{-tD_{\text{inv}}^2}$ is trace-class for all $t > 0$. 
The transverse index problem for Riemannian foliations

Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$. 

Assumptions:
1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$,
2. The Molino sheaf has trivial holonomy on $M$,
3. For all $w \in W$, $K_w$ is connected.

Note: If $M$ is simply-connected then assumptions (1) and (2) are automatic.

Put $W_{\max} = \{ w \in W : K_w \cong T \}$. 
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$.

**Assumptions :**

1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$, 
2. The Molino sheaf has trivial holonomy on $M$, and 
3. For all $w \in W$, $K_w$ is connected.

Note: If $M$ is simply-connected then assumptions (1) and (2) are automatic.
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$.

Assumptions:

1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$,
2. The Molino sheaf has trivial holonomy on $M$, and
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$.

**Assumptions:**

1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$,
2. The Molino sheaf has trivial holonomy on $M$, and
3. For all $w \in W$, $K_w$ is connected.
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$.

**Assumptions**:

1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$,
2. The Molino sheaf has trivial holonomy on $M$, and
3. For all $w \in W$, $K_w$ is connected.

**Note**: If $M$ is simply-connected then assumptions (1) and (2) are automatic.
Let $W$ be the orbit space of $\overline{G}$. It is the same as the space of leaf closures.

A neighborhood of $w \in W$ is homeomorphic to $V_w/K_w$, where $V_w$ is a vector space and $K_w$ is a compact Lie group, whose Lie algebra lies in $\mathfrak{g}$.

**Assumptions**:

1. The Molino Lie algebra $\mathfrak{g}$ is an abelian Lie algebra $\mathbb{R}^k$,
2. The Molino sheaf has trivial holonomy on $M$, and
3. For all $w \in W$, $K_w$ is connected.

**Note**: If $M$ is simply-connected then assumptions (1) and (2) are automatic.

Put $W_{\text{max}} = \{ w \in W : K_w \cong T^k \}$. 
Index theorem

**Theorem**

*Under the assumptions of the previous slide, let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded basic Clifford module on $\mathcal{T}$. Let $D_{\text{inv}}$ be the basic Dirac-type operator on $L^2(\mathcal{T} ; \mathcal{E})^G$. Then*

$$
\text{Index}(D_{\text{inv}}) = \int_{W_{\text{max}}} \hat{A}(TW_{\text{max}}) \, N_{\mathcal{E}}.
$$
Theorem

Under the assumptions of the previous slide, let $\mathcal{E}$ be a $\mathbb{Z}_2$-graded basic Clifford module on $\mathcal{T}$. Let $D_{\text{inv}}$ be the basic Dirac-type operator on $L^2(\mathcal{T}; \mathcal{E})^\mathcal{G}$. Then

$$\text{Index}(D_{\text{inv}}) = \int_{W_{\text{max}}} \hat{\mathcal{A}}(TW_{\text{max}}) \, \mathcal{N}_{\mathcal{E}}.$$

Here $\mathcal{N}_{\mathcal{E}}$ is a characteristic class on $W_{\text{max}}$, which is computed from the normal data of $W_{\text{max}}$, along with $\mathcal{E}$. 
Introduction

Riemannian foliations

Basic index

Compact Lie group actions

Riemannian groupoids

Index theorem

Steps of the proof
Take open coverings \( \{ U_\alpha \} \) and \( \{ U'_\alpha \} \) of \( W \), so

\[
U_\alpha \subset U'_\alpha,
\]
Step 1

Take open coverings \( \{ U_\alpha \} \) and \( \{ U'_\alpha \} \) of \( W \), so

- \( U_\alpha \subset U'_\alpha \),
- \( U'_\alpha = B(V_\alpha)/K_\alpha \), with \( B(V_\alpha) \) a ball in \( V_\alpha \cong \mathbb{R}^q \) and \( K_\alpha = T^k_{\alpha} \), and
Step 1

Take open coverings \( \{ U_\alpha \} \) and \( \{ U'_\alpha \} \) of \( W \), so

\[
U_\alpha \subset U'_\alpha,
\]

\[
U'_\alpha = B(V_\alpha)/K_\alpha, \text{ with } B(V_\alpha) \text{ a ball in } V_\alpha \cong \mathbb{R}^q \text{ and } K_\alpha = T^{k_\alpha}, \text{ and}
\]

\[
B(V_\alpha) \text{ extends to a closed Riemannian manifold } Y_\alpha \text{ with a } T^{k_\alpha}-\text{action}.
\]
Choose

- A subordinate partition of unity \( \{ \eta_\alpha \} \) for \( \{ U_\alpha \} \), and
Choose

- A subordinate partition of unity \( \{ \eta_\alpha \} \) for \( \{ U_\alpha \} \), and
- Functions \( \rho_\alpha \) with support in \( U'_\alpha \) so \( \rho_\alpha \big|_{\text{supp}(\eta_\alpha)} = 1 \).

(Will also denote by \( \eta_\alpha \) and \( \rho_\alpha \) their lifts to \( T \) and \( Y_\alpha \).)

Let \( D_\alpha \) be the Dirac-type operator on \( Y_\alpha \). Let \( D_{\text{inv},\alpha} \) be its restriction to \( T_k \)-invariant sections.

Put

\[
Q_{\text{inv},\alpha} = I - e^{-tD^2_{\text{inv},\alpha}}D_{\text{inv},\alpha} = \int_0^t e^{-sD^2_{\text{inv},\alpha}}D_{\text{inv},\alpha} \, ds.
\]
Choose

- A subordinate partition of unity $\{\eta_\alpha\}$ for $\{U_\alpha\}$, and
- Functions $\rho_\alpha$ with support in $U'_\alpha$ so $\rho_\alpha\big|_{\text{supp}(\eta_\alpha)} = 1$.

(Will also denote by $\eta_\alpha$ and $\rho_\alpha$ their lifts to $T$ and $Y_\alpha$.)
Choose

- A subordinate partition of unity $\{\eta_\alpha\}$ for $\{U_\alpha\}$, and
- Functions $\rho_\alpha$ with support in $U'_\alpha$ so $\rho_\alpha \bigg|_{\text{supp}(\eta_\alpha)} = 1$.

(Will also denote by $\eta_\alpha$ and $\rho_\alpha$ their lifts to $T$ and $Y_\alpha$.)

Let $D_\alpha$ be the Dirac-type operator on $Y_\alpha$. Let $D_{\text{inv},\alpha}$ be its restriction to $T^{k_\alpha}$-invariant sections.
Choose

- A subordinate partition of unity \( \{ \eta_\alpha \} \) for \( \{ U_\alpha \} \), and
- Functions \( \rho_\alpha \) with support in \( U'_\alpha \) so \( \rho_\alpha \bigg|_{\text{supp}(\eta_\alpha)} = 1 \).

(Will also denote by \( \eta_\alpha \) and \( \rho_\alpha \) their lifts to \( T \) and \( Y_\alpha \).)

Let \( D_\alpha \) be the Dirac-type operator on \( Y_\alpha \). Let \( D_{\text{inv}, \alpha} \) be its restriction to \( T^{k_\alpha} \)-invariant sections.

Put

\[
Q_{\text{inv}, \alpha} = \frac{I - e^{-tD^2_{\text{inv}, \alpha}}}{D^2_{\text{inv}, \alpha}} D_{\text{inv}, \alpha}
\]
Choose

- A subordinate partition of unity \( \{ \eta_\alpha \} \) for \( \{ U_\alpha \} \), and
- Functions \( \rho_\alpha \) with support in \( U'_\alpha \) so \( \rho_\alpha \bigg|_{\text{supp}(\eta_\alpha)} = 1. \)

(Will also denote by \( \eta_\alpha \) and \( \rho_\alpha \) their lifts to \( T \) and \( Y_\alpha \).)

Let \( D_\alpha \) be the Dirac-type operator on \( Y_\alpha \). Let \( D_{\text{inv},\alpha} \) be its restriction to \( T^{k_\alpha} \)-invariant sections.

Put

\[
Q_{\text{inv},\alpha} = I - e^{-tD_{\text{inv},\alpha}^2} D_{\text{inv},\alpha}
\]

\[
= \int_0^t e^{-sd_{\text{inv},\alpha}^2} D_{\text{inv},\alpha} \, ds.
\]
Proposition

\[ \sum_\alpha \rho_\alpha Q_{\text{inv},\alpha} \eta_\alpha \text{ is a parametrix for } D_{\text{inv}}. \]
Proposition

\[ \sum_{\alpha} \rho_{\alpha} Q_{\text{inv}, \alpha} \eta_{\alpha} \text{ is a parametrix for } D_{\text{inv}}. \]

Also, for all \( t > 0 \),

\[
\text{Index}(D_{\text{inv}}) = \sum_{\alpha} \text{Tr}_s \left( e^{-tD_{\text{inv}, \alpha}^2} \eta_{\alpha} \right) + \frac{1}{2} \sum_{\alpha} \text{Tr}_s \left( Q_{\text{inv}, \alpha} [D_{\text{inv}, \alpha}, \eta_{\alpha}] \right)
\]
Proposition

\[ \sum_{\alpha} \rho_{\alpha} Q_{\text{inv}, \alpha} \eta_{\alpha} \text{ is a parametrix for } D_{\text{inv}}. \]

Also, for all \( t > 0 \),

\[
\text{Index}(D_{\text{inv}}) = \sum_{\alpha} \text{Tr}_s \left( e^{-tD_{\text{inv}, \alpha}^2} \eta_{\alpha} \right) + \frac{1}{2} \sum_{\alpha} \text{Tr}_s \left( Q_{\text{inv}, \alpha} [D_{\text{inv}, \alpha}, \eta_{\alpha}] \right)
\]

\[
= \sum_{\alpha} \text{Tr}_s \left( e^{-tD_{\text{inv}, \alpha}^2} \eta_{\alpha} \right) + \frac{1}{2} \sum_{\alpha, \beta} \text{Tr}_s \left( \rho_{\alpha} (Q_{\text{inv}, \alpha} - Q_{\text{inv}, \beta}) \eta_{\beta} [D_{\text{inv}, \alpha}, \eta_{\alpha}] \right)
\]
Fix a Haar measure $d\mu_g$ on $g \simeq \mathbb{R}^k$. If $F \in C^\infty(\mathbb{R}^k)$ is a finite sum of periodic functions, put

$$AV_X F = \lim_{R \to \infty} \frac{\int_{B(0,R)} F(X) \, d\mu_g(X)}{\int_{B(0,R)} 1 \, d\mu_g(X)}.$$
Fix a Haar measure $d\mu_g$ on $g \simeq \mathbb{R}^k$. If $F \in C^\infty(\mathbb{R}^k)$ is a finite sum of periodic functions, put

$$AV_X F = \lim_{R \to \infty} \frac{\int_{B(0,R)} F(X) \, d\mu_g(X)}{\int_{B(0,R)} 1 \, d\mu_g(X)}.$$

Given $X \in \mathbb{R}^k$, let $X$ denote the corresponding vector field on $Y_\alpha$. 
Fix a Haar measure $d\mu_g$ on $g \simeq \mathbb{R}^k$. If $F \in C^\infty(\mathbb{R}^k)$ is a finite sum of periodic functions, put

$$AV_X F = \lim_{R \to \infty} \frac{\int_{B(0,R)} F(X) \, d\mu_g(X)}{\int_{B(0,R)} 1 \, d\mu_g(X)}.$$

Given $X \in \mathbb{R}^k$, let $X$ denote the corresponding vector field on $Y_\alpha$.

**Proposition**

$$\text{Index}(D_{\text{inv}}) = AV_X \left( \sum_\alpha \text{Tr}_s \left( e^{-(tD^2_\alpha + \mathcal{L}_X)} \eta_\alpha \right) \right)$$

$$+ \frac{1}{2} \sum_\alpha \int_0^t \text{Tr}_s \left( e^{-(sD^2_\alpha + \mathcal{L}_X)} D_\alpha [D_\alpha, \eta_\alpha] \, ds \right).$$
For $t > 0$ and $\epsilon \in \mathbb{C}$, put

$$D_{\alpha,t,\epsilon} = D_{\alpha} + \epsilon \frac{c(X)}{4t}.$$
For $t > 0$ and $\epsilon \in \mathbb{C}$, put

$$D_{\alpha, t, \epsilon} = D_\alpha + \epsilon \frac{c(X)}{4t}.$$ 

Put

$$F_{t, \epsilon}(X) = \sum_\alpha \text{Tr}_s \left( e^{-(tD^2_{\alpha, t, \epsilon} + \mathcal{L}X)} \eta_\alpha \right)$$

$$+ \frac{1}{2} \sum_\alpha \int_0^t \text{Tr}_s \left( e^{-(sD^2_{\alpha, t, \epsilon} + \mathcal{L}X)} D_{\alpha, t, \epsilon} [D_{\alpha, t, \epsilon}, \eta_\alpha] \, ds \right).$$
Step 3

For \( t > 0 \) and \( \epsilon \in \mathbb{C} \), put

\[
D_{\alpha, t, \epsilon} = D_\alpha + \epsilon \frac{c(X)}{4t}.
\]

Put

\[
F_{t, \epsilon}(X) = \sum_{\alpha} \text{Tr}_s \left( e^{-(tD_{\alpha, t, \epsilon}^2 + \mathcal{L}X)} \eta_\alpha \right)
\]

\[
+ \frac{1}{2} \sum_{\alpha} \int_0^t \text{Tr}_s \left( e^{-(sD_{\alpha, t, \epsilon}^2 + \mathcal{L}X)} D_{\alpha, t, \epsilon} [D_{\alpha, t, \epsilon}, \eta_\alpha] ds \right).
\]

From before,

\[
\text{Index} \left( D_{\text{inv}} \right) = AV_X F_{t, 0}.
\]
Deformations

\[ D_{\alpha,t,\epsilon} = D_{\alpha} + \epsilon \frac{c(X)}{4t}. \]
Deformations

\[ D_{\alpha,t,\epsilon} = D_{\alpha} + \epsilon \frac{c(X)}{4t}. \]

Proposition

\( F_{t,0}(X) \) is independent of \( t \).
Deformations

\[ D_{\alpha,t,\epsilon} = D_{\alpha} + \epsilon \frac{c(X)}{4t}. \]

**Proposition**

\( F_{t,0}(X) \) is independent of \( t \).

**Proposition**

\( F_{t,\epsilon}(X) \) is independent of \( \epsilon \).
Deformations

\[ D_{\alpha,t,\epsilon} = D_{\alpha} + \epsilon \frac{c(X)}{4t}. \]

Proposition

\( F_{t,0}(X) \) is independent of \( t \).

Proposition

\( F_{t,\epsilon}(X) \) is independent of \( \epsilon \).

Proposition

\( F_{t,2}(X) \) has a holomorphic extension to \( X \in \mathbb{C}^k \).
Proposition

If $X \in \mathbb{R}^k$ then

$$\lim_{t \to 0} F_{t,1}(iX) = \sum_{\alpha} \int_{Y_\alpha} \tilde{A}(iX, Y_\alpha) \ ch(iX, \mathcal{E}_\alpha/S) \eta_\alpha.$$
Step 4

**Proposition**

If $X \in \mathbb{R}^k$ then

$$\lim_{t \to 0} F_{t,1}(iX) = \sum_{\alpha} \int_{Y_{\alpha}} \hat{A}(iX, Y_{\alpha}) \, \text{ch}(iX, E_{\alpha}/S) \, \eta_{\alpha}.$$  

**Corollary**

$$\text{Index} \left( D_{\text{inv}} \right) = AV_X F_{t,0} = AV_X F_{t,1}$$
Proposition

If $X \in \mathbb{R}^k$ then

$$\lim_{t \to 0} F_{t,1}(iX) = \sum_{\alpha} \int_{Y_\alpha} \hat{A}(iX, Y_\alpha) \ ch(iX, E_\alpha/S) \ \eta_\alpha.$$ 

Corollary

$$\text{Index} (D_{\text{inv}}) = AV_X F_{t,0} = AV_X F_{t,1} = AV_X \sum_{\alpha} \int_{Y_\alpha} \hat{A}(X, Y_\alpha) \ ch(X, E_\alpha/S) \ \eta_\alpha.$$
Let $C \subset \mathcal{T}$ be the elements $p \in \mathcal{T}$ with isotropy group $\mathcal{G}_p = T^k$. It passes to $\mathcal{W}_{\text{max}} \subset \mathcal{W}$. 
Step 5

Let $C \subset T$ be the elements $p \in T$ with isotropy group $\overline{G}_p = T^k$. It passes to $W_{\text{max}} \subset W$.

For simplicity, assume that each component of $C$ is spin, with normal spinor bundle $S_N$, and that $E = S^T \otimes \mathcal{W}$. Descend $S_N$ and $\mathcal{W}|_C$ to $W_{\text{max}}$. 
Let $C \subset \mathcal{T}$ be the elements $p \in \mathcal{T}$ with isotropy group $G^p = T^k$. It passes to $W_{\text{max}} \subset W$.

For simplicity, assume that each component of $C$ is spin, with normal spinor bundle $S_N$, and that $\mathcal{E} = S^T \otimes \mathcal{W}$. Descend $S_N$ and $\mathcal{W}|_C$ to $W_{\text{max}}$.

**Proposition**

*For any $Q \in \mathbb{C}^k$,*

$$AV_X \sum_{\alpha} \int_{Y_\alpha} \hat{A}(X, Y_\alpha) \ ch(X, \mathcal{E}_\alpha / S) \eta_\alpha =$$

$$AV_X \int_{W_{\text{max}}} \hat{A}(TW_{\text{max}}) \ \frac{ch_{\mathcal{W}}(e^{-X+Q})}{ch_{S_N}(e^{-X+Q})}.$$
For generic $Q \in \mathbb{C}^k$, define $N_{\mathcal{E},Q} \in \Omega^*(W_{\text{max}})$ by

$$N_{\mathcal{E},Q} = AV_X \frac{\text{ch}_W(e^{-X+Q})}{\text{ch}_{S_N}(e^{-X+Q})}.$$
For generic $Q \in \mathbb{C}^k$, define $N_{\varepsilon, Q} \in \Omega^*(W_{\text{max}})$ by

$$N_{\varepsilon, Q} = AV_X \frac{\text{ch}_W(e^{-X+Q})}{\text{ch}_{S_N}(e^{-X+Q})}.$$

As long as $\text{Im}(Q)$ remains outside certain hyperplanes in $\mathbb{R}^k$, what’s being averaged is smooth in $X$. 
For generic $Q \in \mathbb{C}^k$, define $\mathcal{N}_{\mathcal{E},Q} \in \Omega^*(W_{\text{max}})$ by

$$
\mathcal{N}_{\mathcal{E},Q} = A_{V_X} \frac{\text{ch}_{W}(e^{-X}+Q)}{\text{ch}_{S_N}(e^{-X}+Q)}.
$$

As long as $\text{Im}(Q)$ remains outside certain hyperplanes in $\mathbb{R}^k$, what’s being averaged is smooth in $X$.

$\mathcal{N}_{\mathcal{E},Q}$ can be computed using contour integrals.
For generic $Q \in \mathbb{C}^k$, define $N_{\epsilon,Q} \in \Omega^*(W_{\text{max}})$ by

$$N_{\epsilon,Q} = AV_X \frac{\text{ch}_{W}(e^{-X+Q})}{\text{ch}_{S_N}(e^{-X+Q})}.$$ 

As long as $\text{Im}(Q)$ remains outside certain hyperplanes in $\mathbb{R}^k$, what’s being averaged is smooth in $X$.

$N_{\epsilon,Q}$ can be computed using contour integrals.

**Corollary**

$$\text{Index} \left( D_{\text{inv}} \right) = \int_{W_{\text{max}}} \hat{A}(TW_{\text{max}}) N_{\epsilon,Q}$$