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Superconnections in geometry
— commutative and noncommutative.

B smooth manifold
E $\mathbb{Z}_2$-graded vector bundle on B.

Connection on E : $\nabla : C^\infty(B; E_\pm) \to \Omega^1(B; E_\pm)$

$$\nabla(f\xi) = (df)\xi + f\nabla\xi$$

Superconnection on E : $A = A_{[0]} + A_{[1]} + \cdots$
(Quillen)

$A_{[0]} \in C^\infty(B; \mathrm{End}_-(E))$
$A_{[1]} = \nabla^E$; connection
$A_{[2]} \in \Omega^2(B; \mathrm{End}_-(E))$
$A_{[i]} \in \Omega^i(B; \mathrm{End}_{(-1)^{i+1}}(E))$ $i \neq 1$

$$A : C^\infty(B; E) \to \Omega^*(B; E)$$

$$A(f\xi) = (df)\xi + fA(\xi)$$

Extend A : $A : \Omega(B; E) \to \Omega(B; E)$

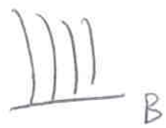
s.t. $A(\omega\xi) = (d\omega)\xi + (-1)^\omega \omega A(\xi)$.

$A^2 \in \Omega(B; \mathrm{End}(E))$ even.

$\mathrm{ch}(A) = \mathrm{tr}_E(e^{-A^2}) \in \Omega^{\mathrm{even}}(B)$. closed.

Example.

$$\begin{array}{c} M \\ \downarrow \\ B \end{array} \quad , \text{ fiber } Z$$



Say $\omega \in \Omega^*(M)$.
Want to decompose this into horizontal and vertical components.

Add $T^H M$, horizontal distribution on M.

$\exists$ vector bundle W on B s.t.

1. $W_b = \Omega^*(Z_b)$
2. $\Omega^*(B; W) \cong \Omega^*(M)$.

$d^M \omega$, decompose into horizontal, vertical pieces

Lemma. $d^M = d^Z + \nabla^W + i_T$

1. $d^Z =$ vertical d
2. $\nabla^W =$ horizontal diff
3. $i_T =$ interior mult. by $T \in \Omega^2(M; TZ)$

on W, $A = A_{[0]} + A_{[1]} + A_{[2]}$.

Riemannian geometry

M compact connected smooth manifold. Fix $K \geq 0$

$\mathfrak{M} = \{$ Riem. metrics g on M with $\|R(M,g)\|_\infty \leq K$, $\operatorname{diam}(M,g) = 1 \} / \operatorname{Diff}(M)$

$\operatorname{diam}(M,g) = \sup\limits_{p,q \in M} d(p,q)$

<u>Question</u>: Does one have uniform estimates on $\mathcal{M}$ for eigenvalues of Δ, Δ_p, $D\!\!\!/$, ...

<u>Ex</u>. In the case of function Laplacian, Cheeger's inequality $\Rightarrow$ $\exists$ uniform $\overbrace{\text{lower}}^{\text{positive}}$ bound on the smallest positive eigenvalue of Δ.

Can one extend this to Δ_p?

$$\Delta_p = dd^* + d^*d \Big|_{\Omega^p(M)}$$

$\mathcal{M}$ noncompact in C^∞ topology

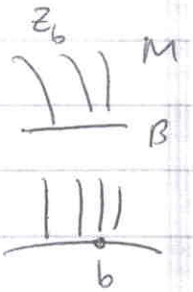
$\mathcal{M} \subset \left(\{\text{compact metric spaces}\}/_{\text{isom}}, d_{GH} \right)$

$\overline{\mathcal{M}}$ is compact

$\partial\overline{\mathcal{M}}$ = lower-dimensional metric spaces to which M can "collapse"

$B \in \partial\overline{\mathcal{M}}$. Assume B is a Riem. mfld.

Cheeger, Fukaya, Gromov: $\exists$ nbhd U of B s.t. if $(M, g) \in U \cap \mathcal{M}$, then M fibers over B, fiber Z is an infranil manifold.



Construct W as before, ∞-dim v. bundle on B.

Put $E_b = \{ \omega \in Z_b : \omega$ is parallel on $Z_b \}$, finite dim v. bundle.

d^M (superconnection on W) restricts to a superconnection A on E.

$\mathrm{diam}(Z) \sim d_{GH}(M, B)$.

Theorem: As $d_{GH}(M,B) \to 0$, the superconn. Laplacian $(AA^* + A^*A)_p$ contains all of the spectrum of Δ_p that doesn't shoot off to ∞.

Theorem. The set of A that so occur is relatively compact.

Theorem. (The number of small eigenvalues for the 1-form Laplacian on M) $\leq b_1(M) + \dim(M)$.

$X = \hat{M}/G$

More precisely, if $\exists j$ small eigenvalues and $j > b_1(M)$ then $j \leq b_1(X) + \dim(M) - \dim(X)$.

Superconn. pf of Connes' foliation index thm.
(joint work with A. Gorokhovsky)

B = algebra over $\mathbb{C}$.

Ω^* = dga with $\Omega^0 = B$.

$\mathcal{E}$ = left B-module, $\mathbb{Z}_2$-graded

$\nabla^E : \mathcal{E}_\pm \longrightarrow \Omega^1 \otimes_B \mathcal{E}_\pm$ s.t. $\nabla(b\xi) = (db)\xi + b(\nabla\xi)$.

Non comm. superconnection

$$A = \sum A_{[i]}$$

$$A_{[1]} = \text{connection } \nabla^E$$

$$A_{[i]} \in \mathcal{H}om_B(\mathcal{E}, \Omega^i \otimes_B \mathcal{E}) \quad \text{odd}, \ i \neq 1$$

$$e^{-A^2} \in \mathcal{H}om_B(\mathcal{E}, \Omega^* \otimes_B \mathcal{E})$$

$$ch(A) = tr(e^{-A^2}).$$

$$\in \Omega^* / [\Omega, \Omega].$$

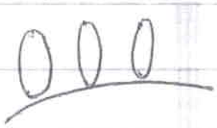
$(M, \mathcal{F})$ foliated mfld.

$\not{D}$ = leafwise Dirac-type operator.

$Ind(\not{D}) \in$ "K^*(leaf space)"

For fiber bdle

$ch(Ind(\not{D})) \in$ "H^*(leaf space)"



index lies in K th of base

Complete transversal $\mathcal{T}$ to $\mathcal{F}$

$$\dim(\mathcal{T}) = \operatorname{codim}(\mathcal{F}).$$

For simplicity, assume no holonomy.

Put $G = \{ (m_1, m_2) \in T \times T : m_1, m_2$ lie on same leaf $\}$.

dim G = dim T, etale groupoid

$\mathcal{B} = C_0^\infty(G)$.

$f_1, f_2 \in \mathcal{B}$.

$$(f_1 f_2)(m_1, m_2) = \sum_{m_3} f_1(m_1, m_3) f_2(m_3, m_2)$$

<u>Def</u> $G^{(n)} = \{ (m_0, \dots, m_n) \in T^{n+1} : m_i$'s all on the same leaf $\}$.

dim $G^{(n)}$ = dim T

$$\Omega^{m,n} = \Omega^m(G^{(n)}).$$

$\Omega^p = \bigoplus_{n+m=p} \Omega^{m,n}$

$d^{1,0}$ = exterior derivative

$$(d^{0,1}\omega)(m_0, \dots, m_{n+1}) = \delta_{m_0 m_1} \omega(m_1, \dots, m_{n+1}).$$

$d = d^{1,0} + d^{0,1}$

$$\begin{array}{c} P \\ \downarrow \\ T \end{array}$$

P_t = leaf that goes through $t \in T$

W_t = spinors on P_t

Define $\mathcal{E} = C^\infty(T; W)$.

Superconnection $A = A_{[0]} + A_{[1]} + A_{[2]}$

$A_{[0]} = \not{D}$

$A_{[1]}$ = a connection on W = $A_{[1]}^{(1,0)} + A_{[1]}^{(0,1)}$

$A_{[2]} = -\frac{1}{4} c(T)$ T: curvature of $\frac{P}{T}$

Theorem. $ch(A) \in \Omega^* / [\Omega^*, \Omega^*]$ represents $ch(\mathrm{Ind}(D))$

Theorem. $ch(A) = \int_M \hat{A}(T\mathcal{F}) \wedge ch(V) \wedge \omega$

ω pullsback from BG.

Thm (Lichnerowicz) M compact, spin, positive scalar curv $\Rightarrow \hat{A}(M) = 0$

Thm (Connes) M compact, foliated, spin leaves, leafwise p.s.c $\Rightarrow \hat{A}(M) = 0$.