$\mathbf{R}/\mathbf{Z}$ Index Theory

John Lott*
Department of Mathematics
University of Michigan
Ann Arbor, MI 48109-1003
USA
lott@math.lsa.umich.edu

Abstract

We define topological and analytic indices in $\mathbf{R}/\mathbf{Z}$ K-theory and show that they are equal.

1 Introduction

The purpose of this paper is to introduce an index theory in which the indices take value in $\mathbf{R}/\mathbf{Z}$. In order to motivate this theory, let us first recall the integral analog, the Atiyah-Singer families index theorem.

Let $Z \to M \to B$ be a smooth fiber bundle whose fiber Z is a closed even-dimensional manifold and whose base B is a compact manifold. Suppose that the vertical tangent bundle TZ has a spin^c-structure. Then there is a topologically defined map $\mathrm{ind}_{top} : K^0(M) \to K^0(B)$ [1], which in fact predates the index theorem. It is a K-theory analog of "integration over the fiber" in de Rham cohomology. Atiyah and Singer construct a map $\mathrm{ind}_{an} : K^0(M) \to K^0(B)$ by analytic means as follows. Given $V \in K^0(M)$, we can consider it to be a virtual vector bundle on M, meaning the formal difference of two vector bundles on M. The base B then parametrizes a

*Partially supported by NSF grant DMS-9101920.

family of Dirac operators on the fibers, coupled to the fiberwise restrictions of V. The kernels of these Dirac-type operators are used to construct a virtual vector bundle $\mathrm{ind}_{an}(V) \in K^0(B)$ on B, and the families index theorem states that $\mathrm{ind}_{an}(V) = \mathrm{ind}_{top}(V)$ [4]. Upon applying the Chern character, one obtains an equality in $H^*(B; \mathbf{Q})$:

$$\mathrm{ch}(\mathrm{ind}_{an}(V)) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}(V), \tag{1}$$

where L_Z is the Hermitian line bundle on M which is associated to the spin^c-structure on TZ.

The Atiyah-Singer families index theorem is an integral theorem, in that $K^0(\mathrm{pt.}) = \mathbf{Z}$. It is conceivable that one could have a more refined index theorem, provided that one considers a restricted class of vector bundles. What is relevant for this paper is the simple observation that from (1), if $\mathrm{ch}(V) = 0$ then $\mathrm{ch}(\mathrm{ind}_{an}(V)) = 0$. Thus it is consistent to restrict oneself to virtual vector bundles with vanishing Chern character.

We will discuss an index theorem which is an $\mathbf{R}/\mathbf{Z}$-theorem, in the sense that it is based on a generalized cohomology theory whose even coefficient groups are copies of $\mathbf{R}/\mathbf{Z}$. To describe this cohomology theory, consider momentarily a single manifold M. There is a notion of $K^*_{\mathbf{C}/\mathbf{Z}}(M)$, the K-theory of M with $\mathbf{C}/\mathbf{Z}$ coefficients, and Karoubi has given a geometric description of $K^{-1}_{\mathbf{C}/\mathbf{Z}}(M)$. In this description, a generator of $K^{-1}_{\mathbf{C}/\mathbf{Z}}(M)$ is given by a complex vector bundle E on M with trivial Chern character, along with a connection on E whose Chern character form is written as an explicit exact form [16, 17]. By adding Hermitian structures to the vector bundles, we obtain a geometric description of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$, the K-theory of M with $\mathbf{R}/\mathbf{Z}$ coefficients. The ensuing generalized cohomology theory has $K^0_{\mathbf{R}/\mathbf{Z}}(\mathrm{pt.}) = \mathbf{R}/\mathbf{Z}$.

One special way of constructing an element of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ is by taking the formal difference of two flat Hermitian vector bundles on M of the same rank. It is well-known that flat Hermitian vector bundles have characteristic classes which take value in $\mathbf{R}/\mathbf{Z}$, and $\mathbf{R}/\mathbf{Z}$-valued K-theory provides a way of extending these constructions to the framework of a generalized cohomology theory. We show that one can detect elements of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ analytically by means of reduced eta-invariants. This extends the results of Atiyah-Patodi-Singer on flat vector bundles [3].

Returning to the fiber bundle situation, under the above assumptions on the fiber bundle $Z \to M \to B$ one can define a map $\mathrm{ind}_{top} : K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to$

$K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$ by topological means. A major point of this paper is the construction of a corresponding analytic index map. Given a cocycle $\mathcal{E}$ for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$, we first define an analytic index $\mathrm{ind}_{an}(\mathcal{E}) \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$ when $\mathcal{E}$ satisfies a certain technical assumption. To define $\mathrm{ind}_{an}(\mathcal{E})$, we endow TZ with a metric and L_Z with a Hermitian connection. The technical assumption is that the kernels of the fiberwise Dirac-type operators form a vector bundle on B. The construction of $\mathrm{ind}_{an}(\mathcal{E})$ involves this vector bundle on B, and the eta-form of Bismut and Cheeger [8, 10]. If $\mathcal{E}$ does not satisfy the technical assumption, we effectively deform it to a cocycle which does, and again define $\mathrm{ind}_{an}(\mathcal{E})$.

We prove that $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$. Our method of proof is to show that one has an equality after pairing both sides of the equation with an arbitrary element of the odd-dimensional K-homology of B. These pairings are given by eta-invariants and the main technical feature of the proof is the computation of adiabatic limits of eta-invariants.

The paper is organized as follows. In Section 2 we define $K^{-1}_{\mathbf{R}/\mathbf{Z}}$, the Chern character on $K^{-1}_{\mathbf{R}/\mathbf{Z}}$, and describe the pairing between $K^{-1}_{\mathbf{R}/\mathbf{Z}}$ and K_{-1} in terms of reduced eta-invariants. Section 3 contains a short digression on the homotopy invariance of eta-invariants, and the vanishing of eta-invariants on manifolds of positive scalar curvature. In Section 4 we define the index maps $\mathrm{ind}_{top}(\mathcal{E})$ and $\mathrm{ind}_{an}(\mathcal{E})$ in $\mathbf{R}/\mathbf{Z}$-valued K-theory, provided that the cocycle $\mathcal{E}$ satisfies the technical assumption. We prove that $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$. In Section 5 we show how to remove the technical assumption. In Section 6 we look at the case when B is a circle and relate ind_{an} to the holonomy of the Bismut-Freed connection on the determinant line bundle. Finally, in Section 7 we briefly discuss the case of odd-dimensional fibers.

I thank Xianzhe Dai and Dan Freed for helpful discussions and the referee for his/her comments. I thank the UC-Berkeley Mathematics Department for its hospitality while this paper was written.

2 R/Z K-Theory

Let M be a smooth compact manifold. Let $\Omega^*(M)$ denote the smooth real-valued differential forms on M.

One way to define $K^0(M)$ (see, for example, [18]) is to say that it is the quotient of the free abelian group generated by complex vector bundles E on

M, by the relations that $E_2 = E_1 + E_3$ if there is a short exact sequence

$$0 \longrightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \longrightarrow 0. \tag{2}$$

Let ∇^E be a connection on a complex vector bundle E. The geometric Chern character of ∇^E, which we will denote by $\mathrm{ch}_{\mathbf{Q}}(\nabla^E) \in \Omega^{even}(M) \otimes \mathbf{C}$, is given by

$$\mathrm{ch}_{\mathbf{Q}}(\nabla^E) = \mathrm{tr}\left(e^{-\frac{(\nabla^E)^2}{2\pi i}}\right). \tag{3}$$

Then $\mathrm{ch}_{\mathbf{Q}}(\nabla^E)$ is a closed differential form which, under the de Rham map, goes to image of the topological Chern character $\mathrm{ch}_{\mathbf{Q}}(E) \in H^{even}(M; \mathbf{Q})$ in $H^{even}(M; \mathbf{C})$.

If ∇_1^E and ∇_2^E are two connections on E, there is a canonically-defined Chern-Simons class $CS(\nabla_1^E, \nabla_2^E) \in (\Omega^{odd}(M) \otimes \mathbf{C})/\mathrm{im}(d)$ [2, Section 4] such that

$$dCS(\nabla_1^E, \nabla_2^E) = \mathrm{ch}_{\mathbf{Q}}(\nabla_1^E) - \mathrm{ch}_{\mathbf{Q}}(\nabla_2^E). \tag{4}$$

To construct $CS(\nabla_1^E, \nabla_2^E)$, let $\gamma(t)$ be a smooth path in the space of connections on E, with $\gamma(0) = \nabla_2^E$ and $\gamma(1) = \nabla_1^E$. Let A be the connection on the vector bundle $[0,1] \times E$, with base $[0,1] \times M$, given by

$$A = dt\, \partial_t + \gamma(t). \tag{5}$$

Then

$$CS(\nabla_1^E, \nabla_2^E) = \int_{[0,1]} \mathrm{ch}_{\mathbf{Q}}(A) \quad (\mathrm{mod}\ \mathrm{im}(d)). \tag{6}$$

One has

$$CS(\nabla_1^E, \nabla_3^E) = CS(\nabla_1^E, \nabla_2^E) + CS(\nabla_2^E, \nabla_3^E). \tag{7}$$

Given a short exact sequence (2) of complex vector bundles on M, choose a splitting map

$$s : E_3 \to E_2. \tag{8}$$

Then

$$i \oplus s : E_1 \oplus E_3 \longrightarrow E_2 \tag{9}$$

is an isomorphism. Suppose that E_1, E_2 and E_3 have connections ∇^{E_1}, ∇^{E_2} and ∇^{E_3}, respectively. We define $CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in (\Omega^{odd}(M) \otimes \mathbf{C})/\mathrm{im}(d)$ by

$$CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = CS\left((i \oplus s)^* \nabla^{E_2}, \nabla^{E_1} \oplus \nabla^{E_3}\right). \tag{10}$$

One can check that $CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$ is independent of the choice of the splitting map s. By construction,

$$dCS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) = \mathrm{ch}_{\mathbf{Q}}(\nabla^{E_2}) - \mathrm{ch}_{\mathbf{Q}}(\nabla^{E_1}) - \mathrm{ch}_{\mathbf{Q}}(\nabla^{E_3}). \qquad (11)$$

Definition 1 *A $\mathbf{C}/\mathbf{Z}$ K-generator of M is a triple*

$$\mathcal{E} = (E, \nabla^E, \omega)$$

where

- *E is a complex vector bundle on M.*
- *∇^E is a connection on E.*
- *$\omega \in (\Omega^{odd}(M) \otimes \mathbf{C})/\mathrm{im}(d)$ satisfies $d\omega = \mathrm{ch}_{\mathbf{Q}}(\nabla^E) - \mathrm{rk}(E)$.*

Definition 2 *A $\mathbf{C}/\mathbf{Z}$ K-relation is given by three $\mathbf{C}/\mathbf{Z}$ K-generators $\mathcal{E}_1$, $\mathcal{E}_2$ and $\mathcal{E}_3$ of M, along with a short exact sequence*

$$0 \longrightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \longrightarrow 0 \qquad (12)$$

such that $\omega_2 = \omega_1 + \omega_3 + CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

Definition 3 *[16, Section 7.5] The group $MK_{\mathbf{C}/\mathbf{Z}}(M)$ is the quotient of the free abelian group generated by the $\mathbf{C}/\mathbf{Z}$ K-generators, by the $\mathbf{C}/\mathbf{Z}$ K-relations $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$. The group $K^{-1}_{\mathbf{C}/\mathbf{Z}}(M)$ is the subgroup of $MK_{\mathbf{C}/\mathbf{Z}}(M)$ consisting of elements of virtual rank zero.*

The group $K^{-1}_{\mathbf{C}/\mathbf{Z}}$ is part of a 2-periodic generalized cohomology theory $K^*_{\mathbf{C}/\mathbf{Z}}$ whose Ω-spectrum $\{G_n\}_{n=-\infty}^{\infty}$ can be described as follows. Consider the map $\mathrm{ch} : BGL \to \prod_{n=1}^{\infty} K(\mathbf{C}, 2n)$ corresponding to the Chern character. Let $\mathcal{G}$ be the homotopy fiber of ch. Then for all $j \in \mathbf{Z}$, $G_{2j} = \mathbf{C}/\mathbf{Z} \times \Omega\mathcal{G}$ and $G_{2j+1} = \mathcal{G}$ [16, Section 7.21].

Definition 4 *We write $K^*_{\mathbf{Z}}(M)$ for the usual K-groups of M, and we put $K^0_{\mathbf{C}}(M) = H^{even}(M; \mathbf{C})$, $K^{-1}_{\mathbf{C}}(M) = H^{odd}(M; \mathbf{C})$.*

There is an exact sequence [16, Section 7.21]

$$\ldots \to K_{\mathbf{Z}}^{-1}(M) \stackrel{\mathrm{ch}}{\to} K_{\mathbf{C}}^{-1}(M) \stackrel{\alpha}{\to} K_{\mathbf{C}/\mathbf{Z}}^{-1}(M) \stackrel{\beta}{\to} K_{\mathbf{Z}}^{0}(M) \stackrel{\mathrm{ch}}{\to} K_{\mathbf{C}}^{0}(M) \to \ldots, \tag{13}$$

where ch is the Chern character,

$$\alpha(\omega) = \Big([\mathbf{C}^N], \nabla^{flat}, \omega\Big) - \Big([\mathbf{C}^N], \nabla^{flat}, 0\Big) \tag{14}$$

and β is the forgetful map.

It will be convenient for us to consider generalized cohomology theories based on Hermitian vector bundles. Let E be a complex vector bundle on M which is equipped with a positive-definite Hermitian metric h^E. A short exact sequence of such Hermitian vector bundles is defined to be a short exact sequence as in (2), with the additional property that $i : E_1 \to E_2$ and $j^* : E_3 \to E_2$ are isometries with respect to the given Hermitian metrics. Then there is an equivalent description of $K^0(M)$ [18, Exercise 6.8, p. 106] as the quotient of the free abelian group generated by Hermitian vector bundles E on M, by the relations $E_2 = E_1 + E_3$ whenever one has a short exact sequence (2) of Hermitian vector bundles. The equivalence essentially follows from the fact that the group of automorphisms of a complex vector bundle E acts transitively on the space of Hermitian metrics h^E.

Hereafter, we will only consider connections ∇^E on E which are compatible with h^E. Then $\mathrm{ch}_{\mathbf{Q}}(\nabla^E) \in \Omega^{even}(M)$, $CS(\nabla_1^E, \nabla_2^E) \in \Omega^{odd}(M)/\mathrm{im}(d)$ and $CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3}) \in \Omega^{odd}(M)/\mathrm{im}(d)$. We can take the splitting map in (8) to be j^*.

Definition 5 *An* $\mathbf{R}/\mathbf{Z}$ *K-generator of M is a quadruple*

$$\mathcal{E} = (E, h^E, \nabla^E, \omega)$$

where

- *E is a complex vector bundle on M.*
- *h^E is a positive-definite Hermitian metric on E.*
- *∇^E is a Hermitian connection on E.*
- *$\omega \in \Omega^{odd}(M)/\mathrm{im}(d)$ satisfies $d\omega = \mathrm{ch}_{\mathbf{Q}}(\nabla^E) - \mathrm{rk}(E)$.*

Definition 6 *An* $\mathbf{R}/\mathbf{Z}$ *K-relation is given by three* $\mathbf{R}/\mathbf{Z}$ *K-generators* $\mathcal{E}_1$, $\mathcal{E}_2$ *and* $\mathcal{E}_3$ *of* M*, along with a short exact sequence of Hermitian vector bundles*

$$0 \longrightarrow E_1 \xrightarrow{i} E_2 \xrightarrow{j} E_3 \longrightarrow 0 \tag{15}$$

such that $\omega_2 = \omega_1 + \omega_3 + CS(\nabla^{E_1}, \nabla^{E_2}, \nabla^{E_3})$.

Definition 7 *The group* $MK_{\mathbf{R}/\mathbf{Z}}(M)$ *is the quotient of the free abelian group generated by the* $\mathbf{R}/\mathbf{Z}$ *K-generators, by the* $\mathbf{R}/\mathbf{Z}$ *K-relations* $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$. *The group* $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ *is the subgroup of* $MK_{\mathbf{R}/\mathbf{Z}}(M)$ *consisting of elements of virtual rank zero.*

A simple extension of the results of [16, Chapter VII] gives that the group $K^{-1}_{\mathbf{R}/\mathbf{Z}}$ is part of a 2-periodic generalized cohomology theory $K^*_{\mathbf{R}/\mathbf{Z}}$ whose Ω-spectrum $\{F_n\}_{n=-\infty}^{\infty}$ is follows. Consider the map $\mathrm{ch} : BU \to \prod_{n=1}^{\infty} K(\mathbf{R}, 2n)$ corresponding to the Chern character. Let $\mathcal{F}$ be the homotopy fiber of ch. Then for all $j \in \mathbf{Z}$, $F_{2j} = \mathbf{R}/\mathbf{Z} \times \Omega\mathcal{F}$ and $F_{2j+1} = \mathcal{F}$.

Definition 8 *We put* $K^0_{\mathbf{R}}(M) = H^{even}(M; \mathbf{R})$ *and* $K^{-1}_{\mathbf{R}}(M) = H^{odd}(M; \mathbf{R})$.

There is an exact sequence

$$\ldots \to K^{-1}_{\mathbf{Z}}(M) \stackrel{\mathrm{ch}}{\to} K^{-1}_{\mathbf{R}}(M) \stackrel{\alpha}{\to} K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \stackrel{\beta}{\to} K^{0}_{\mathbf{Z}}(M) \stackrel{\mathrm{ch}}{\to} K^{0}_{\mathbf{R}}(M) \to \ldots . \tag{16}$$

Remark : As seen above, the Hermitian metrics play a relatively minor role. We would have obtained an equivalent K-theory by taking the generators to be triples (E, ∇^E, ω) where ∇^E is a connection on E with unitary holonomy and ω is as above. That is, ∇^E is consistent with a Hermitian metric, but the Hermitian metric is not specified. The relations would then be given by short exact sequences of complex vector bundles, with the ω's related as above.

It will be useful for us to use $\mathbf{Z}_2$-graded vector bundles. We will take the Chern character of a $\mathbf{Z}_2$-graded Hermitian vector bundle $E = E_+ \oplus E_-$ with Hermitian connection $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ to be

$$\mathrm{ch}_{\mathbf{Q}}(\nabla^E) = \mathrm{ch}_{\mathbf{Q}}(\nabla^{E_+}) - \mathrm{ch}_{\mathbf{Q}}(\nabla^{E_-}). \tag{17}$$

We define the Chern-Simons class $CS(\nabla^E_1, \nabla^E_2)$ similarly.

There is a description of elements of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ by $\mathbf{Z}_2$-graded cocycles, meaning quadruples $\mathcal{E} = (E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega)$ where

- $E = E_+ \oplus E_-$ is a $\mathbf{Z}_2$-graded vector bundle on M.
- $h^E = h^{E_+} \oplus h^{E_-}$ is a Hermitian metric on E.
- $\nabla^E = \nabla^{E_+} \oplus \nabla^{E_-}$ is a Hermitian connection on E.
- $\omega \in \Omega^{odd}(M)/\mathrm{im}(d)$ satisfies $d\omega = \mathrm{ch}_{\mathbf{Q}}(\nabla^E)$.

Given a cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ in the sense of Definition 7, of the form $\sum_i c_i\, \mathcal{E}_i$, one obtains a $\mathbf{Z}_2$-graded cocycle by putting

- $E_\pm = \bigoplus_{\pm c_i > 0} c_i\, E_i$
- $h^{E_\pm} = \bigoplus_{\pm c_i > 0} h^{c_i E_i}$
- $\nabla^{E_\pm} = \bigoplus_{\pm c_i > 0} \nabla^{c_i E_i}$
- $\omega = \sum_i c_i\, \omega_i$.

Conversely, given a $\mathbf{Z}_2$-graded cocycle, let F be a vector bundle on M such that $E_- \oplus F$ is topologically equivalent to the trivial vector bundle $[\mathbf{C}^N]$ for some N. Let (h^F, ∇^F) be a Hermitian metric and Hermitian connection on F. There is a $\Theta \in \Omega^{odd}(M)/\mathrm{im}(d)$ such that $\mathrm{ch}_{\mathbf{Q}}(\nabla^{E_-} \oplus \nabla^F) = N + d\Theta$. Then

$$\left(E_+ \oplus F, h^{E_+} \oplus h^F, \nabla^{E_+} \oplus \nabla^F, \Theta + \omega\right) - \left(E_- \oplus F, h^{E_-} \oplus h^F, \nabla^{E_-} \oplus \nabla^F, \Theta\right)$$

is a cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ in the sense of Definition 7, whose class in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ is independent of the choices made.

An important special type of $\mathbf{Z}_2$-graded cocycle occurs when $\dim(E_+) = \dim(E_-)$, ∇^{E_+} and ∇^{E_-} are flat and $\omega = 0$. In this case, the class of $\mathcal{E}$ in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ lies in the image of a map from algebraic K-theory. (The analogous statement for $\mathbf{C}/\mathbf{Z}$ K-theory is described in detail in [16, Sections 7.9-7.18].) More precisely, let KU^*_{alg} be the generalized cohomology theory whose coefficients are given by the unitary algebraic K-groups of $\mathbf{C}$, and let $\widetilde{KU}^*_{alg}$ be the reduced groups. In particular, $\widetilde{KU}^0_{alg}(M) = [M, BU(\mathbf{C})_\delta{}^+]$, where δ indicates the discrete topology on $U(\mathbf{C})$ and $+$ refers to Quillen's plus construction. The flat Hermitian vector bundle $E_\pm$ on M is classified by a homotopy class of maps $\nu_\pm \in [M, \mathbf{Z} \times BU(\mathbf{C})_\delta]$. There is a homology equivalence

$$\sigma : \mathbf{Z} \times BU(\mathbf{C})_\delta \to \mathbf{Z} \times BU(\mathbf{C})_\delta{}^+$$

and $(\sigma \circ \nu_+ - \sigma \circ \nu_-) \in [M, \mathbf{Z} \times BU(\mathbf{C})_\delta{}^+]$ defines an element $e \in \widetilde{KU}^0_{alg}(M)$. Furthermore, there is a natural transformation $t : \widetilde{KU}^0_{alg}(M) \to K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ and the class of $\mathcal{E}$ in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ is given by $t(e)$.

The spectrum F is a module-spectrum over the K-theory spectrum. The multiplication of $K^0_{\mathbf{Z}}(M)$ on $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ can be described as follows. Let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded cocycle. Let ξ be a vector bundle on M. Let h^ξ be an arbitrary Hermitian metric on ξ and let ∇^ξ be a Hermitian connection on ξ. Put

$$(\xi, h^\xi, \nabla^\xi) \cdot \mathcal{E} = \left(\xi \otimes E_\pm, h^\xi \otimes h^{E_\pm}, (\nabla^\xi \otimes I_\pm) + (I \otimes \nabla^{E_\pm}), \mathrm{ch}_{\mathbf{Q}}(\nabla^\xi) \wedge \omega \right). \tag{18}$$

This extends to a multiplication of $K^0_{\mathbf{Z}}(M)$ on $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$.

There is a homology equivalence $c_{\mathbf{R}/\mathbf{Z}} : \mathcal{F} \to \prod_{n=1}^{\infty} K(\mathbf{R}/\mathbf{Z}, 2n-1)$. Thus one has $\mathbf{R}/\mathbf{Z}$-valued characteristic classes in $\mathbf{R}/\mathbf{Z}$ K-theory. It seems to be difficult to give an explicit description of these classes without using maps to classifying spaces [23]. We will instead describe $\mathbf{R}/\mathbf{Q}$-valued characteristic classes. We will define a map

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}} : K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to H^{odd}(M; \mathbf{R}/\mathbf{Q}) \tag{19}$$

which fits into a commutative diagram

$$\begin{array}{ccccccccc}
\ldots & \to & K^{-1}_{\mathbf{R}}(M) & \stackrel{\alpha}{\to} & K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) & \stackrel{\beta}{\to} & K^0_{\mathbf{Z}}(M) & \to & \ldots \\
 & & -Id. \downarrow & & \mathrm{ch}_{\mathbf{R}/\mathbf{Q}} \downarrow & & \mathrm{ch}_{\mathbf{Q}} \downarrow & & \\
\ldots & \to & H^{odd}(M; \mathbf{R}) & \to & H^{odd}(M; \mathbf{R}/\mathbf{Q}) & \to & H^{even}(M; \mathbf{Q}) & \to & \ldots,
\end{array}$$

where the bottom row is a Bockstein sequence. Upon tensoring everything with $\mathbf{Q}$, it follows from the five-lemma that $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ is a rational isomorphism. (Note that β is rationally zero.)

We define $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ on $MK_{\mathbf{R}/\mathbf{Z}}(M)$. Let $\mathcal{E}$ be an $\mathbf{R}/\mathbf{Z}$ K-generator. Put $N = \mathrm{rk}(E)$. The existence of the form ω implies that the class of $E - [\mathbf{C}^N]$ in $K^0_{\mathbf{Z}}(M)$ has vanishing Chern character. Thus there is a positive integer k such that kE is topologically equivalent to the trivial vector bundle $[\mathbf{C}^{kN}]$ on M. Let ∇^{kE}_0 be a Hermitian connection on kE with trivial holonomy. It follows from the definitions that $\frac{1}{k} CS(k\nabla^E, \nabla^{kE}_0) - \omega$ is an element of $H^{odd}(M; \mathbf{R})$.

Definition 9 *Let* $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E})$ *be the image of* $\frac{1}{k}CS(k\nabla^E, \nabla_0^{kE}) - \omega$ *under the map* $H^{odd}(M;\mathbf{R}) \to H^{odd}(M;\mathbf{R}/\mathbf{Q})$.

Lemma 1 $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E})$ *is independent of the choices of* ∇_0^{kE} *and* k.

Pf. First, let ∇_1^{kE} be another Hermitian connection on kE with trivial holonomy. It differs from ∇_0^{kE} by a gauge transformation specified by a map $g : M \to U(kN)$. We can think of g as specifying a class $[g] \in K_{\mathbf{Z}}^{-1}(M)$. Then $\frac{1}{k}CS(k\nabla^E, \nabla_0^{kE}) - \frac{1}{k}CS(k\nabla^E, \nabla_1^{kE}) = \frac{1}{k}CS(\nabla_1^{kE}, \nabla_0^{kE})$ is the same, up to multiplication by rational numbers, as the image of $\mathrm{ch}_{\mathbf{Q}}([g]) \in H^{odd}(M;\mathbf{Q})$ in $H^{odd}(M;\mathbf{R})$, and so vanishes when mapped into $H^{odd}(M;\mathbf{R}/\mathbf{Q})$. Thus $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E})$ is independent of the choice of ∇_0^{kE}.

Now suppose that k' is another positive integer such that $k'E$ is topologically equivalent to $[\mathbf{C}^{k'N}]$. Let $\nabla_1^{k'E}$ be a Hermitian connection on $k'E$ with trivial holonomy. Then

$$\begin{aligned}
\frac{1}{k}CS(k\nabla^E, \nabla_0^{kE}) - \frac{1}{k'}CS(k'\nabla^E, \nabla_1^{k'E}) &= \frac{1}{kk'}\Big(CS(kk'\nabla^E, k'\nabla_0^{kE}) \\
&\qquad -CS(kk'\nabla^E, k\nabla_1^{k'E})\Big) \\
&= \frac{1}{kk'}CS(k\nabla_1^{k'E}, k'\nabla_0^{kE}). \qquad (20)
\end{aligned}$$

By the previous argument, the image of this in $H^{odd}(M;\mathbf{R}/\mathbf{Q})$ vanishes. Thus $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E})$ is independent of the choice of k. ■

Proposition 1 $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ *extends to a linear map from* $MK_{\mathbf{R}/\mathbf{Z}}(M)$ *to* $H^{odd}(M;\mathbf{R}/\mathbf{Q})$.

Pf. We must show that $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ vanishes on K-relations. Suppose that $\mathcal{E}_2 = \mathcal{E}_1 + \mathcal{E}_3$ is a K-relation. By multiplying the vector bundles by a large enough positive integer, we may assume that E_1, E_2 and E_3 are topologically trivial. Let $\nabla_0^{E_1}$ and $\nabla_0^{E_3}$ be Hermitian connections with trivial holonomy. Using the isometric splitting of E_2 as $E_1 \oplus E_3$, we can take $\nabla_0^{E_2} = \nabla_0^{E_1} \oplus \nabla_0^{E_3}$. It follows that

$$\begin{aligned}
\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}_2) - \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}_1) - \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}_3) &= CS(\nabla^{E_2}, \nabla_0^{E_2}) - CS(\nabla^{E_1}, \nabla_0^{E_1}) \\
&\quad -CS(\nabla^{E_3}, \nabla_0^{E_3}) - \omega_2 + \omega_1 + \omega_3 \\
&= CS(\nabla^{E_2}, \nabla_0^{E_1} \oplus \nabla_0^{E_3}) \\
&\quad -CS(\nabla^{E_1}, \nabla_0^{E_1}) - CS(\nabla^{E_3}, \nabla_0^{E_3}) \\
&\quad -CS(\nabla^{E_2}, \nabla^{E_1} \oplus \nabla^{E_3}) \\
&= 0. \quad ■ \qquad (21)
\end{aligned}$$

One can check that the restriction of $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ to $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ does fit into the commutative diagram, as claimed.

We now describe $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}$ in terms of $\mathbf{Z}_2$-graded cocyles for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$. Let $\mathcal{E} = (E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega)$ be a $\mathbf{Z}_2$-graded cocycle. Let us first assume that E_+ and E_- are topologically equivalent. Let $\mathrm{Isom}(E_+, E_-)$ denote the space of isometries from E_+ to E_-.

Definition 10 *For $j \in \mathrm{Isom}(E_+, E_-)$, put*

$$\mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j) = CS(\nabla^{E_+}, j^* \nabla^{E_-}) - \omega. \tag{22}$$

By construction, $\mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j)$ is an element of $H^{odd}(M; \mathbf{R})$. As above, one can show that it depends on $\mathcal{E}$ only through its class $[\mathcal{E}]$ in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$.

Proposition 2 *We have that $\mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j)$ depends on j only through its class in $\pi_0(\mathrm{Isom}(E_+, E_-))$.*

Pf. : Acting on sections of E_+, we have $j^* \nabla^{E_-} = j^{-1} \nabla^{E_-} j$. Let $j(\epsilon)$ be a smooth 1-parameter family in $\mathrm{Isom}(E_+, E_-)$. From the construction of the Chern-Simons class, we have

$$\begin{aligned}
\frac{d}{d\epsilon} \mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j(\epsilon)) &= \frac{1}{2\pi i} \mathrm{tr} \left(\frac{d}{d\epsilon} (j(\epsilon)^* \nabla^{E_-}) \; e^{-\frac{j(\epsilon)^* (\nabla^{E_-})^2}{2\pi i}} \right) \\
&= \frac{1}{2\pi i} \mathrm{tr} \left((j(\epsilon)^*)^{-1} \frac{d(j(\epsilon)^* \nabla^{E_-})}{d\epsilon} e^{-\frac{(\nabla^{E_-})^2}{2\pi i}} \right) \\
&= \frac{1}{2\pi i} \mathrm{tr} \left([\nabla^{E_-}, \frac{dj(\epsilon)}{d\epsilon} j(\epsilon)^{-1}] e^{-\frac{(\nabla^{E_-})^2}{2\pi i}} \right) \\
&= \frac{1}{2\pi i} \, d \, \mathrm{tr} \left(\frac{dj(\epsilon)}{d\epsilon} j(\epsilon)^{-1} e^{-\frac{(\nabla^{E_-})^2}{2\pi i}} \right).
\end{aligned} \tag{23}$$

Thus $\frac{d}{d\epsilon} \mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j(\epsilon))$ is represented by an exact form and vanishes in $H^{odd}(M; \mathbf{R})$. ■

The topological interpretation of $\mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j)$ is as follows. In terms of (16), the isometry j gives an explicit trivialization of $\beta([\mathcal{E}]) \in K^0_{\mathbf{Z}}(M)$. This lifts $[\mathcal{E}]$ to an element of $K^{-1}_{\mathbf{R}}(M) = H^{odd}(M; \mathbf{R})$, which is given by $\mathrm{ch}_{\mathbf{R}}(\mathcal{E}, j)$.

For a general $\mathbf{Z}_2$-graded cocycle $\mathcal{E} = (E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega)$, there is a positive integer k such that kE_+ is topologically equivalent to kE_-. Let $k\mathcal{E}$

denote the $\mathbf{Z}_2$-graded cocycle $(kE_\pm, kh^{E_\pm}, k\nabla^{E_\pm}, k\omega)$. Choose an isometry $j \in \mathrm{Isom}(kE_+, kE_-)$. Then $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E})$ is the image of $\frac{1}{k}\mathrm{ch}_{\mathbf{R}}(k\mathcal{E}, j)$ under the map $H^{odd}(M; \mathbf{R}) \to H^{odd}(M; \mathbf{R}/\mathbf{Q})$. This is independent of the choices of k and j.

With respect to the product (18), one has

$$\mathrm{ch}_{\mathbf{Q}}(\xi)\, \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}) = \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\xi \cdot \mathcal{E}\right). \tag{24}$$

On general grounds, there is a topological pairing

$$\langle \cdot, \cdot \rangle : K_{-1}(M) \times K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to \mathbf{R}/\mathbf{Z}. \tag{25}$$

We describe this pairing analytically. Recall that cycles for the K-homology group $K_{-1}(M)$ are given by triples $\mathcal{K} = (X, F, f)$ consisting of a smooth closed odd-dimensional spin^c-manifold X, a complex vector bundle F on X and a continuous map $f : X \to M$ [5]. In our case, we may assume that f is smooth. The spin^c-condition on X means that the principal $GL(\dim(X))$-bundle on X has a topological reduction to a principal spin^c-bundle P. There is a Hermitian line bundle L on X which is associated to P. Choosing a soldering form on P [20], we obtain a Riemannian metric on X. Let us choose a Hermitian connection ∇^L on L, a Hermitian metric h^F on F and a Hermitian connection ∇^F on F. Let $\widehat{A}(\nabla^{TX}) \in \Omega^{even}(X)$ be the closed form which represents $\widehat{A}(TX) \in H^{even}(X; \mathbf{Q})$ and let $e^{\frac{c_1(\nabla^L)}{2}} \in \Omega^{even}(X)$ be the closed form which represents $e^{\frac{c_1(L)}{2}} \in H^{even}(X; \mathbf{Q})$. Let S_X denote the spinor bundle of X.

Given a $\mathbf{Z}_2$-graded cocycle $\mathcal{E}$ for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$, let $D_{f^*\nabla^{E_\pm}}$ be the Dirac-type operator acting on L^2-sections of $S_X \otimes F \otimes f^*E_\pm$. Let

$$\overline{\eta}(D_{f^*\nabla^{E_\pm}}) = \frac{\eta(D_{f^*\nabla^{E_\pm}}) + \dim(\mathrm{Ker}(D_{f^*\nabla^{E_\pm}}))}{2} \quad (\mathrm{mod}\, \mathbf{Z}) \tag{26}$$

be its reduced eta-invariant [2, Section 3].

Definition 11 *The reduced eta-invariant of $f^*\mathcal{E}$ on X, an element of $\mathbf{R}/\mathbf{Z}$, is given by*

$$\overline{\eta}(f^*\mathcal{E}) = \overline{\eta}(D_{f^*\nabla^{E_+}}) - \overline{\eta}(D_{f^*\nabla^{E_-}}) - \int_X \widehat{A}(\nabla^{TX}) \wedge e^{\frac{c_1(\nabla^L)}{2}} \wedge \mathrm{ch}_{\mathbf{Q}}(\nabla^F) \wedge f^*\omega. \tag{27}$$

Proposition 3 *Given a cycle $\mathcal{K}$ for $K_{-1}(M)$ and a $\mathbf{Z}_2$-graded cocycle $\mathcal{E}$ for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$, we have*

$$\langle [\mathcal{K}], [\mathcal{E}] \rangle = \overline{\eta}(f^* \mathcal{E}). \tag{28}$$

Pf. The triple $(X, [\mathbf{C}], Id)$ determines a cycle $\mathcal{X}$ for $K_{-1}(X)$, and $[\mathcal{K}] = f_*([F] \cap [\mathcal{X}])$. Then

$$\begin{aligned} \langle [\mathcal{K}], [\mathcal{E}] \rangle &= \langle f_*([F] \cap [\mathcal{X}]), [\mathcal{E}] \rangle = \langle [F] \cap [\mathcal{X}], f^*[\mathcal{E}] \rangle \\ &= \langle [\mathcal{X}], [F] \cdot f^*[\mathcal{E}] \rangle \,. \end{aligned} \tag{29}$$

Without loss of generality, we may assume that $\mathcal{E}$ is defined on X and that F is trivial. We now follow the method of proof of [3, Sections 5-8], where the proposition is proven in the special case when ∇^{E_+} and ∇^{E_-} are flat and ω vanishes. (Theorem 5.3 of [3] is in terms of $K^1(TX)$, but by duality and the Thom isomorphism, this is isomorphic to $K_{-1}(X)$.) By adding a Hermitian vector bundle with connection to both E_+ and E_-, we may assume that E_- is topologically equivalent to a trivial bundle $[\mathbf{C}^N]$. Then E_+ is rationally trivial, and so there is a positive integer k such that both kE_+ and kE_- are topologically equivalent to $[\mathbf{C}^{kN}]$. Choose an isometry $j \in \mathrm{Isom}(kE_+, kE_-)$. As in [2, Section 5], the triple (E_+, E_-, j) defines an element of $K^{-1}_{\mathbf{Z}/k\mathbf{Z}}(X)$, which maps to $K^{-1}_{\mathbf{Q}/\mathbf{Z}}(X)$. The method of proof of [3] is to divide the problem into a real part [3, Section 6] and a torsion part [3, Sections 7-8]. In our case, the torsion part of the proof is the same as in [3, Sections 7-8], and we only have to deal with the modification to [3, Section 6].

Replacing $E_\pm$ by $kE_\pm$, we may assume that E_+ and E_- are topologically trivial, with a fixed isometry j between them. Then $CS(\nabla^{E_+}, j^*\nabla^{E_-}) - \omega$ is an element of $H^{odd}(X; \mathbf{R})$ which, following the notation of [3, p. 89], we write as $b(\mathcal{E}, j)$. As explained in [3, Section 6], under these conditions there is a lifting of $\overline{\eta}(\mathcal{E})$ to an $\mathbf{R}$-valued invariant $\mathrm{ind}(\mathcal{E}, j)$, which vanishes if $\nabla^{E_+} = j^*\nabla^{E_-}$ and $\omega = 0$. Using the variational formula for the eta-invariant [2, Section 4], one finds

$$\mathrm{ind}(\mathcal{E}, j) = \int_X \widehat{A}(\nabla^{TX}) \wedge e^{\frac{c_1(\nabla^L)}{2}} \wedge \Big(CS(\nabla^{E_+}, j^*\nabla^{E_-}) - \omega \Big) \,. \tag{30}$$

Then the analog of [3, Proposition 6.2] holds, and the rest of the proof proceeds as in [3]. ■

Note that if we rationalize (28), we obtain that as elements of $\mathbf{R}/\mathbf{Q}$,

$$\begin{aligned}\overline{\eta}(f^*\mathcal{E}) &= \left\langle \mathrm{ch}_{\mathbf{Q}}([\mathcal{K}]), \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}([\mathcal{E}]) \right\rangle \\ &= \left(\widehat{A}(TX) \cup e^{\frac{c_1(L)}{2}} \cup \mathrm{ch}_{\mathbf{Q}}(F) \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(f^*\mathcal{E}) \right) [X]. \end{aligned} \tag{31}$$

Remark : As mentioned in Definition 3, by removing the Hermitian structures on the vector bundles, one obtains $\mathbf{C}/\mathbf{Z}$-valued K-theory. Although the ensuing Dirac-type operators may no longer be self-adjoint, the reduced eta-invariant can again be defined and gives a pairing $\langle \cdot, \cdot \rangle : K_{-1}(M) \times K^{-1}_{\mathbf{C}/\mathbf{Z}}(M) \to \mathbf{C}/\mathbf{Z}$. In [15], this was used to detect elements of $K_3(R)$ for certain rings R. For analytic simplicity, in this paper we only deal with self-adjoint operators.

3 Homotopy Invariants

Let M be a closed oriented odd-dimensional smooth manifold. Let Γ be a finitely-presented discrete group. As $B\Gamma$ may be noncompact, when discussing a generalized cohomology group of $B\Gamma$ we will mean the representable cohomology, given by homotopy classes of maps to the spectrum, and similarly for generalized homology.

Upon choosing a Riemannian metric g^{TM} on M, the tangential signature operator $\sigma_M = \pm(*d - d*)$ of M defines an element $[\sigma_M]$ of $K_{-1}(M)$ which is independent of the choice of g^{TM}.

Definition 12 *We say that Γ has property (A) if whenever M and M' are manifolds as above, with $f : M' \to M$ an orientation-preserving homotopy equivalence and $\nu \in [M, B\Gamma]$ a homotopy class of maps, there is an equality in $K_{-1}(B\Gamma)$:*

$$\nu_*([\sigma_M]) = (\nu \circ f)_*([\sigma_{M'}]). \tag{32}$$

We say that Γ satisfies the integral Strong Novikov Conjecture ($\mathrm{SNC}_{\mathbf{Z}}$) *if the assembly map*

$$\beta : K_*(B\Gamma) \to K_*(C^*_r\Gamma) \tag{33}$$

is injective, where $C^*_r\Gamma$ is the reduced group C^*-algebra of Γ.

The usual Strong Novikov Conjecture is the conjecture that β is always rationally injective [19, 26]. One knows [19] that

$$\beta(\nu_*([\sigma_M])) = \beta((\nu \circ f)_*([\sigma_{M'}])). \tag{34}$$

Thus $\mathrm{SNC}_{\mathbf{Z}}$ implies property (A). Examples of groups which satisfy $\mathrm{SNC}_{\mathbf{Z}}$ are given by torsion-free discrete subgroups of Lie groups with a finite number of connected components, and fundamental groups of complete Riemannian manifolds of nonpositive curvature [19]. It is conceivable that all torsion-free finitely-presented discrete groups satisfy $\mathrm{SNC}_{\mathbf{Z}}$. Groups with nontrivial torsion elements generally do not have property (A).

Given $\mathcal{E} \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(B\Gamma)$, let $\overline{\eta}_{sig}(\nu^*\mathcal{E}) \in \mathbf{R}/\mathbf{Z}$ denote the reduced eta-invariant of Definition 11, defined using σ_M as the Dirac-type operator.

Proposition 4 *If Γ has property (A) then $\overline{\eta}_{sig}(\nu^*\mathcal{E})$ is an (orientation-preserving) homotopy-invariant of M.*

Pf. This is a consequence of Proposition 3 and Definition 12. ■

Suppose now that M is spin and has a Riemannian metric g^{TM}. Let D_M be the Dirac operator on M, acting on L^2-sections of the spinor bundle. Its class $[D_M]$ in $K_{-1}(M)$ is independent of g^{TM}. Given $\mathcal{E} \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(B\Gamma)$, let $\overline{\eta}_{Dirac}(\nu^*\mathcal{E}) \in \mathbf{R}/\mathbf{Z}$ denote the reduced eta-invariant of Definition 11, defined using D_M.

Proposition 5 *If g^{TM} has positive scalar curvature and Γ satisfies $\mathrm{SNC}_{\mathbf{Z}}$ then $\overline{\eta}_{Dirac}(\nu^*\mathcal{E}) = 0$.*

Pf. From [26], the positivity of the scalar curvature implies that $\beta(\nu_*([D_M]))$ vanishes. Then by the assumption on Γ, we have that $\nu_*([D_M]) = 0$. The proposition now follows from Proposition 3. ■

Let $\rho_\pm : \Gamma \to U(N)$ be two representations of Γ. Let $E_\pm = E\Gamma \times_{\rho_\pm} \mathbf{C}^N$ be the associated flat Hermitian vector bundles on $B\Gamma$. By simplicial methods, one can construct an element $\mathcal{E}$ of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B\Gamma)$ such that $\nu^*\mathcal{E}$ equals the $\mathbf{Z}_2$-graded cocycle on M constructed from the flat Hermitian vector bundles $\nu^*E_\pm$. (If $B\Gamma$ happens to be a manifold then $\mathcal{E}$ can be simply constructed from the flat Hermitian vector bundles $E_\pm$.) Because of the de Rham isomorphism between the kernel of the (twisted) tangential signature operator and the (twisted) cohomology groups of M, in this case one can lift $\overline{\eta}_{sig}(\nu^*\mathcal{E})$ to a real-valued diffeomorphism-invariant $\eta_{sig}(\nu^*\mathcal{E})$ of M [2, Theorem 2.4]. Similarly, let $\mathcal{R}$ denote the space of Riemannian metrics on M and let $\mathcal{R}^+$ denote those with positive scalar curvature. If M is spin then one can lift $\overline{\eta}_{Dirac}(\nu^*\mathcal{E})$ to a real-valued function $\eta_{Dirac}(\nu^*\mathcal{E})$ on $\mathcal{R}$ which is locally constant on $\mathcal{R}^+$ [2, Section 3].

It was shown in [28] that if the L-theory assembly map of Γ is an isomorphism then $\eta_{sig}(\nu^*\mathcal{E})$ is an (orientation-preserving) homotopy-invariant of M. If the assembly map β is an isomorphism (for the maximal group C^*-algebra) then one can show that $\eta_{sig}(\nu^*\mathcal{E})$ is an (orientation-preserving) homotopy-invariant of M, and that $\eta_{Dirac}(\nu^*\mathcal{E})$ vanishes on $\mathcal{R}^+$ [14]. The comparison of these statements with those of Propositions 4 and 5 is the following. Propositions 4 and 5 are more general, in that there may well be elements of $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B\Gamma)$ which do not arise from flat vector bundles. However, when dealing with flat vector bundles the results of [28] and [14] are more precise, as they are statements about unreduced eta-invariants. The results of [28] and [14] can perhaps be best considered to be statements about the terms in the surgery exact sequence [29] and its analog for positive-scalar-curvature metrics [12, 27].

4 Index Maps in R/Z K-Theory

Let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with compact base B, whose fiber Z is even-dimensional and closed. Suppose that TZ has a spin^c-structure. Then π is K-oriented and general methods [11, Chapter 1D] show that there is an Umkehr, or "integration over the fiber", homomorphism

$$\pi_! : K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to K^{-1}_{\mathbf{R}/\mathbf{Z}}(B). \tag{35}$$

To describe $\pi_!$ explicitly, we denote the Thom space of a vector bundle V over a manifold X by X^V, and we denote its basepoint by $*$. Let $i : M \to \mathbf{R}^d$ be an embedding of M. Define an embedding $\widehat{\pi} : M \to B \times \mathbf{R}^d$ by $\widehat{\pi} = \pi \times i$. Let ν be the normal bundle of $\widehat{\pi}(M)$ in $B \times \mathbf{R}^d$. With our assumptions, ν is K-oriented, and as $K_{\mathbf{R}/\mathbf{Z}}$-theory is a module-theory over ordinary K-theory, there is a Thom isomorphism

$$r_1 : K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(M^\nu, *).$$

The collapsing map $B^{B\times\mathbf{R}^d} \to M^\nu$ induces a homomorphism

$$r_2 : K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(M^\nu, *) \to K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(B^{B\times\mathbf{R}^d}, *).$$

Finally, there is a desuspension map

$$r_3 : K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(B^{B\times\mathbf{R}^d}, *) \to K^{-1}_{\mathbf{R}/\mathbf{Z}}(B).$$

The homomorphism $\pi_!$ is the composition

$$K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \xrightarrow{r_1} K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(M^\nu, *) \xrightarrow{r_2} K^{-1+d}_{\mathbf{R}/\mathbf{Z}}(B^{B\times\mathbf{R}^d}, *) \xrightarrow{r_3} K^{-1}_{\mathbf{R}/\mathbf{Z}}(B).$$

For notation, we will also write $\pi_!$ as the topological index :

$$\mathrm{ind}_{top} = \pi_!. \tag{36}$$

Let $\widehat{A}(TZ) \in H^{even}(M;\mathbf{Q})$ be the $\widehat{A}$-class of the vertical tangent bundle TZ. Let $e^{\frac{c_1(L_Z)}{2}} \in H^{even}(M;\mathbf{Q})$ be the characteristic class of the Hermitian line bundle L_Z on M which is associated to the spin^c-structure on TZ. One has

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{ind}_{top}(\mathcal{E})\right) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathcal{E}\right). \tag{37}$$

Give TZ a positive-definite metric g^{TZ}. Let L_Z have a Hermitian connection ∇^{L_Z}. Given a $\mathbf{Z}_2$-graded cocycle $\mathcal{E} = \left(E_\pm, h^{E_\pm}, \nabla^{E_\pm}, \omega\right)$ on M, we have vertical Dirac-type operators $D^Z_{\nabla^{E_\pm}}$. As Z is even-dimensional, for each fiber, the kernels of $D^Z_{\nabla^{E_+}}$ and $D^Z_{\nabla^{E_-}}$ are $\mathbf{Z}_2$-graded vector spaces:

$$\begin{aligned} \mathrm{Ker}(D^Z_{\nabla^{E_+}}) &= \left(\mathrm{Ker}(D^Z_{\nabla^{E_+}})\right)_+ \oplus \left(\mathrm{Ker}(D^Z_{\nabla^{E_+}})\right)_-, \\ \mathrm{Ker}(D^Z_{\nabla^{E_-}}) &= \left(\mathrm{Ker}(D^Z_{\nabla^{E_-}})\right)_+ \oplus \left(\mathrm{Ker}(D^Z_{\nabla^{E_-}})\right)_-. \end{aligned} \tag{38}$$

Assumption 1 *The kernels of $D^Z_{\nabla^{E_\pm}}$ form vector bundles on B.*

That is, we have a $\mathbf{Z}_2$-graded vector bundle Ind on B with

$$\begin{aligned} Ind_+ &= \left(\mathrm{Ker}(D^Z_{\nabla^{E_+}})\right)_+ \oplus \left(\mathrm{Ker}(D^Z_{\nabla^{E_-}})\right)_-, \\ Ind_- &= \left(\mathrm{Ker}(D^Z_{\nabla^{E_+}})\right)_- \oplus \left(\mathrm{Ker}(D^Z_{\nabla^{E_-}})\right)_+. \end{aligned} \tag{39}$$

Then Ind inherits an L^2-Hermitian metric $h^{Ind_\pm}$.

In order to define an analytic index, we put additional structure on the fiber bundle. Let $s \in \mathrm{Hom}(\pi^* TB, TM)$ be a splitting of the exact sequence

$$0 \longrightarrow TZ \longrightarrow TM \longrightarrow \pi^* TB \longrightarrow 0. \tag{40}$$

Putting $T^H M = \mathrm{im}(s)$, we have

$$TM = T^H M \oplus TZ \tag{41}$$

Then there is a canonical metric-compatible connection ∇^{TZ} on TZ [7]. Let $\widehat{A}(\nabla^{TZ}) \in \Omega^{even}(M)$ be the closed form which represents $\widehat{A}(TZ)$. Let $e^{\frac{c_1(\nabla^{L_Z})}{2}} \in \Omega^{even}(M)$ be the closed form which represents $e^{\frac{c_1(L_Z)}{2}}$.

One also has an L^2-Hermitian connection $\nabla^{Ind_\pm}$ on Ind. There is an analytically-defined form $\widetilde{\eta} \in \Omega^{odd}(B)/\mathrm{im}(d)$ such that [8, 10]

$$d\widetilde{\eta} = \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \mathrm{ch}_{\mathbf{Q}}(\nabla^E) - \mathrm{ch}_{\mathbf{Q}}(\nabla^{Ind}). \tag{42}$$

Definition 13 *The analytic index,* $\mathrm{ind}_{an}(\mathcal{E}) \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$*, of* $\mathcal{E}$ *is the class of the* $\mathbf{Z}_2$*-graded cocycle*

$$\mathcal{I} = \left(Ind_\pm, h^{Ind_\pm}, \nabla^{Ind_\pm}, \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega - \widetilde{\eta} \right). \tag{43}$$

It follows from (42) that $\mathcal{I}$ does indeed define a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$. One can show directly that $\mathrm{ind}_{an}(\mathcal{E})$ is independent of the splitting map s. (This will also follow from Corollary 1.)

Proposition 6 *If the* $\mathbf{Z}_2$*-graded cocycle* $\mathcal{E}$ *for* $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ *satisfies Assumption 1 then for all* $x \in K_{-1}(B)$*, we have*

$$\langle x, \mathrm{ind}_{an}(\mathcal{E}) \rangle = \langle x, \mathrm{ind}_{top}(\mathcal{E}) \rangle. \tag{44}$$

Pf. It suffices to show that for all cycles $\mathcal{K} = (X, F, f)$ for $K_{-1}(B)$, we have

$$\langle [\mathcal{K}], \mathrm{ind}_{an}(\mathcal{E}) \rangle = \langle [\mathcal{K}], \mathrm{ind}_{top}(\mathcal{E}) \rangle. \tag{45}$$

As in the proof of Proposition 3, by pulling the fiber bundle and the other structures back to X, by means of f, we may assume that the base of the fiber bundle is X. By changing $\mathcal{E}$ to $(\pi^*F)\cdot\mathcal{E}$, we may assume that F is trivial. That is, $[\mathcal{K}]$ is the fundamental K-homology class x_X of X.

By Proposition 3, we have $\langle x_X, \mathrm{ind}_{an}(\mathcal{E})\rangle = \overline{\eta}(\mathcal{I})$. Let TM have the spin^c-structure which is induced from those of TZ and TX. Let $L_M = L_Z \otimes \pi^* L_X$ be the associated Hermitian line bundle. Let $x_M \in K_{-1}(M)$ be the fundamental K-homology class of M. There is a homomorphism $\pi^! : K_*(X) \to K_*(M)$ which is dual to the Umkehr homomorphism, and one has $\pi^!(x_X) = x_M$. Then

$$\langle x_X, \mathrm{ind}_{top}(\mathcal{E})\rangle = \langle x_X, \pi_!(\mathcal{E})\rangle = \left\langle \pi^!(x_X), \mathcal{E}\right\rangle = \langle x_M, \mathcal{E}\rangle = \overline{\eta}(\mathcal{E}). \tag{46}$$

Thus it suffices to show that as elements of $\mathbf{R}/\mathbf{Z}$, we have

$$\overline{\eta}(\mathcal{I}) = \overline{\eta}(\mathcal{E}). \tag{47}$$

Let g^{TX} be a Riemannian metric on X and let $g^{TM} = g^{TZ} + \pi^* g^{TX}$ be the Riemannian metric on M which is constructed using $T^H M$. Let ∇^{L_X} be a Hermitian connection on L_X and define a Hermitian connection on L_M by

$$\nabla^{L_M} = (\nabla^{L_Z} \otimes I) + (I \otimes \pi^* \nabla^{L_X}). \tag{48}$$

Let $D_{\nabla^{E_\pm}}$ be the Dirac-type operators on M and let $D_{\nabla^{Ind_\pm}}$ be the Dirac-type operators on X. From the definitions, we have

$$\begin{aligned}
\overline{\eta}(\mathcal{E}) &= \overline{\eta}(D_{\nabla^{E_+}}) - \overline{\eta}(D_{\nabla^{E_-}}) - \int_M \widehat{A}\left(\nabla^{TM}\right) \wedge e^{\frac{c_1(\nabla^{L_M})}{2}} \wedge \omega, \\
\overline{\eta}(\mathcal{I}) &= \overline{\eta}(D_{\nabla^{Ind_+}}) - \overline{\eta}(D_{\nabla^{Ind_-}}) - \int_X \widehat{A}\left(\nabla^{TX}\right) \wedge e^{\frac{c_1(\nabla^{L_X})}{2}} \wedge \\
&\quad \left(\int_Z \widehat{A}\left(\nabla^{TZ}\right) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega - \widetilde{\eta}\right).
\end{aligned} \tag{49}$$

Thus

$$\begin{aligned}
\overline{\eta}(\mathcal{E}) - \overline{\eta}(\mathcal{I}) &= \overline{\eta}(D_{\nabla^{E_+}}) - \overline{\eta}(D_{\nabla^{E_-}}) \\
&\quad - \left(\overline{\eta}(D_{\nabla^{Ind_+}}) - \overline{\eta}(D_{\nabla^{Ind_-}}) + \int_X \widehat{A}\left(\nabla^{TX}\right) \wedge e^{\frac{c_1(\nabla^{L_X})}{2}} \wedge \widetilde{\eta}\right) \\
&\quad - \left(\int_M \widehat{A}\left(\nabla^{TM}\right) \wedge e^{\frac{c_1(\nabla^{L_M})}{2}} \wedge \omega - \right. \\
&\quad \left. \int_X \widehat{A}\left(\nabla^{TX}\right) \wedge e^{\frac{c_1(\nabla^{L_X})}{2}} \wedge \int_Z \widehat{A}\left(\nabla^{TZ}\right) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega\right).
\end{aligned} \tag{50}$$

For $\epsilon > 0$, consider a rescaling of the Riemannian metric on X to

$$g_\epsilon^{TX} = \frac{1}{\epsilon^2} g^{TX}. \tag{51}$$

From [10, Theorem 0.1′], in $\mathbf{R}/\mathbf{Z}$ we have

$$\begin{aligned} 0 \quad = \quad & \lim_{\epsilon \to 0} \left[\overline{\eta}(D_{\nabla^{E_+}}) - \overline{\eta}(D_{\nabla^{E_-}}) - \right. \\ & \left. \left(\overline{\eta}(D_{\nabla^{Ind_+}}) - \overline{\eta}(D_{\nabla^{Ind_-}}) + \int_X \widehat{A}\left(\nabla^{TX}\right) \wedge e^{\frac{c_1(\nabla^{L_X})}{2}} \wedge \widetilde{\eta} \right) \right]. \end{aligned} \tag{52}$$

(Theorem $0.1'$ of [10] must be slightly corrected. The correct statement is

$$\lim_{x \to 0} \overline{\eta}(D_x) \equiv \int_B \widehat{A}\left(\frac{R^B}{2\pi}\right) \wedge \widetilde{\eta} + \overline{\eta}(D_B \otimes \mathrm{Ker} D_Y) \quad (\mathrm{mod}\ \mathbf{Z}). \tag{53}$$

This follows from [10, Theorem 0.1] as follows. Following the notation of [10], we have trivially

$$\lim_{x \to 0} \sum_{\lambda_0, \lambda_1 = 0} \mathrm{sign}(\lambda_x) \equiv \lim_{x \to 0} \sum_{\lambda_0, \lambda_1 = 0} 1 \quad (\mathrm{mod}\ 2), \tag{54}$$

and this last term is the number of small nonzero eigenvalues. The total number of small eigenvalues is $\dim(\mathrm{Ker}(D_B \otimes \mathrm{Ker} D_Y))$, and so

$$\lim_{x \to 0} \sum_{\lambda_0, \lambda_1 = 0} \mathrm{sign}(\lambda_x) \equiv \dim(\mathrm{Ker}(D_B \otimes \mathrm{Ker} D_Y)) - \lim_{x \to 0} \dim(\mathrm{Ker}(D_x)) \quad (\mathrm{mod}\ 2).$$

Dividing the result of [10, Theorem 0.1] by 2 and taking the mod $\mathbf{Z}$ reduction yields (53). The stabilization assumption of [10, Theorem 0.1] is not necessary here, as a change in the sign of a small nonzero eigenvalue will change the left-hand-side of (54) by an even number. I thank X. Dai for a discussion of these points.)

Furthermore, in the $\epsilon \to 0$ limit, ∇^{TM} takes an upper-triangular form with respect to the decomposition (41) [8, Section 4a], [10, Section 1.1]. Then the curvature form also becomes upper-triangular. As

$$c_1(\nabla^{L_M}) = c_1(\nabla^{L_Z}) + \pi^* c_1(\nabla^{L_X}), \tag{55}$$

we obtain

$$\begin{aligned} 0 &= \lim_{\epsilon\to 0}\left[\int_M \widehat{A}\left(\nabla^{TM}\right)\wedge e^{\frac{c_1(\nabla^{L_M})}{2}}\wedge\omega - \right. \\ &\left. \int_X \widehat{A}\left(\nabla^{TX}\right)\wedge e^{\frac{c_1(\nabla^{L_X})}{2}}\wedge\int_Z \widehat{A}\left(\nabla^{TZ}\right)\wedge e^{\frac{c_1(\nabla^{L_Z})}{2}}\wedge\omega\right]. \end{aligned} \tag{56}$$

Now $\overline{\eta}(\mathcal{E}) - \overline{\eta}(\mathcal{I})$ is topological in nature, and so is independent of the Riemannian metric on X, and in particular of ϵ. Combining the above equations, (47) follows. ■

Corollary 1 *If the* $\mathbf{Z}_2$*-graded cocycle* $\mathcal{E}$ *for* $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ *satisfies Assumption 1 then* $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$.

Pf. The Universal Coefficient Theorem of [30, eqn. (3.1)] implies that there is a short exact sequence

$$0 \to \mathrm{Ext}(K_{-2}(B), \mathbf{R}/\mathbf{Z}) \to K^{-1}_{\mathbf{R}/\mathbf{Z}}(B) \to \mathrm{Hom}(K_{-1}(B), \mathbf{R}/\mathbf{Z}) \to 0. \tag{57}$$

As $\mathbf{R}/\mathbf{Z}$ is divisible, $\mathrm{Ext}(K_{-2}(B), \mathbf{R}/\mathbf{Z}) = 0$. The corollary follows from Proposition 6. ■

Corollary 2 *If the* $\mathbf{Z}_2$*-graded cocycle* $\mathcal{E}$ *for* $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ *satisfies Assumption 1 then*

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{ind}_{an}(\mathcal{E})\right) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathcal{E}\right). \tag{58}$$

Pf. This follows from Corollary 1 and equation (37). ■

Remark : It follows *a posteriori* from Corollary 1 that if $\mathcal{E}_1$ and $\mathcal{E}_2$ are $\mathbf{Z}_2$-graded cocycles which satisfy Assumption 1 and represent the same class in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ then $\mathrm{ind}_{an}(\mathcal{E}_1) = \mathrm{ind}_{an}(\mathcal{E}_1)$ in $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$.

Remark : Suppose that there is an isometry $j \in \mathrm{Isom}(Ind_+, Ind_-)$. As in Definition 10, we can use j to lift $\mathrm{ind}_{an}(\mathcal{E})$ to $\mathrm{ch}_{\mathbf{R}}(\mathcal{I}, j) \in H^{odd}(B; \mathbf{R})$. In particular, we get a unique such lifting when $Ind_+ = Ind_- = 0$, given by $\int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega - \widetilde{\eta}$.

5 The General Case

In this section we indicate how to remove Assumption 1. The technical trick, taken from [22], is a time-dependent modification of the Bismut superconnection. Let us first discuss eta-invariants and adiabatic limits in general.

Let M be a closed manifold. Let $\mathcal{D}$ be a smooth 1-parameter family of first-order self-adjoint elliptic pseudo-differential operators $D(t)$ on M such that

- There is a $\delta > 0$ and a first-order self-adjoint elliptic pseudo-differential operator D_0 on M such that for $t \in (0, \delta)$, we have $D(t) = \sqrt{t}\, D_0$.
- There is a $\Delta > 0$ and a first-order self-adjoint elliptic pseudo-differential operator D_∞ on M such that for $t > \Delta$, we have $D(t) = \sqrt{t}\, D_\infty$.

For $s \in \mathbf{C}$, $\mathrm{Re}(s) >> 0$, put

$$\eta(\mathcal{D})(s) = \frac{2}{\sqrt{\pi}} \int_0^\infty t^s \,\mathrm{Tr}\left(\frac{dD(t)}{dt} e^{-D(t)^2} \right) dt. \tag{59}$$

Lemma 2 *$\eta(\mathcal{D})(s)$ extends to a meromorphic function on $\mathbf{C}$ which is holomorphic near $s = 0$.*

Pf. Write $\eta(\mathcal{D})(s) = \eta_1(s) + \eta_2(s)$, where

$$\eta_1(s) = \frac{2}{\sqrt{\pi}} \int_0^\infty t^s \,\mathrm{Tr}\left(\frac{D_0}{2\sqrt{t}} e^{-tD_0^2} \right) dt \tag{60}$$

and

$$\begin{aligned} \eta_2(s) &= \frac{2}{\sqrt{\pi}} \int_0^\infty t^s \,\mathrm{Tr}\left(\frac{dD(t)}{dt} e^{-D(t)^2} - \frac{D_0}{2\sqrt{t}} e^{-tD_0^2} \right) dt \\ &= \frac{2}{\sqrt{\pi}} \int_\delta^\infty t^s \,\mathrm{Tr}\left(\frac{dD(t)}{dt} e^{-D(t)^2} - \frac{D_0}{2\sqrt{t}} e^{-tD_0^2} \right) dt. \end{aligned} \tag{61}$$

It is known [13] that $\eta_1(s)$ extends to a meromorphic function on $\mathbf{C}$ which is holomorphic near $s = 0$. It is not hard to see that $\eta_2(s)$ extends to a holomorphic function on $\mathbf{C}$. ■

Define the eta-invariant of $\mathcal{D}$ by

$$\eta(\mathcal{D}) = \eta(\mathcal{D})(0) \tag{62}$$

and the reduced eta-invariant of $\mathcal{D}$ by

$$\overline{\eta}(\mathcal{D}) = \frac{\eta(\mathcal{D}) + \dim(\mathrm{Ker}(D_\infty))}{2} \quad (\mathrm{mod}\ \mathbf{Z}). \tag{63}$$

Lemma 3 *$\eta(\mathcal{D})$ only depends on D_0 and D_∞, and $\overline{\eta}(\mathcal{D})$ only depends on D_0.*

Pf. For $x \in \mathbf{R}$, define

$$\mathrm{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{64}$$

Then $\mathrm{erf}(0) = 0$ and $\mathrm{erf}(\pm\infty) = \pm 1$.

Let $\mathcal{D}_1$ and $\mathcal{D}_2$ be two families such that $(D_1)_0 = (D_2)_0 = D_0$. We may assume that there is a $\delta > 0$ such that for $t \in (0, \delta)$, $D_1(t) = D_2(t) = \sqrt{t} D_0$. Formally, we have

$$\begin{aligned}
\eta(\mathcal{D}_2) - \eta(\mathcal{D}_1) &= \lim_{s \to 0} \frac{2}{\sqrt{\pi}} \int_0^\infty t^s \, \mathrm{Tr} \left(\frac{dD_2(t)}{dt} e^{-D_2(t)^2} - \frac{dD_1(t)}{dt} e^{-D_1(t)^2} \right) dt \\
&= \lim_{s \to 0} \frac{2}{\sqrt{\pi}} \int_\delta^\infty t^s \, \mathrm{Tr} \left(\frac{dD_2(t)}{dt} e^{-D_2(t)^2} - \frac{dD_1(t)}{dt} e^{-D_1(t)^2} \right) dt \\
&= \frac{2}{\sqrt{\pi}} \int_\delta^\infty \mathrm{Tr} \left(\frac{dD_2(t)}{dt} e^{-D_2(t)^2} - \frac{dD_1(t)}{dt} e^{-D_1(t)^2} \right) dt \\
&= \lim_{x \to \infty} \frac{2}{\sqrt{\pi}} \int_\delta^x \mathrm{Tr} \left(\frac{dD_2(t)}{dt} e^{-D_2(t)^2} - \frac{dD_1(t)}{dt} e^{-D_1(t)^2} \right) dt \\
&= \lim_{x \to \infty} \int_\delta^x \frac{d}{dt} \mathrm{Tr} \left(\mathrm{erf}(D_2(t)) - \mathrm{erf}(D_1(t)) \right) dt \\
&= \lim_{x \to \infty} \mathrm{Tr} \left(\mathrm{erf}(D_2(x)) - \mathrm{erf}(D_1(x)) \right) \\
&= \lim_{x \to \infty} \mathrm{Tr} \left(\mathrm{erf}(\sqrt{x}\, (D_2)_\infty) - \mathrm{erf}(\sqrt{x}\, (D_1)_\infty) \right).
\end{aligned} \tag{65}$$

It is not hard to justify the formal manipulations in (65). The first statement of the lemma follows. For the second statement, as $(D_1)_\infty$ and $(D_2)_\infty$ can both be joined to D_0 by a smooth 1-parameter family of first-order self-adjoint elliptic pseudo-differential operators, it follows that there is a smooth 1-parameter family $\{T(\epsilon)\}_{\epsilon\in[1,2]}$ of such operators with $T(1) = (D_1)_\infty$ and $T(2) = (D_2)_\infty$, which can even be taken to be an analytic family. Then

$$\operatorname{Tr}\left(\operatorname{erf}(\sqrt{x}\,(D_2)_\infty) - \operatorname{erf}(\sqrt{x}\,(D_1)_\infty)\right) = \int_1^2 \sqrt{x}\,\operatorname{Tr}\left(\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon. \quad (66)$$

For $\mu > 0$, let $P_\epsilon(\mu)$ be the spectral projection onto the eigenfunctions $\psi_i(\epsilon)$ of $T(\epsilon)$ with eigenvalue $|\lambda_i(\epsilon)| \le \mu$. Then

$$\begin{aligned}\int_1^2 \sqrt{x}\,\operatorname{Tr}\left(\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon &= \int_1^2 \sqrt{x}\,\operatorname{Tr}\left((I - P_\epsilon(\mu))\,\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon \\ &+ \int_1^2 \sqrt{x}\,\operatorname{Tr}\left(P_\epsilon(\mu)\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon. \quad (67)\end{aligned}$$

From the spectral decomposition of $T(\epsilon)$, we have

$$\lim_{x\to\infty}\int_1^2 \sqrt{x}\,\operatorname{Tr}\left((I - P_\epsilon(\mu))\,\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon = 0, \quad (68)$$

showing that

$$\eta(\mathcal{D}_2) - \eta(\mathcal{D}_1) = \lim_{x\to\infty}\int_1^2 \sqrt{x}\,\operatorname{Tr}\left(P_\epsilon(\mu)\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon. \quad (69)$$

From eigenvalue perturbation theory,

$$\int_1^2 \sqrt{x}\,\operatorname{Tr}\left(P_\epsilon(\mu)\frac{dT(\epsilon)}{d\epsilon}e^{-xT(\epsilon)^2}\right)d\epsilon = \int_1^2 \sum_{|\lambda_i(\epsilon)|\le\mu}\frac{d}{d\epsilon}\operatorname{erf}\left(\sqrt{x}\,\lambda_i(\epsilon)\right)d\epsilon. \quad (70)$$

Define the spectral flow of the family $\{T(\epsilon)\}_{\epsilon\in[1,2]}$ as in [3, Section 7]. Taking μ sufficiently small, we see from (69) and (70) that $\eta(\mathcal{D}_2) - \eta(\mathcal{D}_1)$ equals $\dim(\operatorname{Ker}((D_1)_\infty)) - \dim(\operatorname{Ker}((D_2)_\infty))$ plus twice the spectral flow. As the spectral flow is an integer, the lemma follows. ■

In the special case when $D(t) = \sqrt{t}\,D_0$ for all $t > 0$, $\eta(\mathcal{D})$ and $\overline{\eta}(\mathcal{D})$ are the usual eta-invariant and reduced eta-invariant of D_0.

Now let X be a closed spin^c-manifold with a Riemannian metric g^{TX}. Let ∇^L be a Hermitian connection on the associated Hermitian line bundle L. Let S_X be the spinor bundle on X. Let V be a $\mathbf{Z}_2$-graded Hermitian vector bundle on X and let A be a superconnection on V [25, 6]. Explicitly,

$$A = \sum_{j=0}^{\infty} A_{[j]}, \tag{71}$$

where

- A_1 is a grading-preserving connection on V.
- For $k \geq 0$, $A_{[2k]}$ is an element of $\Omega^{2k}(X; \mathrm{End}^{odd}(V))$.
- For $k > 0$, $A_{[2k+1]}$ is an element of $\Omega^{2k+1}(X; \mathrm{End}^{even}(V))$.

We also require that A be Hermitian in an appropriate sense. Let $\overline{A}$ be the self-adjoint Dirac-type operator obtained by "quantizing" A [6, Section 3.3]. This is a linear operator on $C^\infty(X; S_X \otimes V)$ which is essentially given by replacing the Grassmann variables in A by Clifford variables. For $t > 0$, define a rescaled superconnection A_t by

$$A_t = \sum_{j=0}^{\infty} t^{\frac{1-j}{2}} A_{[j]}. \tag{72}$$

Let $\mathcal{A}$ be a smooth 1-parameter family of superconnections $A(t)$ on V. Suppose that

- There is a $\delta > 0$ and a superconnection A_0 on V such that for $t \in (0, \delta)$, we have $A(t) = (A_0)_t$.
- There is a $\Delta > 0$ and a superconnection A_∞ on V such that for $t > \Delta$, we have $A(t) = (A_\infty)_t$.

Suppose that $(A_\infty)_{[0]}$ is invertible. Let $\mathcal{R} : \Omega^*(X) \to \Omega^*(X)$ be the linear operator which acts on a homogeneous form ω by

$$\mathcal{R}\, \omega = (2\pi i)^{-\frac{\deg(\omega)}{2}} \omega. \tag{73}$$

For $s \in \mathbf{C}$, $\mathrm{Re}(s) >> 0$, define $\widetilde{\eta}(\mathcal{A})(s) \in \Omega^{odd}(X)/\mathrm{im}(d)$ by

$$\widetilde{\eta}(\mathcal{A})(s) = (2\pi i)^{-\frac{1}{2}}\, \mathcal{R} \int_0^\infty t^s \, \mathrm{tr}_s \left(\frac{dA(t)}{dt} e^{-A(t)^2} \right) dt. \tag{74}$$

Lemma 4 *$\widetilde{\eta}(\mathcal{A})(s)$ extends to a meromorphic vector-valued function on* $\mathbf{C}$ *with simple poles. Its residue at zero vanishes in $\Omega^{odd}(X)/\mathrm{im}(d)$.*

Pf. As the s-singularities in (74) are a small-t phenomenon, it follows that the poles and residues of $\widetilde{\eta}(\mathcal{A})(s)$ are the same as those of

$$(2\pi i)^{-\frac{1}{2}}\mathcal{R}\int_0^\delta t^s \mathrm{tr}_s \left(\frac{dA(t)}{dt} e^{-A(t)^2} \right) dt = (2\pi i)^{-\frac{1}{2}}\mathcal{R}\int_0^\delta t^s \mathrm{tr}_s \left(\frac{d(A_0)_t}{dt} e^{-(A_0)_t^2} \right) dt. \tag{75}$$

It is known that the right-hand-side of (75) satisfies the claims of the lemma [8, (A.1.5-6)]. ■

Define the eta-form of $\mathcal{A}$ by

$$\widetilde{\eta}(\mathcal{A}) = \widetilde{\eta}(\mathcal{A})(0). \tag{76}$$

As in Lemma 3, $\widetilde{\eta}(\mathcal{A})$ only depends on A_0 and A_∞.

For $\epsilon > 0$, define a family of operators $\mathcal{D}_\epsilon$ by

$$\mathcal{D}_\epsilon(t) = \sqrt{\epsilon t}\ \overline{A(t)_{\frac{1}{\epsilon t}}}. \tag{77}$$

Then a generalization of [8, eqn. (A.1.7)], which we will not prove in detail here, gives

$$\lim_{\epsilon \to 0} \eta(\mathcal{D}_\epsilon) = \int_X \widehat{A}(\nabla^{TX}) \wedge e^{\frac{c_1(\nabla^L)}{2}} \wedge \widetilde{\eta}(\mathcal{A}). \tag{78}$$

Example : Suppose that B is a superconnection on V with $B_{[0]}$ invertible and put $A(t) = B_t$ for all $t > 0$. Then

$$D_\epsilon(t) = \sqrt{\epsilon t}\ \overline{B_{\frac{1}{\epsilon}}}. \tag{79}$$

It follows that

$$\eta(\mathcal{D}_\epsilon) = \eta(\sqrt{\epsilon}\ \overline{B_{\frac{1}{\epsilon}}}), \tag{80}$$

where the right-hand-side of (80) is the eta-invariant of the operator $\sqrt{\epsilon}\ \overline{B_{\frac{1}{\epsilon}}}$ in the usual sense. Similarly, $\widetilde{\eta}(\mathcal{A})$ is the eta-form of the superconnection B in the usual sense. Thus (78) becomes

$$\lim_{\epsilon \to 0} \eta(\sqrt{\epsilon}\ \overline{B_{\frac{1}{\epsilon}}}) = \int_X \widehat{A}(\nabla^{TX}) \wedge e^{\frac{c_1(\nabla^L)}{2}} \wedge \widetilde{\eta}(B), \tag{81}$$

which is the same as [8, eqn. (A.1.7)].
End of Example

Now let $Z \to M \xrightarrow{\pi} X$ be a smooth fiber bundle whose fiber is even-dimensional and closed. Suppose that TZ has a spin^c-structure. As in Section 4, we endow TZ with a positive-definite metric g^{TZ} and L_Z with a Hermitian connection ∇^{L_Z}. Let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded $\mathbf{R}/\mathbf{Z}$-cocycle on M and let $D^Z_{\nabla^E}$ be the vertical Dirac-type operators on the fiber bundle. We no longer suppose that Assumption 1 is satisfied. Let $W = W_+ \oplus W_-$ be the infinite-dimensional $\mathbf{Z}_2$-graded Hermitian vector bundle $\pi_*(S_M \otimes E)$ over X. A standard result in index theory [21] says that there are smooth finite-dimensional subbundles $F_\pm$ of $W_\pm$ and complementary subbundles $G_\pm$ such that $D^Z_{\nabla^E}$ is diagonal with respect to the decomposition $W_\pm = G_\pm \oplus F_\pm$, and writing $D^Z_{\nabla^E} = D_G \oplus D_F$, in addition $D_{G\pm} : C^\infty(G_\pm) \to C^\infty(G_\mp)$ is L^2-invertible. The vector bundle F acquires a Hermitian metric h^F from W. Let ∇^F be a grading-preserving Hermitian connection on F.

Let $T^H M$ be a horizontal distribution on M. One has the Bismut superconnection A_B on W [7], [6, Chapter 10]. Symbolically,

$$A_B = D^Z_{\nabla^E} + \nabla^W - \frac{1}{4} c(T), \tag{82}$$

where ∇^W is a certain Hermitian connection on W and $c(T)$ is Clifford multiplication by the curvature 2-form of the fiber bundle. Put

$$H_\pm = W_\pm \oplus F_\mp = G_\pm \oplus F_\pm \oplus F_\mp. \tag{83}$$

Let $\phi(t) : [0, \infty] \to [0, 1]$ be a smooth bump function such that there exist $\delta, \Delta > 0$ satisfying

- $\phi(t) = 0$ if $t \in (0, \delta)$.
- $\phi(t) = 1$ if $t > \Delta$.

For $\alpha \in \mathbf{R}$, define $R_\pm(t) : C^\infty(H_\pm) \to C^\infty(H_\mp)$ by

$$R_\pm(t) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \alpha\phi(t) \\ 0 & \alpha\phi(t) & 0 \end{pmatrix}. \tag{84}$$

Define a family $\mathcal{A}$ of superconnections on H by

$$A(t) = ((A_B \oplus \nabla^F) + R(t))_t. \tag{85}$$

Put

$$A_0 = A_B \oplus \nabla^F, \quad A_\infty = (A_B \oplus \nabla^F) + R(\infty). \tag{86}$$

Then for $t \in (0, \delta)$,

$$A(t) = (A_0)_t \tag{87}$$

and for $t > \Delta$,

$$A(t) = (A_\infty)_t. \tag{88}$$

Furthermore, $(A_\infty)_{[0]\pm} : C^\infty(H_\pm) \to C^\infty(H_\mp)$ is given by

$$(A_\infty)_{[0]\pm} = \begin{pmatrix} D_{G\pm} & 0 & 0 \\ 0 & D_{F\pm} & \alpha \\ 0 & \alpha & 0 \end{pmatrix}. \tag{89}$$

If α is sufficiently large then $(A_\infty)_{[0]}$ is L^2-invertible. We will assume hereafter that α is so chosen.

We are now formally in the setting described previously in this section. The only difference is that the finite-dimensional vector bundle V is replaced by the infinite-dimensional vector bundle H. Nevertheless, as in [8, Section 4], equations (74)-(78) all carry through to the present setting.

Let g_ϵ^{TX} be the rescaled metric of (51). Let g_ϵ^{TM} be the corresponding metric on M. Let D_{∇^E} be the Dirac-type operator on M, defined using the metric g_ϵ^{TM}. Let D_{∇^F} be the Dirac-type operator on X, defined using the metric g_ϵ^{TX}. Putting

$$D_0 = D_{\nabla^E} \oplus D_{\nabla^F}, \tag{90}$$

we see from (77) that for $t \in (0, \delta)$,

$$D_\epsilon(t) = \sqrt{t}\, D_0. \tag{91}$$

Furthermore, there is a first-order self-adjoint elliptic pseudo-differential operator D_∞ on $M \cup X$ such that for $t > \Delta$,

$$D_\epsilon(t) = \sqrt{t}\, D_\infty. \tag{92}$$

As $\overline{\eta}(\mathcal{D})$ only depends on D_0, it follows that

$$\overline{\eta}(\mathcal{D}_\epsilon) = \overline{\eta}(D_{\nabla^{E_+}}) - \overline{\eta}(D_{\nabla^{E_-}}) - \left(\overline{\eta}(D_{\nabla^{F_+}}) - \overline{\eta}(D_{\nabla^{F_-}})\right), \tag{93}$$

where the terms on the right-hand-side are ordinary reduced eta-invariants. Then equation (78) becomes

$$\lim_{\epsilon \to 0} \left[\overline{\eta}(D_{\nabla^{E_+}}) - \overline{\eta}(D_{\nabla^{E_-}}) - \left(\overline{\eta}(D_{\nabla^{F_+}}) - \overline{\eta}(D_{\nabla^{F_-}})\right) \right] = \int_X \widehat{A}(\nabla^{TX}) \wedge e^{\frac{c_1(\nabla^L)}{2}} \wedge \widetilde{\eta}(\mathcal{A}), \quad (\text{mod } \mathbf{Z}) \tag{94}$$

which is the replacement for (52).

One has

$$d\widetilde{\eta}(\mathcal{A}) = \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \mathrm{ch}_{\mathbf{Q}}(\nabla^E) - \mathrm{ch}_{\mathbf{Q}}(\nabla^F), \tag{95}$$

which is the replacement for equation (42).

Definition 14 *The analytic index,* $\mathrm{ind}_{an}(\mathcal{E}) \in K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$*, of* $\mathcal{E}$ *is the class of the* $\mathbf{Z}_2$*-graded cocycle*

$$\mathcal{I} = \left(F_\pm, h^{F_\pm}, \nabla^{F_\pm}, \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega - \widetilde{\eta}(\mathcal{A}) \right). \tag{96}$$

It follows from (95) that $\mathcal{I}$ does indeed define a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(B)$.

Proposition 7 *For all* $x \in K_{-1}(B)$*, we have*

$$\langle x, \mathrm{ind}_{an}(\mathcal{E}) \rangle = \langle x, \mathrm{ind}_{top}(\mathcal{E}) \rangle. \tag{97}$$

Pf. The proof is virtually the same as that of Proposition 6. ■

Corollary 3 $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$.

Pf. The proof is virtually the same as that of Corollary 1. ■

Corollary 4 *We have*

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{ind}_{an}(\mathcal{E})\right) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathcal{E}). \tag{98}$$

Pf. The proof is virtually the same as that of Corollary 2. ■

6 Circle Base

We now consider the special case of a circle base. Fixing its orientation, S^1 has a unique spin^c-structure. There is an isomorphism $i : K^{-1}_{\mathbf{R}/\mathbf{Z}}(S^1) \to \mathbf{R}/\mathbf{Z}$ which is given by pairing with the fundamental K-homology class of S^1. More explicitly, let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(S^1)$. Then ω is a 1-form on S^1 (mod Im(d)) and E_+ and E_- are both topologically equivalent to a trivial vector bundle $[\mathbf{C}^N]$ on S^1. Choose an isometry $j \in \mathrm{Isom}(E_+, E_-)$. Then

$$i\left([\mathcal{E}]\right) = \int_{S^1} \left(-\frac{1}{2\pi i} \mathrm{tr}(\nabla^{E_+} - j^* \nabla^{E_-}) - \omega \right) \quad (\mathrm{mod}\, \mathbf{Z}). \tag{99}$$

Let $Z \to M \to S^1$ be a fiber bundle as before and let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$. In this special case of a circle base, we can express $\mathrm{ind}_{an}(\mathcal{E})$ in an alternative way. For simplicity, suppose that Assumption 1 is satisfied. There is a determinant line bundle $\mathrm{DET} = (\Lambda^{max}(Ind_+))^* \otimes (\Lambda^{max}(Ind_-))$ on S^1, which is a complex line bundle with a canonical Hermitian metric h^{DET} and compatible Hermitian connection ∇^{DET} [24, 9], [6, Section 9.7]. Let $\mathrm{hol}(\nabla^{DET}) \in U(1)$ be the holonomy of ∇^{DET} around the circle. Explicitly,

$$\mathrm{hol}(\nabla^{DET}) = e^{-\int_{S^1} \nabla^{DET}}. \tag{100}$$

As $\mathrm{ch}_{\mathbf{Q}}(E_+) = \mathrm{ch}_{\mathbf{Q}}(E_-)$, it follows from the Atiyah-Singer index theorem that $\dim(Ind_+) = \dim(Ind_-)$.

Proposition 8 *In* $\mathbf{R}/\mathbf{Z}$*, we have*

$$i\left(\mathrm{ind}_{an}(\mathcal{E})\right) = -\frac{1}{2\pi i} \ln\, \mathrm{hol}(\nabla^{DET}) - \int_M \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega. \tag{101}$$

Pf. Choose an isometry $j \in \mathrm{Isom}(Ind_+, Ind_-)$. From the definition of $\mathrm{ind}_{an}(\mathcal{E})$, in $\mathbf{R}/\mathbf{Z}$ we have

$$i\left(\mathrm{ind}_{an}(\mathcal{E})\right) = \int_{S^1} \left(-\frac{1}{2\pi i} \mathrm{tr}(\nabla^{Ind_+} - j^* \nabla^{Ind_-}) - \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega + \widetilde{\eta} \right). \tag{102}$$

Let ∇^{L^2} denote the L^2-connection on DET. Then

$$-\frac{1}{2\pi i} \int_{S^1} \mathrm{tr}(\nabla^{Ind_+} - j^* \nabla^{Ind_-}) = -\frac{1}{2\pi i} \ln\, \mathrm{hol}(\nabla^{L^2}) \quad (\mathrm{mod}\, \mathbf{Z}). \tag{103}$$

Following the notation of [8], one computes

$$\widetilde{\eta} = -\frac{1}{2}\frac{1}{2\pi i}\int_0^\infty Tr_s\left([\nabla, D_{\nabla^E}]D_{\nabla^E}e^{-uD^2_{\nabla^E}}\right)du. \tag{104}$$

On the other hand,

$$\nabla^{DET} = \nabla^{L^2} + \frac{1}{4}\,d\left(\ln\,\det{}'(D^2_{\nabla^E})\right) - \frac{1}{2}\int_0^\infty Tr_s\left([\nabla, D_{\nabla^E}]D_{\nabla^E}e^{-uD^2_{\nabla^E}}\right)du. \tag{105}$$

Thus

$$-\frac{1}{2\pi i}\,\ln\,\mathrm{hol}(\nabla^{DET}) = -\frac{1}{2\pi i}\,\ln\,\mathrm{hol}(\nabla^{L^2}) + \int_{S^1}\widetilde{\eta} \quad (\mathrm{mod}\,\mathbf{Z}). \tag{106}$$

The proposition follows. ■

The fact that $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$ is now a consequence of the holonomy theorem for ∇^{DET} [9, Theorem 3.16]. Proposition 8 remains true if Assumption 1 is not satisfied.

7 Odd-Dimensional Fibers

Let $Z \to M \xrightarrow{\pi} B$ be a smooth fiber bundle with compact base B, whose fiber Z is odd-dimensional and closed. Suppose that the vertical tangent bundle TZ has a spin^c-structure. As before, there is a topological index map

$$\mathrm{ind}_{top} : K^{-1}_{\mathbf{R}/\mathbf{Z}}(M) \to K^0_{\mathbf{R}/\mathbf{Z}}(B). \tag{107}$$

One can define a Chern character $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}} : K^0_{\mathbf{R}/\mathbf{Z}}(B) \to H^{even}(B; \mathbf{R}/\mathbf{Q})$, and one has

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{ind}_{top}(\mathcal{E})\right) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathcal{E}\right). \tag{108}$$

Let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$. Due to well-known difficulties in constructing analytic indices in the odd-dimensional case, we will not try to define an analytic index $\mathrm{ind}_{an}(\mathcal{E}) \in K^0_{\mathbf{R}/\mathbf{Z}}(B)$, but will instead say what its Chern character should be. Let g^{TZ} be a positive-definite metric on TZ and let ∇^{L_Z} be a Hermitian connection on L_Z. For simplicity, suppose that Assumption 1 is satisfied. Give M a horizontal distribution $T^H M$. Let $\widetilde{\eta} \in \Omega^{even}(B)/\mathrm{im}(d)$ be the difference of the eta-forms associated to (E_+, ∇^{E_+}) and (E_-, ∇^{E_-}). We have [8, 10]

$$d\widetilde{\eta} = \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \mathrm{ch}_{\mathbf{Q}}(\nabla^E). \tag{109}$$

It follows from (109) that $\widetilde{\eta} - \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega$ is an element of $H^{even}(B; \mathbf{R})$.

Definition 15 *The Chern character of the analytic index,* $\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}(\mathrm{ind}_{an})$*, is the image of* $\widetilde{\eta} - \int_Z \widehat{A}(\nabla^{TZ}) \wedge e^{\frac{c_1(\nabla^{L_Z})}{2}} \wedge \omega$ *in* $H^{even}(B; \mathbf{R}/\mathbf{Q})$.

Making minor modifications to the proof of Corollary 2 gives

Proposition 9 *If the* $\mathbf{Z}_2$*-graded cocycle* $\mathcal{E}$ *for* $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$ *satisfies Assumption 1 then*

$$\mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathrm{ind}_{an}(\mathcal{E})\right) = \int_Z \widehat{A}(TZ) \cup e^{\frac{c_1(L_Z)}{2}} \cup \mathrm{ch}_{\mathbf{R}/\mathbf{Q}}\left(\mathcal{E}\right). \tag{110}$$

Consider now the special case when B is a point. There is an isomorphism $i : K^0_{\mathbf{R}/\mathbf{Z}}(\mathrm{pt.}) \to \mathbf{R}/\mathbf{Z}$. Let $\mathcal{E}$ be a $\mathbf{Z}_2$-graded cocycle for $K^{-1}_{\mathbf{R}/\mathbf{Z}}(M)$. Using the Dirac operator corresponding to the fundamental K-homology class of M, define the analytic index $\mathrm{ind}_{an}(\mathcal{E}) \in K^0_{\mathbf{R}/\mathbf{Z}}(\mathrm{pt.})$ of $\mathcal{E}$ by

$$i(\mathrm{ind}_{an}(\mathcal{E})) = \overline{\eta}(\mathcal{E}). \tag{111}$$

Proposition 3 implies that $\mathrm{ind}_{an}(\mathcal{E}) = \mathrm{ind}_{top}(\mathcal{E})$.

References

[1] M. Atiyah and F. Hirzebruch, "The Riemann-Roch Theorem for Analytic Embeddings", Topology 1, p. 151-166 (1962)

[2] M. Atiyah, V. Patodi and I. Singer, "Spectral Asymmetry and Riemannian Geometry II", Math. Proc. Camb. Phil. Soc. 78, p. 405-432 (1975)

[3] M. Atiyah, V. Patodi and I. Singer, "Spectral Asymmetry and Riemannian Geometry III", Math. Proc. Camb. Phil. Soc. 79, p. 71-99 (1976)

[4] M. Atiyah and I. Singer, "The Index of Elliptic Operators IV", Ann. Math. 93, p. 119-138 (1971)

[5] P. Baum and R. Douglas, "K-Homology and Index Theory", in Operator Algebras and Applications I, Proc. Symp. Pure Math. 38, ed. R. Kadison, AMS, Providence, p. 117-173 (1982)

[6] N. Berline, E. Getzler and M. Vergne, Heat Kernels and Dirac Operators, Springer-Verlag, Berlin (1992)

[7] J.-M. Bismut, "The Index Theorem for Families of Dirac Operators: Two Heat Equation Proofs", Inv. Math. 83, p. 91-151 (1986)

[8] J.-M. Bismut and J. Cheeger, "η-Invariants and Their Adiabatic Limits", J. Amer. Math. Soc. 2, p. 33-70 (1989)

[9] J.-M. Bismut and D. Freed, "The Analysis of Elliptic Families II", Comm. Math. Phys. 107, p. 103-163 (1986)

[10] X. Dai, "Adiabatic Limits, Non-Multiplicativity of Signature and Leray Spectral Sequence", J. Amer. Math. Soc. 4, p. 265-321 (1991)

[11] E. Dyer, Cohomology Theories, Benjamin, New York (1969)

[12] P. Gajer, "Concordances of Metrics of Positive Scalar Curvature", Pac. J. of Math. 157, p. 257-268 (1993)

[13] P. Gilkey, "The Residue of the Global Eta Function at the Origin", Adv. Math. 40, p. 290-307 (1981)

[14] N. Higson, to appear

[15] J. Jones and B. Westbury, "Algebraic K-Theory, Homology Spheres and the η-Invariant", Warwick preprint (1993)

[16] M. Karoubi, "Homologie Cyclique et K-Théorie", Astérisque 149 (1987)

[17] M. Karoubi, "Théorie Générale des Classes Caractéristiques Secondaires", K-Theory 4, p. 55-87 (1990)

[18] M. Karoubi, K-Theory, Springer-Verlag, Berlin (1978)

[19] G. Kasparov, "Equivariant KK-Theory and the Novikov Conjecture", Inv. Math. 91, p. 147-201 (1988)

[20] S. Kobayashi and K. Nomizu, "Foundations of Differential Geometry I", Interscience, New York (1963)

[21] A. Mischenko and A. Fomenko, "The Index of Elliptic Operators over C^*-Algebras", Izv. Akad. Nauk SSSR, Ser. Mat. 43, p. 831-859 (1979)

[22] R. Melrose and P. Piazza, "Families of Dirac Operators, Boundaries and the b-Calculus", to appear

[23] O. Pekonen, "Invariants Secondaires des Fibrés Plats", C. R. Acad. Sci. Paris, Série I, t. 304, p. 13-14 (1987)

[24] D. Quillen, "Determinants of Cauchy-Riemann Operators over a Riemann Surface", Funct. Anal. Appl. 19, p. 31-34 (1985)

[25] D. Quillen, "Superconnections and the Chern character", Topology 24, p. 89-95 (1985)

[26] J. Rosenberg, "C^*-Algebras, Positive Scalar Curvature and the Novikov Conjecture", Publ. Math. IHES 58, p. 197-212 (1983)

[27] S. Stolz, to appear

[28] S. Weinberger, "Homotopy Invariance of η-Invariants", Proc. Natl. Acad. Sci. USA 85, p. 5362-5363 (1988)

[29] S. Weinberger, The Topological Classification of Stratified Spaces, University of Chicago Press, Chicago, to appear

[30] Z. Yosimura, "Universal Coefficient Sequences for Cohomology Theories of CW-Spectra", Osaka J. Math. 12, p. 305-323 (1975)