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On the long-time behavior of type-III Ricci flow solutions

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Abstract We show that three-dimensional homogeneous Ricci flow solutions that admit finite-volume quotients have long-time limits given by expanding solitons. We show that the same is true for a large class of four-dimensional homogeneous solutions. We give an extension of Hamilton's compactness theorem that does not assume a lower injectivity radius bound, in terms of Riemannian groupoids. Using this, we show that the long-time behavior of type-III Ricci flow solutions is governed by the dynamics of an $\mathbb{R}^+$-action on a compact space.

1 Introduction

A type-III Ricci flow solution is a 1-parameter family $\{g(t)\}_{t \in (0,\infty)}$ of Riemannian metrics on a manifold M that satisfy the Ricci flow equation and have sectional curvatures that decay at least as fast as t^{-1}, i.e. $\sup_{t \in (0,\infty)} t \parallel \mathrm{Riem}(g_t) \parallel_\infty < \infty$.

In three dimensions Perelman has given important information about the long-time behavior of Ricci flow solutions [22–24], which is especially relevant for topological purposes, but the precise behavior is largely unknown. All known compact three-dimensional Ricci flow solutions that exist for all $t \in (0, \infty)$ are type-III, but it is not known whether this is always the case. Hamilton had shown earlier that the geometrization conjecture holds for such manifolds [14].

This paper is concerned with the long-time behavior of n-dimensional Ricci flow solutions, which we assume to be type-III. Given a Ricci flow solution $g(\cdot)$ and a parameter $s > 0$, there is another Ricci flow solution $g_s(\cdot)$ given by $g_s(t) =$

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$s^{-1} g(st)$. The time interval $[a, b]$ for g_s corresponds to the time interval $[sa, sb]$ for g. Understanding the behavior of $g(t)$ for large t amounts to understanding the behavior of $g_s(\cdot)$ as $s \to \infty$.

We show that in many examples there is a limit as $s \to \infty$ of $g_s(\cdot)$, given by an expanding soliton $g_\infty(\cdot)$. An expanding soliton has the scaling property that $g_\infty(t)$ differs from $t\, g_\infty(1)$ only by the action of a diffeomorphism η_t. That the limit metric is expanding by the factor t may seem contradictory to the fact that there are compact Ricci flow solutions that collapse, so we must explain in what sense there is a limit and where it lives.

For concreteness, let us first discuss the case of a locally homogeneous finite-volume 3-manifold. The lifted flow $\widetilde{g}(\cdot)$ on the universal cover M has been extensively studied; see Isenberg–Jackson [15] and Knopf-McLeod [18]. In order to obtain a limit $\widetilde{g}_\infty(\cdot) = \lim_{s\to\infty} \widetilde{g}_s(\cdot)$ we use *pointed convergence* of Ricci flows. Roughly speaking, instead of comparing metrics on M with respect to a fixed coordinate system, we allow ourselves to transform the metric $\widetilde{g}_s(t)$ by a s-dependent diffeomorphism. In effect, we are choosing coordinates based on what an observer inside of the manifold sees.

Theorem 1.1 *If $\widetilde{g}(\cdot)$ is a homogeneous Ricci flow solution on a three-dimensional simply-connected homogeneous manifold that admits finite-volume quotients, which exists for all $t \in (0, \infty)$, then there is a limit Ricci flow $\widetilde{g}_\infty(\cdot) = \lim_{s\to\infty} \widetilde{g}_s(\cdot)$ which is an expanding soliton solution.*

For each of the three-dimensional homogeneous classes there is a unique limit soliton $\widetilde{g}_\infty(\cdot)$. It may be in a different homogeneity class than the initial metric. The expanding solitons that we find are of type $\mathbb{R}^3$, $\mathbb{R} \times H^2$, H^3, Sol and Nil. If we start with an initial metric of type $\widetilde{\mathrm{Isom}}^+(\mathbb{R}^2)$ or $\widetilde{\mathrm{SL}(2, \mathbb{R})}$ then we end up with an expanding soliton of type $\mathbb{R}^3$ or $\mathbb{R} \times H^2$, respectively. In Sect. 3.4 we extend Theorem 1.1 to the four-dimensional homogeneous metrics considered by Isenberg–Jackson–Lu [16]. Again we find that there are limits $\widetilde{g}_\infty(\cdot) = \lim_{s\to\infty} \widetilde{g}_s(\cdot)$ given by expanding solitons.

In these examples, the metric $\widetilde{g}_\infty(t)$ gives M the structure of a Riemannian submersion whose fiber is a nilpotent Lie group and whose holonomy preserves the affine-flat structure of the fiber. The diffeomorphisms η_t act fiberwise by means of Lie group automorphisms. This is related to the Nil-structure described by Cheeger–Fukaya–Gromov [4] for collapse with bounded sectional curvature, and suggests that the expanding solitons which are relevant for type-III solutions may have a special structure. Based on this, in Sect. 4 we consider the expanding soliton equation in the simplest case of a Nil-structure, namely when a manifold M has a free isometric $\mathbb{R}^N$-action.

Theorem 1.2 *Let M be the total space of a flat $\mathbb{R}^N$-vector bundle over a Riemannian manifold B, with flat Riemannian metrics on the fibers. Suppose that the fiberwise volume forms are preserved by the flat connection. Let $V(t)$ be the fiberwise radial vector field $\frac{1}{2t}\sum_{i=1}^N x^i \frac{\partial}{\partial x_i}$. Then the expanding soliton equation on M becomes the equation for a harmonic map $G : B \to \mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$ along with the equation*

$$R_{\alpha\beta} - \frac{1}{4}\,\mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right) + \frac{1}{2t}\, g_{\alpha\beta} = 0 \tag{1.3}$$

on B.

In writing (1.3) we think of G as taking value in positive-definite symmetric $(N \times N)$-matrices. Using this result, we give relevant examples of expanding solitons.

The results mentioned so far mostly concern limit Ricci flows on noncompact manifolds, which may arise from Ricci flows on compact manifolds upon taking the universal cover. One would also like to construct a limit flow for the compact manifold. Hamilton's compactness theorem gives sufficient conditions for a sequence $\{g_k(\cdot)\}_{k=1}^{\infty}$ of pointed Ricci flow solutions to have a convergent subsequence [13]. However, in order to apply it one needs a uniform lower injectivity radius bound $\mathrm{inj}_{g_k(t_0)}(p_k) \geq i_0 > 0$. In our case this precludes the collapsing situation. In order to obtain a limit flow in the collapsing case one must consider Ricci flow on a larger class of spaces than smooth manifolds. One might try to define Ricci flow on a Gromov–Hausdorff limit space, but this is not very convenient. Instead we will allow the limit Ricci flow to live on a space which in a sense has the same dimension as the original manifold but which takes the collapsing symmetry into account. A convenient language is that of Riemannian groupoids. A Riemannian groupoid is an étale groupoid equipped with an invariant Riemannian metric. Riemannian groupoids have a history in foliation theory, where they are used to describe the transverse structure of Riemannian foliations; see Haefliger [12] and references therein. More recently a similar notion was introduced by Petrunin and Tuschmann under the name "megafold" [25, Appendix], with application to collapsing in Riemannian geometry. Two definitions were given in [25, Appendix], one in terms of topoi and one in terms of pseudogroups. We prefer the Riemannian groupoid language, but all three definitions are essentially equivalent. We give an extension of Hamilton's compactness theorem to the case when there is no positive lower bound on the injectivity radius. The limit Ricci flow will not be on a manifold but rather on a groupoid.

Theorem 1.4 *Let $\{(M_i, p_i, g_i(\cdot))\}_{i=1}^{\infty}$ be a sequence of Ricci flow solutions on pointed n-dimensional manifolds (M_i, p_i). We assume that there are numbers $-\infty \leq A < 0$ and $0 < \Omega \leq \infty$ so that*

1. *The Ricci flow solution $(M_i, p_i, g_i(\cdot))$ is defined on the time interval (A, Ω).*
2. *For each $t \in (A, \Omega)$, $g_i(t)$ is a complete Riemannian metric on M_i.*
3. *For each compact interval $I \subset (A, \Omega)$ there is some $K_I < \infty$ so that $|\mathrm{Riem}(g_i)(x, t)| \leq K_I$ for all $x \in M_i$ and $t \in I$.*

Then after passing to a subsequence, the Ricci flow solutions $g_i(\cdot)$ converge smoothly to a Ricci flow solution $g_\infty(\cdot)$ on a pointed n-dimensional étale groupoid $\left(G_\infty, O_{x_\infty}\right)$, defined again for $t \in (A, \Omega)$.

A result in this direction was proven by Glickenstein [8] who constructed a limit flow on a ball in a single tangent space; groupoids give a way of piecing these limits together for various tangent spaces. Using the results of Sect. 3.3, we show that if $g(\cdot)$ is a Ricci flow on a finite-volume locally homogeneous three-dimensional manifold, that exists for all $t \in \infty$, then $\lim_{s\to\infty} g_s(\cdot)$ exists and is an expanding soliton on a three-dimensional étale groupoid.

Given $K > 0$, the space of pointed n-dimensional Ricci flow solutions on manifolds with $\sup_{t\in(0,\infty)} t \parallel \mathrm{Riem}(g_t) \parallel_\infty \leq K$ is precompact among Ricci flows on pointed n-dimensional étale groupoids. The closure $\mathcal{S}_{n,K}$ has an $\mathbb{R}^+$-action that sends g to g_s. Understanding the long-time behaviour of n-dimensional type-III Ricci flow solutions translates to understanding the dynamics of the $\mathbb{R}^+$-action on $\mathcal{S}_{n,K}$, which seems to be an interesting problem.

The organization of the paper is as follows. In Sect. 2 we give some basic results about expanding solitons. In Sect. 3 we consider the long-time behavior of Ricci flow on homogeneous spaces of dimension one through four. In Sect. 4 we look at the expanding soliton equation on a space with a free isometric $\mathbb{R}^N$-action and reduce it to the harmonic-Einstein equations. In Sect. 5 we recall basic facts about Riemannian groupoids and give the extension of Hamilton's compactness theorem. More detailed descriptions are at the beginnings of the sections.

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2 Expanding solitons

In this section we recall some basic properties of expanding solitons. We also recall the definition of pointed convergence of a sequence of Ricci flows. We define the rescaling $g_s(\cdot)$ of a Ricci flow solution $g(\cdot)$ defined for $t \in (0, \infty)$. We show that if $\{g_s(\cdot)\}_{s>0}$ has a limit as $s \to \infty$ then the limit $g_\infty(\cdot)$ is an expanding soliton.

2.1 Definitions

An *expanding soliton* on a manifold M is a special type of Ricci flow solution on a time interval (t_0, ∞). For convenience, we take $t_0 = 0$. Then the equation for the time-dependent Riemannian metric $g(t)$ and the time-dependent vector field $V(t)$ is

$$\mathrm{Ric} + \frac{\mathcal{L}_V g}{2} + \frac{g}{2t} = 0. \tag{2.1}$$

Also, $V(t) = \frac{1}{t} V(1)$. The corresponding Ricci flow is given by

$$g(t) = t\, \eta_t^* g(1), \tag{2.2}$$

where $\{\eta_t\}_{t>0}$ is the 1-parameter family of diffeomorphisms generated by $\{V(t)\}_{t>0}$, normalized by $\eta_1 = \mathrm{Id}$. (If M is noncompact then we assume that V is such that we can solve for the 1-parameter family.) Conversely, given a solution to the time-independent equation

$$\mathrm{Ric} + \frac{\mathcal{L}_V g}{2} + \frac{g}{2} = 0, \tag{2.3}$$

put $V(t) = \frac{1}{t} V$, solve for $\{\eta_t\}_{t>0}$ and put $g(t) = t\, \eta_t^* g$. Then $(g(t), V(t))$ satisfies (2.1).

If $(M_1, g_1(\cdot))$ and $(M_2, g_2(\cdot))$ are two expanding soliton solutions, with associated diffeomorphisms $\{\eta_t^{(1)}\}_{t>0}$ and $\{\eta_t^{(2)}\}_{t>0}$, then the product flow $(M_1 \times M_2, g_1(\cdot) + g_2(\cdot))$ is an expanding soliton solution with $\eta_t = \left(\eta_t^{(1)}, \eta_t^{(2)}\right)$.

Let $g(\cdot)$ be an expanding soliton on M. Suppose that Γ is a discrete group that acts on M freely, properly discontinuously and isometrically (with respect to the metrics $g(\cdot)$). Then there is an quotient Ricci flow solution $\overline{g}(\cdot)$ on M/Γ. If Γ also preserves the vector fields $V(\cdot)$ then $\overline{g}(\cdot)$ is an expanding soliton, but this does not have to be the case.

2.2 Expanding solitons as long-time limits

Let (M, p) be a connected manifold with a basepoint p. Let $\{g(t)\}_{t\in(0,\infty)}$ be a Ricci flow solution on M. We assume that for each $t > 0$, the pair $(M, g(t))$ is a complete Riemannian manifold. If $\{(M_i, p_i, g_i(\cdot))\}_{i=1}^{\infty}$ is a sequence of such Ricci flow solutions then there is a notion of *pointed convergence* to a limit Ricci flow solution $(M_\infty, p_\infty, g_\infty(\cdot))$, as considered in [13]. In our case, this means that one has

1. A sequence of open subsets $\{U_j\}_{j=1}^{\infty}$ of M_∞ containing p_∞, so that any compact subset of M_∞ eventually lies in all U_j, and
2. Time-independent diffeomorphisms $\phi_{i,j} : U_j \to V_{i,j}$ from U_j to open subsets $V_{i,j} \subset M_i$, with $\phi_{i,j}(p_\infty) = p_i$, so that
3. For all j, $\lim_{i\to\infty} \phi_{i,j}^* g_i(\cdot) = g_\infty(\cdot)\big|_{U_j}$ smoothly on $U_j \times [j^{-1}, j]$.

The compactness theorem of [13] implies the following. Suppose that

1. For each compact interval $I \subset (0, \infty)$ there is some $K_I < \infty$ so that $|\operatorname{Riem}|(x, t) \leq K_I$ for all $x \in M_i$ and $t \in I$.
2. There are some $t_0 > 0$ and $i_0 > 0$ so that for all i, $\operatorname{inj}_{g_i(t_0)}(p_i) \geq i_0$.

Then $\{g_i(\cdot)\}_{i=1}^{\infty}$ has a convergent subsequence.

Given a 1-parameter family $\{M, p, g_s(\cdot)\}_{s>0}$ of Ricci flow solutions, there is an analogous notion of convergence as $s \to \infty$, i.e. for any sequence $\{s_j\}_{j=1}^{\infty}$ converging to infinity the sequence $\{M, p, g_{s_j}(\cdot)\}_{j=1}^{\infty}$ converges and the limit is independent of the choice of $\{s_j\}_{j=1}^{\infty}$. Hence it makes sense to talk about having a limit solution $\lim_{s\to\infty}(M, p, g_s(\cdot)) = (M_\infty, p_\infty, g_\infty(\cdot))$

Now suppose that we have a type-III Ricci flow solution $(M, p, g(\cdot))$, meaning that $\sup_{t\in(0,\infty)} t \;\| \operatorname{Riem}(g_t) \|_\infty < \infty$. For any $s > 0$, there is a rescaled Ricci flow solution $(M, p, g_s(\cdot))$ given by $g_s(t) = s^{-1} g(st)$. We will consider the convergence or subconvergence of $(M, p, g_s(\cdot))$ as $s \to \infty$. It is important to note that although all of the Ricci flow solutions $(M, p, g_s(\cdot))$ live on the same manifold M, the notion of convergence is not that of smooth metrics on M. Instead, we are interested in *pointed convergence* as defined above.

Lemma 2.4 *If* $\liminf_{t\to\infty} t^{-\frac{1}{2}} \operatorname{inj}_{g(t)}(p) > 0$ *then any sequence* $\{s_i\}_{i=1}^{\infty}$ *converging to infinity has a subsequence, which we again denote by* $\{s_i\}_{i=1}^{\infty}$*, so that* $\lim_{i\to\infty} (M, p, g_{s_i}(\cdot)) = (M_\infty, p_\infty, g_\infty(\cdot))$ *for some Ricci flow solution* $(M_\infty, p_\infty, g_\infty(\cdot))$ *defined for* $t \in (0, \infty)$.

Proof This is an immediate consequence of Hamilton's compactness theorem. □

We now consider what happens if there actually is a limit.

Proposition 2.5 *If* $\lim_{s\to\infty}(M, p, g_s(\cdot)) = (M_\infty, p_\infty, g_\infty(\cdot))$ *then* $(M_\infty, g_\infty(\cdot))$ *is an expanding soliton.*

Proof Let $\mathcal{M}$ denote the space of pointed Riemannian metrics on M_∞, with the topology of smooth convergence on compact subsets. The Ricci flow solution $g_\infty(\cdot)$ defines a smooth curve in $\mathcal{M}$. Given $t, \alpha > 0$, we can formally write (modulo diffeomorphisms)

$$g_\infty(\alpha t) = \lim_{s\to\infty} s^{-1}\, g(s\alpha t) = \lim_{s\to\infty} \alpha\, s^{-1}\, g(st) = \alpha\, g_\infty(t). \tag{2.6}$$

More precisely, for any $R > 0$ and $\epsilon > 0$ there is a pointed (= basepoint-preserving) diffeomorphism $\phi_{R,\epsilon}$ from the time-αt ball $B_R(p_\infty) \subset M_\infty$ to a subset $V_{R,\epsilon} \subset M_\infty$ such that $\alpha\phi^*_{R,\epsilon} g_\infty(t)\big|_{V_{R,\epsilon}}$ is ϵ-close in the smooth topology to $g_\infty(\alpha t)\big|_{B_R(p_\infty)}$. Taking the limit of an appropriate sequence of the $\phi_{R,\epsilon}$'s, we obtain a pointed diffeomorphism $\phi : M_\infty \to M_\infty$ such that $\alpha\, \phi^* g_\infty(t) = g_\infty(\alpha t)$.

Let $\mathrm{Diff}_{p_\infty}(M_\infty)$ denote the pointed diffeomorphisms of M_∞, again with the topology of smooth convergence on compact subsets. We have shown that for all $t > 0$, the metric $t^{-1}\, g_\infty(t)$ lies in the $\mathrm{Diff}_{p_\infty}(M_\infty)$-orbit of $g_\infty(1)$. As in [2], the $\mathrm{Diff}_{p_\infty}(M_\infty)$-orbit of $g_\infty(1)$ is the image of a proper embedding of the smooth infinite-dimensional manifold $\mathrm{Diff}_{p_\infty}(M_\infty)/\,\mathrm{Isom}_{p_\infty}(g_\infty(1))$ in $\mathcal{M}$. (Strictly speaking the paper [2] deals with compact manifolds.) Hence the smooth curve $t \to t^{-1}\, g_\infty(t)$ defines a smooth curve in $\mathrm{Diff}_{p_\infty}(M_\infty)/\,\mathrm{Isom}_{p_\infty}(g_\infty(1))$, which we can lift to a smooth curve in $\mathrm{Diff}_{p_\infty}(M_\infty)$.

Thus we have found a smooth 1-parameter family of pointed diffeomorphisms $\{\eta_t\}_{t>0}$ so that (2.2) is satisfied for $g_\infty(\cdot)$. Letting $\{V(t)\}_{t>0}$ be the generator of $\{\eta_t\}_{t>0}$, equation (2.1) is satisfied. Substituting (2.2) into (2.1) gives $t\, \mathcal{L}_{V(t)} g_\infty(1) = \mathcal{L}_{V(1)} g_\infty(1)$. Hence we may assume that $V(t) = \frac{1}{t}\, V(1)$ and redefine $\{\eta_t\}_{t>0}$. This proves the proposition. □

3 Homogeneous solutions

In this section we consider the Ricci flow on simply-connected homogeneous Riemannian manifolds of dimension one through four that admit finite-volume quotients and exist for all $t \in (0, \infty)$. In dimensions one through three we show that in all cases there is a limit Ricci flow solution $g_\infty(\cdot) = \lim_{s\to\infty} \phi_s^* g_s(\cdot)$ given by an expanding soliton. We compute the soliton metric explicitly. In dimension four we show that this is also true for the homogeneous metrics considered in [16]. The main task in all of these cases is to construct appropriate diffeomorphisms ϕ_s.

A pointed Gromov–Hausdorff limit of a sequence of homogeneous manifolds is still homogeneous [9, Corollary on p. 66]. Hence if $(M, p, g(\cdot))$ is a homogeneous Ricci flow solution then assuming that the limit exists, we know that $(M_\infty, p_\infty, g_\infty(\cdot)) =$

$\lim_{s\to\infty}(M, p, g_s(\cdot))$ is also homogeneous. However, the isometry group may change in the limit.

We now examine the long-time limits for homogeneous Ricci flow solutions of dimensions one through four. The manifolds that we consider are simply-connected homogeneous spaces $M = G/H$, where G is a transitive group of diffeomorphisms of M and H is the isotropy subgroup, assumed to be compact. We will assume that G is connected and unimodular, i.e. has a bi-invariant Haar measure. This will be the case if M admits finite-volume quotients. We take the basepoint p to be the identity coset eH. The Riemannian metrics that we consider on G/H will be left-invariant.

Given the manifold M, there are various groups $G \subset \mathrm{Diff}(M)$ that act transitively on M with compact isotropy group. We wish to take *minimal* such groups, i.e. no proper subgroup of G acts transitively on M. This allows for the widest class of Ricci flow solutions. However, we must note that a compact quotient of M may be of the form $\Gamma\backslash M$ where Γ is a freely-acting discrete subgroup of some larger such group G' containing G. For this reason, for the purposes of the geometrization conjecture one generally takes G to be a *maximal* element among the groups of diffeomorphisms of M that act transitively with compact isotropy group [27, Sect. 5], [28, Chap. 3].

Given a homogeneous Ricci flow solution $g(\cdot)$, the question is whether we can find pointed diffeomorphisms $\{\phi_s\}_{s>0}$ so that there is a limit Ricci flow solution $g_\infty(\cdot) = \lim_{s\to\infty} \phi_s^* g_s(\cdot)$, where $g_s(t) = \frac{1}{s} g(st)$. By Proposition 2.5, the limit will necessarily be an expanding homogeneous soliton solution.

Remark 3.1 We will see examples of expanding solitons on Lie groups G with the property that the rescaling diffeomorphisms $\{\eta_t\}_{t>0}$ arise from a 1-parameter group $\{a_t\}_{t>0}$ of automorphisms of G, by $\eta_t = a_{t^{-1}}$. If so, let Γ be a discrete subgroup of G. Then $\Gamma\backslash G$ with the quotient metric $\overline{g}(t)$ is isometric to the result of quotienting $(G, tg(1))$ on the left by the subgroup $a_{t^{-1}}(\Gamma)$. Thus we can basically either think of the metric as evolving, or of the discrete group as evolving.

3.1 One dimension

The manifold M is $\mathbb{R}$, with $(G, H) = (\mathbb{R}, \{e\})$. The basepoint is $0 \in \mathbb{R}$. The Ricci flow solution $g(t)$ is constant in t, equaling a flat metric g_0. Then $g_s(t) = s^{-1} g_0$. Let ϕ_s be multiplication by $\sqrt{s}$ on $\mathbb{R}$. Then $\phi_s^* g_s(t) = g_0$. Hence there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$ given by $g_\infty(t) = g_0$. We note that this is an expanding soliton solution, with η_t being multiplication on $\mathbb{R}$ by $t^{-\frac{1}{2}}$.

The quotient $S^1 = \mathbb{Z}\backslash\mathbb{R}$ has the constant Ricci flow solution $(S^1, \overline{g}(t))$. We can consider $(S^1, \overline{g}(t))$ to be isometric to the quotient of $(\mathbb{R}, tg_0)$ by $a_{t^{-1}}(\mathbb{Z})$, where a_t is the automorphism of $\mathbb{R}$ given by multiplication by $\sqrt{t}$.

3.2 Two dimensions

The possible homogeneous spaces are S^2, $\mathbb{R}^2$ and H^2. Their pairs (G, H) are $(\mathrm{SO}(3), \mathrm{SO}(2))$, $(\mathbb{R}^2, \{e\})$ and $(\mathrm{Isom}^+(H^2), \mathrm{SO}(2))$. The Ricci flow on S^2 has finite extinction time, so we do not consider it further. The case $\mathbb{R}^2$ is a product case, and so has already been covered.

For the H^2 case, let g_0 be a complete constant-curvature metric on the plane with $\mathrm{Ric}\,(g_0) = -cg_0$ for some $c > 0$. The Ricci flow solution starting at g_0 is given by $g\,(t) = (1 + 2ct)\, g_0$. Then $g_s(t) = s^{-1}\,(1 + 2cst)\, g_0$. Taking $\phi_s = \mathrm{Id}$, there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by $g_\infty(t) = 2ctg_0$. This is independent of c and is an expanding soliton solution with $V = 0$.

3.3 Three dimensions

A homogeneous Ricci flow on S^3 or $S^2 \times \mathbb{R}$ has finite extinction time, so we do not consider it further. The homogeneous spaces $\mathbb{R}^3$ and $H^2 \times \mathbb{R}$ are product cases. By the previous discussion, after appropriate rescaling their Ricci flows have expanding soliton limits.

We now list the cases $M = G/H$ by the group G.

3.3.1 $G = \mathrm{Isom}^+(H^3)$

The group G is the connected component of the identity in SO(3, 1). The subgroup H is SO(3). Let g_0 be a complete constant-curvature metric on $\mathbb{R}^3$ with $\mathrm{Ric}\,(g_0) = -cg_0$ for some $c > 0$. The Ricci flow solution starting at g_0 is given by $g\,(t) = (1 + 2ct)\, g_0$. Then $g_s(t) = s^{-1}\,(1 + 2cst)\, g_0$. Taking $\phi_s = \mathrm{Id}$, there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by $g_\infty(t) = 2ctg_0$. This is independent of c and is an expanding soliton solution with $V = 0$.

The remaining cases have trivial isotropy group H, i.e. $M = G$. It is known that M admits a Milnor frame, i.e. a left-invariant orthonormal frame field $\{X_1, X_2, X_3\}$ so that $[X_i, X_j] = \sum_k c^k{}_{ij} X_k$ with $c^k{}_{ij}$ vanishing unless i, j and k are mutually distinct. In this basis, the nonzero components of the curvature tensor are of the form K_{ijij}. If $\{\theta^1, \theta^2, \theta^3\}$ is the dual orthonormal coframe then the Ricci flow solution can be written in the form

$$g\,(t) = A\,(t) \left(\theta^1\right)^2 + B\,(t) \left(\theta^2\right)^2 + C\,(t) \left(\theta^3\right)^2. \tag{3.2}$$

We write $A\,(0) = A_0$, $B\,(0) = B_0$ and $C\,(0) = C_0$.

In what follows, we use computations from [15] and [18]. We note that the metrics in [15] and [18] different by a constant. Our normalizations will be those of [15]. However, we will use a Milnor basis as in [18]. The simpler solutions are listed first.

3.3.2 $G = \mathrm{Sol}$

The group G is a semidirect product $\mathbb{R}^2 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by $z \cdot (x, y) = (e^z x, e^{-z} y)$. The subgroup H is trivial. After a change of basis, the Lie algebra relations are $[X_2, X_3] = X_1$, $[X_3, X_1] = 0$ and $[X_1, X_2] = -\,X_3$. The $\mathbb{R}^2$-factor is spanned by X_1 and X_3, and the $\mathbb{R}$-factor is spanned by X_2.

The metric is

$$g(t) = A(t)(\theta^1)^2 \; + \; B(t)(\theta^2)^2 \; + \; C(t)(\theta^3)^2, \tag{3.3}$$

where

$$d\theta^1 = -\,\theta^2 \wedge \theta^3, \quad d\theta^2 = 0, \quad d\theta^3 = \theta^1 \wedge \theta^2. \tag{3.4}$$

The sectional curvatures are

$$\begin{aligned} K_{12} &= \frac{(A-C)^2 - 4C^2}{4ABC} \\ K_{23} &= \frac{(A-C)^2 - 4A^2}{4ABC} \\ K_{31} &= \frac{(A+C)^2}{4ABC}. \end{aligned} \tag{3.5}$$

The Ricci flow is given by

$$\begin{aligned} \frac{dA}{dt} &= \frac{C^2 - A^2}{BC} \\ \frac{dB}{dt} &= \frac{(A+C)^2}{AC} \\ \frac{dC}{dt} &= \frac{A^2 - C^2}{AB}. \end{aligned} \tag{3.6}$$

From [18] the large-t asymptotics are $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} C(t) = \sqrt{A_0 C_0}$ and $B(t) \sim 4t$. Then

$$g_s(t) \sim s^{-1} \sqrt{A_0 C_0} \left((\theta^1)^2 \; + \; (\theta^3)^2 \right) \; + \; 4t \, (\theta^2)^2. \tag{3.7}$$

We take coordinates (x, y, z) for G in which $\theta^1 + \theta^3 = e^{-z}\, dx$, $\theta^1 - \theta^3 = e^z\, dy$ and $\theta^2 = dz$. Define diffeomorphisms $\phi_s : \mathbb{R}^3 \to G$ by

$$\phi_s(x, y, z) = \left((A_0 C_0)^{-\frac{1}{4}} \sqrt{s} x, \, (A_0 C_0)^{-\frac{1}{4}} \sqrt{s} y, \, z \right). \tag{3.8}$$

Then there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by

$$g_\infty(t) = (\theta^1)^2 \; + \; (\theta^3)^2 \; + \; 4t \, (\theta^2)^2. \tag{3.9}$$

This is an expanding soliton solution with $\eta_t(x, y, z) = \left(t^{-\frac{1}{2}} x, t^{-\frac{1}{2}} y, z \right)$. Its geometry is a Sol-geometry. We note that it is *not* a gradient expanding soliton. The equation for the soliton also appeared in [1].

Example 3.10 An example of a Sol-manifold is given by the total space of a T^2-bundle over S^1 whose monodromy is a hyperbolic element of $\mathrm{SL}(2,\mathbb{Z})$. Geometrically, the long-time asymptotics of its Ricci flow amount to shrinking the T^2 fiber by a factor of $\sqrt{t}$ and then multiplying the overall metric by t.

3.3.3 Nil

The group G is a nontrivial central $\mathbb{R}$-extension of $\mathbb{R}^2$. The subgroup H is trivial. The Lie algebra relations are $[X_2, X_3] = -X_1$ and $[X_3, X_1] = [X_1, X_2] = 0$. The $\mathbb{R}$-factor is spanned by X_1, and the $\mathbb{R}^2$-factor is spanned by X_2 and X_3.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2, \tag{3.11}$$

where

$$d\theta^1 = \theta^2 \wedge \theta^3, \quad d\theta^2 = 0, \quad d\theta^3 = 0. \tag{3.12}$$

The sectional curvatures are

$$\begin{aligned} K_{12} &= \frac{A}{4BC} \\ K_{23} &= -\frac{3A}{4BC} \\ K_{31} &= \frac{A}{4BC}. \end{aligned} \tag{3.13}$$

The Ricci flow is given by

$$\begin{aligned} \frac{dA}{dt} &= -\frac{A^2}{BC} \\ \frac{dB}{dt} &= \frac{A}{C} \\ \frac{dC}{dt} &= \frac{A}{B}. \end{aligned} \tag{3.14}$$

The solution is

$$\begin{aligned} A &= A_0 \left(1 + \frac{3A_0}{B_0C_0} t\right)^{-1/3} \\ B &= B_0 \left(1 + \frac{3A_0}{B_0C_0} t\right)^{1/3} \\ C &= C_0 \left(1 + \frac{3A_0}{B_0C_0} t\right)^{1/3}. \end{aligned} \tag{3.15}$$

Then

$$g_s(t) = s^{-1}A_0\left(1+\frac{3A_0}{B_0C_0}st\right)^{-1/3}\left(\theta^1\right)^2 + s^{-1}B_0\left(1+\frac{3A_0}{B_0C_0}st\right)^{1/3}\left(\theta^2\right)^2 + s^{-1}C_0\left(1+\frac{3A_0}{B_0C_0}st\right)^{1/3}\left(\theta^3\right)^2 \sim \left(\frac{A_0^2B_0C_0}{3}\right)^{1/3} s^{-\frac{4}{3}}\, t^{-\frac{1}{3}}\left(\theta^1\right)^2 + \left(\frac{3A_0B_0^2}{C_0}\right)^{1/3} s^{-\frac{2}{3}}\, t^{\frac{1}{3}}\left(\theta^2\right)^2 + \left(\frac{3A_0C_0^2}{B_0}\right)^{1/3} s^{-\frac{2}{3}}\, t^{\frac{1}{3}}\left(\theta^3\right)^2. \tag{3.16}$$

We take coordinates (x, y, z) for G in which $\theta^1 = dx + \frac{1}{2}y\,dz - \frac{1}{2}z\,dy, \theta^2 = dy$ and $\theta_3 = dz$. Define diffeomorphisms $\phi_s : \mathbb{R}^3 \to G$ by

$$\phi_s(x, y, z) = \left(\left(9A_0^2B_0C_0\right)^{-1/6} s^{2/3}x, \left(\frac{3A_0B_0^2}{C_0}\right)^{-1/6} s^{1/3}y, \left(\frac{3A_0C_0^2}{B_0}\right)^{-1/6} s^{1/3}z\right). \tag{3.17}$$

Then there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by

$$g_\infty(t) = \frac{1}{3t^{1/3}}\left(\theta^1\right)^2 + t^{1/3}\left(\left(\theta^2\right)^2 + \left(\theta^3\right)^2\right). \tag{3.18}$$

This is an expanding soliton solution with $\eta_t(x, y, z) = \left(t^{-\frac{2}{3}}x, t^{-\frac{1}{3}}y, t^{-\frac{1}{3}}z\right)$. Its geometry is a Nil-geometry. The equation for the soliton also appeared in [1] and, implicitly, in [19].

Example 3.19 If Γ is a lattice in Nil, consider any locally homogeneous Ricci flow $\overline{g}(\cdot)$ on $M = \Gamma\backslash\,\text{Nil}$. As $t \to \infty$, $(M, \overline{g}(t))$ will approach the quotient of $(\text{Nil}, tg_\infty(1))$ by the subgroup $a_{t^{-1}}(\Gamma')$, where a_t is the automorphism of Nil given by $a_t(x, y, z) = \left(t^{\frac{2}{3}}x, t^{\frac{1}{3}}y, t^{\frac{1}{3}}z\right)$ and Γ' is a subgroup of Nil that is isomorphic to Γ.

3.3.4 $G = \widetilde{\text{Isom}^+(\mathbb{R}^2)}$

The group G is the universal cover of the orientation-preserving isometries of $\mathbb{R}^2$. It is a semidirect product $\mathbb{R}^2 \widetilde{\rtimes} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^2$ by rotation. The subgroup H is trivial. The Lie algebra relations are $[X_2, X_3] = X_1$, $[X_3, X_1] = X_2$ and $[X_1, X_2] = 0$. The $\mathbb{R}^2$-factor is spanned by X_1 and X_2, and the $\mathbb{R}$-factor is spanned by X_3.

The compact quotients of G, as smooth manifolds, admit flat metrics. Because of this, the group G is generally not considered with regard to the geometrization conjecture. Nevertheless, it is relevant for homogeneous Ricci flow solutions.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2, \tag{3.20}$$

where

$$d\theta^1 = -\,\theta^2 \wedge \theta^3, \quad d\theta^2 = -\,\theta^3 \wedge \theta^1, \quad d\theta^3 = 0. \tag{3.21}$$

The sectional curvatures are

$$\begin{aligned} K_{23} &= \frac{(A+B)^2 - 4A^2}{4ABC} \\ K_{31} &= \frac{(A+B)^2 - 4B^2}{4ABC} \\ K_{12} &= \frac{(A-B)^2}{4ABC}. \end{aligned} \tag{3.22}$$

The Ricci flow is given by

$$\begin{aligned} \frac{dA}{dt} &= -\frac{A^2 - B^2}{BC} \\ \frac{dB}{dt} &= -\frac{B^2 - A^2}{AC} \\ \frac{dC}{dt} &= \frac{(A-B)^2}{AB}. \end{aligned} \tag{3.23}$$

From [18], there are limits $\lim_{t\to\infty} A(t) = \lim_{t\to\infty} B(t) = A_*$ and $\lim_{t\to\infty} C(t) = C_*$, where $A_* = \sqrt{A_0 B_0}$ and $C_* = \frac{C_0}{2}\left(\sqrt{\frac{A_0}{B_0}} + \sqrt{\frac{B_0}{A_0}}\right)$. Then

$$g_s(t) \sim s^{-1} A_* \left(\left(\theta^1\right)^2 + \left(\theta^2\right)\right)^2 + s^{-1} C_* \left(\theta^3\right)^2. \tag{3.24}$$

Define a diffeomorphism $\phi_s : \mathbb{R}^3 \to G$ by

$$\phi_s(x, y, z) = \alpha_s(x, y)\, \beta_s(z), \tag{3.25}$$

where $\alpha_s(x, y) = e^{\sqrt{s}(xX_1 + yX_2)}$ and $\beta_s(z) = e^{\sqrt{s}zX_3}$. Letting $h^{-1}dh$ denote the Maurer–Cartan form on G, we have

$$\begin{aligned} \phi_s^*(h^{-1}dh) = \beta_s^{-1}\,\alpha_s^{-1}\, d\alpha_s\, \beta_s + \beta_s^{-1}\, d\beta_s &= \sqrt{s}\,\beta_s^{-1}\,(dx\, X_1 + dy\, X_2)\,\beta_s \\ + \sqrt{s}\, dz\, X_3 &= \sqrt{s}\,\left(\cos(\sqrt{s}z)dx + \sin(\sqrt{s}z)dy\right)\, X_1 \\ &+ \sqrt{s}\left(-\sin(\sqrt{s}z)dx + \cos(\sqrt{s}z)dy\right)\, X_2 + \sqrt{s}\, dz\, X_3. \end{aligned} \tag{3.26}$$

If $\cdot_i$ denotes the X_i-component of an element of the Lie algebra then

$$\phi_s^* g_s\,(t) = s^{-1}\,A(st)\,\left(\phi_s^*(h^{-1}dh)\right)_1^2 + s^{-1}\,B(st)\,\left(\phi_s^*(h^{-1}dh)\right)_2^2 + s^{-1}\,C(st)\,\left(\phi_s^*(h^{-1}dh)\right)_3^2. \tag{3.27}$$

We see that there is a limit as $s \to \infty$ of $\phi_s^* g_s\,(\cdot)$, given by

$$g_\infty(t) = A_*\,(dx^2 + dy^2) + C_*\,dz^2. \tag{3.28}$$

This is a flat metric on $\mathbb{R}^3$ and, as we have seen, is an expanding soliton solution.

3.3.5 $\widetilde{\mathrm{SL}\,(2, \mathbb{R})}$

The group G is the universal cover of $\mathrm{SL}(2, \mathbb{R})$. The subgroup H is trivial. The Lie algebra relations are $[X_2, X_3] = -X_1$, $[X_3, X_1] = X_2$ and $[X_1, X_2] = X_3$.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2, \tag{3.29}$$

where

$$d\theta^1 = \theta^2 \wedge \theta^3, \quad d\theta^2 = -\,\theta^3 \wedge \theta^1, \quad d\theta^3 = -\,\theta^1 \wedge \theta^2. \tag{3.30}$$

The sectional curvatures are

$$\begin{aligned} K_{23} &= \frac{(B-C)^2 - A(3A+2B+2C)}{4ABC} \\ K_{31} &= \frac{[A-(B-C)]^2 - 4B(B-C)}{4ABC} \\ K_{12} &= \frac{[A+(B-C)]^2 + 4C(B-C)}{4ABC}. \end{aligned} \tag{3.31}$$

The Ricci flow is given by

$$\begin{aligned} \frac{dA}{dt} &= \frac{(B-C)^2 - A^2}{BC} \\ \frac{dB}{dt} &= \frac{(C+A)^2 - B^2}{AC} \\ \frac{dC}{dt} &= \frac{(A+B)^2 - C^2}{AB}. \end{aligned} \tag{3.32}$$

From [18], the large-t asymptotics are given by $\lim_{t\to\infty} A(t) = A_* > 0$, $B(t) \sim 2t$ and $C(t) \sim 2t$. Then

$$g_s(t) \sim s^{-1} A_* \left(\theta^1\right)^2 + 2t \left(\left(\theta^2\right)^2 + \left(\theta^3\right)^2\right). \tag{3.33}$$

In order to find appropriate diffeomorphisms ϕ_s to extract a limit, we use a bit of the geometry of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Consider the $\mathbb{R}$-subgroup of $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ generated by X_1. For any $t > 0$, the metric on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ given by (3.33) is the total space of a Riemannian submersion $\pi : \widetilde{\mathrm{SL}(2,\mathbb{R})} \to \mathbb{R}\backslash\widetilde{\mathrm{SL}(2,\mathbb{R})}$, with $\mathbb{R}\backslash\widetilde{\mathrm{SL}(2,\mathbb{R})}$ having a metric of constant negative curvature. Furthermore, the fibers of the submersion are totally geodesic lines. The idea is to construct something like Fermi coordinates around the fiber through the identity element $e \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$.

Lemma 3.34 *Any element $h \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$ can be written uniquely as $h = e^{aX_2+bX_3} e^{cX_1}$ for some $a, b, c \in \mathbb{R}^3$.*

Proof Choosing the metric $\left(\theta^1\right)^2 + \left(\theta^2\right)^2 + \left(\theta^3\right)^2$ on $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ for concreteness, one can check that for any $a, b \in \mathbb{R}$, the curve $v \to e^{avX_2+bvX_3}$ is a geodesic in $\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Clearly it is horizontal at the identity element $e \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$. Hence it is horizontal for all v and represents the horizontal lift of a geodesic in $\mathbb{R}\backslash\widetilde{\mathrm{SL}(2,\mathbb{R})}$. Now given an element $h \in \widetilde{\mathrm{SL}(2,\mathbb{R})}$, consider the unique geodesic $\gamma : [0,1] \to \mathbb{R}\backslash\widetilde{\mathrm{SL}(2,\mathbb{R})}$ with $\gamma(0) = \pi(e)$ and $\gamma(1) = \pi(h)$. Lift it to a horizontal geodesic $\widehat{\gamma}(v) = e^{avX_2+bvX_3}$. As $\pi(\widehat{\gamma}(1)) = \pi(h)$, there is a unique $c \in \mathbb{R}$ so that $h = e^{cX_1}\, \widehat{\gamma}(1)$.

This shows that h can be written as uniquely as $e^{cX_1} e^{aX_2+bX_3}$ for some $a, b, c \in \mathbb{R}$. As

$$e^{cX_1} e^{aX_2+bX_3} = e^{(\cos(c)a - \sin(c)b)X_2 + (\sin(c)a+\cos(c)b)X_3}\, e^{cX_1}, \tag{3.35}$$

the lemma follows. □

Define a diffeomorphism $\phi_s : \mathbb{R}^3 \to G$ by

$$\phi_s(x, y, z) = \alpha(y, z)\, \beta_s(x), \tag{3.36}$$

where $\alpha(y, z) = e^{yX_2+zX_3}$ and $\beta_s(x) = e^{\sqrt{\frac{s}{A_*}}\, xX_1}$. Letting $h^{-1}dh$ denote the Maurer–Cartan form on G, we have

$$\begin{aligned}\phi_s^*(h^{-1}dh) &= \beta_s^{-1}\, \alpha^{-1}\, d\alpha\, \beta_s + \beta_s^{-1}\, d\beta_s \\ &= \beta_s^{-1}\, \alpha^{-1}\, d\alpha\, \beta_s + \sqrt{\frac{s}{A_*}}\, dx\, X_1.\end{aligned} \tag{3.37}$$

If $\cdot_i$ denotes the X_i-component of an element of the Lie algebra then

$$\begin{aligned}\phi_s^* g_s(t) &= s^{-1}A(st)\left(\phi_s^*(h^{-1}dh)\right)_1^2 + s^{-1}B(st)\left(\phi_s^*(h^{-1}dh)\right)_2^2\\ &\quad + s^{-1}C(st)\left(\phi_s^*(h^{-1}dh)\right)_3^2\\ &= s^{-1}A(st)\left((\alpha^{-1}d\alpha)_1 + \sqrt{\frac{s}{A_*}}\,dx\right)^2 + s^{-1}B(st)\left(\beta_s^{-1}\alpha^{-1}d\alpha\,\beta_s\right)_2^2\\ &\quad + s^{-1}C(st)\left(\beta_s^{-1}\alpha^{-1}d\alpha\,\beta_s\right)_3^2. \end{aligned}\tag{3.38}$$

As conjugation by β_s amounts to a rotation in the (y, z)-plane, we see that there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by

$$g_\infty(t) = dx^2 + 2t\left(\left(\alpha^{-1}d\alpha\right)_2^2 + \left(\alpha^{-1}d\alpha\right)_3^2\right). \tag{3.39}$$

The pullback under $\pi \circ \alpha$ of the metric on $\mathbb{R}\backslash\widetilde{\mathrm{SL}(2,\mathbb{R})}$ is the same as $\left(\alpha^{-1}d\alpha\right)_2^2 + \left(\alpha^{-1}d\alpha\right)_3^2$. Hence $g_\infty(\cdot)$ is the expanding soliton on $\mathbb{R} \times H^2$, with $\eta_t(x, y, z) = \left(t^{-\frac{1}{2}}x, y, z\right)$.

Example 3.40 An example of a locally homogeneous $\widetilde{\mathrm{SL}(2,\mathbb{R})}$-geometry is the unit sphere bundle $S^1\Sigma$ of a closed hyperbolic surface Σ. Let $\overline{g}(\cdot)$ be its Ricci flow. In the most direct picture, the manifolds $\left(S^1\Sigma, t^{-1}\overline{g}(\cdot)\right)$ have a Gromov–Hausdorff limit, as $t \to \infty$, given by the rescaling of Σ which has constant sectional curvature $-\frac{1}{2}$. For a more refined picture, let $\pi : S^1\Sigma \to \Sigma$ be the projection map. Given $p' \in \Sigma$, let B be a small ball around p' in Σ. Then $\pi^{-1}(B)$ is diffeomorphic to $S^1 \times B$. Consider the restriction of $t^{-1}\,\overline{g}(t)$ to $\pi^{-1}(B)$ and then its pullback to the universal cover $\widetilde{\pi^{-1}(B)}$. Taking a basepoint $p \in \widetilde{\pi^{-1}(B)}$ over p', the pointed limit as $t \to \infty$ of the metric on $\widetilde{\pi^{-1}(B)}$ will be isometric to $\mathbb{R}$ times a ball of constant sectional curvature $-\frac{1}{2}$. In effect, as the circle fibers of the manifold $\left(S^1\Sigma, t^{-1}\,\overline{g}(t)\right)$ shrink, the local geometry becomes more and more product-like.

3.4 Four dimensions

We first list the four-dimensional simply-connected homogeneous spaces G/H with *maximal* groups G (acting transitively with compact isotropy group) that admit finite-volume quotients [29]. Besides product cases, they are

G	H	G/H
SO(5)	SO(4)	S^4

$$\begin{array}{rll} \mathrm{SU}(3) & U(2) & \mathbb{C}P^2 \\ \mathrm{Isom}^+(H^4) & \mathrm{SO}(4) & H^4 \\ \mathrm{SU}(2,1) & U(2) & \mathbb{C}H^2 \\ \mathbb{R}^2 \widetilde{\times}\, \mathrm{SL}(2,\mathbb{R}) & \mathrm{SO}(2) & F_4 \\ (\mathbb{C}\times\mathbb{R})\widetilde{\times}\mathbb{C}^* & \mathrm{SO}(2) & \mathrm{Sol}_0 \\ \mathrm{Nil}^4 & \{e\} & \mathrm{Nil}^4 \\ \mathrm{Sol}^4_{m,n} & \{e\} & \mathrm{Sol}^4_{m,n} \\ \mathrm{Sol}^4_1 & \{e\} & \mathrm{Sol}^4_1. \end{array} \tag{3.41}$$

In the definition of Sol_0, the action of $\mathbb{C}^*$ on $\mathbb{C}\times\mathbb{R}$ is given by $\lambda\cdot(a,b) = (\lambda a, |\lambda|^{-2}b)$. The group Nil^4 is the semidirect product $\mathbb{R}^3\widetilde{\times}\mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $\beta(r) = \begin{pmatrix} 1 & r & \frac{r^2}{2} \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix}$. The group $\mathrm{Sol}^4_{m,n}$ is the semidirect product $\mathbb{R}^3\widetilde{\times}\mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $r\cdot(x,y,z) = \left(e^{ar}x, e^{br}y, e^{cr}z\right)$. Here $a > b > c$ are real, $a+b+c=0$ and e^a, e^b, e^c are the roots of $\lambda^3 - m\lambda^2 + n\lambda - 1 = 0$ with $m,n\in\mathbb{Z}^+$. Finally, $\mathrm{Sol}^4_1 = \left\{ \begin{pmatrix} 1 & b & c \\ 0 & \alpha & a \\ 0 & 0 & 1 \end{pmatrix} : \alpha, a, b, c \in \mathbb{R},\ \alpha > 0 \right\}$.

There is also a list of nonmaximal geometries [29, Theorem 3.1] but we do not consider it here.

A homogeneous Ricci flow on S^4 or $\mathbb{C}P^2$ has finite extinction time, so we do not consider it further.

3.4.1 $G = \mathrm{Isom}^+(H^4)$

The group G is the connected component of the identity in $\mathrm{SO}(4,1)$. The subgroup H is $\mathrm{SO}(4)$. Let g_0 be a complete constant-curvature metric on $\mathbb{R}^4$ with $\mathrm{Ric}\,(g_0) = -cg_0$ for some $c > 0$. The Ricci flow solution starting at g_0 is given by $g\,(t) = (1+2ct)\,g_0$. Then $g_s(t) = s^{-1}\,(1+2cst)\,g_0$. Taking $\phi_s = \mathrm{Id}$, there is a limit as $s\to\infty$ of $\phi_s^* g_s(\cdot)$, given by $g_\infty(t) = 2ctg_0$. This is independent of c and is an expanding soliton solution with $V=0$.

3.4.2 $G = \mathrm{SU}(2,1)$

The subgroup H is $U(2)$. Let g_0 be a complete metric on $\mathbb{C}^2$ with constant holomorphic sectional curvature and with $\mathrm{Ric}\,(g_0) = -cg_0$ for some $c > 0$. The Ricci flow solution starting at g_0 is given by $g\,(t) = (1+2ct)\,g_0$. Then $g_s(t) = s^{-1}\,(1+2cst)\,g_0$. Taking $\phi_s = \mathrm{Id}$, there is a limit as $s\to\infty$ of $\phi_s^* g_s(\cdot)$, given by $g_\infty(t) = 2ctg_0$. This is independent of c and is an expanding soliton solution with $V=0$.

3.4.3 $G = \mathbb{R}^2 \widetilde{\times} \mathrm{SL}(2, \mathbb{R})$

The quotient space $M = G/H$ fibers homogeneously over H^2, with fiber $\mathbb{R}^2$. Any left-invariant metric defines a homogeneous Riemannian submersion $M \to H^2$. The isotropy group SO(2) acts isometrically on the Riemannian submersion, rotating the $\mathbb{R}^2$-fiber containing the basepoint $\star$. By the rotational symmetry, the curvature tensor of the Riemannian submersion, a horizontal 2-form with values in the vertical tangent bundle, must vanish at $*$. Then by homogeneity, it must vanish everywhere. Thus the horizontal space is integrable. It follows that the Riemannian submersion is of the type considered in Sect. 4, so we can apply the results of that section; see Example 4.28. In particular, the homogeneous metric on M is specified by the relative size of the fiberwise metric g_{ij} on $\mathbb{R}^2$ and the relative size of the base metric $g_{\alpha\beta}$ on H^2. The Ricci curvature calculation of (4.12) shows that under the Ricci flow, g_{ij} is constant in t and $g_{\alpha\beta}$ increases linearly in t. Taking ϕ_s to be multiplication by $\sqrt{s}$ in the $\mathbb{R}^2$-fibers, there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by the expanding soliton solution on F4. It satisfies the harmonic-Einstein equations of Proposition 4.4.

The remaining cases can be seen as Ricci flows on certain unimodular Lie groups. For example, the case Sol_0 can be viewed as Ricci flow on the Lie group $\mathbb{R}^3 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $\delta(r) = \begin{pmatrix} e^r & 0 & 0 \\ 0 & e^r & 0 \\ 0 & 0 & e^{-2r} \end{pmatrix}$. In [16] the Ricci flow was considered on a class of metrics on four-dimensional unimodular Lie groups that have the property that their Ricci flow "diagonalizes". The groups are listed as A1–A10 in [16]. They include some that do not admit finite-volume quotients. In what follows we will use the calculations of [16]. We now go through the cases A1–A10 in order.

3.4.4 A1

This is flat $\mathbb{R}^4$.

3.4.5 A2

The group G is a semidirect product $\mathbb{R}^3 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $r \cdot (x, y, z) = \left(e^r x, e^{kr} y, e^{-(k+1)r} z\right)$. Here k is a free parameter. Special case are Sol_0^4 and $\mathrm{Sol}_{m,n}^4$. The nonzero Lie algebra relations are $[X_1, X_4] = X_1$, $[X_2, X_4] = kX_2$ and $[X_3, X_4] = -(k+1)X_3$. The $\mathbb{R}$-factor is spanned by X_4.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.42}$$

where

$$d\theta^1 = -\theta^1 \wedge \theta^4, \quad d\theta^2 = -k\,\theta^2 \wedge \theta^4, \quad d\theta^3 = (k+1)\,\theta^3 \wedge \theta^4, \quad d\theta^4 = 0. \tag{3.43}$$

We note that here, and in the cases that follow, the metric (3.42) is not the most general homogeneous metric on G/H. However, it is a metric for which the Ricci flow "diagonalizes".

The Ricci flow is given by

$$\begin{aligned} A(t) &= A_0, \\ B(t) &= B_0, \\ C(t) &= C_0, \\ D(t) &= D_0 + 4(k^2+k+1)t. \end{aligned} \tag{3.44}$$

Then

$$\begin{aligned} g_s(t) \sim\, & s^{-1} A_0(\theta^1)^2 + s^{-1} B_0(\theta^2)^2 + s^{-1} C_0(\theta^3)^2 \\ & +4(k^2+k+1)t\,(\theta^4)^2. \end{aligned} \tag{3.45}$$

Define diffeomorphisms $\phi_s : \mathbb{R}^4 \to G$ by $\phi_s(x,y,z,r) = \left(A_0^{-\frac{1}{2}}\sqrt{s}x, B_0^{-\frac{1}{2}}\sqrt{s}y, C_0^{-\frac{1}{2}}\sqrt{s}z, r\right)$. Then there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$ given by

$$g_\infty(t) = (\theta^1)^2 + (\theta^2)^2 + (\theta^1)^2 + 4(k^2+k+1)t\,(\theta^4)^2. \tag{3.46}$$

This is an expanding soliton solution with $\eta_t(x,y,z,r) = \left(t^{-\frac{1}{2}}x, t^{-\frac{1}{2}}y, t^{-\frac{1}{2}}z, r\right)$.

3.4.6 A3

The group G is a semidirect product $\mathbb{R}^3 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $\epsilon(r) = \begin{pmatrix} e^{kr}\cos(r) & e^{kr}\sin(r) & 0 \\ -e^{kr}\sin(r) & e^{kr}\cos(r) & 0 \\ 0 & 0 & e^{-2kr} \end{pmatrix}$. The nonzero Lie algebra relations are $[X_1, X_4] = kX_1 + X_2$, $[X_2, X_4] = -X_1 + kX_2$ and $[X_3, X_4] = -2kX_3$. Here k is a nonzero number. The $\mathbb{R}$-factor is spanned by X_4.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.47}$$

where

$$\begin{aligned} & d\theta^1 = -k\theta^1 \wedge \theta^4 + \theta^2 \wedge \theta^4, \quad d\theta^2 = -\theta^1 \wedge \theta^4 - k\theta^2 \wedge \theta^4, \\ & d\theta^3 = 2k\,\theta^3 \wedge \theta^4, \quad d\theta^4 = 0. \end{aligned} \tag{3.48}$$

The Ricci flow asymptotics are

$$\begin{aligned} A(t) &\sim \sqrt{A_0 B_0}, \\ B(t) &\sim \sqrt{A_0 B_0}, \\ C(t) &= C_0 \\ D(t) &\sim 12k^2 t. \end{aligned} \tag{3.49}$$

Then

$$g_s(t) \sim s^{-1} \sqrt{A_0 B_0} \left((\theta^1)^2 + (\theta^2)^2 \right) + s^{-1} C_0 (\theta^3)^2 + 12k^2 t (\theta^4)^2. \tag{3.50}$$

Define a diffeomorphism $\phi_s : \mathbb{R}^4 \to G$ by

$$\phi_s(x, y, z, r) = \alpha_s(x, y, z) \beta(r), \tag{3.51}$$

where $\alpha_s(x, y, z) = e^{(A_0 B_0)^{-\frac{1}{4}} \sqrt{s}(x X_1 + y X_2) + C_0^{-\frac{1}{2}} \sqrt{s} z X_3}$ and $\beta(r) = e^{k^{-1} r X_4}$. Letting $h^{-1} dh$ denote the Maurer–Cartan form on G, we have

$$\begin{aligned} \phi_s^*(h^{-1} dh) = \beta^{-1} \alpha_s^{-1} d\alpha_s \beta + \beta^{-1} d\beta &= (A_0 B_0)^{-\frac{1}{4}} \sqrt{s} \beta^{-1} \\ &\quad \times (dx\, X_1 + dy\, X_2) \beta + C_0^{-\frac{1}{2}} \sqrt{s}\, dz\, \beta^{-1} X_3 \beta + k^{-1} dr\, X_4 \\ &= (A_0 B_0)^{-\frac{1}{4}} \sqrt{s}\, e^r \left(\cos(k^{-1} \sqrt{s} z) dx - \sin(k^{-1} \sqrt{s} z) dy \right) X_1 \\ &\quad + (A_0 B_0)^{-\frac{1}{4}} \sqrt{s}\, e^r \left(\sin(k^{-1} \sqrt{s} z) dx + \cos(k^{-1} \sqrt{s} z) dy \right) X_2 \\ &\quad + C_0^{-\frac{1}{2}} \sqrt{s}\, e^{-2r} dz\, X_3 + k^{-1} dr\, X_4. \end{aligned} \tag{3.52}$$

If $\cdot_i$ denotes the X_i-component of an element of the Lie algebra then

$$\begin{aligned} \phi_s^* g_s(t) = s^{-1} A(st) \left(\phi_s^*(h^{-1} dh) \right)_1^2 + s^{-1} B(st) \left(\phi_s^*(h^{-1} dh) \right)_2^2 + s^{-1} C(st) \\ \times \left(\phi_s^*(h^{-1} dh) \right)_3^2 + s^{-1} D(st) \left(\phi_s^*(h^{-1} dh) \right)_4^2. \end{aligned} \tag{3.53}$$

We see that there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by

$$g_\infty(t) = e^{2r} (dx^2 + dy^2) + e^{-4r} dz^2 + 12t\, dr^2. \tag{3.54}$$

This is an expanding soliton with $\eta_t(x, y, z, r) = \left(t^{-\frac{1}{2}} x, t^{-\frac{1}{2}} y, t^{-\frac{1}{2}} z, r \right)$. It has Sol_0^4-symmetry.

3.4.7 A4

This is a product case $G = \mathrm{Nil}^3 \times \mathbb{R}$.

3.4.8 A5

The group G is a semidirect product $\mathbb{R}^3 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on $\mathbb{R}^3$ by $\phi(r) = \begin{pmatrix} e^{-\frac{r}{2}} & re^{-\frac{r}{2}} & 0 \\ 0 & e^{-\frac{r}{2}} & 0 \\ 0 & 0 & e^r \end{pmatrix}$. The nonzero Lie algebra relations are $[X_1, X_4] = -\frac{1}{2} X_1 + X_2$, $[X_2, X_4] = -\frac{1}{2} X_2$ and $[X_3, X_4] = X_3$. The $\mathbb{R}$-factor is spanned by X_4.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.55}$$

where

$$\begin{aligned} d\theta^1 &= \frac{1}{2}\,\theta^1 \wedge \theta^4, \quad d\theta^2 = -\,\theta^1 \wedge \theta^4 + \frac{1}{2}\,\theta^2 \wedge \theta^4, \\ d\theta^3 &= -\,\theta^3 \wedge \theta^4, \quad d\theta^4 = 0. \end{aligned} \tag{3.56}$$

The Ricci flow asymptotics are

$$\begin{aligned} A(t) &\sim 2\lambda(\ln t)^{\frac{1}{2}}, \\ B(t) &\sim 3\lambda(\ln t)^{-\frac{1}{2}}, \\ C(t) &= C_0 \\ D(t) &\sim 3t. \end{aligned} \tag{3.57}$$

Then

$$\begin{aligned} g_s(t) \sim\ & 2\lambda s^{-1}\,(\ln(st))^{\frac{1}{2}}\,(\theta^1)^2 + 3\lambda s^{-1}\,(\ln(st))^{-\frac{1}{2}}(\theta^2)^2 \\ & + s^{-1}\,C_0\,(\theta^3)^2 + 3t\,(\theta^4)^2. \end{aligned} \tag{3.58}$$

We take coordinates (x, y, z, r) for G in which $\theta^1 = dx + \frac{1}{2}\,xdr$, $\theta^2 = dy + \left(\frac{y}{2} - x\right) dr$, $\theta^3 = dz - zdr$ and $\theta^4 = dr$. Define $\phi_s : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi_s(x, y, z, r) = \left(\left(\frac{s}{2\lambda(\ln s)^{\frac{1}{2}}} \right)^{\frac{1}{2}} x, \left(\frac{s(\ln s)^{\frac{1}{2}}}{3\lambda} \right)^{\frac{1}{2}} y, \left(\frac{s}{C_0} \right)^{\frac{1}{2}} z,\ r \right). \tag{3.59}$$

Then $\lim_{s\to\infty} s^{-1}\phi_s^* g(st) = g_\infty(t)$, where

$$g_\infty(t) = (dx + \frac{1}{2}\,x\,dr)^2 + (dy + \frac{1}{2}\,y\,dr)^2 + (dz - zdr)^2 + 3t\,dr^2. \tag{3.60}$$

This is an expanding soliton with $\eta_t(x, y, z, r) = \left(t^{-\frac{1}{2}}x, t^{-\frac{1}{2}}y, t^{-\frac{1}{2}}z, r\right)$. It has Sol_0^4-symmetry.

3.4.9 A6

The group G is the four-dimensional nilpotent Lie group whose nonzero Lie algebra relations are $[X_1, X_4] = X_2$ and $[X_2, X_4] = X_3$.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.61}$$

where

$$d\theta^1 = 0, \quad d\theta^2 = -\theta^1 \wedge \theta^4, \quad d\theta^3 = -\theta^2 \wedge \theta^4, \quad d\theta^4 = 0. \tag{3.62}$$

Put $E_0 = \frac{B_0}{A_0 D_0}$ and $F_0 = \frac{C_0}{B_0 D_0}$. The Ricci flow solution is

$$\begin{aligned} A(t) &= A_0 (3E_0 t + 1)^{\frac{1}{3}}, \\ B(t) &= B_0 (3E_0 t + 1)^{-\frac{1}{3}} (3F_0 t + 1)^{\frac{1}{3}}, \\ C(t) &= C_0 (3F_0 t + 1)^{-\frac{1}{3}} \\ D(t) &= D_0 (3E_0 t + 1)^{\frac{1}{3}} (3F_0 t + 1)^{\frac{1}{3}}. \end{aligned} \tag{3.63}$$

Then

$$\begin{aligned} g_s(t) \sim &\left(\frac{3A_0^2 B_0}{D_0}\right)^{\frac{1}{3}} s^{-\frac{2}{3}} t^{\frac{1}{3}} (\theta^1)^2 + (A_0 B_0 C_0)^{\frac{1}{3}} s^{-1} (\theta^2)^2 \\ &+\left(\frac{B_0 C_0^2 D_0}{3}\right)^{\frac{1}{3}} s^{-\frac{4}{3}} t^{-\frac{1}{3}} (\theta^3)^2 + \left(\frac{9C_0 D_0}{A_0}\right)^{\frac{1}{3}} s^{-\frac{1}{3}} t^{\frac{2}{3}} (\theta^4)^2. \end{aligned} \tag{3.64}$$

We take coordinates (x, y, z, r) for G in which $\theta^1 = dx$, $\theta^2 = dy - xdr$, $\theta^3 = dz - ydr$ and $\theta^4 = dr$. Define $\phi_s : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi_s(x, y, z, r) = \left(\left(\frac{D_0 s^2}{A_0^2 B_0}\right)^{\frac{1}{6}} x, \left(\frac{s^3}{A_0 B_0 C_0}\right)^{\frac{1}{6}} y, \left(\frac{s^4}{B_0 C_0^2 D_0}\right)^{\frac{1}{6}} z, \left(\frac{A_0 s}{C_0 D_0}\right)^{\frac{1}{6}} r\right). \tag{3.65}$$

Then $\lim_{s\to\infty} s^{-1}\phi_s^* g(st) = g_\infty(t)$, where

$$g_\infty(t) = 3^{\frac{1}{3}} t^{\frac{1}{3}} (\theta^1)^2 + (\theta^2)^2 + 3^{-\frac{1}{3}} t^{-\frac{1}{3}} (\theta^3)^2 + 3^{\frac{2}{3}} t^{\frac{2}{3}} (\theta^4)^2 \tag{3.66}$$

This is an expanding soliton with $\eta_t(x, y, z, r) = \left(t^{-\frac{2}{6}}x, t^{-\frac{3}{6}}y, t^{-\frac{4}{6}}z, t^{-\frac{1}{6}}r\right)$.

3.4.10 A7(i)

The group G is the group Sol_1^4 mentioned above. The nonzero Lie algebra relations are $[X_2, X_3] = X_4$, $[X_3, X_1] = X_2$ and $[X_1, X_2] = -X_3$.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.67}$$

where

$$d\theta^1 = 0, \quad d\theta^2 = -\theta^3 \wedge \theta^1, \quad d\theta^3 = \theta^1 \wedge \theta^2, \quad d\theta^4 = -\theta^2 \wedge \theta^3. \tag{3.68}$$

The Ricci flow asymptotics are

$$\begin{aligned} A(t) &\sim 4t, \\ B(t) &\sim (9B_0C_0D_0^2)^{\frac{1}{6}}\, t^{\frac{1}{3}}, \\ C(t) &\sim (9B_0C_0D_0^2)^{\frac{1}{6}}\, t^{\frac{1}{3}}, \\ D(t) &= D_0 \left(1 + \frac{3D_0}{B_0C_0} t\right)^{-\frac{1}{3}}. \end{aligned} \tag{3.69}$$

Then

$$g_s(t) \sim 4t\, (\theta^1)^2 + (9B_0C_0D_0^2)^{\frac{1}{6}}\, s^{-\frac{2}{3}}\, t^{\frac{1}{3}} \left((\theta^2)^2 + (\theta^3)^2\right) + \left(\frac{B_0C_0D_0^2}{3}\right)^{\frac{1}{3}} s^{-\frac{4}{3}}\, t^{-\frac{1}{3}}\, (\theta^4)^2. \tag{3.70}$$

We take coordinates (x, y, z, r) for G in which $\theta^1 = dr$, $\theta^2 + \theta^3 = dx - xdr$, $\theta^2 - \theta^3 = dy + ydr$ and $\theta^4 = dz + \frac{1}{4}(xdy - ydx) + \frac{1}{2}xydr$. Define $\phi_s : \mathbb{R}^4 \to \mathbb{R}^4$ by

$$\phi_s(x, y, z, r) = \left((B_0C_0D_0^2)^{-\frac{1}{12}}\, s^{\frac{1}{3}}\, x,\, (B_0C_0D_0^2)^{-\frac{1}{12}}\, s^{\frac{1}{3}}\, y,\, (B_0C_0D_0^2)^{-\frac{1}{6}}\, s^{\frac{2}{3}}\, z,\, r\right). \tag{3.71}$$

Then $\lim_{s\to\infty} s^{-1}\phi_s^* g(st) = g_\infty(t)$, where

$$g_\infty(t) = 4t\, (\theta^1)^2 + 3^{\frac{1}{3}}\, t^{\frac{1}{3}} \left((\theta^2)^2 + (\theta^3)^2\right) + 3^{-\frac{1}{3}}\, t^{-\frac{1}{3}}\, (\theta^4)^2. \tag{3.72}$$

This is an expanding soliton with $\eta_t(x, y, z, r) = \left(t^{-\frac{1}{3}}x, t^{-\frac{1}{3}}y, t^{-\frac{2}{3}}z, r\right)$.

3.4.11 A8

The group G is a semidirect product $\mathrm{Nil}^3 \widetilde{\times} \mathbb{R}$, where $\mathbb{R}$ acts on Nil^3, in appropriate coordinates, by $r \cdot (x, y, z) = (\cos(r)x + \sin(r)y, -\sin(r)x + \cos(r)y, z)$. The nonzero Lie algebra relations are $[X_2, X_3] = -X_4$, $[X_3, X_1] = X_2$, $[X_1, X_2] = X_3$. The $\mathbb{R}$-factor is spanned by X_1.

The metric is

$$g(t) = A(t)(\theta^1)^2 + B(t)(\theta^2)^2 + C(t)(\theta^3)^2 + D(t)(\theta^4)^2, \tag{3.73}$$

where

$$d\theta^1 = 0, \quad d\theta^2 = -\theta^3 \wedge \theta^1, \quad d\theta^3 = -\theta^1 \wedge \theta^2, \quad d\theta^4 = \theta^2 \wedge \theta^3. \tag{3.74}$$

The Ricci flow asymptotics are

$$\begin{aligned} A(t) &\sim \frac{D_0}{2}, \\ B(t) &\sim (9B_0C_0D_0^2)^{\frac{1}{6}}\, t^{\frac{1}{3}}, \\ C(t) &\sim (9B_0C_0D_0^2)^{\frac{1}{6}}\, t^{\frac{1}{3}}, \\ D(t) &= D_0 \left(1 + \frac{3D_0}{B_0C_0} t\right)^{-\frac{1}{3}}. \end{aligned} \tag{3.75}$$

Then

$$\begin{aligned} g_s(t) \sim\ & \frac{D_0}{2}\, s^{-1}\, (\theta^1)^2 + (9B_0C_0D_0^2)^{\frac{1}{6}}\, s^{-\frac{2}{3}}\, t^{\frac{1}{3}} \left((\theta^2)^2 + (\theta^3)^2\right) \\ & + \left(\frac{B_0C_0D_0^2}{3}\right)^{\frac{1}{3}} s^{-\frac{4}{3}}\, t^{-\frac{1}{3}}\, (\theta^4)^2. \end{aligned} \tag{3.76}$$

Define a diffeomorphism $\phi_s : \mathbb{R}^4 \to G$ by

$$\phi_s(x, y, z, r) = \alpha_s(x, y, z)\, \beta(r), \tag{3.77}$$

where $\alpha_s(x, y, z) = e^{(B_0C_0D_0^2)^{-\frac{1}{12}} s^{\frac{1}{3}} (xX_2 + yX_3) + (B_0C_0D_0^2)^{-\frac{1}{6}} s^{\frac{2}{3}} zX_4}$ and $\beta(r) = e^{D_0^{-\frac{1}{2}} s^{\frac{1}{2}} rX_1}$. Letting $h^{-1}dh$ denote the Maurer–Cartan form on G, we have

$$\phi_s^*(h^{-1}dh) = \beta_s^{-1}\, \alpha_s^{-1}\, d\alpha_s\, \beta_s + \beta_s^{-1}\, d\beta_s. \tag{3.78}$$

As conjugation by β_s acts isometrically on $\alpha_s^{-1}\, d\alpha_s$, we see that there is a limit as $s \to \infty$ of $\phi_s^* g_s(\cdot)$, given by the expanding soliton $g_\infty(\cdot)$ on $\mathbb{R} \times \mathrm{Nil}^3$; see Sect. 3.3.3.

3.4.12 A9

This is a product case $G = \widetilde{\mathrm{SL}(2, \mathbb{R})} \times \mathbb{R}$.

3.4.13 A10

This is a product case $G = \mathrm{SU}(2) \times \mathbb{R}$.

4 Expanding solitons on vector bundles

In this section we consider the expanding soliton equation in the case of a family $g(\cdot)$ of $\mathbb{R}^N$-invariant metrics on the total space M of a flat $\mathbb{R}^N$-vector bundle over a manifold B, with the property that the fiberwise volume forms are preserved by the flat connection. The vector field V is assumed to be the standard radial vector field along the fibers. We show that the expanding soliton equation on M becomes two equations on B : a harmonic map equation $G : B \to \mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$ and an equation that relates dG to Ric_B. We give examples of expanding soliton solutions with $\dim(B) = 1$, which are generalized Sol-solutions, and an example with $\dim(B) = 2$. We then show that if a rescaling limit $\lim_{k\to\infty} g_{s_k}(\cdot)$ exists for such an $\mathbb{R}^N$-invariant Ricci flow then the limit satisfies the harmonic-Einstein equations.

The expanding solitons $(M_\infty, g_\infty(t))$ of Sect. 3 all have a certain fibration structure. Namely, there is a Riemannian submersion $\pi : M_\infty \to B_\infty$ whose fibers are diffeomorphic to a nilpotent Lie group $\mathcal{N}$ and whose holonomy preserves the natural flat linear connection ∇^{aff} on $\mathcal{N}$. (The connection ∇^{aff} has the property that left-invariant vector fields are parallel.) The diffeomorphisms $\{\eta_t\}_{t>0}$ act fiberwise and arise from a 1-parameter group $\{a_t\}_{t>0}$ of automorphisms of $\mathcal{N}$, by $\eta_t = a_{t^{-1}}$. We list below the relevant groups $\mathcal{N}$ that appeared in Sect. 3, along with the subsection in which they appeared. (There are also some product cases that we omit.)

$$\begin{array}{rl} \{e\} & 3.2, 3.3.1, 3.4.1, 3.4.2 \\ \mathbb{R} & 3.1, 3.3.5 \\ \mathbb{R}^2 & 3.3.2, 3.4.3, 3.4.12 \\ \mathbb{R}^3 & 3.3.4, 3.4.5, 3.4.6, 3.4.8 \\ \mathbb{R}^4 & 3.4.4 \\ \mathrm{Nil}^3 & 3.3.3, 3.4.7, 3.4.10, 3.4.11 \\ \mathrm{Nil}^4 & 3.4.9 \end{array} \tag{4.1}$$

These special fibration structures are related to the results of Cheeger–Fukaya–Gromov on collapsing with bounded sectional curvature [4]. Namely, any sufficiently collapsed manifold can be slightly perturbed to have a so-called Nil-structure, where "Nil" refers to a local nilpotent Lie algebra of Killing vector fields. It will follow from Sect. 5.6 that if $(M, p, g(\cdot))$ is a pointed type-III Ricci flow solution then there is a sequence $\{s_j\}_{j=1}^\infty$ tending to infinity so that there is a limit flow $g_\infty(\cdot) = \lim_{j\to\infty} g_{s_j}(\cdot)$ in an appropriate sense which, if $\lim_{t\to\infty} t^{-\frac{1}{2}} \mathrm{inj}_{g(t)}(p) = 0$, will have a Nil-structure.

This suggests looking for expanding soliton solutions with a special fibration structure of the type mentioned above, with the action of the diffeomorphisms $\{\eta_t\}_{t>0}$ being compatible with the fibration structure. In terms of the fiber, it is known that there are many nilpotent Lie groups that do admit Ricci soliton metrics and also some that do not [19].

Based on these considerations, in this section we look at what the expanding soliton equation becomes if we assume compatibility with the simplest type of Nil-structure. Namely, we consider the expanding soliton equation on the total space of an $\mathbb{R}^N$-vector bundle $\pi : M \to B$. We assume that

1. π is a Riemannian submersion.
2. For each $b \in B$, there is a neighborhood U_b of b in B so that there is a free isometric $\mathbb{R}^N$-action on $\pi^{-1}(U_b)$ which acts by translation on the fibers.
3. The diffeomorphism η_t is fiberwise multiplication by $t^{-\frac{1}{2}}$.

Let $s : B \to M$ be the zero-section. The $\mathbb{R}^N$-action on $\pi^{-1}(U_b)$ gives a local trivialization of the vector bundle by $(b', \vec{v}) \to s(b') + \vec{v}$. In view of this, it is natural to reduce the data to

1. A vector bundle on B with a flat vector bundle connection ∇,
2. A Riemannian metric on B and
3. Flat Riemannian metrics on the fibers.

There is a corresponding canonical metric on M. We do not assume that ∇ preserves the fiberwise metrics.

For simplicity of notation, in this section we write $\overline{g}_{IJ}$ for the metric on M, $\overline{R}_{IJ}$ for the Ricci tensor on M, etc. We write $g_{\alpha\beta}$ for the metric on B, $R_{\alpha\beta}$ for the Ricci tensor on B, etc. We let Greek indices denote horizontal directions and we let lower case Roman indices denote vertical directions. In terms of local coordinates $\{x^\alpha, x^i\}$, we can write the metric on M as

$$\begin{aligned} \overline{g}_{\alpha\beta} &= g_{\alpha\beta}(b) \\ \overline{g}_{i\alpha} &= 0 \\ \overline{g}_{ij} &= g_{ij}(b). \end{aligned} \tag{4.2}$$

We will use the Einstein summation convention freely.

Hereafter we assume that ∇ preserves the fiberwise volume forms, as this is what arises in the examples of Sect. 3. We write

$$g_{ij;\alpha\beta} = g_{ij,\alpha\beta} - \Gamma^{\sigma}{}_{\alpha\beta}\, g_{ij,\sigma}. \tag{4.3}$$

Proposition 4.4 *The expanding soliton equation becomes the pair of equations*

$$R_{\alpha\beta} - \frac{1}{4}\, g^{ij}\, g_{jk,\alpha}\, g^{kl}\, g_{li,\beta} + \frac{1}{2t}\, g_{\alpha\beta} = 0 \tag{4.5}$$

and

$$g^{\alpha\beta}\, g_{ij;\alpha\beta} - g^{\alpha\beta}\, g_{ik,\alpha}\, g^{kl}\, g_{lj,\beta} = 0. \tag{4.6}$$

Proof The nonzero Christoffel symbols are

$$\begin{aligned}
\overline{\Gamma}^{\alpha}_{\ \beta\gamma} &= \Gamma^{\alpha}_{\ \beta\gamma} \\
\overline{\Gamma}^{\alpha}_{\ ij} &= -\frac{1}{2} g^{\alpha\beta} g_{ij,\beta} \\
\overline{\Gamma}^{i}_{\ j\alpha} = \overline{\Gamma}^{i}_{\ \alpha j} &= \frac{1}{2} g^{ik} g_{kj,\alpha}.
\end{aligned} \tag{4.7}$$

The nonzero components of the curvature tensor are

$$\begin{aligned}
\overline{R}^{\alpha}_{\ \beta\gamma\delta} &= R^{\alpha}_{\ \beta\gamma\delta} \\
\overline{R}^{\alpha}_{\ i\beta j} &= -\frac{1}{2} g^{\alpha\gamma} g_{ij;\gamma\beta} + \frac{1}{4} g^{\alpha\gamma} g_{ik,\beta} g^{kl} g_{lj,\gamma} \\
\overline{R}^{i}_{\ j\alpha\beta} &= -\frac{1}{4} g^{ik} g_{kl,\alpha} g^{lm} g_{mj,\beta} + \frac{1}{4} g^{ik} g_{kl,\beta} g^{lm} g_{mj,\alpha} \\
\overline{R}^{i}_{\ jkl} &= -\frac{1}{4} g^{ir} g_{rk,\alpha} g^{\alpha\beta} g_{jl,\beta} + \frac{1}{4} g^{ir} g_{rl,\alpha} g^{\alpha\beta} g_{jk,\beta}.
\end{aligned} \tag{4.8}$$

The Ricci tensor is

$$\begin{aligned}
\overline{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{2} g^{ij} g_{ij;\alpha\beta} + \frac{1}{4} g^{ij} g_{jk,\alpha} g^{kl} g_{li,\beta} \\
\overline{R}_{\alpha i} &= 0 \\
\overline{R}_{ij} &= -\frac{1}{2} g^{\alpha\beta} g_{ij;\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g_{ik,\alpha} g^{kl} g_{lj,\beta} - \frac{1}{4} g^{\alpha\beta} g^{kl} g_{kl,\alpha} g_{ij,\beta}.
\end{aligned} \tag{4.9}$$

As ∇ preserves the fiberwise volume forms,

$$g^{ij} g_{ij,\alpha} = 0 \tag{4.10}$$

and

$$g^{ij} g_{ij;\alpha\beta} = g^{ij} g_{jk,\beta} g^{kl} g_{li,\alpha}. \tag{4.11}$$

Then

$$\begin{aligned}
\overline{R}_{\alpha\beta} &= R_{\alpha\beta} - \frac{1}{4} g^{ij} g_{jk,\alpha} g^{kl} g_{li,\beta} \\
\overline{R}_{\alpha i} &= 0 \\
\overline{R}_{ij} &= -\frac{1}{2} g^{\alpha\beta} g_{ij;\alpha\beta} + \frac{1}{2} g^{\alpha\beta} g_{ik,\alpha} g^{kl} g_{lj,\beta}.
\end{aligned} \tag{4.12}$$

If $\{V(t)\}_{t>0}$ is the vector field generating $\{\eta_t\}_{t>0}$ then

$$\begin{aligned}(\mathcal{L}_V g)_{\alpha\beta} &= 0\\ (\mathcal{L}_V g)_{\alpha i} &= 0\\ (\mathcal{L}_V g)_{ij} &= -\frac{1}{t}\, g_{ij}.\end{aligned} \tag{4.13}$$

The proposition follows. □

Remark 4.14 Without assuming that ∇ preserves the fiberwise volume forms, it follows from (4.9) that $g^{ij}\,\overline{R}_{ij} = -\frac{1}{\sqrt{|G|}}\, g^{\alpha\beta}\,\left(\sqrt{|G|}\right)_{;\alpha\beta}$, where $|G| = \det\left(g_{ij}\right)$. Then under the Ricci flow,

$$\frac{\partial\sqrt{|G|}}{\partial t} = -\sqrt{|G|}\, g^{ij}\,\overline{R}_{ij} = g^{\alpha\beta}\,\left(\sqrt{|G|}\right)_{;\alpha\beta}. \tag{4.15}$$

Hence it is consistent to assume that $|G|$ is spatially constant, i.e. that ∇ preserves the fiberwise volume forms.

We will call equations (4.5)–(4.6) the *harmonic-Einstein* equations.

Now consider the space $\mathcal{S}$ of positive-definite symmetric matrices $\{G_{ij}\}$ on $\mathbb{R}^N$ with a fixed determinant. An element $A \in \mathrm{SL}(N,\mathbb{R})$ acts on $\mathcal{S}$ by sending G to AGA^T. This identifies $\mathcal{S}$ with $\mathrm{SL}(N,\mathbb{R})/\,\mathrm{SO}(N)$. The corresponding Riemannian metric on $\mathcal{S}$ can be written informally as $\mathrm{Tr}\,\left(G^{-1}dG\right)^2$. That is, for a symmetric matrix $K \in T_G\mathcal{S}$

$$\langle K, K\rangle_G = G^{ij}\,K_{jk}\,G^{kl}\,K_{li}. \tag{4.16}$$

Proposition 4.17 *Equation (4.6) is the local expression for a harmonic map from B to $\mathcal{S}$.*

Proof The energy of a map $G : B \to \mathcal{S}$ is

$$E(G) = \frac{1}{2}\int_B g^{\alpha\beta}\,\mathrm{Tr}\left(G^{-1}\,G_{,\alpha}\,G^{-1}\,G_{,\beta}\right)\,\mathrm{dvol}. \tag{4.18}$$

Consider a variation of G of the form $\delta G = KG$ with $\mathrm{Tr}\,K = 0$. The variation of E is

$$\delta E = \int_B g^{\alpha\beta}\,\mathrm{Tr}\left(G^{-1}G_{,\alpha}G^{-1}K_{,\beta}G\right)\mathrm{dvol} = \int_B g^{\alpha\beta}\,\mathrm{Tr}\left(G_{,\alpha}G^{-1}K_{,\beta}\right)\mathrm{dvol}. \tag{4.19}$$

If K is compactly supported then integration by parts gives

$$\delta E = -\int_B g^{\alpha\beta}\,\mathrm{Tr}\left(\left(G_{,\alpha}\,G^{-1}\right)_{;\beta}\,K\right)\,\mathrm{dvol}. \tag{4.20}$$

If this vanishes for all such K then

$$g^{\alpha\beta}\left(G_{,\alpha}\,G^{-1}\right)_{;\beta} = \sigma\, I \tag{4.21}$$

for some function σ on B. On the other hand, as G has constant determinant,

$$\operatorname{Tr}\left(G_{,\alpha}\,G^{-1}\right) = 0 \tag{4.22}$$

and so

$$\operatorname{Tr}\left(G_{,\alpha}\,G^{-1}\right)_{;\beta} = 0. \tag{4.23}$$

Tracing (4.21) gives $\sigma = 0$, so the variational equation is

$$g^{\alpha\beta}\left(G_{,\alpha}\,G^{-1}\right)_{;\beta} = g^{\alpha\beta}\,G_{;\alpha\beta}\,G^{-1} - g^{\alpha\beta}\,G_{,\alpha}\,G^{-1}\,G_{,\beta}\,G^{-1} = 0. \tag{4.24}$$

Equivalently,

$$g^{\alpha\beta}\,G_{;\alpha\beta} - g^{\alpha\beta}\,G_{,\alpha}\,G^{-1}\,G_{,\beta} = 0, \tag{4.25}$$

which is the same as (4.6). □

The equations (4.5) and (4.6) are defined locally. To give them global meaning, let $\rho : \pi_1(B) \to \mathrm{SL}(N,\mathbb{R})$ be the holonomy representation of the flat connection ∇. Let $\widetilde{B}$ be the universal cover of B. The flat vector bundle M over B can be written as $\widetilde{B} \times_{\pi_1(B)} \mathbb{R}^N$, where $\pi_1(B)$ is represented on $\mathbb{R}^N$ via ρ. Then the family of fiberwise metrics $\{g_{ij}(b)\}_{b\in B}$ corresponds to a $\pi_1(B)$-equivariant map $G : \widetilde{B} \to \mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)$, where $\pi_1(B)$ acts on $\mathrm{SL}(N,\mathbb{R})/\mathrm{SO}(N)$ via left multiplication by ρ. Equation (4.6) says that G is a harmonic map. As $\overline{R}_{ij} = 0$, the metrics $\{g_{ij}(b)\}_{b\in B}$ are constant in time and so the map G is time-independent. The metric $g_{\alpha\beta}$ on B is proportionate to t. Equation (4.5) relates the metric on B to the harmonic map G.

If, after an appropriate choice of basis, the representation ρ takes value in $\mathrm{SL}(N,\mathbb{Z})$ then we can quotient M fiberwise by $\mathbb{Z}^N$. The resulting space $\widetilde{M}/(\mathbb{Z}^N \widetilde{\times} \pi_1(B))$ is a flat torus bundle over B and carries a quotient Ricci flow metric.

Example 4.26 If B is compact and ρ is trivial then G must be a point map (see, for example, [17, Sect. 1.2]). Thus the expanding soliton metric $(M, \overline{g}(t))$ is just a product metric $(B, tg_B) \times (\mathbb{R}^N, g_{flat})$ for an Einstein metric g_B on B with Einstein constant $-\frac{1}{2}$.

Example 4.27 Suppose that $B = \mathbb{R}$. Let X be a real diagonal $(N \times N)$-matrix with vanishing trace. Then there is a geodesic $G : \mathbb{R} \to \mathcal{S}$ given by $G(s) = e^{sX}$. Equation (4.5) is satisfied by the metric $\frac{t}{2}\operatorname{Tr}(X^2)\, ds^2$ on B. This generalizes the Sol-solutions of Sects. 3.3.2 and 3.4.5.

If $e^{\frac{X}{2}}$ is conjugate to an integer matrix $A \in \mathrm{SL}(N,\mathbb{Z})$ then one obtains a quotient Ricci flow solution on the total space of a flat T^N-bundle over S^1 with holonomy A.

Example 4.28 Take $B = \mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$. The identity map G from B to $\mathrm{SL}(N, \mathbb{R})/\mathrm{SO}(N)$ is harmonic. Equation (4.5) is satisfied by the canonical metric on B, after appropriate normalization. As B is the moduli space for constant-volume inner products on $\mathbb{R}^N$, the manifold M is the corresponding universal Euclidean bundle over B. If Γ is a finite-index torsion-free subgroup of $\mathrm{SL}(N, \mathbb{Z})$ then there is a quotient T^n-bundle over $\Gamma \backslash \mathrm{SL}(N, \mathbb{Z})/\mathrm{SO}(N)$, with a quotient Ricci flow solution.

In the case $N = 2$, we can identify $\mathrm{SL}(2, \mathbb{R})/\mathrm{SO}(2)$ with the space of translation-invariant complex structures on $\mathbb{R}^2$. The manifold M is the resulting universal complex line bundle. It has a homogeneous complex geometry, called F4 in [29]. The quotient of M by $\mathbb{Z}^2 \widetilde{\times} \mathrm{SL}(2, \mathbb{Z})$ is the universal curve of complex structures on T^2, which we equip with constant volume.

Turning from the expanding soliton equation, we now consider Ricci flow on a 1-parameter family of metrics $\overline{g}(\cdot)$ on M of the type considered above. We ask for sufficient conditions to ensure that the limit of a convergent subsequence of rescaled flows $s_j^{-1}\, \overline{g}(s_j \cdot)$ (modulo diffeomorphisms) is in fact an expanding soliton. This is a different question than that addressed in Proposition 2.5 where we assumed that there is an actual limit as $s \to \infty$ of $s^{-1}\, g(s\cdot)$, instead of just a convergent subsequence.

Suppose first that $N = 0$, i.e. we just have Ricci flow on B. Given a type-III Ricci flow $g(\cdot)$ on a compact manifold B, suppose that there is a sequence $\{s_j\}_{j=1}^{\infty}$ of positive numbers converging to infinity and diffeomorphisms $\{\phi_j\}_{j=1}^{\infty}$ of B so that $\left\{s_j^{-1}\phi_j^* g(s_j\cdot)\right\}_{j=1}^{\infty}$ converges to a Ricci flow $g_\infty(\cdot)$ on B. Then $g_\infty(t) = t\, g_E$ for a Einstein metric g_E on B satisfying $\mathrm{Ric}(g_E) = -\frac{1}{2}\, g_E$ [6, Theorem 1.3],[14, Sect. 7]. The hypotheses are equivalent to saying that we have a type-III Ricci flow solution $g(\cdot)$ on a manifold B such that $\limsup_{t\to\infty}\, t^{-\frac{1}{2}}\, \mathrm{diam}(B, g(t)) < \infty$ and $\liminf_{t\to\infty}\, t^{-\frac{1}{2}}\, \mathrm{inj}(B, g(t)) > 0$.

We now consider the case $N > 0$. An automorphism of a flat vector bundle (W, ∇^{flat}) over B is an invertible vector bundle map $\widehat{\phi} : W \to W$ with $\widehat{\phi}^* \nabla^{flat} = \nabla^{flat}$. It covers a diffeomorphism ϕ of B.

Proposition 4.29 *Fix a flat $\mathbb{R}^N$-vector bundle M on a compact manifold B. Let $\{\overline{g}(t)\}_{t\in(0,\infty)}$ be a 1-parameter family of metrics on M of the type considered above. Suppose that $\overline{g}(\cdot)$ satisfies the Ricci flow equation. Suppose that*

1. *There is a sequence $\{s_j\}_{j=1}^{\infty}$ of positive numbers converging to infinity and*
2. *There are automorphisms $\{\widehat{\phi}_j\}_{j=1}^{\infty}$ of the flat vector bundle M so that*
3. $\left\{s_j^{-1}\widehat{\phi}_j^*\overline{g}(s_j\cdot)\right\}_{j=1}^{\infty}$ *converges to a Ricci flow solution $\overline{g}_\infty(\cdot)$.*

Then $\overline{g}_\infty(\cdot)$ satisfies the harmonic-Einstein equations (4.5)–(4.6).

Proof The proof is an adaptation of one of the proofs of [6, Theorem 1.3], this particular proof being due to Hamilton. We write G_{ij} for g_{ij}. On M we have the equation

$$\frac{\partial \overline{R}}{\partial t} = \overline{\triangle}\, \overline{R} + 2\, |\overline{R}_{IJ}|^2. \tag{4.30}$$

Translating this to an equation on B gives

$$\frac{\partial}{\partial t}\left(R-\frac{1}{4}\, g^{\alpha\beta}\ \mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)\right)=\triangle\left(R-\frac{1}{4}\, g^{\alpha\beta}\ \mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)\right)$$
$$+2\left|R_{\alpha\beta}-\frac{1}{4}\ \mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)\right|^2$$
$$+2\ \mathrm{Tr}\left(-\frac{1}{2}\, g^{\alpha\beta}\, G^{-1}\, G_{;\alpha\beta}\ +\ \frac{1}{2}\, g^{\alpha\beta}\, G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)^2. \tag{4.31}$$

The maximum principle implies that

$$R-\frac{1}{4}\, g^{\alpha\beta}\ \mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)\ +\ \frac{n}{2t}\ \geq\ 0, \tag{4.32}$$

where $n=\dim(B)$. On the other hand, putting $\tilde{V}(t)=t^{-\frac{n}{2}}\int_B \mathrm{dvol}_B$, one has

$$\frac{d\tilde{V}}{dt}=-\,t^{-\frac{n}{2}}\int_B\left(R-\frac{1}{4}\, g^{\alpha\beta}\ \mathrm{Tr}\left(G^{-1}\, G_{,\alpha}\, G^{-1}\, G_{,\beta}\right)\ +\ \frac{n}{2t}\right)\mathrm{dvol}_B\,. \tag{4.33}$$

Then $\tilde{V}(t)$ is nonincreasing in t and the corresponding quantity $\tilde{V}_\infty(t)$ for $\overline{g}_\infty(\cdot)$ must be constant in t. Equation (4.33), applied to $\overline{g}_\infty(\cdot)$, implies

$$R_\infty-\frac{1}{4}\, g_\infty^{\alpha\beta}\ \mathrm{Tr}\left(G_\infty^{-1}\, G_{\infty,\alpha}\, G_\infty^{-1}\, G_{\infty,\beta}\right)\ +\ \frac{n}{2t}=0. \tag{4.34}$$

Plugging this into (4.31) (applied to $\overline{g}_\infty$) gives

$$\frac{n}{2t^2}\ =\ 2\left|R_{(\infty)\alpha\beta}-\frac{1}{4}\ \mathrm{Tr}\left(G_\infty^{-1}\, G_{\infty,\alpha}\, G_\infty^{-1}\, G_{\infty,\beta}\right)\right|^2$$
$$+2\ \mathrm{Tr}\left(-\frac{1}{2}\, g_\infty^{\alpha\beta}\, G_\infty^{-1}\, G_{\infty;\alpha\beta}+\frac{1}{2}\, g_\infty^{\alpha\beta}\, G_\infty^{-1}\, G_{\infty,\alpha}\, G_\infty^{-1}\, G_{\infty,\beta}\right)^2, \tag{4.35}$$

or

$$0\ =\ 2\left|R_{(\infty)\alpha\beta}-\frac{1}{4}\ \mathrm{Tr}\left(G_\infty^{-1}\, G_{\infty,\alpha}\, G_\infty^{-1}\, G_{\infty,\beta}\right)\ +\ \frac{1}{2t}\, g_{\infty,\alpha\beta}\right|^2$$
$$+2\,\mathrm{Tr}\left(-\frac{1}{2}g_\infty^{\alpha\beta}G_\infty^{-1}G_{\infty;\alpha\beta}+\frac{1}{2}g_\infty^{\alpha\beta}G_\infty^{-1}G_{\infty,\alpha}G_\infty^{-1}G_{\infty,\beta}\right)^2. \tag{4.36}$$

The proposition follows. □

Based on Proposition 4.29, one can speculate the expanding solitons that arise in type-III Ricci flows again involve some sort of harmonic-Einstein equations. Of course, the relevant nilpotent Lie groups may be more complicated than $\mathbb{R}^N$ and they may act with orbits of varying dimensions. Nevertheless, a Nil-structure has a quotient space with bounded geometry in a certain sense [4, Appendix 1], which is where the harmonic-Einstein equations would live.

5 Ricci flow on étale groupoids

An étale groupoid is a mathematical object which in some sense combines the notions of topological spaces and discrete groups. A Riemannian groupoid is an étale groupoid equipped with an invariant Riemannian metric. The relevance for us comes from the Cheeger–Fukaya–Gromov theory of bounded curvature collapse, which implies that when a Riemannian manifold collapses with bounded sectional curvature, it asymptotically obtains local symmetries.

In this section we recall some basic definitions about Riemannian groupoids. A good source for background information is [12, Sect. 7]. Further references are [3, Chap. IIIG], [11] and [26, Appendix D]. We then prove an extension of Hamilton's compactness theorem, not assuming a lower bound on the injectivity radius. Although it takes a bit of time to set up the right framework, once the framework is in place then the proof is almost the same as in Hamilton's paper [13]. We discuss how the long-time behavior of type-III Ricci flow solutions becomes a problem about the dynamics of the $\mathbb{R}^+$-action on a compact space $\mathcal{S}_{n,K}$. We list the Riemannian groupoids that arise in the long-time behavior of finite-volume locally homogeneous three-dimensional Ricci flow solutions.

5.1 Etale groupoids

A *groupoid* G consists of

1. Sets $G^{(0)}$ and $G^{(1)}$,
2. An injection $e : G^{(0)} \to G^{(1)}$ (with which we will think of the "units" $G^{(0)}$ as a subset of $G^{(1)}$),
3. "Source" and "range" maps $s, r : G^{(1)} \to G^{(0)}$ with $s \circ e = r \circ e = \mathrm{Id}\big|_{G^{(0)}}$ and
4. A partially-defined multiplication $G^{(1)} \times G^{(1)} \to G^{(1)}$

so that

1. The product $\gamma\gamma'$ of $\gamma, \gamma' \in G^{(1)}$ is defined if and only if $s(\gamma) = r(\gamma')$, and then $s(\gamma\gamma') = s(\gamma')$ and $r(\gamma\gamma') = r(\gamma)$.
2. $(\gamma\gamma')\gamma'' = \gamma(\gamma'\gamma'')$ whenever the two sides are defined.
3. $\gamma s(\gamma) = r(\gamma)\gamma = \gamma$.
4. For all $\gamma \in G^{(1)}$, there is an element $\gamma^{-1} \in G^{(1)}$ so that $\gamma\gamma^{-1} = r(\gamma)$ and $\gamma^{-1}\gamma = s(\gamma)$.

A *morphism* $m : G_1 \to G_2$ between two groupoids is given by maps $m^{(1)} : G_1^{(1)} \to G_2^{(1)}$ and $m^{(0)} : G_1^{(0)} \to G_2^{(0)}$ that satisfy obvious compatibility conditions. An *isomorphism* between G_1 and G_2 is an invertible morphism.

Given $x \in G^{(0)}$, we write $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $G^x_x = G^x \cap G_x$, the latter being the *isotropy group* of x. The *orbit* of x is the set $O_x = s(r^{-1}(x))$. There is an *orbit space* $\mathcal{O}$.

A *pointed* groupoid (G, O_x) is a groupoid G equipped with a preferred orbit O_x. A morphism $m : (G_1, O_{x_1}) \to (G_2, O_{x_2})$ of pointed groupoids will be assumed to have the property that $m^{(0)}$ sends O_{x_1} to O_{x_2}.

A groupoid G is *smooth* if $G^{(1)}$ and $G^{(0)}$ are smooth manifolds, e is a smooth embedding, s and r are submersions, and the structure maps are all smooth. (For example, multiplication is supposed to be a smooth map from $\{(\gamma, \gamma') \in G^{(1)} \times G^{(1)} : s(\gamma) = r(\gamma')\}$ to $G^{(1)}$.) Morphisms between smooth groupoids are assumed to be smooth. A smooth groupoid is *étale* if s and r are local diffeomorphisms. Hereafter, unless otherwise stated, groupoids will be smooth and étale. We do not assume that $G^{(1)}$ has a countable basis, although in the cases of interest $G^{(0)}$ will have a countable basis.

An étale groupoid G is *Hausdorff* if $G^{(1)}$ is Hausdorff and whenever $c : [0, 1] \to G^{(1)}$ is a continuous path such that $\lim_{t\to 1} s(c(t))$ and $\lim_{t\to 1} r(c(t))$ exist, there is a limit $\lim_{t\to 1} c(t)$ in $G^{(1)}$. Hereafter we assume that G is Hausdorff.

Example 5.1 If M is a smooth manifold then there is an étale groupoid G with $G^{(1)} = G^{(0)} = M$, where s and r are the identity maps. The product $m \cdot m'$ is defined if and only if $m = m'$, in which case the product is m. We will call this the groupoid M.

Example 5.2 If a discrete group Γ acts smoothly on M, define the cross-product groupoid $G = M \rtimes \Gamma$ as follows. Put $G^{(1)} = M \times \Gamma$ and $G^{(0)} = M$, with $r(m, \gamma) = m$ and $s(m, \gamma) = m\gamma$. The product $(m, \gamma) \cdot (m', \gamma')$ is defined if $m\gamma = m'$, in which case the product is $(m, \gamma\gamma')$.

For example, we can take $\mathbb{Z}$ acting on S^1 by rotations. Or we can take SO(2) acting on S^1 by rotations. Note that in the latter case we give SO(2) the discrete topology so that G will be étale.

5.2 Equivalence of étale groupoids

If $\mathcal{U} = \{U_i\}_{i\in I}$ is an open cover of $G^{(0)}$ then there is a new étale groupoid $G_{\mathcal{U}}$, called the *localization* of G to $\mathcal{U}$. It has $G^{(1)}_{\mathcal{U}} = \{(i, \gamma, j) : s(\gamma) \in U_j, r(\gamma) \in U_i\}$ and $G^{(0)}_{\mathcal{U}} = \{(i, \gamma, i) : \gamma \in U_i\}$. The source and range maps send (i, γ, j) to $(j, s(\gamma), j)$ and $(i, r(\gamma), i)$, respectively. The product is $(i, \gamma, j) \cdot (j, \gamma', k) = (i, \gamma\gamma', k)$.

Two étale groupoids G_1 and G_2 are *equivalent* if there are open covers $\mathcal{U}_1$ and $\mathcal{U}_2$ of $G^{(0)}_1$ and $G^{(0)}_2$, respectively, so that the localizations $(G_1)_{\mathcal{U}_1}$ and $(G_2)_{\mathcal{U}_2}$ are isomorphic. Other ways of expressing this are given in [3, p. 597, 599–601] but the above definition is good enough for our purposes.

Example 5.3 Let $G = M$ be the groupoid of Example 5.1. Let $\{U_i\}_{i=1}^{\infty}$ be an open cover of M. Then $G^{(1)}_{\mathcal{U}}$ is the disjoint union of the U_i's and $G^{(1)}_{\mathcal{U}}$ consists of pairs of points $(m_i, m_j) \in U_i \times U_j$ that get identified to the same point in M. By definition, $G_{\mathcal{U}}$ is equivalent to $G = M$.

Example 5.4 In the setup of Example 5.2, suppose that Γ acts freely and properly discontinuously on M. Then the cross-product groupoid $M \rtimes \Gamma$ is equivalent to the groupoid M/Γ.

Remark 5.5 We wish to identify equivalent groupoids. A more intrinsic approach is to consider instead the category $\mathcal{B}G$ of G-sheaves. Here a G-sheaf is a local homeomorphism $\sigma : E \to G^{(0)}$ equipped with a continuous right G-action $E \times_{G^{(0)}} G^{(1)} \to E$. Pulling back the differentiable structure to E, we may assume that σ is a local diffeomorphism. Then $\mathcal{B}G$ is a topos and equivalent groupoids give rise to equivalent topoi [20,21]. However, we will not pursue this approach.

5.3 Riemannian groupoids

A *smooth path* c in G consists of a partition $0 = t_0 \leq t_1 \leq \cdots \leq t_k = 1$, and a sequence $c = (\gamma_0, c_1, \gamma_1, \ldots, c_k, \gamma_k)$ where $c_i : [t_{i-1}, t_i] \to G^{(0)}$ is smooth, $\gamma_i \in G^{(1)}$, $c_i(t_{i-1}) = s(\gamma_{i-1})$ and $c_i(t_i) = r(\gamma_i)$. The path is said to go from $r(\gamma_0)$ to $s(\gamma_k)$. The groupoid G is *path connected* if any $x, y \in G^{(0)}$ can be joined by a smooth path. We will generally assume that G is path connected.

Given $\gamma \in G^{(1)}$, there is a neighborhood U of γ in $G^{(1)}$ so that $\{(s(\gamma'), r(\gamma')) : \gamma' \in U\}$ is the graph of a diffeomorphism $h : V \to W$ between neighborhoods (in $G^{(0)}$) of $s(\gamma)$ and $r(\gamma)$. In this way, G gives rise to a *pseudogroup* of diffeomorphisms of $G^{(0)}$. Conversely, if $\mathcal{P}$ is a pseudogroup of diffeomorphisms of $G^{(0)}$ then there is a corresponding étale groupoid G, with $G^{(1)}$ consisting of the germs of elements of $\mathcal{P}$. We say that an étale groupoid G is *effective* when the germ of an element $\gamma \in G^{(1)}$ is trivial if and only if $\gamma \in G^{(0)}$. An example of a noneffective groupoid comes from a nontrivial discrete group Γ, with $G^{(1)} = \Gamma$ and $G^{(0)} = \{e\}$. Hereafter the étale groupoids will be assumed to be effective.

An étale groupoid is *Riemannian* if there is a Riemannian metric g on $G^{(0)}$ so that the germs of elements of $G^{(1)}$ act isometrically. There is an obvious notion of the length of a smooth path in G. There is a pseudometric d on the orbit space $\mathcal{O}$, given by saying that for $x, y \in G^{(0)}$, $d(O_x, O_y)$ is the infimum of the lengths of the smooth paths joining x to y. The Riemannian groupoid G is *complete* if $(\mathcal{O}, d)$ is a complete pseudometric space. Hereafter the Riemannian groupoids will be assumed to be complete. The *diameter* of G is the diameter of the pseudometric space $(\mathcal{O}, d)$. If (G, O_x) is a pointed Riemannian groupoid then we write $B_R(O_x) = \bigcup\{O_y : d(O_y, O_x) < R\} \subset G^{(0)}$. Two Riemannian groupoids are (pointed) *isometrically equivalent* if there is a (pointed) isometric equivalence between them, as defined in terms of localizations.

Let J_1 be the groupoid of 1-jets of local diffeomorphisms of $G^{(0)}$. With the natural topology on $J_1^{(1)}$, it is a smooth nonétale groupoid with $J_1^{(0)} = G^{(0)}$. There is a continuous morphism $j_1 : G \to J_1$ which is injective, as the germ of an isometry is determined by its 1-jet. Taking the closure of $j_1\left(G^{(1)}\right)$ in $J_1^{(1)}$ gives a space that can be written as the embedding $\overline{j}_1 : \overline{G} \to J_1$ of a unique Riemannian groupoid $\overline{G}$ with $\overline{G}^{(0)} = G^{(0)}$. It is called the *closure* of G. In effect, $\overline{G}^{(1)}$ is obtained by taking the closure of $j_1\left(G^{(1)}\right)$ and changing the topology to give it the structure of an étale groupoid.

The orbits of $\overline{G}$ are closed submanifolds of $G^{(0)}$. The orbit space of $\overline{G}$ is Hausdorff. There is a locally constant sheaf $\underline{\mathfrak{g}}$ on $G^{(0)}$ of finite-dimensional Lie algebras, called the *structure sheaf*, with the following properties :

1. $\underline{\mathfrak{g}}$ is a $\overline{G}$-sheaf.
2. $\underline{\mathfrak{g}}$ is a sheaf of germs of Killing vector fields.
3. The elements of $\overline{G}$ that are close to units in J_1 are germs of local isometries $\exp(\xi)$, where ξ is a local section ξ of $\underline{\mathfrak{g}}$ that is close to zero.

Hereafter we assume that the Riemannian groupoids are closed, unless otherwise stated.

Example 5.6 If $G = S^1 \rtimes \mathbb{Z}$ is the groupoid of Example 5.2, with the generator of $\mathbb{Z}$ acting by an irrational rotation, then $\overline{G} = S^1 \rtimes \mathrm{SO}(2)$ and $\underline{\mathfrak{g}}$ is the constant $\mathbb{R}$-sheaf on S^1.

Example 5.7 Let M be a complete Riemannian manifold with sectional curvatures between $-K$ and K, for some $K > 0$. Given $r < \frac{\pi}{\sqrt{K}}$, for any $m \in M$ the exponential map $\exp_m : T_m M \to M$ restricts to a local diffeomorphism from the r-ball $B_r^{(m)}(0) \subset T_m M$ to $B_r(m) \subset M$. Put the metric $(\exp_m)^* g$ on $B_r^{(m)}(0)$.

Let $\{m_i\}_{i \in I}$ be points in M so that $\{B_r(m_i)\}_{i \in I}$ covers M. Define a Riemannian groupoid G with $G^{(1)} = \bigsqcup_{i,j \in I} \{(v_i, v_j) \in B_r^{(m_i)}(0) \times B_r^{(m_j)}(0) : \exp_{m_i}(v_i) = \exp_{m_j}(v_j)\}$ and $G^{(0)} = \bigsqcup_{i \in I} B_r^{(m_i)}(0)$ by $r(v_i, v_j) = v_i$, $s(v_i, v_j) = v_j$ and $(v_i, v_j) \cdot (v_j, v_k) = (v_i, v_k)$. Then G is isometrically equivalent to the groupoid M.

5.4 Convergence of Riemannian groupoids

Definition 5.8 Let $\{(G_i, O_{x_i})\}_{i=1}^{\infty}$ be a sequence of pointed n-dimensional Riemannian groupoids. Let (G_∞, O_{x_∞}) be a pointed Riemannian groupoid. Let J_1 be the groupoid of 1-jets of local diffeomorphisms of $G_\infty^{(0)}$. We say that $\lim_{i \to \infty} (G_i, O_{x_i}) = (G_\infty, O_{x_\infty})$ in the pointed smooth topology if for all $R > 0$,

1. There are pointed diffeomorphisms $\phi_{i,R} : B_R(O_{x_\infty}) \to B_R(O_{x_i})$, defined for large i, from the pointed R-ball in $G_\infty^{(0)}$ to the pointed R-ball in $G_i^{(0)}$, so that $\lim_{i \to \infty} \phi_{i,R}^* \, g_i \big|_{B_R(O_{x_i})} = g_\infty \big|_{B_R(O_{x_\infty})}$.
2. After conjugating by $\phi_{i,R}$, the images of $s_i^{-1}(\overline{B_{R/2}(O_{x_i})}) \cap r_i^{-1}(\overline{B_{R/2}(O_{x_i})})$ in J_1 converge in the Hausdorff sense to the image of $s_\infty^{-1}(\overline{B_{R/2}(O_{x_\infty})}) \cap r_\infty^{-1}(\overline{B_{R/2}(O_{x_\infty})})$ in J_1.

This definition is similar to [25, Definition A.4]. The paper [25] considers Lipschitz convergence instead of smooth convergence.

We will allow ourselves to freely replace a Riemannian groupoid by an isometrically equivalent one, without saying so explicitly.

Proposition 5.9 *Let $\{(M_i, p_i)\}_{i=1}^{\infty}$ be a sequence of pointed complete n-dimensional Riemannian manifolds. Suppose that for each $a \in \mathbb{Z}^{\geq 0}$ and $R > 0$, there is some*

$K_{a,R} < \infty$ so that for all i, one has $\| \nabla^a \operatorname{Riem}(M_i) \|_\infty \leq K_{a,R}$ on $B_R(p_i)$. Then there is a subsequence of $\{(M_i, p_i)\}_{i=1}^\infty$ that converges to some pointed n-dimensional Riemannian groupoid (G_∞, O_{x_∞}) in the pointed smooth topology.

Proof Put $r(j) = \frac{\pi}{2\sqrt{K_{0,2j}}}$. With reference to Example 5.7, for each $j \in \mathbb{Z}^+$ there is a number N_j so that we can find points $\{x_{i,j,k}\}_{k=1}^{N_j}$ in $B_j(p_i) - B_{j-1}(p_i)$ with the property that $\bigcup_{j=1}^\infty \bigcup_{k=1}^{N_j} B_{r(j)}(x_{i,j,k})$ covers M_i. As in Example 5.7, we form the corresponding Riemannian groupoid G_i with $G_i^{(0)} = \bigsqcup_{j=1}^\infty \bigsqcup_{k=1}^{N_j} B_{r(j)}^{(x_{i,j,k})}(0)$. It is isometrically equivalent to the Riemannian groupoid M_i. After passing to a subsequence, we may assume that $\{G_i^{(0)}\}_{i=1}^\infty$ converges smoothly to some $G_\infty^{(0)}$ in the sense of Definition 5.8.1. The rest of the argument is basically the same as in [7, Pf. of Theorem 0.5], which in turn uses ideas from [10, Chap. 8C and 8D]. Namely, after passing to a further subsequence, we can construct $G_\infty^{(1)}$ as a pointed Hausdorff limit, in the sense of Definition 5.8.2, of the images of $G_i^{(1)}$ in $J_1^{(1)}$, where J_1 is the groupoid of 1-jets of local diffeomorphisms of $G_\infty^{(0)}$. (Because of the convergence of the metrics in the sense of Definition 5.8.1, the images of $G_i^{(1)}$ in $J_1^{(1)}$ come closer and closer to taking value in the 1-jets of local isometries.) □

From the construction of G_∞, there is a G_∞-invariant sheaf $\underline{\mathfrak{g}}$ on $G_\infty^{(0)}$ consisting of local Killing vector fields that generate the collapsing directions. From [7, Sect. 4], $\underline{\mathfrak{g}}$ is a sheaf of nilpotent Lie algebras.

The orbit space of G_∞ is the same as the pointed Gromov-Hausdorff limit of the convergent subsequence of $\{(M_i, p_i)\}_{i=1}^\infty$.

Proposition 5.9 is essentially equivalent to [25, Theorem A.5(i)], except that we use smooth convergence instead of Lipschitz convergence. Ricci flow will provide the needed smoothness.

Example 5.10 Let M_i be the circle of radius i^{-1}. Then $\lim_{i\to\infty} M_i$ is the cross-product groupoid $\mathbb{R} \rtimes \mathbb{R}$ of Example 5.2.

Example 5.11 As in Example 3.40, let Σ be a closed hyperbolic surface and let $S^1\Sigma$ be its unit sphere bundle. Let M_i be $S^1\Sigma$ equipped with $\frac{1}{i}$ times the time-i Ricci flow metric. Then $\lim_{i\to\infty}(M_i, p)$ is the cross-product groupoid $(\mathbb{R} \times \Sigma) \rtimes \mathbb{R}$.

5.5 Convergence of Ricci flows on étale groupoids

Let G be a complete Riemannian groupoid. We can construct its Ricci tensor $\operatorname{Ric}(g)$ as a symmetric covariant 2-tensor field on $G^{(0)}$ which is invariant in the sense that it is preserved by the germs of elements of $G^{(1)}$.

Let $\{g(t)\}$ be a smooth 1-parameter family of Riemannian metrics on the étale groupoid G, where smoothness in t can be checked locally on $G^{(0)}$. Then $\{g(t)\}$ satisfies the Ricci flow equation if $\frac{\partial g}{\partial t} = -2\operatorname{Ric}$. We say that it is an *expanding soliton* if the flow is defined for $t \in (0, \infty)$ and $g(t)$ is isometrically equivalent to $t\, g(1)$ for all $t \in (0, \infty)$.

We now state an analog of [13, Theorem 1.2], except without the assumption of a lower bound on the injectivity radius. We define convergence of a sequence of Ricci flow solutions as in Sect. 2.2, except that now we allow the limit to be a Ricci flow on an étale groupoid.

Theorem 5.12 *Let $\{(M_i, p_i, g_i(\cdot))\}_{i=1}^{\infty}$ be a sequence of Ricci flow solutions on pointed n-dimensional manifolds (M_i, p_i). We assume that there are numbers $-\infty \leq A < 0$ and $0 < \Omega \leq \infty$ so that*

1. *The Ricci flow solution $(M_i, p_i, g_i(\cdot))$ is defined on the time interval (A, Ω).*
2. *For each $t \in (A, \Omega)$, $g_i(t)$ is a complete Riemannian metric on M_i.*
3. *For each compact interval $I \subset (A, \Omega)$ there is some $K_I < \infty$ so that $|\mathrm{Riem}(g_i)(x, t)| \leq K_I$ for all $x \in M_i$ and $t \in I$.*

Then after passing to a subsequence, the Ricci flow solutions $g_i(\cdot)$ converge smoothly to a Ricci flow solution $g_\infty(\cdot)$ on a pointed n-dimensional étale groupoid $\left(G_\infty, O_{x_\infty}\right)$, defined again for $t \in (A, \Omega)$.

Proof The proof is virtually the same as that of [13, Theorem 1.2]. From local derivative estimates, the pointed Riemannian manifolds $\{(M_i, p_i, g_i(0))\}_{i=1}^{\infty}$ satisfy the assumptions of Proposition 5.9. Then after passing to a subsequence, we can assume that $\{(M_i, p_i, g_i(0))\}_{i=1}^{\infty}$ converges to a pointed Riemannian groupoid $\left(G_\infty, O_{x_\infty}, g_\infty(0)\right)$. In terms of the proof of Proposition 5.9, we have pointed time-0 convergence of the Riemannian groupoids G_i to G_∞. The remaining step is to get convergence on the whole time interval (A, Ω), after passing to a further subsequence. The argument for this is as in [13, Sect. 2]. Namely, for any $R > 0$, if i is sufficiently large then one can use $\phi_{i,R}$ to transfer the time-t metric $g_i(t)$ on $B_R\left(O_{x_i}\right) \subset G_i^{(0)}$ to the time-t metric $\phi_{i,R}^* g_i(t)$ on the time-zero set $B_R\left(O_{x_\infty}\right) \subset G_\infty^{(0)}$. As in [13, Lemma 2.4], for any compact subinterval $I \subset (A, \Omega)$ one has uniform bounds on the norms (with respect to g_∞) of the spatial and temporal derivatives of $\phi_{i,R}^* g_i(\cdot)$ on $I \times B_R\left(O_{x_\infty}\right)$. Then after passing to a further subsequence, one obtains a limiting metric $g_\infty(\cdot)$ on G_∞ which will necessarily satisfy the Ricci flow equation. □

Remark 5.13 If one considers individual tangent spaces $T_{p_i}M_i$ instead of groupoids then an analog of [13, Theorem 1.2] was proven in [8]. The result of [8] was used in [5] to study three-dimensional type-II Ricci flow solutions.

Corollary 5.14 *Let $(M, p, g(\cdot))$ be a type-III Ricci flow solution. Then for any sequence $s_i \to \infty$, there are a subsequence (which we relabel as $\{s_i\}_{i=1}^{\infty}$) and a Ricci flow solution $\left(G_\infty, O_{x_\infty}, g_\infty(\cdot)\right)$, defined for $t \in (0, \infty)$, so that $\lim_{i\to\infty}(M, p, g_{s_i}(\cdot)) = \left(G_\infty, O_{x_\infty}, g_\infty(\cdot)\right)$.*

Corollary 5.15 *Given $K > 0$, the space of pointed n-dimensional Ricci flow solutions with $\sup_{t\in(0,\infty)} t \parallel \mathrm{Riem}(g_t) \parallel_\infty \leq K$ is relatively compact among Ricci flows on pointed n-dimensional étale groupoids, defined for $t \in (0, \infty)$.*

5.6 Compactification of type-III Ricci flow solutions

With reference to Corollary 5.15, let $\mathcal{S}_{n,K}$ denote the closure of the pointed n-dimensional Ricci flow solutions on manifolds with $\sup_{t\in(0,\infty)} t \parallel \mathrm{Riem}(g_t) \parallel_\infty$

$\leq K$. Given $g(\cdot) \in \mathcal{S}_{n,K}$, there is a rescaled Ricci flow solution $g_s(\cdot) \in \mathcal{S}_{n,K}$ given by $g_s(t) = s^{-1}\, g(st)$. This means that there is an $\mathbb{R}^+$-action on $\mathcal{S}_{n,K}$ where $s \in \mathbb{R}^+$ sends g to g_s. Understanding the long-time behavior of a type-III Ricci flow solution g boils down to understanding the dynamics of its orbit in the compact set $\mathcal{S}_{n,K}$. We make some elementary comments about $\mathcal{S}_{n,K}$.

First, a Ricci flow in the boundary of $\mathcal{S}_{n,K}$ necessarily has a collapsing structure, i.e. a nontrivial sheaf of nilpotent Lie algebras that act as local Killing vector fields on the space of units. For simple examples of Ricci flows in the boundary of $\mathcal{S}_{n,K}$, consider first the Ricci flow of a generic flat metric on the j-torus. Its rescaling limit is the constant Ricci flow on the cross-product groupoid $\mathbb{R}^j \rtimes \mathbb{R}^j$, where the first $\mathbb{R}^j$ factor has a flat metric g_0. Now if $(\widehat{M}, \widehat{g}(\cdot))$ is any pointed Ricci flow on an $(n-j)$-dimensional manifold $\widehat{M}$ with $\sup_{t\in(0,\infty)} t \ \parallel \mathrm{Riem}(\widehat{g}_t) \parallel_\infty \leq K$ then the product flow $\widehat{g}(\cdot) + g_0$ on the groupoid $(\widehat{M} \times \mathbb{R}^j) \rtimes \mathbb{R}^j$ is an element of the boundary of $\mathcal{S}_{n,K}$, as it is a limit of Ricci flows on $\widehat{M} \times T^j$. The $\mathbb{R}^+$-action commutes with the inclusion $\mathcal{S}_{n-j,K} \to \partial \mathcal{S}_{n,K}$.

Now let $\mathcal{E}_{n,K}$ be the n-dimensional pointed Einstein metrics g with Einstein constant $-\frac{1}{2}$ and $\parallel \mathrm{Riem}(g) \parallel_\infty \leq K$, modulo pointed diffeomorphisms. We can identify $\mathcal{E}_{n,K}$ with a set of Ricci flow solutions, with the Einstein metric g being the time-1 metric of the solution. In this way there is an inclusion $\mathcal{E}_{n,K} \subset \mathcal{S}_{n,K}$. We remark that by moving the basepoint on a finite-volume noncompact manifold with constant sectional curvature $-\frac{1}{2(n-1)}$, we obtain examples where $\overline{\mathcal{E}_{n,K}}$ intersects $\partial \mathcal{S}_{n,K}$.

Consider a pointed n-dimensional compact Ricci flow solution $(M, p, g(\cdot))$ with $\sup_{t\in(0,\infty)} t \ \parallel \mathrm{Riem}(g_t) \parallel_\infty \leq K$. Suppose that for large s, the corresponding orbit $\{g_s(\cdot)\}$ in $\mathcal{S}_{n,K}$ stays away from the boundary. This is equivalent to saying that $\liminf_{t\to\infty} t^{-\frac{1}{2}} \operatorname{inj}_{g(t)}(p) > 0$. If there is a pointed limit $\lim_{j\to\infty} g_{s_j}(\cdot)$, with $\{s_j\}_{j=1}^\infty$ a sequence tending to infinity, then the limit is a Ricci flow on a finite-volume Einstein manifold with negative Einstein constant [14, Sect. 7],[23, Sect. 7.1]. (The proof in [14, Sect. 7] is for the normalized Ricci flow while the proof in [23, Sect. 7.1] is for the unnormalized Ricci flow. Both proofs are for $n = 3$ but extend to general n.) It follows that as $s \to \infty$, the orbit $\{g_s(\cdot)\}$ approaches $\mathcal{E}_{n,K}$. However, it does not immediately follow that the orbit approaches a fixed-point.

To understand the asymptotics of the orbits that do not stay away from the boundary of $\mathcal{S}_{n,K}$, a first question is whether the orbit approaches the boundary, i.e. whether Ricci flow favors the formation of continuous symmetries.

Independent of this question, one can ask about the dynamics of the $\mathbb{R}^+$-action on the boundary. As the boundary elements have a collapsing structure, in principle one can use this to help analyze the Ricci flow equations. An example is Proposition 4.29, where a monotonicity formula was used. An overall question is whether the orbits of the $\mathbb{R}^+$-action on $\mathcal{S}_{n,K}$ approach fixed points, i.e. expanding soliton solutions.

In the case of a finite-volume locally homogeneous three-manifold, the results of Sect. 3 imply that the orbit approaches a fixed-point $g_\infty(\cdot)$. The next proposition makes this explicit. In the statement of the proposition we allow the fundamental group Γ to be a discrete subgroup of a *maximal* element G among groups of diffeomorphisms of the universal cover that act transitively with compact isotropy group, and we take the

Riemannian metric to be G-invariant; see [27, Sects. 4 and 5] and [28, Chap. 3] for the description of such groups.

Proposition 5.16 *Let $(M, p, g(\cdot))$ be a finite-volume pointed locally homogeneous three-dimensional Ricci flow solution that exists for all $t \in (0, \infty)$. Then $\lim_{s\to\infty} (M, p, g_s(\cdot))$ exists and is an expanding soliton on a pointed three-dimensional étale groupoid G_∞. Put $\Gamma = \pi_1(M, p)$. The groupoid G_∞ and its metric $g_\infty(t)$ are given as follows.*
1. If $(M, g(0))$ has constant negative curvature then G_∞ is the cross-product groupoid $H^3 \rtimes \Gamma$ (which is equivalent to M) and $g_\infty(t)$ has constant sectional curvature $-\frac{1}{4t}$.
2. If $(M, g(0))$ has $\mathbb{R}^3$-geometry, there is a homomorphism $\alpha : \mathrm{Isom}(\mathbb{R}^3) \to \mathrm{Isom}(\mathbb{R}^3)/\mathbb{R}^3$. (Here $\mathbb{R}^3$ is the translation subgroup of $\mathrm{Isom}(\mathbb{R}^3)$. The quotient $\mathrm{Isom}(\mathbb{R}^3)/\mathbb{R}^3$ is isomorphic to $O(3)$.) Put $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$. Then G_∞ is the cross-product groupoid $\mathbb{R}^3 \rtimes \Gamma_{\mathbb{R}}$ and $g_\infty(t)$ is the constant flat metric.
3. If $(M, g(0))$ has Sol-geometry, there is a homomorphism $\alpha : \mathrm{Isom}(\mathrm{Sol}) \to \mathrm{Isom}(\mathrm{Sol})/\mathbb{R}^2$. (Here $\mathbb{R}^2$ is a normal subgroup of Sol, which is a normal subgroup of $\mathrm{Isom}(\mathrm{Sol})$. The quotient $\mathrm{Isom}(\mathrm{Sol})/\mathbb{R}^2$ has $\mathbb{R}$ as a normal subgroup of index 8.) Put $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$. Then G_∞ is the cross-product groupoid $\mathrm{Sol} \rtimes \Gamma_{\mathbb{R}}$, with $g_\infty(t)$ given by (3.9).
4. If $(M, g(0))$ has Nil-geometry, there is a homomorphism $\alpha : \mathrm{Isom}(\mathrm{Nil}) \to \mathrm{Isom}(\mathrm{Nil})/\mathrm{Nil}$. (Here $\mathrm{Nil} \subset \mathrm{Isom}(\mathrm{Nil})$ acts by left multiplication. The quotient $\mathrm{Isom}(\mathrm{Nil})/\mathrm{Nil}$ is isomorphic to $O(2)$.) Put $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$. Then G_∞ is the cross-product groupoid $\mathrm{Nil} \rtimes \Gamma_{\mathbb{R}}$, with $g_\infty(t)$ given by (3.18).
5. If $(M, g(0))$ has $(\mathbb{R} \times H^2)$-geometry, there is a homomorphism $\alpha : \mathrm{Isom}\left(\mathbb{R} \times H^2\right) \to \mathrm{Isom}\left(\mathbb{R} \times H^2\right)/\mathbb{R}$. (Here $\mathrm{Isom}\left(\mathbb{R} \times H^2\right)/\mathbb{R}$ is isomorphic to $\mathbb{Z}_2 \times \mathrm{Isom}(H^2)$.) Put $\Gamma_{\mathbb{R}} = \alpha^{-1}(\alpha(\Gamma))$. Then G_∞ is the cross-product groupoid $(\mathbb{R} \times H^2) \rtimes \Gamma_{\mathbb{R}}$ and $g_\infty(t) = g_{\mathbb{R}} + g_{H^2}(t)$, where $g_{H^2}(t)$ has constant sectional curvature $-\frac{1}{2t}$.
6. If $(M, g(0))$ has $\widetilde{\mathrm{SL}_2(\mathbb{R})}$-geometry, there is a homomorphism $\alpha : \mathrm{Isom}\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}\right) \to \mathrm{Isom}\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}\right)/\mathbb{R}$. (Here $\mathrm{Isom}\left(\widetilde{\mathrm{SL}_2(\mathbb{R})}\right)/\mathbb{R}$ is isomorphic to $\mathrm{Isom}(H^2)$.) Then G_∞ is the cross-product groupoid $(\mathbb{R} \times H^2) \rtimes (\mathbb{R} \widetilde{\times} \alpha(\Gamma))$ and $g_\infty(t) = g_{\mathbb{R}} + g_{H^2}(t)$, where $g_{H^2}(t)$ has constant sectional curvature $-\frac{1}{2t}$. In writing $\mathbb{R} \widetilde{\times} \alpha(\Gamma)$, the group $\alpha(\Gamma) \subset \mathrm{Isom}(H^2)$ acts linearly on $\mathbb{R}$ via the orientation homomorphism $\alpha(\Gamma) \to \mathbb{Z}_2$.

Proof This follows from the results of Sect. 3.3. We just give two examples.

In case 4, suppose for simplicity that $\Gamma \subset \mathrm{Nil}$ and $M = \Gamma \backslash \mathrm{Nil}$ is compact. Fix a basepoint $p \in M$. Following the proof of Theorem 5.12, we will first construct a limiting time-1 Riemannian groupoid $(G_\infty, O_{x_\infty}) = \lim_{s\to\infty} (M, p, g_s(1))$. From (3.15), we have $\lim_{s\to\infty} \mathrm{diam}(M, g_s(1)) = 0$. For any $r > 0$, if s is sufficiently large then the exponential map $\exp_p(s) : B_r^{(p)}(0) \to M$ for M, with the metric $g_s(1)$, is a surjective local diffeomorphism. From the calculations in Sect. 3.3.3, after performing appropriate diffeomorphisms the pullback metrics $\exp_p(s)^* g_s(1)$ will approach $\exp_{p_\infty}^* g_\infty(1)$, where $g_\infty(1)$ is the metric of (3.18) and $\exp_{p_\infty} : B_r^{(p_\infty)}(0) \to M_\infty$ is the corresponding exponential map on the r-ball in the tangent space at the basepoint $p_\infty \in M_\infty$. The limit groupoid G_∞ consists of germs of local isometries of $B_r^{(p_\infty)}(0)$, the latter being equipped with the Riemannian metric $\exp_{p_\infty}^* g_\infty(1)$. This groupoid is isometrically equivalent to the cross-product groupoid Nil $\rtimes$ Nil, with metric $g_\infty(1)$. The extension to times t other than 1 gives the Ricci flow on the cross-product groupoid Nil $\rtimes$ Nil with metric $g_\infty(\cdot)$.

In case 6, suppose that M is the pointed unit sphere bundle $S^1\Sigma$ of an oriented hyperbolic surface Σ. The pointed Gromov–Hausdorff limit $\lim_{s\to\infty}(M, p, g_s(1))$ is the surface Σ with the metric rescaled to have sectional curvature $-\frac{1}{2}$. We choose points $x_{j,k}$ as in the proof of Proposition 5.9; we can choose them to be independent of the parameter s (which replaces i). In the limit we obtain a groupoid (G_∞, O_{x_∞}) whose unit space $G_\infty^{(0)}$ consists of a disjoint union of manifolds, each isometrically equivalent to a domain in $\mathbb{R} \times H^2$; see Example 3.40. After making the remaining identifications, this groupoid is isometrically equivalent to $(\mathbb{R} \times \Sigma) \rtimes \mathbb{R}$, equipped with the product metric $g_{\mathbb{R}} + g_\Sigma(1)$. The extension to times t other than 1 gives the Ricci flow on $(\mathbb{R} \times \Sigma) \rtimes \mathbb{R}$ with metric $g_{\mathbb{R}} + g_\Sigma(t)$, where $g_\Sigma(t)$ has constant sectional curvature $-\frac{1}{2t}$. This, in turn, is isometrically equivalent to the cross-product groupoid $(\mathbb{R} \times H^2) \rtimes (\mathbb{R} \times \alpha(\Gamma))$ with the metric $g_{\mathbb{R}} + g_{H^2}(t)$. $\square$

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