THE ETA FUNCTION AND SOME NEW ANOMALIES

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We wish to point out that two related anomalies [1,2] can be easily understood in terms of the \( \eta \) function of spectral geometry. For simplicity we take our manifolds to be closed, riemannian and with a spin structure and consider an elliptic self-adjoint differential operator \( H \) acting on fields (cross sections of a vector bundle) over the \( m \)-dimensional manifold \( M \). Let the eigenvalues of \( H \) be \( \{ \lambda_i \} \). Then [3]

\[
\eta_H(s) = \sum_{\lambda_i \neq 0} \text{sign} \lambda_i |\lambda_i|^{-s}.
\]

This converges for \( \Re s > m/(\text{differential order of } H) \) and can be analytically continued to the complex \( s \)-plane. Amazingly, \( \eta_H(0) \) is always finite [3,4]. If \( H(\epsilon) \) is a one-parameter family of invertible \( H \)'s then

\[
\frac{d}{d\epsilon} \eta_H(s) = \frac{d}{d\epsilon} \text{Tr} H(H^2)^{(-s-1)/2} = -s \text{Tr} \frac{dH}{d\epsilon} H(H^2)^{(-s-1)/2}.
\]

Thus to find \( \frac{d\eta_H(s)}{d\epsilon} \) as \( s \to 0 \) it is only necessary to find the pole term in \( (H^2)^{(-s-1)/2} \). Let \( \text{tr} \) denote the trace over the matrix part of the fields (the fiber trace on \( \text{End}(V) \)). For \( \Re s \) large, we can write the operator trace as the integral over \( M \) of a local matrix trace giving, in the case that \( dH/d\epsilon \) is a zeroth order operator,

\[
-s \text{Tr} \frac{dH}{d\epsilon} (H^2)^{(-s-1)/2} = -s \int_M \text{tr} \frac{dH}{d\epsilon} (H^2)^{(-s-1)/2}(x,x) \sqrt{g} \, d^m x,
\]

where \( g \) is the determinant of the metric tensor. Now

\[
(H^2)^{(-s-1)/2}(x,x) = \frac{1}{\Gamma((s+1)/2)} \int_0^\infty T^{(s-1)/2} \times e^{-TH^2(x,x)} \, dT.
\]

There is an asymptotic expansion

\[
e^{-TH^2(x,x)} \sim T^{-m/2} \sum_{i=0}^\infty a_i(x,x)T^i,
\]

which gives

\[
\lim_{s \to 0} s(H^2)^{(-s-1)/2}(x,x)
\]

\[
= \lim_{s \to 0} \frac{1}{\sqrt{\pi}} \int_0^1 T^{(s-m-1)/2} \sum_{i=0}^\infty a_i(x,x)T^i \, dT
\]

\[
= (2/\sqrt{\pi}) a_{(m-1)/2}(x,x).
\]

Thus

\[
\frac{d}{d\epsilon} \eta_H(s) = -2 \frac{1}{\sqrt{\pi}} \int_M \text{tr} \left( \frac{dH}{d\epsilon} (x) a_{(m-1)/2}(x,x) \right) \sqrt{g} \, d^m x.
\]

The \( a_{(m-1)/2} \) are computable and have been tabulated for \( m \leq 7 \) [5]. That is, variations of \( \eta_H(0) \) are computable for invertible \( H \)'s as local expressions.

If, on the other hand, \( H(\epsilon) \) passes through a nonin-
vertible $H$ then an eigenvalue can jump from, say, $\lambda < 0$ to $\lambda > 0$ as, say, $e$ goes from $0^-$ to $0^+$. Because $\eta_H(0)$ essentially adds the eigenvalues by sign, the jump in $\eta_H(0)$ from $e = 0^-$ to $e = 0^+$ is $2$. Similarly if an eigenvalue goes from $\lambda > 0$ to $\lambda < 0$ as $e$ goes from $0^-$ to $0^+$ then the jump in $\eta_H(0)$ at $e = 0$ is $-2$. The preceding is all well known to mathematicians.

The first anomaly is that of induced vacuum charge in an even-dimensional space [1]. The renormalized charge for a quantized spinor field in a classical background field is $Q = -\frac{1}{2}\eta_H(0)$, $H$ being the spinor hamiltonian [6]. Because $a_{(m-1)/2} = 0$ for $m$ even, $Q$ is given solely by the integer jumps of above. Take $H = -i\gamma^5(\nabla_j + A_j) + \phi \gamma^5$. For $\phi = 0$ there is a conjugation symmetry and $\eta_H(0) = 0$. Write the kernel of $H$ as $\text{Ker} H = W^+ \oplus W^-$ such that for $V \in W^\pm$, $\gamma^5 V = \pm V$.

For $\phi = 0^+$ the vectors in $W^+$ will acquire a positive energy while those in $W^-$ will acquire a negative energy, giving

$$Q = -\frac{1}{2}(\dim W^+ - \dim W^-)$$

$$= -\frac{1}{2}\text{Index}(-i\gamma^5(\nabla_j + A_j)),$$

the last operator mapping positive chirality spinors to negative chirality spinors.

From the Atiyah–Singer index theorem [7],

$$Q = -\frac{1}{2} \int (\text{ch} F) \hat{A}(g),$$

the integral of the product of the Chern character of the gauge field and the $\hat{A}$ character of the metric. On a flat space

$$Q = -\frac{1}{2} \frac{1}{(2\pi)^{m/2}} \frac{1}{(m/2)!} \int_M \text{tr} F^{m/2},$$

which is the result of ref. [11]. The $\phi \gamma^5$ term in $H$ comes from the use of a regulator mass in the lagrangian approach.

The second anomaly is a parity-violating current in odd-dimensional space-time [2]. Let the wave equation for a massless spinor be $\Box \psi = 0$ and let $\Box$ have (real) spectrum $\{\lambda_i\}$. Formally define the effective action to be

$$\Gamma = \ln \det \Box \equiv \frac{1}{2} \ln \det \Box^2 + (2n + 1)i\pi$$

$\times (\text{number of } \lambda_i < 0).$

Under a parity transformation,

$$\Gamma \to \Gamma_p = \ln \det(-\Box) =$$

$$\equiv \frac{1}{2} \ln \det \Box^2 + (2n + 1)i\pi \left(\text{number of } \lambda_i > 0\right).$$

We define

$$\Gamma - \Gamma_p = -(2n + 1)i\pi \eta_\Box(0).$$

If only gauge fields are present then the Atiyah–Patodi–Singer theorem [3] implies that

$$\frac{1}{2}(\eta_\Box(0) + \dim \text{Ker } \Box) = \text{CS}(M) \pmod{Z},$$

the evaluation of the Chern–Simons secondary characteristic class (CS) with values in $\mathbb{R}/\mathbb{Z}$ [8]. This is the result of ref. [2]. If a mass term $\phi$ is added to $\Box$ then, with $\{\lambda_i\}$ being the new spectrum, a parity transformation takes $\lambda_i$ to $2\phi - \lambda_i$. Then

$$\text{Im}(\Gamma - \Gamma_p) = (\text{formally}) (2n + 1)i\pi$$

$$\times (\text{number of } \lambda_i < 0) - (\text{number of } 2\phi - \lambda_i < 0)$$

$$= (2n + 1)i\pi \left(\text{number of } \lambda_i \in (0, 2\phi)\right) - \eta_\Box(0).$$

By the methods of the first paragraph $\frac{1}{2}(\eta_\Box(0) + \dim \text{Ker } \Box) \pmod{Z}$ can be found. For $m = 3$,

$$\text{Im}(\Gamma - \Gamma_p) = (2n + 1)i\pi \left(\text{number of } \lambda_i \in (0, 2\phi)\right)$$

$$+ \dim \text{Ker } \Box - 2\text{CS}(M)$$

$$+ \frac{1}{3} \int \phi^3 \sqrt{g} \, d^m x \pmod{2i\pi Z}.$$}

On an open manifold the last term may be infinite, but the currents are finite.

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References