REMARK ABOUT SCALAR CURVATURE ON CERTAIN NONCOMPACT MANIFOLDS

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Abstract. We give a sufficient condition to rule out complete Riemannian metrics with nonnegative scalar curvature on the interiors of handlebodies. In higher dimensions, we give examples of ends of manifolds with positive scalar curvature metrics.

1. Introduction

Let $\overline{M}$ be an orientable three dimensional handlebody, i.e. a repeated boundary connected sum of solid tori. Equivalently, $\overline{M}$ is homeomorphic to a regular neighborhood of a bouquet of circles in $\mathbb{R}^3$. We say that $\overline{M}$ has genus g if $\partial \overline{M}$ is a surface of genus g.

Let M be the interior of $\overline{M}$. If $g = 0$ then M is diffeomorphic to $\mathbb{R}^3$ and has a complete Riemannian metric of uniformly positive scalar curvature. If $g = 1$ then M is diffeomorphic to $S^1 \times \mathbb{R}^2$ and has a complete Riemannian metric of nonuniformly positive scalar curvature. If $g > 1$ then it is unclear whether M admits a complete Riemannian metric of nonnegative scalar curvature.

Chodosh, Lai and Xu used the inverse mean curvature flow to show that this is not the case under the condition that the metric has bounded sectional curvature and positive injectivity radius [1]. Their proof actually shows that the metric cannot have nonnegative scalar curvature outside of a compact set, indicating that there is an end obstruction. We give another condition to rule out nonnegative scalar curvature outside of a compact subset. The condition roughly says that there are surfaces going out the end whose areas do not increase too fast.

Theorem 1. *Let $\overline{M}$ be an orientable three dimensional handlebody. Let M be the interior of $\overline{M}$. Suppose that M has a complete Riemannian metric with nonnegative scalar curvature outside of a compact subset. Choose a basepoint $m_0 \in M$. Given $r > 0$, let $A(r)$ be the infimum of the areas of surfaces in M that are homologous to $\partial \overline{M}$ in $\overline{M}$, and lie outside of $B(m_0, r)$. Suppose that $\liminf_{r \to \infty} r^{-2} A(r) < \frac{12}{\pi}$. Then the genus of $\overline{M}$ is at most one.*

The proof of Theorem 1 indicates that a possible end obstruction depends on whether the boundary at infinity has almost nonnegative scalar curvature, in a suitable sense. The next result, which is valid in any dimension, gives a sufficient condition to have positive scalar curvature on an end.

Date: January 13, 2026.

Theorem 2. *Let N be a compact manifold that admits a Riemannian metric with nonnegative scalar curvature. Let X be the total space of an S^1-bundle over N. Then $[0,\infty) \times X$ admits a complete Riemannian metric with positive scalar curvature.*

The organization of the paper is as follows. In Section 2 we prove Theorem 1. We also state a band width inequality. In Section 3 we prove Theorem 2. Section 4 has a discussion.

I thank Otis Chodosh, Yi Lai and Kai Xu for conversations.

2. Proof of Theorem 1

The proof uses μ-bubbles, for which references are [2, Section 3] and [3, Section 5]. We find quantitative negative lower bounds on the total curvatures for a sequence of μ-bubbles exiting the end. One ingredient is upper area bounds for the μ-bubbles.

By assumption, there are some $\epsilon > 0$ and a sequence $r_i \to \infty$ so that for each i, there is a surface $\Sigma_i \subset M$ with

- $\Sigma_i \subset M - B(m_0, r_i)$,
- Σ_i is homologous in $\overline{M}$ to $\partial \overline{M}$, and
- $\mathrm{Area}(\Sigma_i) \le \left(\frac{12}{\pi} - \epsilon\right) r_i^2$.

Let $\epsilon' > 0$ be a parameter that will be independent of i. For the moment, we drop the index i. Choose $\epsilon'' > 0$ (which can depend on i). Put $D = \max_{p \in \Sigma} d(m_0, p)$. The surface Σ is separating. Let d_Σ denote the signed distance from Σ, with $d_\Sigma(m)$ going to $-\infty$ as m exits the end of M. There is some $\delta > 0$ so that d_Σ is smooth in the δ-neighborhood of Σ. Furthermore, we can find some $\widehat{d_\Sigma} \in C^\infty(M)$ so that on $B(m_0, 3D)$,

- $|\widehat{d_\Sigma} - d_\Sigma| \le \epsilon''$,
- $|\nabla \widehat{d_\Sigma}| \le 1 + \epsilon''$, and
- $\widehat{d_\Sigma}$ equals d_Σ on the $\frac{\delta}{2}$-neighborhood of Σ.

By Sard's theorem, we can find $L \in [(1-\epsilon')r - \epsilon'', (1-\epsilon')r]$ so that $-L$ and L are regular values for $\widehat{d_\Sigma}$. Put

- $N = \widehat{d_\Sigma}^{-1}[-L, L]$,
- $\partial_+ N = \widehat{d_\Sigma}^{-1}(L)$ and $\partial_- N = \widehat{d_\Sigma}^{-1}(-L)$, and
- $\Omega_0 = \widehat{d_\Sigma}^{-1}[0, L]$.

The boundary component $\partial_- N$ is farther out the end than $\partial_+ N$.

If i is large and ϵ'' is small then N has nonnegative scalar curvature. Put $h = \frac{(1+\epsilon'')2\pi}{3L} \tan\left(\frac{\pi}{2L} \widehat{d_\Sigma}\right)$ on the interior of N. Then $h(m)$ approaches $\pm\infty$ as m approaches $\partial_\pm N$.

Let Ω be a codimension-zero submanifold-with-boundary of N so that the symmetric difference $\Omega \Delta \Omega_0$ has compact support in the interior of N. In particular, $\partial_+ N \subset \partial \Omega$. Put $\partial_* \Omega = \partial \Omega - \partial_+ N$. Define the functional

$$\mathcal{A}(\Omega) = A(\partial_* \Omega) - \int_N (\chi_\Omega - \chi_{\Omega_0}) h \, \mathrm{dvol}_N \, . \tag{2.1}$$

There is a minimizer Ω for $\mathcal{A}$. The mean curvature of $\partial_* \Omega$ is $h\Big|_{\partial_*\Omega}$. The second variation formula for $\mathcal{A}$ implies that for any $\psi \in C^\infty(\partial_* \Omega)$, we have

$$0 \le \int_{\partial_*\Omega} \left(|\nabla_{\partial_*\Omega} \psi|^2 - \frac{1}{2} \left(R_M - R_{\partial_*\Omega} + \frac{3}{2} h^2 + 2 \langle \nabla h, \nu \rangle \right) \psi^2 \right) dA, \tag{2.2}$$

where ν is the outward pointing normal vector. In particular, if C is a connected component of $\partial_* \Omega$ then taking $\psi = \chi_C$ gives

$$\int_C R_C \, dA \ge \int_C \left(\frac{3}{2} h^2 - 2 |\nabla h| \right) dA. \tag{2.3}$$

Now

$$\begin{aligned} \frac{3}{2} h^2 - 2 |\nabla h| &= \frac{2(1+\epsilon'')^2 \pi^2}{3L^2} \tan^2 \left(\frac{\pi}{2L} \widehat{d_\Sigma} \right) - \frac{2(1+\epsilon'')\pi^2}{3L^2} |\nabla \widehat{d_\Sigma}| \sec^2 \left(\frac{\pi}{2L} \widehat{d_\Sigma} \right) \\ &\ge \frac{2(1+\epsilon'')^2 \pi^2}{3L^2} \tan^2 \left(\frac{\pi}{2L} \widehat{d_\Sigma} \right) - \frac{2(1+\epsilon'')\pi^2}{3L^2} (1+\epsilon'') \sec^2 \left(\frac{\pi}{2L} \widehat{d_\Sigma} \right) \\ &= - \frac{2(1+\epsilon'')^2 \pi^2}{3L^2}. \end{aligned} \tag{2.4}$$

Finally, $\mathcal{A}(\Omega) \le \mathcal{A}(\Omega_0) = A(\Sigma)$, so

$$\begin{aligned} A(\partial_* \Omega) &\le A(\Sigma) + \int_N (\chi_\Omega - \chi_{\Omega_0}) h \, \mathrm{dvol}_N \\ &= A(\Sigma) + \int_{\Omega - (\Omega_0 \cap \Omega)} h \, \mathrm{dvol}_N - \int_{\Omega_0 - (\Omega_0 \cap \Omega)} h \, \mathrm{dvol}_N \, . \end{aligned} \tag{2.5}$$

If $\epsilon'' \ll \delta$ then h is nonpositive on $\Omega - (\Omega_0 \cap \Omega)$ and nonnegative on $\Omega_0 - (\Omega_0 \cap \Omega)$, so $A(\partial_* \Omega) \le A(\Sigma)$. In particular, $A(C) \le A(\Sigma)$.

Equations (2.3) and (2.4) now imply that $\int_C R_C \, dA \ge - \frac{2(1+\epsilon'')^2\pi^2}{3L^2} A(\Sigma)$. Taking $\epsilon'' \to 0$ gives $\int_C R_C \, dA \ge - \frac{2\pi^2}{3(1-\epsilon')^2 r^2} A(\Sigma)$.

Restoring the index i shows that for i large, any connected component C of $\partial_* \Omega_i$ satisfies

$$\int_C R_C \, dA \ge - \frac{2\pi^2}{3(1-\epsilon')^2 r_i^2} A(\Sigma_i) \ge - \frac{2\pi^2}{3(1-\epsilon')^2} \left(\frac{12}{\pi} - \epsilon \right). \tag{2.6}$$

If ϵ' is sufficiently small then $\int_C R_C \, dA > - \frac{2}{3} \pi^2 \cdot \frac{12}{\pi} = -8\pi$. By the Gauss-Bonnet theorem, $\int_C R_C \, dA = 4\pi \chi(C)$. Hence C is diffeomorphic to S^2 or T^2.

By construction, $\partial_* \Omega_i$ is homologous to Σ_i. For fixed $\epsilon' > 0$, there is a subset S of $\overline{M}$, diffeomorphic to $[0,1] \times \partial \overline{M}$ and containing $\partial \overline{M}$, so that $\partial_* \Omega_i$ lies in S for all large i. Let $e_i : \partial_* \Omega_i \to S$ be the embedding and let $\alpha : S \to \partial \overline{M}$ be the projection map. By assumption, $\alpha \circ e_i$ gives an isomorphism on the second homology group. Hence it is a degree one map from $\partial_* \Omega_i$ to $\partial \overline{M}$, when everything is consistently oriented. Let $\{C_{i,j}\}$ be the connected components of $\partial_* \Omega_i$. Then $\alpha \circ e_i$ restricts to $C_{i,j}$ to give a map from $C_{i,j}$ to $\partial \overline{M}$, of some degree $d_{i,j}$. Since $\sum_j d_{i,j} = 1$, some component $C_{i,j}$ has a map to $\partial \overline{M}$ of nonzero degree. As $C_{i,j}$ is diffeomorphic to S^2 or T^2, the genus of $\partial\overline{M}$ is at most one. This proves the theorem.

Remark 1. The n-dimensional analog is the following. Put $h = \frac{(1+\epsilon'')(n-1)\pi}{nL}\tan\left(\frac{\pi}{2L}\widehat{d}_\Sigma\right)$. Let ψ be the eigenfunction corresponding to the lowest eigenvalue λ of the operator $-\triangle + \frac{1}{2}(R_\Sigma - R_M - \frac{n}{n-1}h^2 + 2|\nabla h|)$. Then λ is nonnegative and ψ is positive. Put $R_1 = R_\Sigma - 2\frac{\triangle\psi}{\psi}$; it is the modified scalar curvature R_q associated to the pair $(g_\Sigma, \psi\,\mathrm{dvol}_\Sigma)$ when $q = 1$, in the sense of [5]. Then $R_1 \geq -\frac{(n-1)(1+\epsilon'')^2\pi^2}{nL^2}$.

Remark 2. The proof of Theorem 1 gives the following band width inequality. Suppose that M is diffeomorphic to $[-1,1]\times V$, where V is a compact orientable surface of genus $g \geq 1$. Put $\Sigma_x = \{x\}\times V$. Suppose that $R_M \geq r_0 > -\frac{8\pi(g-1)}{A(\Sigma_0)}$. Then

$$\min(\mathrm{dist}(\Sigma_0, \Sigma_{-1}), \mathrm{dist}(\Sigma_0, \Sigma_1)) \leq \pi\sqrt{\frac{2A(\Sigma_0)}{24\pi(g-1) + 3r_0A(\Sigma_0)}}. \tag{2.7}$$

One sees that if $g > 1$ then this has a weaker assumption and stronger conclusion than the $g = 1$ case, the latter of which is area independent.

If instead M is diffeomorphic to $[0,1]\times V$, and Σ_0 has nonnegative mean curvature, then a doubling argument gives the same inequality for $\mathrm{dist}(\Sigma_0, \Sigma_1)$.

3. Proof of Theorem 2

Let N^{n-2} be compact with a fixed Riemannian metric h and scalar curvature $R_h \geq 0$. We can assume that N is connected. Let $\pi : X \to N$ be an S^1-bundle. We can reduce the structure group of X from $\mathrm{Diff}(S^1)$ to $\mathrm{O}(2)$. After passing to a double cover if necessary and working equivariantly, we can assume that the structure group is $\mathrm{SO}(2)$. Then X is a principal $\mathrm{SO}(2)$-bundle over N. Let θ be a connection 1-form on X, with curvature

$$d\theta = \pi^*\Omega, \qquad \Omega \in \Omega^2(N). \tag{3.1}$$

Put $\Omega_{ij} = \Omega(\partial_i, \partial_j)$ and $|\Omega|_h^2 = \frac{1}{2}h^{ik}h^{jl}\Omega_{ij}\Omega_{kl}$. Define

$$\Omega_\infty = \max_N |\Omega|_h < \infty. \tag{3.2}$$

For functions $a(t) > 0$ and $b(t) > 0$ on $[0,\infty)$, consider on $[0,\infty)\times X$ the metric

$$\bar{g} \;=\; dt^2 + g(t), \qquad g(t) = a(t)^2\,\theta^2 + b(t)^2\,\pi^*h. \tag{3.3}$$

All norms $|\cdot|_h$ below are computed using h, and primes denote t-derivatives. The formulas below were generated with artificial intelligence and checked with natural perseverance.

The scalar curvature of $\bar{g}$ on $[0,\infty)\times X$ is

$$R_{\bar{g}} = \frac{1}{b^2}R_h - \frac{a^2}{4b^4}|\Omega|_h^2 - 2\frac{a''}{a} - 2(n-2)\frac{b''}{b} - 2(n-2)\frac{a'}{a}\frac{b'}{b} - (n-2)(n-3)\left(\frac{b'}{b}\right)^2. \tag{3.4}$$

Since $R_h \geq 0$, for lower bounds one may discard the nonnegative term $\frac{1}{b^2}R_h$.

We now separate into the cases $n = 2$, $n = 3$, $n = 4$, $n = 5$ and $n \geq 6$.

n = 2. Here $\dim N = 0$, so $R_h \equiv 0$ and $\Omega \equiv 0$. The metric is $\bar{g} = dt^2 + a(t)^2\, d\phi^2$ and

$$R_{\bar{g}}(t) = -2\,\frac{a''(t)}{a(t)}. \tag{3.5}$$

Example:

$$a(t) = 1 + \sqrt{1+t} \quad\Longrightarrow\quad R_{\bar{g}}(t) = \frac{1}{2(1+t)^{3/2}\big(1+\sqrt{1+t}\big)} \;>\; 0. \tag{3.6}$$

n = 3. Now $\dim N = 1$, so $R_h \equiv 0$ and $\Omega \equiv 0$. Taking $b(t) \equiv 1$ gives again

$$R_{\bar{g}}(t) = -2\,\frac{a''(t)}{a(t)}. \tag{3.7}$$

Equation (3.6) applies.

n = 4. Choose

$$b(t) = (1+t)^{3/5}, \qquad a(t) \equiv a_0. \tag{3.8}$$

Then

$$R_{\bar{g}}(t,x) = (1+t)^{-6/5} R_h(x) \;-\; \frac{a_0^2}{4}(1+t)^{-12/5}|\Omega|_h^2(x) \;+\; \frac{6}{25}(1+t)^{-2}. \tag{3.9}$$

Using $R_h \geq 0$ and $|\Omega|_h \leq \Omega_\infty$ gives

$$R_{\bar{g}}(t,x) \;\geq\; \frac{6}{25}\frac{1}{(1+t)^2} \;-\; \frac{a_0^2}{4}\,\Omega_\infty^2\,\frac{1}{(1+t)^{12/5}}. \tag{3.10}$$

If $\Omega_\infty = 0$ then $R_{\bar{g}} > 0$ for all $t \geq 0$. If $\Omega_\infty > 0$ and

$$0 < a_0 < \frac{2\sqrt{6}}{5\,\Omega_\infty} \tag{3.11}$$

then $R_{\bar{g}} > 0$ for all $t \geq 0$.

n = 5. Choose

$$b(t) = (1+t)^{2/5}, \qquad a(t) = \varepsilon(1+t)^{-2/5}. \tag{3.12}$$

Then

$$R_{\bar{g}}(t,x) = (1+t)^{-4/5} R_h(x) \;-\; \frac{\varepsilon^2}{4}(1+t)^{-12/5}|\Omega|_h^2(x) \;+\; \frac{8}{25}(1+t)^{-2}. \tag{3.13}$$

and the uniform lower bound is

$$R_{\bar{g}}(t,x) \;\geq\; \frac{8}{25}\frac{1}{(1+t)^2} \;-\; \frac{\varepsilon^2}{4}\,\Omega_\infty^2\,\frac{1}{(1+t)^{12/5}}. \tag{3.14}$$

If $\Omega_\infty = 0$ then $R_{\bar{g}} > 0$ for all $t \geq 0$. If $\Omega_\infty > 0$ and

$$0 < \varepsilon < \frac{4\sqrt{2}}{5\,\Omega_\infty} \tag{3.15}$$

then $R_{\bar{g}} > 0$ for all $t \geq 0$.

$\mathbf{n \geq 6}$. Choose

$$b(t) = (1+t)^{\frac{2}{n-1}}, \qquad a(t) = \varepsilon(1+t)^{-\frac{n-5}{n-1}}. \tag{3.16}$$

Then

$$R_{\bar{g}}(t,x) = (1+t)^{-\frac{4}{n-1}} R_h(x) \; - \; \frac{\varepsilon^2}{4}(1+t)^{-2}|\Omega|_h^2(x) \; + \; \frac{4(n-5)}{(n-1)^2}(1+t)^{-2}. \tag{3.17}$$

Therefore the uniform lower bound is

$$R_{\bar{g}}(t,x) \; \geq \; \left(\frac{4(n-5)}{(n-1)^2} - \frac{\varepsilon^2}{4}\Omega_\infty^2 \right) \frac{1}{(1+t)^2}. \tag{3.18}$$

If $\Omega_\infty = 0$ then $R_{\bar{g}} > 0$ for all $t \geq 0$. If $\Omega_\infty > 0$ and

$$0 < \varepsilon < \frac{4\sqrt{n-5}}{(n-1)\,\Omega_\infty} \tag{3.19}$$

then $R_{\bar{g}} > 0$ for all $t \geq 0$.

4. Discussion

With reference to Theorem 1, from the point of view of index theory there isn't much difference whether the genus is one or the genus is greater than one. In both cases $\partial \overline{M}$ is aspherical and enlargeable, and its fundamental group satisfies the Strong Novikov Conjecture. And in both cases there is no index theoretic obstruction to having a Riemannian metric on $\overline{M}$ with positive scalar curvature and positive mean curvature boundary, since such metrics exist. Either there is no end obstruction to nonnegative scalar curvature, or index theory is not suited to find it, or there is some new aspect to be discovered.

In general dimension, if X is compact and there is a complete Riemannian metric on $[0,\infty) \times X$, with *uniformly* positive scalar curvature, then μ-bubbles show that there is a sequence $\{H_i\}_{i=1}^\infty$ of hypersurfaces exiting the end so that each H_i admits a Riemannian metric of positive scalar curvature, and there is a degree one map from H_i to X; c.f. [6]. In contrast, in the proof of Theorem 1 we obtain surfaces with a quantitative negative lower bound on their total scalar curvatures.

If a compact manifold X has a Riemannian metric of nonnegative scalar curvature then a warped product construction shows that $[0,\infty) \times X$ has a Riemannian metric of positive scalar curvature. However, the converse is not true, as shown by Theorem 2 when X is a three dimensional nilmanifold. In general, the most one could expect is that X has almost nonnegative scalar curvature in some sense.

To elaborate on this, a compact orientable three dimensional manifold X has almost nonnegative scalar curvature, relative to the volume, if and only if it is a graph manifold, i.e. has no hyperbolic piece in its geometric decomposition [4, Proposition 93.10]. As to when $[0,\infty) \times X$ has a metric of positive scalar curvature, Theorem 2, or more precisely its proof, implies that this is the case when X has Thurston type $\mathbb{R}^3$, S^3, $\mathbb{R} \times S^2$ or Nil. If X has Thurston type H^3, $\mathbb{R} \times H^2$, $\widetilde{\mathrm{SL}(2,\mathbb{R})}$ or Sol then we were unable to find such a metric.

The manifold X in Theorem 2 has almost nonnegative scalar curvature in the sense that with reference to the metric g in (3.3), as $a \to 0$, i.e. as the circle fiber shrinks, the scalar curvature has a negative lower bound that approaches zero.

References

[1] O. Chodosh, Y. Lai and K. Xu, "3-Manifolds with positive scalar curvature and bounded geometry", preprint, https://arxiv.org/abs/2502.09727 (2025)

[2] O. Chodosh and C. Li, "Generalized soap bubbles and the topology of manifolds with positive scalar curvature", Ann. of Math. 199, p. 707-740 (2024)

[3] M. Gromov, "Four lectures on scalar curvature", in Perspectives in Scalar Curvature, Vol. 1, World Scientific, Singapore (2023)

[4] B. Kleiner and J. Lott, "Notes on Perelman's papers", Geom. Top. 12, p. 2587-2855 (2008)

[5] J. Lott, "Remark about scalar curvature and Riemannian submersions", Proc. of the AMS 135, p. 3375-3381 (2007)

[6] Y. Shi, J. Wang, R. Wu and J. Zhu, "On open manifolds admitting no complete metric with positive scalar curvature", to appear, Annales de l'Institut Fourier

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