A DOLBEAULT-HILBERT COMPLEX FOR A VARIETY WITH ISOLATED SINGULAR POINTS

JOHN LOTT

Abstract. Given a compact Hermitian complex space with isolated singular points, we construct a Dolbeault-type Hilbert complex whose cohomology is isomorphic to the cohomology of the structure sheaf. We show that the corresponding K-homology class coincides with the one constructed by Baum-Fulton-MacPherson.

1. Introduction

The program of doing index theory, or more generally elliptic theory, on singular varieties goes back at least to Singer's paper [26, §4]. This program takes various directions, for example the relation between L^2-cohomology and intersection homology. In this paper we consider a somewhat different direction, which is related to the arithmetic genus. This is motivated by work of Baum-Fulton-MacPherson [7, 8].

Let X be a projective algebraic variety and let $\mathcal{S}$ be a coherent sheaf on X. In [8], the authors associated to $\mathcal{S}$ an element $[\mathcal{S}]_{BFM} \in \mathrm{K}_0(X)$ of the topological K-homology of X. This class enters into their Riemann-Roch theorem for singular varieties. In particular, under the map $p : X \to \mathrm{pt}$, the image $p_*[\mathcal{S}]_{BFM} \in \mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$ is expressed in terms of sheaf cohomology by $\sum_i (-1)^i \dim(\mathrm{H}^i(X;\mathcal{S}))$.

In view of the isomorphism between topological K-homology and analytic K-homology [6, 9], the class $[\mathcal{S}]_{BFM}$ can be represented by an "abstract elliptic operator" in the sense of Atiyah [3]. This raised the question of how to find an explicit cycle in analytic K-homology, even if X is singular, that represents $[\mathcal{S}]_{BFM}$. The most basic case is when $\mathcal{S}$ is the structure sheaf $\mathcal{O}_X$. If X is smooth then the operator representing $[\mathcal{O}_X]_{BFM}$ is $\overline{\partial} + \overline{\partial}^*$. Hence we are looking for the right analog of this operator when X may be singular.

A second related question is to find a Hilbert complex, in the sense of [11], whose cohomology is isomorphic to $\mathrm{H}^\star(X;\mathcal{O}_X)$. We want the complex to be intrinsic to X. We also want to have a subcomplex consisting of the smooth compactly supported $(0,\star)$-forms on the regular part X_{reg}, on which the differential is $\overline{\partial}$. In this paper, we answer these questions when X has isolated singular points.

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To see the nature of the problem, suppose that X is a complex curve, whose normalization has genus g. In this case, the Riemann-Roch theorem says

$$\dim(\mathrm{H}^0(X;\mathcal{O}_X)) - \dim(\mathrm{H}^1(X;\mathcal{O}_X)) = 1 - g - \sum_{x\in X_{sing}} \delta_x, \tag{1.1}$$

where δ_x is a certain positive integer attached to the singular point x [16, p. 298]. To find the appropriate Hilbert complex, it is natural to start with the Dolbeault complex $\Omega^{0,0}_c(X_{reg}) \xrightarrow{\overline{\partial}} \Omega^{0,1}_c(X_{reg})$ of smooth compactly supported forms on X_{reg} and look for a closed operator extension, where X_{reg} is endowed with the induced Riemannian metric from its projective embedding. For the minimal closure $\overline{\partial}_s$, one finds $\mathrm{Index}(\overline{\partial}_s) = 1 - g$. Taking a different closure can only make the index go up [12], whereas in view of (1.1) we want the index to go down. (Considering complete Riemannian metrics on X_{reg} does not help.) However, on the level of indices, we can get the right answer by enhancing the codomain by $\bigoplus_{x\in X_{sing}} \mathbb{C}^{\delta_x}$.

Now let X be a compact Hermitian complex space of pure dimension n. For technical reasons, we assume that the singular set X_{sing} consists of isolated singularities. (In the bulk of the paper we allow coupling to a holomorphic vector bundle, but in this introduction we only discuss the case when the vector bundle is trivial.) Let $\overline{\partial}_s$ be the minimal closed extension of the $\overline{\partial}$-operator on $X_{reg} = X - X_{sing}$. Its domain $\mathrm{Dom}(\overline{\partial}_s^{0,\star})$ can be localized to a complex of sheaves $\underline{\mathrm{Dom}}(\overline{\partial}_s^{0,\star})$. Let $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_s)$ denote the cohomology, a sum of skyscraper sheaves on X if $\star > 0$. We write $\mathcal{O}_s$ for $\underline{\mathrm{H}}^{0,0}(\overline{\partial}_s)$, which is the sheaf of weakly holomorphic functions on X. Then $\mathcal{O}_s/\mathcal{O}_X$ is also a sum of skyscraper sheaves on X. Its vector space of global sections will be written as $(\mathcal{O}_s/\mathcal{O}_X)(X)$. Both $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_s)$ and $\mathcal{O}_s/\mathcal{O}_X$ can be computed using a resolution of X [25, Corollary 1.2].

Define vector spaces T^* by

$$\begin{aligned} T^0 &= \mathrm{Dom}(\overline{\partial}_s^{0,0}), \\ T^1 &= \mathrm{Dom}(\overline{\partial}_s^{0,1}) \oplus (\mathcal{O}_s/\mathcal{O}_X)(X), \\ T^\star &= \mathrm{Dom}(\overline{\partial}_s^{0,\star}) \oplus (\underline{\mathrm{H}}^{0,\star-1}(\overline{\partial}_s))(X), \text{ if } 2 \le \star \le n. \end{aligned} \tag{1.2}$$

To define a differential on T^*, let $\triangle_s^{0,\star}$ be the Laplacian associated to $\overline{\partial}_s$. Let $P_{\mathrm{Ker}(\triangle_s^{0,\star})}$ be orthogonal projection onto the kernel of $\triangle_s^{0,\star}$. As the elements of $\mathrm{Ker}(\triangle_s^{0,\star})$ are $\overline{\partial}_s$-closed, for each $x \in X_{sing}$ there is a well-defined map $\mathrm{Ker}(\triangle_s^{0,\star}) \to (\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_s))_x$ to the stalk of $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_s)$ at x. For $\star > 0$, putting these together for all $x \in X_{sing}$, and precomposing with $P_{\mathrm{Ker}(\triangle_s^{0,\star})}$, gives a linear map $\gamma : \mathrm{Dom}(\overline{\partial}_s^{0,\star}) \to (\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_s))(X)$. For $\star = 0$, we similarly define $\gamma : \mathrm{Dom}(\overline{\partial}_s^{0,0}) \to (\mathcal{O}_s/\mathcal{O}_X)(X)$. Define a differential $d : T^* \to T^{*+1}$ by

$$\begin{aligned} d(\omega) &= (\overline{\partial}_s\omega, \gamma(\omega)), \text{ if } \star = 0, \\ d(\omega, a) &= (\overline{\partial}_s\omega, \gamma(\omega)), \text{ if } \star > 0. \end{aligned} \tag{1.3}$$

Theorem 1.4. *The cohomology of* (T, d) *is isomorphic to* $\mathrm{H}^*(X;\mathcal{O}_X)$.

Theorem 1.4 can be seen as an extension of [25, Corollary 1.3] by Ruppenthal, which implies the result when X is normal and has rational singularities. To prove Theorem 1.4, we construct a certain resolution of $\mathcal{O}_X$ by fine sheaves. The cohomology of the complex $(\widetilde{T}, \widetilde{d})$ of global sections is then isomorphic to $\mathrm{H}^*(X; \mathcal{O}_X)$. The complex $(\widetilde{T}, \widetilde{d})$ is not quite the same as (T, d) but we show that they are cochain-equivalent, from which the theorem follows.

The spectral triple $(C(X), T, d + d^*)$ defines an element $[\mathcal{O}_X]_{an} \in \mathrm{K}_0(X)$ of the analytic K-homology of X.

Theorem 1.5. *If X is a projective algebraic variety with isolated singularities then $[\mathcal{O}_X]_{an} = [\mathcal{O}_X]_{BFM}$ in $\mathrm{K}_0(X)$.*

There has been interesting earlier work on the questions addressed in this paper. In [1], Ancona and Gaveau gave a resolution of the structure sheaf of a normal complex space X, assuming that the singular locus is smooth, in terms of differential forms on a resolution of X. The construction depended on the choice of resolution. In [14], Fox and Haskell discussed using a perturbed Dolbeault operator on an ambient manifold to represent the K-homology class of the structure sheaf. In [2], Andersson and Samuelsson gave a resolution of the structure sheaf by certain currents on X, that are smooth on X_{reg}. After this paper was written, Bei and Piazza posted [10], which also has a proof of Proposition 5.1.

The structure of the paper is the following. In Section 2, given a holomorphic vector bundle V on X, we recall the definition of the minimal closure $\overline{\partial}_{V,s}$ and show that $\overline{\partial}_{V,s} + \overline{\partial}_{V,s}^*$ gives an element of the analytic K-homology group $\mathrm{K}_0(X)$, in the unbounded formalism for the Kasparov KK-group $\mathrm{KK}(C(X); \mathbb{C})$. In Section 3 we construct a resolution of the sheaf $\underline{V}$ by fine sheaves. Their global sections give a Hilbert complex. In Section 4 we deform this to the complex (T_V, d_V). Section 5 has the proof of Theorem 1.5. More detailed descriptions appear at the beginning of the sections.

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2. Minimal closure and compact resolvent

In this section we consider a holomorphic vector bundle V on a compact complex space X with isolated singularities. We define the minimal closure $\overline{\partial}_{V,s}$. We show that the spectral triple $(C(X), \overline{\partial}_{V,s} + \overline{\partial}_{V,s}^*, \Omega_{L^2}^{0,*}(X_{reg}; V))$ gives a well-defined element of the analytic K-homology group $\mathrm{K}_0(X)$, in the unbounded formalism. The main issue is to show that $\overline{\partial}_{V,s} + \overline{\partial}_{V,s}^*$ has a compact resolvent. When V is trivial, this was shown in [22].

Let X be a reduced compact complex space of pure dimension n. For each $x \in X$, there is a neighborhood U of x with an embedding of U into some domain $U' \subset \mathbb{C}^N$, as the zero set of a finite number of holomorphic functions on U'.

Let $\mathcal{O}_X$ be the analytic structure sheaf of X. Let X_{sing} be the set of singular points of X and put $X_{reg} = X - X_{sing}$.

We equip X with a Hermitian metric g on X_{reg} that satisfies the following property: For each $x \in X$, there are U and U' as above, along with a smooth Hermitian metric G on U', so that $g\Big|_{X_{reg}\cap U} = G\Big|_{X_{reg}\cap U}$.

Let V be a finite dimensional holomorphic vector bundle on X or, equivalently, a locally free sheaf $\underline{V}$ of $\mathcal{O}_X$-modules. For each $x \in X$, there are U and U' as above so that $V\Big|_U$ is the restriction of a trivial holomorphic bundle $U' \times \mathbb{C}^N$ on U'. Let h be a Hermitian inner product on $V\Big|_{X_{reg}}$ that satisfies the following property: For each $x \in X$, there are such U and U' so that $h\Big|_{X_{reg}\cap U}$ is the restriction of a smooth Hermitian metric on $U' \times \mathbb{C}^N$.

Let $\overline{\partial}_{V,s}$ be the minimal closed extension of the $\overline{\partial}_V$-operator on X_{reg}. That is, the domain of $\overline{\partial}_{V,s}$ is the set of $\omega \in \Omega^{0,*}_{L^2}(X_{reg}; V)$ so that there are a sequence of compactly supported smooth forms $\omega_i \in \Omega^{0,*}(X_{reg}; V)$ on X_{reg} and some $\eta \in \Omega^{0,*+1}_{L^2}(X_{reg}; V)$ such that $\lim_{i\to\infty} \omega_i = \omega$ in $\Omega^{0,*}_{L^2}(X_{reg}; V)$, and $\lim_{i\to\infty} \overline{\partial}_{V,s}\omega_i = \eta$ in $\Omega^{0,*+1}_{L^2}(X_{reg}; V)$. We then put $\overline{\partial}_{V,s}\omega = \eta$, which is uniquely defined.

Hereafter we assume that X_{sing} is finite.

Proposition 2.1. *The spectral triple* $(C(X), \overline{\partial}_{V,s} + \overline{\partial}^*_{V,s}, \Omega^{0,*}_{L^2}(X_{reg}; V))$ *gives a well-defined element of the analytic* K*-homology group* $\mathrm{K}_0(X)$.

Proof. Put $D_V = \overline{\partial}_{V,s} + \overline{\partial}^*_{V,s}$, with dense domain $\mathrm{Dom}(\overline{\partial}_{V,s}) \cap \mathrm{Dom}(\overline{\partial}^*_{V,s})$. Put $D = \overline{\partial}_s + \overline{\partial}^*_s$, the case when V is the trivial complex line bundle. Put

$$\mathcal{A} = \{f \in C(X) \; : \; f(\mathrm{Dom}(D_V)) \subset \mathrm{Dom}(D_V) \text{ and } [D_V, f] \text{ is bounded.}\} \tag{2.2}$$

Using the local trivializations of V, it follows that

$$\mathcal{A} = \{f \in C(X) \; : \; f(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D) \text{ and } [D, f] \text{ is bounded.}\} \tag{2.3}$$

To satisfy the definitions of unbounded analytic K-homology [5, 13, 20], we first need to show that $\mathcal{A}$ is dense in $C(X)$.

Given $F \in C(X)$ and $\epsilon > 0$, we can construct $f \in C(X)$ so that

- For each $x_j \in X_{sing}$, there is a neighborhood $U_j \subset X$ of x_j on which f is constant, with $f(x_j) = F(x_j)$.
- F is smooth on X_{reg}.
- $\sup_{x\in X} |f(x) - F(x)| < \epsilon$.

Then $f(\mathrm{Dom}(D)) \subset \mathrm{Dom}(D)$ and $\|[D, f]\| \leq \mathrm{const.} \, \|\nabla_h f\|_\infty < \infty$. It follows that $\mathcal{A}$ is dense in $C(X)$.

To prove the proposition, it now suffices to prove the following lemma.

Lemma 2.4. $(D_V + i)^{-1}$ *is compact*

Proof. If V is trivial then the lemma is true [22]. We will use a parametrix construction to prove it for general V.

We first prove the lemma for a special inner product h' on V. Write $X_{sing} = \{x_j\}_{j=1}^r$. For each j, let U_j be a neighborhood of x_j on which V is trivialized as above, with $\overline{U_j} \cap \overline{U_k} = \emptyset$ for $j \neq k$. Choose open sets with smooth boundary $x_j \in Z_j \subset Y_j \subset W_j \subset U_j$, with $\overline{Z_j} \subset Y_j$, $\overline{Y_j} \subset W_j$ and $\overline{W_j} \subset U_j$. Let $\phi_j \in C(X)$ be identically one on Y_j, with support in W_j, and smooth on $U_j - Y_j$. Let $\eta_j \in C(X)$ be identically one on W_j, with support in U_j, and smooth on $U_j - Y_j$, so that η_j is one on the support of ϕ_j.

Define an inner product h' on V by first taking it to be a trivial inner product on each U_j, in terms of our given trivializations, and then extending it smoothly to the rest of X_{reg}. Let V_j be the extension of the trivialization $U_j \times \mathbb{C}^N$ to a product bundle on $X \times \mathbb{C}^N$ on X, as a smooth vector bundle with trivial inner product. Let $D_{V_j} = D \otimes I_N$ be the corresponding operator. As $(D+i)^{-1}$ is compact [22], the same is true for D_{V_j}. Let D_{APS} be the operator $\overline{\partial}_V + \overline{\partial}_V^*$ on $X - \bigcup_j Z_j$, with Atiyah-Patodi-Singer boundary conditions [4]. (The paper [4] assumes a product structure near the boundary, but this is not necessary.) Then $(D_{APS} + i)^{-1}$ is compact. Put $\phi_0 = 1 - \sum_j \phi_j$, with support in $X - \bigcup_j \overline{Z_j}$. Pick $\eta_0 \in C(X)$ with support in $X - \bigcup_j \overline{Z_j}$, and smooth on X_{reg}, such that η_0 is one on the support of ϕ_0.

For $\omega \in \Omega^{0,*}_{L^2}(X_{reg}; V)$, put

$$Q\omega = \eta_0 (D_{APS} + i)^{-1}(\phi_0 \omega) + \sum_j \eta_j (D_{V_j} + i)^{-1}(\phi_j \omega). \tag{2.5}$$

Then Q is compact and

$$(D_V + i) Q\omega = \omega + [D, \eta_0](D_{APS} + i)^{-1}(\phi_0 \omega) + \sum_j [D, \eta_j](D_{V_j} + i)^{-1}(\phi_j \omega), \tag{2.6}$$

so

$$(D_V + i)^{-1} = Q - (D_V + i)^{-1} \left([D, \eta_0](D_{APS} + i)^{-1}\phi_0 + \sum_j [D, \eta_j](D_{V_j} + i)^{-1}\phi_j \right). \tag{2.7}$$

As $[D, \eta_0]$, $[D, \eta_j]$ and $(D_V + i)^{-1}$ are bounded, it follows that $(D_V + i)^{-1}$ is compact.

As $(D_V + i)^{-1}$ (for the inner product h') is compact, the spectral theorem for compact operators and the functional calculus imply that $(I + D_V^2)^{-1}$ is compact. Writing $\triangle_{V,s} = D_V^2$, there is then a Hodge decomposition

$$\Omega^{0,*}_{L^2}(X_{reg}; V) = \mathrm{Ker}(\triangle^{0,\star}_{V,s}) \oplus \mathrm{Im}(\overline{\partial}_{V,s}) \oplus \mathrm{Im}(\overline{\partial}^*_{V,s}) \tag{2.8}$$

where the right-hand side is a sum of orthogonal closed subspaces. In particular,

1. $\mathrm{Im}(\overline{\partial}_{V,s})$ is closed,
2. $\mathrm{Ker}(\overline{\partial}_{V,s}) / \mathrm{Im}(\overline{\partial}_{V,s})$ is finite dimensional and
3. The map $\overline{\partial}_{V,s} : \Omega^{0,*}_{L^2}(X_{reg}; V) / \mathrm{Ker}(\overline{\partial}_{V,s}) \to \mathrm{Im}(\overline{\partial}_{V,s})$ is invertible and the inverse is compact, i.e. sends bounded sets to precompact sets.

(The inverse map $\mathrm{Im}(\overline{\partial}_{V,s}) \to \Omega^{0,*}_{L^2}(X_{reg}; V) / \mathrm{Ker}(\overline{\partial}_{V,s}) \cong \mathrm{Im}(\overline{\partial}^*_{V,s})$ is DG, where G is the Green's operator for $\triangle_{V,s}$.) As the L^2-inner products on $\Omega^{0,*}_{L^2}(X_{reg}; V)$ coming from h' and h are relatively bounded, the above three properties also hold for h. It follows that there is a Hodge decomposition relative to the inner product h, and $(I+D_V^2)^{-1}$ is compact. Hence $(D_V+i)^{-1}$ is compact. □

This proves the proposition. □

3. Resolution

In this section we construct a certain resolution of the sheaf of holomorphic sections of a holomorphic vector bundle V on X. To begin, we define a sheaf $\underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star})$ on X, following [25, Section 2.1].

Given an open set $U \subset X$ and a compact subset $K \subset U$, we write U_{reg} for $U \cap X_{reg}$ and K_{reg} for $K \cap X_{reg}$.

Let V be a finite dimensional holomorphic vector bundle on X equipped with a Hermitian metric, in the sense of Section 2. There is a sheaf $\underline{\Omega}^{0,\star}_{V,L^2_{loc}}$ on X whose sections over an open set $U \subset X$ are the locally square integrable V-valued forms of degree $(0,\star)$ on U_{reg}, i.e. they are square integrable on K_{reg} for any compact set $K \subset U$. Convergence will mean L^2-convergence on each such K_{reg}. By definition, the sections of $\underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star})$ over U are the elements $\omega \in \Omega^{0,\star}_{L^2_{loc}}(U_{reg};V)$ so that there are

- A sequence $f_i \in \Omega^{0,\star}_{C^\infty_c}(U_{reg};V)$ and
- Some $\eta \in \Omega^{0,\star+1}_{L^2_{loc}}(U_{reg};V)$

such that for any compact $K \subset U$, we have

- $\lim_{i\to\infty} f_i = \omega$ in $\Omega^{0,\star}_{L^2}(K_{reg};V)$ and
- $\lim_{i\to\infty} \overline{\partial}_V f_i = \eta$ in $\Omega^{0,\star+1}_{L^2}(K_{reg};V)$.

Then we put $\overline{\partial}_V\omega = \eta$.

This gives a complex of fine sheaves

$$\ldots \xrightarrow{\overline{\partial}_V} \underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star-1}) \xrightarrow{\overline{\partial}_V} \underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star}) \xrightarrow{\overline{\partial}_V} \underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star+1}) \xrightarrow{\overline{\partial}_V} \ldots \tag{3.1}$$

The cohomology of the complex is the sheaf $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s})$. For $\star > 0$, it is a direct sum of skyscraper sheaves, with support in X_{sing}. We write $\underline{V}_s$ for $\underline{\mathrm{H}}^{0,0}(\overline{\partial}_{V,s})$, i.e. the kernel of $\overline{\partial}_V$ acting on $\underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,0})$. Then $\underline{V}_s/\underline{V}$ is also a direct sum of skyscraper sheaves with support in X_{sing}.

Althought we will not need it here, there is a description of these skyscraper sheaves in terms of a resolution of X. Suppose that $\pi : M \to X$ is a resolution. From [25, Corollary 1.2], if $x \in X$ then we can identify the stalk $(\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s}))_x$ with $V_x \otimes (R^q\pi_*\mathcal{O}_M)_x$. In particular, we can identify $\underline{V}_s$ with $\underline{V} \otimes_{\mathcal{O}_X} \pi_*\mathcal{O}_M$ or, more intrinsically, with the sheaf of weakly holomorphic sections of V, i.e. bounded holomorphic sections of $V\Big|_{X_{reg}}$.

There is a quotient morphism of sheaves: $q : \underline{\mathrm{Ker}}(\overline{\partial}_{V,s}^{0,\star}) \to \underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s})$. As $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s})$ is an injective sheaf for $\star > 0$, we can extend q to a morphism $\alpha : \underline{\mathrm{Dom}}(\overline{\partial}_{V,s}^{0,\star}) \to \underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s})$. More specifically, if x is a singular point then the stalk $(\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s}))_x$ is a finite dimensional complex vector space, so we are extending the quotient map $q_x : (\underline{\mathrm{Ker}}(\overline{\partial}^{0,\star}_{V,s}))_x \to (\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s}))_x$ from the germs of $\overline{\partial}_V$-closed V-valued forms at x, to the germs of forms in the domain of $\overline{\partial}_{V,s}$.

Considering $\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s})$ to be a complex of sheaves with zero differential, α is a morphism of complexes that is an isomorphism on cohomology in degree $\star > 0$ by construction. Let $\underline{\mathrm{cone}}(\alpha_V)$ be the mapping cone of α_V, with $\underline{\mathrm{cone}}^{0,\star}(\alpha_V) = \underline{\mathrm{Dom}}(\overline{\partial}^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\overline{\partial}_{V,s})$ and differential $d_{\mathrm{cone}}(\omega, h) = (\overline{\partial}_V \omega, \alpha_V(\omega))$. It has vanishing cohomology in degree $\star > 1$. Define a complex of sheaves $\underline{\mathcal{C}}^{0,\star}_V$ by

$$\underline{\mathcal{C}}^{0,\star}_V = \begin{cases} \underline{\mathrm{Dom}}(\overline{\partial}^{0,0}_{V,s}), & \star = 0 \\ \underline{\mathrm{Dom}}(\overline{\partial}^{0,1}_{V,s}), & \star = 1 \\ \underline{\mathrm{Dom}}(\overline{\partial}^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\overline{\partial}_{V,s}), & \star > 1 \end{cases} \tag{3.2}$$

where the differential in degree $\star = 0$ is $\overline{\partial}_V$, the differential in degree $\star = 1$ is $(\overline{\partial}_V, \alpha_V)$, and the differential in degrees $\star > 1$ is d_{cone}. Then $\underline{\mathcal{C}}_V$ is a resolution of $\underline{V}_s$ by fine sheaves.

There is a short exact sequence of sheaves

$$0 \longrightarrow \underline{V} \longrightarrow \underline{V}_s \longrightarrow \underline{V}_s/\underline{V} \longrightarrow 0. \tag{3.3}$$

We can think of $\underline{V}_s/\underline{V}$ as a resolution of itself, when concentrated in degree zero. Together with the resolution of $\underline{V}_s$ from (3.2), we can construct a resolution of $\underline{V}$ as follows. As $\underline{V}_s/\underline{V}$ is a finite sum of skyscraper sheaves, we can extend the quotient map $\underline{V}_s \to \underline{V}_s/\underline{V}$ to a morphism $\beta_V : \underline{\mathrm{Dom}}(\overline{\partial}^{0,0}_{V,s}) \to \underline{V}_s/\underline{V}$. Define a complex of sheaves $\widetilde{\underline{\mathcal{C}}}_V$ by

$$\widetilde{\underline{\mathcal{C}}}^{0,\star}_V = \begin{cases} \underline{\mathrm{Dom}}(\overline{\partial}^{0,0}_{V,s}), & \star = 0 \\ \underline{\mathrm{Dom}}(\overline{\partial}^{0,1}_{V,s}) \oplus \underline{V}_s/\underline{V}, & \star = 1 \\ \underline{\mathrm{Dom}}(\overline{\partial}^{0,\star}_{V,s}) \oplus \underline{\mathrm{H}}^{0,\star-1}(\overline{\partial}_{V,s}), & \star > 1 \end{cases} \tag{3.4}$$

where the differential in degree $\star = 0$ is $(\overline{\partial}_V, \beta_V)$, the differential in degree $\star = 1$ sends (ω, v) to $(\overline{\partial}_V \omega, \beta_V(\omega))$, and the differential in degrees $\star > 1$ is d_{cone}. Then $\widetilde{\underline{\mathcal{C}}}_V$ is a resolution of $\underline{V}$ by fine sheaves; c.f. [19, Pf. of Proposition I.6.10].

Taking global sections of $\widetilde{\underline{\mathcal{C}}}^{0,\star}_V$ gives a cochain complex $(\widetilde{T}_V, \widetilde{d}_V)$:

$$\begin{aligned} 0 \to \mathrm{Dom}(\overline{\partial}^{0,0}_{V,s}) \to \mathrm{Dom}(\overline{\partial}^{0,1}_{V,s}) \oplus (\underline{V}_s/\underline{V})(X) \to \mathrm{Dom}(\overline{\partial}^{0,2}_{V,s}) \oplus (\underline{\mathrm{H}}^{0,1}(\overline{\partial}_{V,s}))(X) \to \\ \ldots \to \mathrm{Dom}(\overline{\partial}^{0,n}_{V,s}) \oplus (\underline{\mathrm{H}}^{0,n-1}(\overline{\partial}_{V,s}))(X) \to 0. \end{aligned} \tag{3.5}$$

For the last term, we use the fact that in terms of a resolution $\pi : M \to X$, we have $(\underline{\mathrm{H}}^{0,n}(\overline{\partial}_{V,s}))_x = V_x \otimes (R^n \pi_* \mathcal{O}_M)_x = 0$.

Proposition 3.6. *The cohomology of* $(\widetilde{T}_V, \widetilde{d}_V)$ *is isomorphic to* $\mathrm{H}^*(X; V)$*.*

Proof. This holds because $\widetilde{\underline{\mathcal{C}}}_V$ is a resolution of $\underline{V}$ by fine sheaves. □

Put arbitrary inner products on the finite dimensional vector spaces $(\underline{V}_s/\underline{V})(X)$ and $(\underline{\mathrm{H}}^{0,*}(\overline{\partial}_{V,s}))(X)$.

4. Hilbert complex

The differential $\widetilde{d}_V$ in the Hilbert complex $(\widetilde{T}_V, \widetilde{d}_V)$ of the previous section involved somewhat arbitrary choices of α_V and β_V. In this section we replace $(\widetilde{T}_V, \widetilde{d}_V)$ by a more canonical Hilbert complex (T_V, d_V).

For brevity of notation, we put

$$A_V^* = \begin{cases} (\underline{V}_s/\underline{V})(X), & \star = 0 \\ (\underline{\mathrm{H}}^{0,\star}(\overline{\partial}_{V,s}))(X), & \star > 0. \end{cases} \tag{4.1}$$

Then the complex $\widetilde{T}_V$ has entries $\widetilde{T}_V^{0,\star} = \mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1}$ Combining α_V and β_V, we have constructed a linear map $\gamma_V : \mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star}) \to A_V^\star$ so that the differential of $\widetilde{T}_V$ is given by

$$\widetilde{d}_V(\omega, a) = (\partial_V \omega, \gamma_V(\omega)). \tag{4.2}$$

Note that $\gamma_V \circ \overline{\partial}_{V,s} = 0$.

Let $P_{\mathrm{Ker}(\triangle_{V,s}^{0,\star})}$ be orthogonal projection onto $\mathrm{Ker}(\triangle_{V,s}^{0,\star}) \subset \Omega_{L^2}^{0,\star}(X_{reg}; V)$. Define a new differential d_V on $\widetilde{T}_V$ by

$$d_V(\omega, a) = (\partial_V \omega, \gamma_V(P_{\mathrm{Ker}(\triangle_{V,s}^{0,\star})} \omega)). \tag{4.3}$$

Call the resulting cochain complex (T_V, d_V).

As in (2.8), there is a Hodge decomposition

$$\mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star}) = \mathrm{Ker}(\triangle_{V,s}^{0,\star}) \oplus \mathrm{Im}(\overline{\partial}_{V,s}) \oplus \mathrm{Im}(\overline{\partial}_{V,s}^*). \tag{4.4}$$

Here the terms on the right-hand side of (4.4) are the intersections of $\mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star})$ with the corresponding terms in (2.8). In particular, $\mathrm{Ker}(\triangle_{V,s}^{0,\star})$ and $\mathrm{Im}(\overline{\partial}_{V,s})$ are the same, while the elements of $\mathrm{Im}(\overline{\partial}_{V,s}^*)$ now lie in an H^1-space. Put

$$\mathcal{I}_V = \overline{\partial}_{V,s}\big|_{\mathrm{Im}(\overline{\partial}_{V,s}^*)} \; : \; \mathrm{Im}(\overline{\partial}_{V,s}^*) \to \mathrm{Im}(\overline{\partial}_{V,s}), \tag{4.5}$$

an isomorphism.

Define a linear map $m_V : \mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1} \to \mathrm{Dom}(\overline{\partial}_{V,s}^{0,\star}) \oplus A_V^{\star-1}$ by saying that if

$$(h, \omega_1, \omega_2, a) \in \mathrm{Ker}(\triangle_{V,s}^{0,\star}) \oplus \mathrm{Im}(\overline{\partial}_{V,s}) \oplus \mathrm{Im}(\overline{\partial}_{V,s}^*) \oplus A_V^{\star-1} \tag{4.6}$$

then

$$m_V(h, \omega_1, \omega_2, a) = (h, \omega_1, \omega_2, a + \gamma_V(\mathcal{I}^{-1}(\omega_1))). \tag{4.7}$$

Its inverse is given by

$$m_V^{-1}(h, \omega_1, \omega_2, a) = (h, \omega_1, \omega_2, a - \gamma_V(\mathcal{I}^{-1}(\omega_1))). \tag{4.8}$$

Proposition 4.9. *The linear maps m_V and m_V^{-1} are chain maps between (T_V, d_V) and $(T_V, \widetilde{d}_V)$, i.e. $m_V \circ d_V = \widetilde{d}_V \circ m_V$ and $m_V^{-1} \circ \widetilde{d}_V = d_V \circ m_V^{-1}$.*

Proof. We will check that $m_V \circ d_V = \widetilde{d}_V \circ m_V$; the proof that $m_V^{-1} \circ \widetilde{d}_V = d_V \circ m_V^{-1}$ is similar.

Given $(h, \omega_1, \omega_2, a)$ as in (4.6), we have

$$\begin{aligned} d_V(h, \omega_1, \omega_2, a) =& (0, \overline{\partial}_V \omega_2, 0, \gamma_V(h)), \\ m_V(d_V(h, \omega_1, \omega_2, a)) =& (0, \overline{\partial}_V \omega_2, 0, \gamma_V(h) + \gamma_V(\omega_2)), \\ m_V(h, \omega_1, \omega_2, a) =& (h, \omega_1, \omega_2, a + \gamma_V(\mathcal{I}^{-1}(\omega_1))), \\ \widetilde{d}_V(m_V(h, \omega_1, \omega_2, a)) =& (0, \overline{\partial}_V \omega_2, 0, \gamma_V(h) + \gamma_V(\omega_2)). \end{aligned} \tag{4.10}$$

This proves the proposition. □

Theorem 4.11. *The cohomology of (T_V, d_V) is isomorphic to $\mathrm{H}^*(X; V)$.*

Proof. This follows from Propositions 3.6 and 4.9. □

We can now reprove a result from [15, Example 18.3.3 on p. 362].

Proposition 4.12. *In terms of a resolution $\pi : M \to X$, we have*

$$\begin{aligned} \sum_{i=0}^{n} (-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X)) = \int_M \mathrm{Td}(TM) - \dim((\pi_* \mathcal{O}_M / \mathcal{O}_X)(X)) + \\ \sum_{i=1}^{n} (-1)^{i-1} \dim((R^i \pi_* \mathcal{O}_M)(X)). \end{aligned} \tag{4.13}$$

Proof. Let (T_1, d_1) denote the complex (T_V, d_V) when the vector bundle V is the trivial bundle. From Theorem 4.11, the left-hand side of (4.13) is the index of $d_1 + d_1^*$. We can deform the chain complex (T_1, d_1) to make the differential equal to $\overline{\partial}_s \oplus 0$ without changing the index. The new index is

$$\sum_{i=0}^{n} (-1)^i \dim(\mathrm{H}^i(\overline{\partial}_s)) - \dim((\mathcal{O}_s / \mathcal{O}_X)(X)) + \sum_{i=1}^{n-1} (-1)^i \dim((\underline{\mathrm{H}}^{0,i}(\overline{\partial}_s))(X)). \tag{4.14}$$

From [23], we have $\mathrm{H}^i(\overline{\partial}_s) \cong \mathrm{H}^{0,i}(M)$, so

$$\sum_{i=0}^{n} (-1)^i \dim(\mathrm{H}^i(\overline{\partial}_s)) = \sum_{i=0}^{n} (-1)^i \dim(\mathrm{H}^{0,i}(M)) = \int_M \mathrm{Td}(TM). \tag{4.15}$$

From [25, Corollary 1.2], we have $\mathcal{O}_s \cong \pi_* \mathcal{O}_M$ and $\underline{\mathrm{H}}^{0,i}(\overline{\partial}_s) \cong R^i \pi_* \mathcal{O}_M$. The proposition follows. □

Remark 4.16. We can write $\int_M \mathrm{Td}(TM) = \int_X \pi_* \mathrm{Td}(TM)$, where we are integrating a top-degree form on X_{reg}. It is not so clear what the relevant theory of characteristic classes on X should be, for which this would be an example. We note that there is a rational homology class $\pi_*(PD[\mathrm{Td}(TM)])$ on X, where $PD[\mathrm{Td}(TM)] \in \mathrm{H}_{even}(M;\mathbb{Q})$ is the Poincaré dual of $[\mathrm{Td}(TM)] \in \mathrm{H}^{even}(M;\mathbb{Q})$, and if X is connected then $\int_M \mathrm{Td}(TM)$ can be identified with the degree-zero component of $\pi_*(PD[\mathrm{Td}(TM)])$.

5. K-homology class

In this section we prove Theorem 1.5. We first show that if $\pi : M \to X$ is a resolution of singularities, with a simple normal crossing divisor, then the K-homology class $[\overline{\partial}_s + \overline{\partial}_s^*] \in \mathrm{K}_0(X)$, from Proposition 2.1 with V trivial, equals the pushforward $\pi_*[\overline{\partial}_M + \overline{\partial}_M^*]$. We then prove Theorem 1.5.

Proposition 5.1. *Let $\pi : M \to X$ be a resolution of singularities, with $\pi^{-1}(X_{sing})$ being a simple normal crossing divisor. Then $[\overline{\partial}_s + \overline{\partial}_s^*] = \pi_*[\overline{\partial}_M + \overline{\partial}_M^*]$.*

Proof. The method of proof comes from [17]. Consider the following part of the K-homology exact sequence for the pair (X, X_{sing}):

$$\mathrm{K}_0(X_{sing}) \xrightarrow{\alpha} \mathrm{K}_0(X) \xrightarrow{\beta} \mathrm{K}_0(X, X_{sing}). \tag{5.2}$$

Lemma 5.3. *We have $\beta([\overline{\partial}_s + \overline{\partial}_s^*]) = \beta(\pi_*[\overline{\partial}_M + \overline{\partial}_M^*])$ in $\mathrm{K}_0(X, X_{sing})$.*

Proof. Put $D = \pi^{-1}(X_{sing}) \subset M$. Since it has simple normal crossings, there will be a small regular neighborhood of D whose closure C' is homotopy equivalent to D. We can also assume that $C = \pi(C')$ is homotopy equivalent to X_{sing} [21, Theorem 2.10]. As $[\overline{\partial}_M + \overline{\partial}_M^*]$ is independent of the choice of Hermitian metric on M, we can choose a Hermitian metric on M so that π restricts to an isometry from $M - C'$ to $X - C$.

Consider the commutative diagram

$$\begin{array}{ccccccc} \mathrm{K}_0(M) & \to & \mathrm{K}_0(M,D) & \cong & \mathrm{K}_0(M,C') & \cong & \mathrm{KK}(C_0(M-C');\mathbb{C}) \\ \pi_* \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{K}_0(X) & \xrightarrow{\beta} & \mathrm{K}_0(X,X_{sing}) & \cong & \mathrm{K}_0(X,C) & \cong & \mathrm{KK}(C_0(X-C);\mathbb{C}). \end{array} \tag{5.4}$$

Starting with $[\overline{\partial}_M + \overline{\partial}_M^*] \in \mathrm{K}_0(M)$ and going along the top row, its image in $\mathrm{KK}(C_0(M - C');\mathbb{C})$ is the restriction of the analytic K-homology class, i.e. one only acts by functions that vanish on C'. The right vertical arrow of the diagram is an isomorphism coming from the bijection between $M - C'$ and $X - C$. By the commutativity of the diagram, we now know what $\beta(\pi_*[\overline{\partial}_M + \overline{\partial}_M^*])$ is as an element of $\mathrm{KK}(C_0(X - C);\mathbb{C})$. However, this is isomorphic to the restriction of $[\overline{\partial}_s + \overline{\partial}_s^*] \in \mathrm{K}_0(X)$ to an element of $\mathrm{KK}(C_0(X - C);\mathbb{C})$ (since π gives an isometry between $M - C'$ and $X - C$). The latter restriction is the same as $\beta([\overline{\partial}_s + \overline{\partial}_s^*])$. This proves the lemma. □

To continue with the proof of Proposition 5.1, we know now that $[\overline{\partial}_s + \overline{\partial}_s^*]) - \pi_*[\overline{\partial}_M + \overline{\partial}_M^*]$ lies in the kernel of β, and so lies in the image of α. For the purpose of the proof, we can assume that X is connected. Let $a : \mathrm{pt} \to X$ be an arbitrary fixed embedding and let $a_* : \mathrm{K}_0(\mathrm{pt}) \to \mathrm{K}_0(X)$ be the induced homomorphism. The connectedness of X implies that $\mathrm{Im}(\alpha) = \mathrm{Im}(a_*)$. Let $b : X \to \mathrm{pt}$ be the unique point map. Consider $\mathrm{pt} \xrightarrow{a} X \xrightarrow{b} \mathrm{pt}$

and the induced homomorphisms $\mathrm{K}_0(\mathrm{pt}) \stackrel{a_*}{\to} \mathrm{K}_0(X) \stackrel{b_*}{\to} \mathrm{K}_0(\mathrm{pt})$. Then the map b_* restricts to an isomorphism between $\mathrm{Im}(a_*)$ and $\mathrm{K}_0(\mathrm{pt})$. Hence to prove the proposition, it suffices to show that $b_*[\overline{\partial}_s + \overline{\partial}_s^*] = b_*(\pi_*[\overline{\partial}_M + \overline{\partial}_M^*])$ in $\mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$.

Now $b_*[\overline{\partial}_s + \overline{\partial}_s^*]$ is the index of $\overline{\partial}_s + \overline{\partial}_s^*$, i.e. $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(\overline{\partial}_s))$, while $b_*(\pi_*[\overline{\partial}_M + \overline{\partial}_M^*])$ is the index of $\overline{\partial}_M + \overline{\partial}_M^*$, i.e. $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(\overline{\partial}_M))$. From [23], these are equal term-by-term. This proves the proposition. □

We now prove Theorem 1.5. Suppose that X is a connected projective algebraic variety. In terms of the resolution $\pi : M \to X$, it was pointed out in [7, p. 104] that there is an identity in $\mathrm{K}_0(X)$:

$$[\mathcal{O}_X]_{BFM} - \pi_*[\mathcal{O}_M]_{BFM} = \sum_j n_j [\mathcal{O}_{V_j}]_{BFM}. \tag{5.5}$$

Here the n_j's are certain integers and the V_j's are irreducible subvarieties of the singular locus of X. In our case of isolated singularities, the V_j's are just the points x_j in X_{sing}. As $[\mathcal{O}_M]_{BFM} = [\overline{\partial}_M + \overline{\partial}_M^*]$, Proposition 5.1 implies that

$$[\mathcal{O}_X]_{BFM} = [\overline{\partial}_s + \overline{\partial}_s^*] + \sum_j n_j [\mathcal{O}_{V_j}]_{BFM}. \tag{5.6}$$

Let (T_1, d_1) denote the complex (T_V, d_V) when the vector bundle V is the trivial bundle. Let $[\mathcal{O}_X]_{an} \in \mathrm{K}_0(X)$ be the K-homology class coming from the operator $d_1 + d_1^*$. We can deform the chain complex (T_1, d_1) to make the differential equal to $\overline{\partial}_s \oplus 0$ without changing the K-homology class arising from the complex. Then (5.6) implies that $[\mathcal{O}_X]_{an}$ and $[\mathcal{O}_X]_{BFM}$ have the same image in $\mathrm{K}_0(X, X_{sing})$; c.f. the proof of Lemma 5.3. Let $b : X \to \mathrm{pt}$ be the unique point map. As in the proof of Proposition 5.1, to conclude that $[\mathcal{O}_X]_{an} = [\mathcal{O}_X]_{BFM}$ in $\mathrm{K}_0(X)$, it now suffices to show that $b_*[\mathcal{O}_X]_{an} = b_*[\mathcal{O}_X]_{BFM}$ in $\mathrm{K}_0(\mathrm{pt}) \cong \mathbb{Z}$. Now $b_*[\mathcal{O}_X]_{an}$ is the index of $d_1 + d_1^*$ which, from Theorem 4.11, equals the arithmetic genus $\sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X))$. On the other hand, from [8, Section 3], we also have $b_*[\mathcal{O}_X]_{BFM} = \sum_{i=0}^n (-1)^i \dim(\mathrm{H}^i(X; \mathcal{O}_X))$. This proves the theorem.

Remark 5.7. We mention some of the issues involved in extending the present paper to nonisolated singularities. First, it seems to be open whether $\overline{\partial}_s + \overline{\partial}_s^*$ has compact resolvient, so the unbounded KK-formalism may not be applicable. However, it is known that the unreduced cohomology of the $\overline{\partial}_s$-complex is finite dimensional, being isomorphic to the cohomology of a resolution [23]. Hence the $\overline{\partial}_s$-complex is Fredholm and one could use the bounded KK-description of K-homology, although it would be more cumbersome.

We expect that Proposition 5.1 still holds if X has nonisolated singularities. It is known that taking resolutions $\pi : M \to X$, the pushforward $\pi_*[\overline{\partial}_M + \overline{\partial}_M^*] \in \mathrm{K}_0(X)$ is independent of the choice of resolution [18].

One could ask for an extension of Theorem 4.11 to the case of nonisolated singularities. As an indication, one would expect that taking products of complex spaces would lead to tensor products of the cochain complexes. In particular, suppose that Z is a smooth

Hermitian manifold and X has isolated singular points. Then the cochain complex for $Z \times X$ would have contributions from differential forms along the singular locus.

In a related vein, in principle one can apply (5.5) inductively to get an expression for $[\mathcal{O}_X]_{BFM}$.

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Department of Mathematics, University of California, Berkeley, Berkeley, CA 94720-3840, USA

E-mail address: `lott@berkeley.edu`