

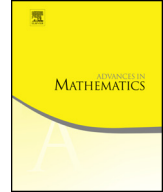


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A Hilbert bundle description of differential K -theory [☆]

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ABSTRACT

We give an infinite dimensional description of the differential K -theory of a manifold M . The generators are triples $[\mathcal{H}, \mathbb{A}, \omega]$ where \mathcal{H} is a \mathbb{Z}_2 -graded Hilbert bundle on M , \mathbb{A} is a superconnection on \mathcal{H} and ω is a differential form on M . The relations involve eta forms. We show that the ensuing group is the differential K -group $\check{K}^0(M)$. In addition, we construct the pushforward of a finite dimensional cocycle under a proper submersion with a Riemannian structure. We give the analogous description of the odd differential K -group $\check{K}^1(M)$. Finally, we give a model for twisted differential K -theory.

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1. Introduction

Differential K -groups are invariants of smooth manifolds that combine K -theory with differential forms. As shown in [18], many results from local index theory fit into the framework of differential K -theory. For background and history about differential K -theory, we refer to the introduction of [18].

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Just as K -theory has different but equivalent descriptions, so does differential K -theory. The primary goal of this paper is to give a new description of differential K -theory, based on Hilbert bundles, that unifies other descriptions. A secondary goal is to provide a functional analytic framework for superconnections on Hilbert bundles.

We recall the generators for some of the descriptions of the K -group $K^0(M)$ of a compact manifold M :

- (1) Vector bundles on M [1].
- (2) Maps from M to the space of Fredholm operators [1, Appendix].
- (3) Maps $p : Z \rightarrow M$ where Z is compact and p is K -oriented [17].
- (4) \mathbb{Z}_2 -graded Hilbert $C(M)$ -modules equipped with certain bounded operators that commute with $C(M)$ [25,28].
- (5) \mathbb{Z}_2 -graded Hilbert $C(M)$ -modules equipped with certain possibly-unbounded operators that commute with $C(M)$ [2].

Of these descriptions, perhaps the last one, based on unbounded KK -cycles, is the most encompassing one.

For the first three descriptions of K -theory, there are corresponding models for differential K -theory:

- (1) Vector bundles with connections, as in [18,34].
- (2) The Hopkins–Singer model [21].
- (3) The geometric families of Bunke–Schick [12].

All of these descriptions give isomorphic groups, which we denote by $\check{K}_{stan}^*(M)$. In this paper we give a new model for the differential K -theory of M , extending the description of K -theory using unbounded KK -cycles. Our model is in terms of Hilbert bundles on M equipped with superconnections. One motivation for our model is that it unifies earlier models, as described below. The main result of the paper is the following.

Theorem 1. *The differential K -groups $\check{K}^*(M)$, as defined using Hilbert bundles and superconnections, are isomorphic to $\check{K}_{stan}^*(M)$.*

Given a finite dimensional Hermitian vector bundle on M with compatible connection, we can think of it as a Hilbert bundle on M with a very special superconnection. Hence our Hilbert bundle model includes the standard description of differential K -theory using finite dimensional vector bundles with connection. Similarly, given a geometric family in the sense of [12, Section 2], there is an ensuing Hilbert bundle equipped with the Bismut superconnection [4]. Hence our model also includes the description of differential K -theory using geometric families. However, we do not have an obvious way to construct a Hilbert bundle, with superconnection, from a Hopkins–Singer cocycle [21, Section 4.4].

To motivate the use of superconnections, we recall that in the vector bundle description of $\check{K}_{stan}^0(M)$, the generators are triples $[E, \nabla, \omega]$, where E is a finite dimensional Hermitian \mathbb{Z}_2 -graded vector bundle on M , ∇ is a compatible connection and $\omega \in \Omega^{odd}(M)/\text{Im}(d)$. The relations involve Chern–Simons forms. There is an equivalent description whose generators are triples $[E, \mathbb{A}, \omega]$, where \mathbb{A} is a compatible superconnection on E in the sense of Quillen [32], and whose relations involve eta forms. When we pass to infinite dimensional vector bundles, the Chern character construction using connections no longer make sense. However, under suitable hypotheses, we show that the construction using superconnections does make sense.

Hence one goal of this paper is to find the right setting for superconnections on Hilbert bundles. As an indication, such a setting should allow for Bismut superconnections. A first question is what the structure group G of such a Hilbert bundle should be. The answer to this is not immediately evident. One remark is that to deal with a geometric family whose fiber is a compact manifold Z , there should be a smooth homomorphism from $\text{Diff}(Z)$ to G coming from the action of $\text{Diff}(Z)$ on the Hilbert space H of square-integrable half-densities on Z . However, with the norm topology on $U(H)$, such an action is not even continuous (as seen in the action on the circle on itself by rotations).

To construct G in general, we use the data of a Hilbert space H and an unbounded self-adjoint operator D on H that is θ -summable for all $\theta > 0$ (such as a Dirac-type operator). Using D , in Section 2 we define Sobolev spaces H^s and pseudodifferential operators op^k that map H^s to H^{s-k} , following Connes and Moscovici [16, Appendix B]. As a set, we take $G = U(H) \cap op^0$. To put a smooth structure on G , we can use the fact that we only care about Hilbert bundles over finite dimensional manifolds, as opposed to more general base spaces. Hence it suffices to say what a smooth map, from a domain in Euclidean space to G , should be. This is the underlying idea of diffeological smooth structures [22]. In our case, we say that such maps are smooth if they are compatible in a certain sense with the Fréchet topologies on H^s and op^k .

Given a Hilbert bundle \mathcal{H} on M , with such a structure group, in Section 3 we develop the theory of superconnections \mathbb{A} on \mathcal{H} . We construct their Chern characters and eta forms. In Section 4 we give our generators and relations for $\check{K}^0(M)$. The generators are triples $[\mathcal{H}, \mathbb{A}, \omega]$ where \mathcal{H} is a \mathbb{Z}_2 -graded Hilbert bundle on M , \mathbb{A} is a superconnection on \mathcal{H} and $\omega \in \Omega^{odd}(M)/\text{Im}(d)$. There are three relations. The first relation is about taking direct sums. The second relation arises when the degree-0 part $\mathbb{A}_{[0]}$ of the superconnection is invertible, and involves an eta form. The third relation says what happens when one changes $\mathbb{A}_{[0]}$ by a family of operators in op^0 , and involves a relative eta form.

If c is a generator for $\check{K}^0(M)$ then we construct a generator $q(c)$ for $\check{K}_{stan}^0(M)$, based on certain choices. We show that the class of $q(c)$ in $\check{K}_{stan}^0(M)$ is independent of the choices. We prove that q passes to a map $q : \check{K}^0(M) \rightarrow \check{K}_{stan}^0(M)$. We then show that q is an isomorphism, thereby proving Theorem 1.

One advantage of a Hilbert bundle approach to differential K -groups is that, as in [12], the pushforward becomes essentially tautological. We recall that using the finite dimensional model \check{K}_{stan}^0 for differential K -theory, in [18] two pushforwards were defined,

called the analytic index and the topological index. The definition of the analytic index there involved some perturbations. The main theorem of [18] said that the two indices coincide. Given a fiber bundle $\pi : M \rightarrow B$ with even dimensional compact fibers and a Riemannian structure, in Section 5 we define the pushforward $\pi_*[\mathcal{H}, \mathbb{A}, \omega]$ of a finite dimensional representative $[\mathcal{H}, \mathbb{A}, \omega]$ for $\check{K}^0(M)$.

Theorem 2. *The pushforward π_* on cocycles passes to a map $\pi_* : \check{K}^0(M) \rightarrow \check{K}^0(B)$. It coincides with the analytic index of [18].*

The proof of Theorem 2 partially uses results from [10,26].

Another advantage of using Hilbert bundles is that it allows a unified treatment of even and odd differential K -groups. (By way of contrast, in [18, Section 9] and [35], the odd differential K -groups were constructed based on the model of odd K -theory coming from maps to unitary groups.) In Section 6 we indicate how the results of the preceding sections extend to the odd differential K -group $\check{K}^1(M)$.

Yet another advantage of using Hilbert bundles is that it allows for a simple model of twisted differential K -theory. We recall that ordinary K -theory can be twisted by an element of $H^3(M; \mathbb{Z})$. The corresponding twisted differential K -theory has been considered in many papers, including [11,13,14,31]. Some applications to mathematical physics are in [24]. In Section 7 we give the basic definitions for a Hilbert bundle model $\check{K}_{\mathcal{L}}^0(M)$ of twisted differential K -theory, where \mathcal{L} is a unitary gerbe on M . If one restricts to finite dimensional vector bundles in the definition then one can only deal with twistings by torsion elements of $H^3(M; \mathbb{Z})$.

There are many directions for further study, including the following.

1. Extend the pushforward to infinite dimensional cocycles for \check{K}^0 ; see Remark 4.
2. Show that the twisted differential K -groups $\check{K}_{\mathcal{L}}^0(M)$ satisfy an axiomatic characterization, as outlined in [13, Section 7].
3. Construct a topological pushforward in \check{K}^0 . We recall that the topological pushforward in KK -theory involves a Kasparov product with a $\bar{\partial}$ -operator [25].

The paper has an appendix in which we prove a formula for the Chern character of a superconnection in relative cohomology. There is an application to eta forms.

More detailed descriptions of the content of the paper appear at the beginnings of the sections.

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2. Pseudodifferential calculus

This section is devoted to functional analytic preliminaries. Given a \mathbb{Z}_2 -graded Hilbert space and an odd self-adjoint operator D that is θ -summable for all $\theta > 0$, in Subsec-

tion 2.1 we define order- s Sobolev spaces H^s and order- k pseudodifferential operators op^k . We prove basic composition properties of the pseudodifferential operators. In Subsection 2.2 we consider a space \mathcal{P} of “Dirac-type” order-1 operators. We prove in particular that \mathcal{P} is preserved by the addition of order-0 operators. Subsection 2.3 shows that for any $\epsilon > 0$, the map that sends P to $e^{-\epsilon P^2}$ is smooth, in the diffeological sense, as a map from \mathcal{P} to the space of trace-class operators. The heart of the proof is to justify a Duhamel formula in this setting.

2.1. Operator norms

Let $H = H^+ \oplus H^-$ be a \mathbb{Z}_2 -graded Hilbert space (possibly finite dimensional) with inner product $\langle \cdot, \cdot \rangle_H$. Let $\mathcal{L}(H)$ denote the bounded operators on H , with operator norm $\| \cdot \|$. Let $\mathcal{L}^1(H)$ be the trace ideal of H , with its norm $\| \cdot \|_{\mathcal{L}^1}$. Let D be an odd (with respect to the \mathbb{Z}_2 -grading) self-adjoint operator on H , possibly unbounded, which is θ -summable for all $\theta > 0$, i.e.

$$\text{Tr } e^{-\theta D^2} < \infty. \tag{2.1}$$

In particular, D^2 has a discrete spectrum. Let $P_{\text{Ker}(D^2)}$ be orthogonal projection onto $\text{Ker}(D^2)$. Define

$$|D| = \sqrt{D^2} + P_{\text{Ker}(D^2)}. \tag{2.2}$$

If D is invertible then $|D|$ has the usual meaning, but for us $|D|$ is always a strictly positive operator.

For $s \in \mathbb{Z}$ nonnegative, put $H^s = \text{Dom}(|D|^s)$, with the inner product

$$\langle v_1, v_2 \rangle_{H^s} = \langle |D|^s v_1, |D|^s v_2 \rangle_H. \tag{2.3}$$

For $s \in \mathbb{Z}$ negative, put $H^s = (H^{-s})^*$. Put $H^\infty = \bigcap_{s \geq 0} H^s$, a dense subspace of H .

Following [16, Appendix B], let op^k be the closed operators F such that

- (1) $H^\infty \subset \text{Dom}(F)$,
- (2) $F(H^\infty) \subset H^\infty$, and
- (3) For all $s \in \mathbb{Z}$, the operator $F : H^\infty \rightarrow H^\infty$ extends to a bounded operator from H^s to H^{s-k} .

Let $|F|_{k,s}$ be the operator norm for $F : H^s \rightarrow H^{s-k}$. Then op^k is a Fréchet space with respect to the norms $|F|_{k,s}$. We take a product of operators to act from right to left. Using the isometric isomorphism $|D|^{-s} : H^0 \rightarrow H^s$, if $F \in op^k$ then

$$|F|_{k,s} = \| |D|^{s-k} F |D|^{-s} \| . \tag{2.4}$$

Let \mathcal{L}_∞^{fr} be the ideal of finite rank operators, i.e. the set of operators T on H that can be expressed as

$$T(v) = \sum_i \xi_i \langle \eta_i, v \rangle_H, \tag{2.5}$$

where the sum is finite and $\xi_i, \eta_i \in H^\infty$. Then $\mathcal{L}_\infty^{fr} \subset \bigcap_{k \in \mathbb{Z}} op^k$.

Lemma 1.

(1) If $F_1 \in op^{k_1}$ and $F_2 \in op^{k_2}$ then $F_1 F_2 \in op^{k_1+k_2}$ and

$$|F_1 F_2|_{k_1+k_2, s} \leq |F_1|_{k_1, s-k_2} |F_2|_{k_2, s}. \tag{2.6}$$

(2) If $F \in op^k$, and $F : H^s \rightarrow H^{s-k}$ is an isomorphism for each $s \in \mathbb{Z}$, then $F^{-1} \in op^{-k}$.

(3) If $F \in op^0$ then its adjoint F^* in $B(H)$ satisfies $F^* \in op^0$.

Proof. (1). The proof is straightforward.

(2). By the bounded inverse theorem, $F^{-1} : H^{s-k} \rightarrow H^s$ is bounded for each $s \in \mathbb{Z}$. In particular, $H^\infty \subset \text{Dom}(F^{-1})$ and $F^{-1}(H^\infty) \subset H^\infty$.

(3). If $v \in H^s$ then for all $w \in H^\infty$, we have

$$\langle Fw, v \rangle_H = \langle (|D|^{-s} F |D|^s) |D|^{-s} w, |D|^s v \rangle_H, \tag{2.7}$$

showing that

$$F^* v = |D|^{-s} (|D|^{-s} F |D|^s)^* |D|^s v. \tag{2.8}$$

In particular,

$$|D|^s F^* v = (|D|^{-s} F |D|^s)^* |D|^s v \in H, \tag{2.9}$$

showing that $F^*(H^s) \subset H^s$, with

$$F^* = |D|^{-s} (|D|^{-s} F |D|^s)^* |D|^s \tag{2.10}$$

being a bounded operator on H^s . \square

Example 1. Suppose that H is finite dimensional. Then $H^\infty = H$ and $H^s = H$ for all $s \in \mathbb{Z}$. Also, $op^k = B(H)$ for all $k \in \mathbb{Z}$.

Example 2. Let Z be a compact Riemannian manifold. Let V be a Clifford module over Z , equipped with a compatible connection. Let $\mathcal{D}^{\frac{1}{2}}$ be the half-density line bundle on Z . Put $H = L^2(Z; \mathcal{D}^{\frac{1}{2}} \otimes V)$. Let D be the Dirac-type operator on H . Then $H^\infty = C^\infty(Z; \mathcal{D}^{\frac{1}{2}} \otimes V)$

and $H^s = H^s(Z; \mathcal{D}^{\frac{1}{2}} \otimes V)$. A pseudodifferential operator of order k gives an element of op^k .

2.2. *The space of Dirac-type operators*

Definition 1. \mathcal{P} is the space of odd self-adjoint operators $P \in op^1$ such that $|P|^{-1} \in op^{-1}$.

In Section 3, the elements of \mathcal{P} will become the possible degree-0 terms of the super-connection.

Recall that $|P|$ is defined to be 1 on $\text{Ker}(P^2)$. If $P \in \mathcal{P}$ then $P^2|P|^{-1} \in op^1$. As $|P| - P^2|P|^{-1} \in \mathcal{L}^{fr}_{\infty}$, it follows that $|P| \in op^1$. Then for any $s \in \mathbb{Z}$, the operators $|D|^s|P|^{-s}$ and $|P|^s|D|^{-s}$ lie in op^0 and, in particular, are bounded on H . It follows that if we defined H^s using P instead of D then we would get the same spaces. If we think of D as being a given Dirac operator then we can think of \mathcal{P} as being a collection of Dirac-type operators. For example, if D is the operator of Example 2, and P is the operator arising from a different Riemannian metric on Z and a different Clifford connection on V , then $P \in \mathcal{P}$.

Lemma 2.

- (1) Any $P \in \mathcal{P}$ is θ -summable for all $\theta > 0$.
- (2) If $P \in \mathcal{P}$, and $Q \in op^0$ is odd and self-adjoint, then $P + Q$ is θ -summable for all $\theta > 0$. More precisely, for any $\epsilon \in (0, 1)$, we have

$$\text{Tr } e^{-\theta(P+Q)^2} \leq e^{\theta(\epsilon^{-2}-1)\|Q\|^2} \cdot \text{Tr } e^{-\theta(1-\epsilon^2)P^2}. \tag{2.11}$$

- (3) Given $P \in \mathcal{P}$, $F_1 \in op^{k_1}$, $F_2 \in op^{k_2}$ and $\epsilon > 0$, for every $t \geq \epsilon$ we have

$$\|F_1 e^{-tP^2} F_2\|_{\mathcal{L}^1} \leq C(\epsilon, P, k_1, k_2) |F_1|_{k_1, k_1} |F_2|_{k_2, 0} \tag{2.12}$$

and

$$\text{Tr} \left(F_1 e^{-tP^2} F_2 \right) = \text{Tr} \left(e^{-tP^2} F_2 F_1 \right) = \text{Tr} \left(F_2 F_1 e^{-tP^2} \right). \tag{2.13}$$

Proof. (1). Since $|P|^{-1}D^2|P|^{-1}$ lies in op^0 , it is bounded on H . Hence there is some $C < \infty$ so that $D^2 \leq C(I + P^2)$. Thus $P^2 \geq C^{-1}D^2 - I$, so P is θ -summable.

(2). We follow the method of proof of [19, Theorem C]. For any $\epsilon > 0$, we have

$$0 \leq (\epsilon P + \epsilon^{-1}Q)^2 = \epsilon^2 P^2 + (PQ + QP) + \epsilon^{-2}Q^2. \tag{2.14}$$

If $\epsilon \in (0, 1)$ then

$$-\theta(P + Q)^2 = -\theta(P^2 + PQ + QP + Q^2) \tag{2.15}$$

$$\begin{aligned} &\leq -\theta(1 - \epsilon^2)P^2 + \theta(\epsilon^{-2} - 1)Q^2 \\ &\leq -\theta(1 - \epsilon^2)P^2 + \theta(\epsilon^{-2} - 1)\|Q\|^2. \end{aligned}$$

The claim follows.

(3). We have

$$\begin{aligned} F_1 e^{-tP^2} F_2 &= (F_1 |D|^{-k_1}) \cdot (|D|^{k_1} |P|^{-k_1}) \cdot e^{-\frac{\epsilon}{2}P^2}. \\ &\quad \left(|P|^{k_1} e^{-(t-\frac{\epsilon}{2})P^2} |P|^{k_2} \right) \cdot (|P|^{-k_2} |D|^{k_2}) \cdot \\ &\quad (|D|^{-k_2} F_2). \end{aligned} \tag{2.16}$$

By assumption, each of the six factors in (2.16) is bounded on H . Part (1) shows that $e^{-\frac{\epsilon}{2}P^2}$ is trace class. Hence the product is trace class. Using (2.4), we obtain

$$\begin{aligned} \|F_1 e^{-tP^2} F_2\|_{\mathcal{L}^1} &\leq \|F_1\|_{k_1, k_1} \cdot \| |D|^{k_1} |P|^{-k_1} \| \cdot \\ &\quad \| e^{-\frac{\epsilon}{2}P^2} \|_{\mathcal{L}^1} \cdot \| |P|^{k_1} e^{-(t-\frac{\epsilon}{2})P^2} |P|^{k_2} \| \cdot \\ &\quad \| |P|^{-k_2} |D|^{k_2} \| \cdot \|F_2\|_{k_2, 0}. \end{aligned} \tag{2.17}$$

From the spectral theorem,

$$\| |P|^{k_1} e^{-(t-\frac{\epsilon}{2})P^2} |P|^{k_2} \| \leq \sup_{r \in \mathbb{R}} \left((1 + r^2)^{\frac{k_1+k_2}{2}} e^{-\frac{\epsilon}{2}r^2} \right) < \infty. \tag{2.18}$$

Next, we can write

$$\begin{aligned} \text{Tr} \left(F_1 e^{-tP^2} F_2 \right) &= \text{Tr} \left(F_1 e^{-\frac{\epsilon}{4}P^2} \cdot e^{-(t-\frac{\epsilon}{2})P^2} \cdot e^{-\frac{\epsilon}{4}P^2} F_2 \right) \\ &= \text{Tr} \left(e^{-(t-\frac{\epsilon}{2})P^2} \cdot e^{-\frac{\epsilon}{4}P^2} F_2 \cdot F_1 e^{-\frac{\epsilon}{4}P^2} \right) \\ &= \text{Tr} \left(e^{-\frac{\epsilon}{4}P^2} e^{-(t-\frac{\epsilon}{2})P^2} e^{-\frac{\epsilon}{4}P^2} F_2 F_1 \right) \\ &= \text{Tr} \left(e^{-tP^2} F_2 F_1 \right). \end{aligned} \tag{2.19}$$

Similarly,

$$\begin{aligned} \text{Tr} \left(F_1 e^{-tP^2} F_2 \right) &= \text{Tr} \left(F_1 e^{-\frac{\epsilon}{4}P^2} \cdot e^{-(t-\frac{\epsilon}{2})P^2} \cdot e^{-\frac{\epsilon}{4}P^2} F_2 \right) \\ &= \text{Tr} \left(e^{-\frac{\epsilon}{4}P^2} F_2 \cdot F_1 e^{-\frac{\epsilon}{4}P^2} \cdot e^{-(t-\frac{\epsilon}{2})P^2} \right) \\ &= \text{Tr} \left(F_2 F_1 e^{-\frac{\epsilon}{4}P^2} e^{-(t-\frac{\epsilon}{2})P^2} e^{-\frac{\epsilon}{4}P^2} \right) \\ &= \text{Tr} \left(F_2 F_1 e^{-tP^2} \right). \end{aligned} \tag{2.20}$$

This proves the claim. \square

Proposition 1. *An odd self-adjoint operator $P \in op^1$ is in \mathcal{P} if and only if there is some odd self-adjoint operator $Q \in op^{-1}$ with $PQ - I \in op^{-1}$ and $QP - I \in op^{-1}$.*

To prove Proposition 1, we begin with some lemmas.

Lemma 3. *Suppose that $P \in op^1$ is an odd self-adjoint Fredholm operator. Then the spectrum of P is disjoint from $[-\epsilon, 0) \cup (0, \epsilon]$, for some $\epsilon > 0$. Put $R = f(P)$, where*

$$f(t) = \begin{cases} 1/t & \text{if } |t| > \epsilon, \\ 1 & \text{otherwise.} \end{cases} \tag{2.21}$$

(This is independent of the choice of such ϵ .) Then $|P|^{-1} \in op^{-1}$ if and only if $R \in op^{-1}$.

Proof. Suppose that $|P|^{-1} \in op^{-1}$. Then $P|P|^{-2} \in op^{-1}$. As $R - P|P|^{-2} \in \mathcal{L}_\infty^{fr}$, it follows that $R \in op^{-1}$.

Now suppose that $R \in op^{-1}$. Since $R^{-1} - P \in \mathcal{L}_\infty^{fr}$, it follows that $R^{-1} \in op^1$. Given $s \in \mathbb{Z}$, write

$$|D|^{s+1}|P|^{-1}|D|^{-s} = (|D|^{s+1}R^{s+1}) \cdot (R^{-s-1}|P|^{-1}R^s) \cdot (R^{-s}|D|^{-s}). \tag{2.22}$$

As $|D|^{s+1}R^{s+1} \in op^0$ and $R^{-s}|D|^{-s} \in op^0$, they are bounded operators. Since $R = f(P)$ we have $R^{-s-1}|P|^{-1}R^s = R^{-1}|P|^{-1}$, which is bounded. Hence $|P|^{-1} \in op^{-1}$. \square

Lemma 4. *If Q_0 is an odd self-adjoint element of op^{-1} such that $Q_0P - 1 \in \mathcal{L}_\infty^{fr}(H)$, then $R - Q_0 \in \mathcal{L}_\infty^{fr}(H)$.*

Proof. First, $Q_0PR - R = (Q_0P - I)R \in \mathcal{L}_\infty^{fr}(H)$. Note that $I - PR$ is the projection on the kernel of P , hence in \mathcal{L}_∞^{fr} . Then $Q_0 - Q_0PR = Q_0(I - PR) \in \mathcal{L}_\infty^{fr}$. The lemma follows. \square

Lemma 5. *If $A \in op^k$ is an even operator, with $k < 0$, then A is a limit of even elements of \mathcal{L}_∞^{fr} , in the op^0 -topology.*

Proof. Write $A = B|D|^k$, with $B \in op^0$. Put

$$\chi_n(t) = \begin{cases} 1 & \text{if } t \leq n, \\ 0 & \text{otherwise.} \end{cases} \tag{2.23}$$

Put $A_n = A\chi_n(|D|)$. Then for every $s \in \mathbb{Z}$,

$$\begin{aligned} \|A - A_n\|_{0,s} &= \| |D|^s B |D|^k \cdot (1 - \chi_n)(|D|) \cdot |D|^{-s} \| \\ &= \| |D|^s B |D|^{-s} \cdot |D|^k (1 - \chi_n)(|D|) \| \end{aligned} \tag{2.24}$$

$$\begin{aligned} &\leq \| |D|^s B |D|^{-s} \| \cdot \| |D|^k (1 - \chi_n)(|D|) \| \\ &= |B|_{0,s} \cdot \| |D|^k (1 - \chi_n)(|D|) \| \leq |B|_{0,s} n^k. \end{aligned}$$

Hence $\lim_{n \rightarrow \infty} |A - A_n|_{0,s} = 0$. This proves the lemma. \square

Proof of Proposition 1. Suppose that $P \in \mathcal{P}$. Then we can take Q to be the operator R of Lemma 3.

Conversely, suppose that we have an odd self-adjoint operator $P \in op^1$, and $Q \in op^{-1}$ is an odd self-adjoint operator such that $PQ - I \in op^{-1}$ and $QP - I \in op^{-1}$. It follows that P is Fredholm, and we therefore can define R as in (2.21). By the proof of Lemma 5, for every $N \in \mathbb{Z}^+$, we can write

$$QP - I = F_N - A_N, \tag{2.25}$$

where F_N and A_N are even operators, $F_N \in \mathcal{L}_\infty^{fr}(H)$ and $C_N = \max\{|A_N|_{0,s} : |s| \leq N\} < 1$. Then $I - A_N$ is an even invertible operator on H and for all $s \in \mathbb{Z}$ with $|s| \leq N$,

$$|(I - A_N)^{-1}|_{0,s} = \left| \sum_{m=0}^\infty A_N^m \right|_{0,s} \leq \frac{1}{1 - C_N}. \tag{2.26}$$

Equation (2.25) implies that

$$(I - A_N)R - Q = Q(PR - I) - F_N R. \tag{2.27}$$

Multiplying on the left by $(I - A_N)^{-1}$ shows that

$$R - (1 - A_N)^{-1}Q = (I - A_N)^{-1}(Q(PR - I) - F_N R). \tag{2.28}$$

Note that $Q(PR - I) - F_N R \in \mathcal{L}_\infty^{fr}$. Given $s \in \mathbb{Z}$, if N is sufficiently large then the right-hand side of (2.28) is a bounded operator from H^s to H^{s+1} . Hence the operator

$$|D|^{s+1}R|D|^{-s} - (|D|^{s+1}(1 - A_N)^{-1}|D|^{-s-1}) (|D|^{s+1}Q|D|^{-s}), \tag{2.29}$$

defined originally on H^∞ , extends to an odd bounded operator on H . For N sufficiently large, we know that $|D|^{s+1}(I - A_N)^{-1}|D|^{-s-1}$ extends to a bounded operator on H . Also, for any $s \in \mathbb{Z}$, the operator $|D|^{s+1}Q|D|^{-s}$ extends to a bounded operator on H . Therefore $|D|^{s+1}R|D|^{-s}$ extends to an odd bounded operator on H and $R \in op^{-1}$. Lemma 3 now implies the proposition. \square

Corollary 1. *If $P \in \mathcal{P}$, and $A \in op^0$ is an odd self-adjoint operator, then $P + A \in \mathcal{P}$.*

Proof. From Proposition 1, there is some odd self-adjoint $Q \in op^{-1}$ so that $PQ - I \in op^{-1}$ and $QP - I \in op^{-1}$. Then $(P + A)Q - I \in op^{-1}$ and $Q(P + A) - I \in op^{-1}$. The corollary now follows from Proposition 1. \square

2.3. Duhamel formula

We say that a map from \mathbb{R}^n to \mathcal{P} is smooth if the composite map $\mathbb{R}^n \rightarrow \mathcal{P} \subset op^1$ is smooth with respect to the Fréchet topology on op^1 .

Proposition 2. *If $f: \mathbb{R}^n \rightarrow \mathcal{P}$ is smooth, then $x \mapsto e^{-\epsilon f(x)^2}$ is a smooth map from \mathbb{R}^n to $\mathcal{L}^1(H)$.*

Proof. Consider first the case when $n = 1$. Let $f: \mathbb{R} \rightarrow \mathcal{P}$ be a smooth map, parametrized by $u \in \mathbb{R}$. We claim that for $u_1 < u_2$, we have

$$e^{-\epsilon f(u_2)^2} - e^{-\epsilon f(u_1)^2} = - \int_{u_1}^{u_2} \int_0^\epsilon e^{-\sigma f(v)^2} \frac{df(v)^2}{dv} e^{-(\epsilon-\sigma)f(v)^2} d\sigma dv. \tag{2.30}$$

To give meaning to the integral over σ , we rewrite it as

$$\begin{aligned} & \int_0^\epsilon e^{-\sigma f(v)^2} \frac{df(v)^2}{dv} e^{-(\epsilon-\sigma)f(v)^2} d\sigma = \\ & \int_0^{\frac{\epsilon}{2}} e^{-\sigma f(v)^2} \left(\frac{df(v)^2}{dv} e^{-\frac{\epsilon}{2}f(v)^2} \right) e^{-(\frac{\epsilon}{2}-\sigma)f(v)^2} d\sigma + \\ & \int_{\frac{\epsilon}{2}}^\epsilon e^{-(\sigma-\frac{\epsilon}{2})f(v)^2} \left(e^{-\frac{\epsilon}{2}f(v)^2} \frac{df(v)^2}{dv} \right) e^{-(\epsilon-\sigma)f(v)^2} d\sigma. \end{aligned} \tag{2.31}$$

Using Lemma 2.(3), the integrands in the last two integrals are continuous as maps from $[0, \frac{\epsilon}{2}]$ (or $[\frac{\epsilon}{2}, \epsilon]$) to $\mathcal{L}^1(H)$.

To prove (2.30), we first prove the corresponding statement for resolvents. For $\lambda \in \mathbb{C} - \mathbb{R}^{\geq 0}$, and $-\infty < v_1 < v_2 < \infty$, we have

$$\begin{aligned} & (\lambda - f(v_2)^2)^{-1} - (\lambda - f(v_1)^2)^{-1} = \\ & (\lambda - f(v_2)^2)^{-1} \cdot (f(v_2)^2 - f(v_1)^2) \cdot (\lambda - f(v_1)^2)^{-1}. \end{aligned} \tag{2.32}$$

Then

$$\begin{aligned} & \frac{(\lambda - f(v_2)^2)^{-1} - (\lambda - f(v_1)^2)^{-1}}{v_2 - v_1} = \\ & (\lambda - f(v_2)^2)^{-1} \cdot \left. \frac{df(v)^2}{dv} \right|_{v=v_1} \cdot (\lambda - f(v_1)^2)^{-1} = \\ & (\lambda - f(v_2)^2)^{-1} \cdot \left(\frac{f(v_2)^2 - f(v_1)^2}{v_2 - v_1} - \left. \frac{df(v)^2}{dv} \right|_{v=v_1} \right) \cdot \\ & (\lambda - f(v_1)^2)^{-1}. \end{aligned} \tag{2.33}$$

By assumption, $f : \mathbb{R} \rightarrow op^1$ is differentiable, so $f^2 : \mathbb{R} \rightarrow op^2$ is differentiable. Using the fact that H^s can be defined using $f(v) \in \mathcal{P}$ instead of D , it follows that $(\lambda - f(v)^2)^{-1} \in op^{-2}$. From (2.32),

$$(\lambda - f(\cdot)^2)^{-1} : \mathbb{R} \rightarrow op^{-2} \tag{2.34}$$

is continuous. Then (2.33) implies that (2.34) is differentiable and the derivative is given by

$$\frac{d}{dv}(\lambda - f(v)^2)^{-1} = (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1}. \tag{2.35}$$

Hence

$$\begin{aligned} &(\lambda - f(u_2)^2)^{-1} - (\lambda - f(u_1)^2)^{-1} = \\ &\int_{u_1}^{u_2} (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1} dv, \end{aligned} \tag{2.36}$$

where the integrand is a continuous map from $[u_1, u_2]$ to op^{-2} .

Put $\Gamma = \{|t| - 1, t\} : t \in \mathbb{R}$, a parametrized curve in the complex plane. By the spectral theorem,

$$e^{-\epsilon f(u)^2} = \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon \lambda} (\lambda - f(u)^2)^{-1} d\lambda. \tag{2.37}$$

Then

$$\begin{aligned} &e^{-\epsilon f(u_2)^2} - e^{-\epsilon f(u_1)^2} = \\ &\frac{1}{2\pi i} \int_{\Gamma} \int_{u_1}^{u_2} e^{-\epsilon \lambda} (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1} dv d\lambda = \\ &\frac{1}{2\pi i} \int_{\Gamma} \int_{u_1}^{u_2} e^{-\epsilon \lambda} \frac{I + f(v)^2}{\lambda - f(v)^2} (I + f(v)^2)^{-1} \frac{df(v)^2}{dv} (I + f(v)^2)^{-1} \cdot \\ &\frac{I + f(v)^2}{\lambda - f(v)^2} \cdot dv d\lambda \end{aligned} \tag{2.38}$$

The spectral theorem gives a uniform bound on $\left\| \frac{I+f(v)^2}{\lambda-f(v)^2} \right\|$ for $v \in \mathbb{R}$ and $\lambda \in \Gamma$. Also, $\|I + f(v)^2\|^{-1} \frac{df(v)^2}{dv} (I + f(v)^2)^{-1}$ is uniformly bounded for $v \in [u_1, u_2]$. Combined with the exponential decay of $e^{-\epsilon \lambda} = e^{-\epsilon(|t|-1+it)}$ as $t \rightarrow \pm\infty$, one can justify switching the order of integration to obtain

$$e^{-\epsilon f(u_2)^2} - e^{-\epsilon f(u_1)^2} = \tag{2.39}$$

$$\frac{1}{2\pi i} \int_{u_1}^{u_2} \int_{\Gamma} e^{-\epsilon \lambda} (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1} d\lambda dv.$$

We claim that

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon \lambda} (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1} d\lambda = \\ & - \int_0^{\epsilon} e^{-\sigma f(v)^2} \frac{df(v)^2}{dv} e^{-(\epsilon-\sigma)f(v)^2} d\sigma. \end{aligned} \tag{2.40}$$

To see this, let e_1 and e_2 be eigenfunctions of $f(v)^2$, with eigenvalues λ_1 and λ_2 , respectively. Then

$$\begin{aligned} & \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon \lambda} \langle e_1, (\lambda - f(v)^2)^{-1} \frac{df(v)^2}{dv} (\lambda - f(v)^2)^{-1} e_2 \rangle d\lambda = \\ & \frac{1}{2\pi i} \int_{\Gamma} e^{-\epsilon \lambda} (\lambda - \lambda_1)^{-1} (\lambda - \lambda_2)^{-1} \langle e_1, \frac{df(v)^2}{dv} e_2 \rangle d\lambda = \\ & \frac{e^{-\epsilon \lambda_2} - e^{-\epsilon \lambda_1}}{\lambda_2 - \lambda_1} \langle e_1, \frac{df(v)^2}{dv} e_2 \rangle, \end{aligned} \tag{2.41}$$

where $\frac{e^{-\epsilon \lambda_2} - e^{-\epsilon \lambda_1}}{\lambda_2 - \lambda_1}$ is taken to be $-\epsilon e^{-\epsilon \lambda_1}$ if $\lambda_2 = \lambda_1$. On the other hand,

$$\begin{aligned} & \int_0^{\epsilon} \langle e_1, e^{-\sigma f(v)^2} \frac{df(v)^2}{dv} e^{-(\epsilon-\sigma)f(v)^2} e_2 \rangle d\sigma = \\ & \int_0^{\epsilon} e^{-\sigma \lambda_1} e^{-(\epsilon-\sigma)\lambda_2} \langle e_1, \frac{df(v)^2}{dv} e_2 \rangle d\sigma = \\ & - \frac{e^{-\epsilon \lambda_2} - e^{-\epsilon \lambda_1}}{\lambda_2 - \lambda_1} \langle e_1, \frac{df(v)^2}{dv} e_2 \rangle, \end{aligned} \tag{2.42}$$

which proves the claim.

This proves (2.30). It follows that $u \rightarrow e^{-\epsilon f(u)^2}$ is differentiable as a map into $\mathcal{L}^1(H)$, with derivative

$$\frac{d}{du} e^{-\epsilon f(u)^2} = - \int_0^{\epsilon} e^{-\sigma f(u)^2} \frac{df(u)^2}{du} e^{-(\epsilon-\sigma)f(u)^2} d\sigma. \tag{2.43}$$

If $f : \mathbb{R}^n \rightarrow \mathcal{P}$ is a smooth map then precomposing with smooth maps $\mathbb{R} \rightarrow \mathbb{R}^n$, we see that $x \rightarrow e^{-\epsilon f(x)^2}$ is differentiable as a map from \mathbb{R}^n to $\mathcal{L}^1(H)$. By a similar argument, we can take more derivatives and see that $x \rightarrow e^{-\epsilon f(x)^2}$ is a smooth map from \mathbb{R}^n to $\mathcal{L}^1(H)$. \square

3. Superconnections

In this section we develop the theory of superconnections on Hilbert bundles. In Subsection 3.1 we define the relevant class of Hilbert bundles and specify, in particular, the structure group. In Subsection 3.2 we define superconnections on such Hilbert bundles, using the pseudodifferential operators of the previous section. We construct Chern characters and eta forms. We prove additivity results for eta forms.

3.1. Structure group

Put $G = U(H) \cap op^0$.

Lemma 6. G is a group.

Proof. If $g \in G$ then Lemma 1.(3) gives that $g^* \in op^0$, so g^* is an inverse of g in G . \square

We now put a smooth structure on G . Since we will be considering principal G -bundles over finite dimensional manifolds, it suffices to give a notion of smooth maps from open subsets of Euclidean spaces to G , i.e. plots in the sense of diffeology. A reference for diffeology is the book [22]. A brief introduction is in [8, Appendix A].

The smooth structure on G that we take is such that the adjoint action of G on op^* is smooth and the action of G on H^* is smooth. The precise definition is the following. (We define smooth maps to op^k using the Fréchet structure on op^k , and to H^s using the Hilbert space structure on H^s .)

In the rest of the paper, we fix a number $K \in \mathbb{N}$; its role will eventually be to bound the pseudodifferential order of the connection form. Taking K large allows more flexibility. The results of the paper, such as Theorems 1 and 2, will hold independent of K .

Definition 2. If U is an open subset of \mathbb{R}^n then a map $g : U \rightarrow G$ is a plot if

- (1) For any smooth map $F : U \rightarrow op^k$, the maps $gFg^{-1} : U \rightarrow op^k$ and $g^{-1}Fg : U \rightarrow op^k$ are smooth.
- (2) For any smooth map $v : U \rightarrow H^s$, the maps $gv : U \rightarrow H^s$ and $g^{-1}v : U \rightarrow H^s$ are smooth.
- (3) There is a smooth map $X : U \rightarrow (\mathbb{R}^n)^* \otimes op^K$ so that for any smooth map $v : U \rightarrow H^s$, we have $g^{-1}d(gv) = dv + Xv$ in $\Omega^1(U; H^{s-K})$.

It is straightforward to see that this defines a diffeology on G . Let M be a manifold and let $q : P \rightarrow M$ be a smooth principal G -bundle [22, Chapter 8.11]. Form the associated

Hilbert bundle $\mathcal{H} = P \times_G H$ [22, Chapter 8.16]. We can find a covering $\{U_\alpha\}$ of M by open sets, diffeomorphic to open subsets of \mathbb{R}^n , so that $q^{-1}(U_\alpha)$ is G -diffeomorphic to $U_\alpha \times G$ [22, Chapter 8.13]. Let $g_{\alpha\beta} : U_\alpha \cap U_\beta \rightarrow G$ be the transition map. By a connection on \mathcal{H} , we will mean a collection of 1-forms $A_\alpha \in \Omega^1(U_\alpha; op^K)$ satisfying

$$A_\alpha = g_{\alpha\beta}^{-1} A_\beta g_{\alpha\beta} + X_{\alpha\beta}, \tag{3.1}$$

where $X_{\alpha\beta}$ comes from Definition 2.(3). Because of our notion of smooth structure on G , we can talk about the space $\bigoplus_{p \geq 0, k \in \mathbb{Z}} \Omega^p(M; op^k(\mathcal{H}))$ of smooth op^* -valued differential forms on M . Since G preserves the space \mathcal{P} from Definition 1, and acts smoothly on it, we can also talk about $\Omega^0(M; \mathcal{P}(\mathcal{H}))$.

Example 3. If H is finite dimensional then $G = U(N)$ and \mathcal{H} is a finite dimensional unitary vector bundle.

Example 4. Let Z be a compact manifold. Let $\mathcal{D}^{\frac{1}{2}}$ be the half-density line bundle on Z . Let V be a Hermitian vector bundle on Z . Let L be the group of Hermitian isomorphisms of V to itself. We do not assume that the elements of L cover the identity diffeomorphism of Z . Put $H = L^2(Z; \mathcal{D}^{\frac{1}{2}} \otimes V)$. Then there is a homomorphism $L \rightarrow U(H)$. We give L the smooth topology. Note that if $\dim(Z) > 0$ then the homomorphism will not be continuous if we give $U(H)$ the topology coming from the norm topology on $B(H)$.

Suppose now that Z is even dimensional and V is a Clifford module. In particular, Z acquires a Riemannian metric. Let D be the associated Dirac-type operator. Putting $G = U(H) \cap op^0$ as before, with its diffeological structure, there is a homomorphism $\rho : L \rightarrow G$ that is smooth in the sense that if U is an open subset of \mathbb{R}^n , and $\alpha : U \rightarrow L$ is a smooth map, then $\rho \circ \alpha$ is a plot for G .

Now let $\pi : Q \rightarrow M$ be a fiber bundle with connected base M and compact even dimensional fibers. Let E be a Hermitian vector bundle on Q . Given $m \in M$, put $Z_m = \pi^{-1}(m)$ and put $V_m = E|_{Z_m}$. Choose $m_0 \in M$ and let L be the Hermitian isomorphisms of V_{m_0} . Then (Q, E) is associated to some principal L -bundle $P \rightarrow M$, using the action of L on V_{m_0} .

Suppose that E is a Clifford module with connection, in the sense of [3, Section 10.2]. Construct H and D as above, using Z_{m_0} and V_{m_0} . Put $\mathcal{H}_m = L^2(Z_m; \mathcal{D}_m^{\frac{1}{2}} \otimes V_m)$. Then $\{\mathcal{H}_m\}_{m \in M}$ are the fibers of a Hilbert bundle \mathcal{H} associated to P using the representation ρ . The smooth sections of $\mathcal{H} \rightarrow M$ are the same as the smooth sections of $(\mathcal{D}^V Q)^{\frac{1}{2}} \otimes E \rightarrow Q$, where $(\mathcal{D}^V Q)^{\frac{1}{2}}$ is the line bundle on Q of vertical half-densities.

3.2. Chern character and eta form

Definition 3. Let $\mathcal{H} = \mathcal{H}^+ \oplus \mathcal{H}^-$ be a \mathbb{Z}_2 -graded Hilbert bundle on M , in the sense of the previous subsection. A superconnection on \mathcal{H} is a sum

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \mathbb{A}_{[2]} + \dots \tag{3.2}$$

where

- (1) $\mathbb{A}_{[0]} \in \Omega^0(M; \mathcal{P})$,
- (2) $\mathbb{A}_{[1]}$ is a connection on \mathcal{H} and
- (3) For $i \geq 2$, we have $\mathbb{A}_{[i]} \in \Omega^i(M, op^{k_i}(\mathcal{H}))$ for some $k_i \in \mathbb{Z}$, and $\mathbb{A}_{[i]}$ has total parity -1 (including the parity $(-1)^i$ of the exterior algebra part).

Example 5. In the fiber bundle setting of [Example 4](#), the Bismut superconnection \mathbb{A}_{Bismut} [[3, Section 10.3](#)] is an example of a superconnection on \mathcal{H} .

As usual, we define $e^{-\mathbb{A}^2}$ by doing a Duhamel expansion around $e^{-\mathbb{A}_{[0]}^2}$. Because of the nilpotency of $\Omega^{\geq 1}(M)$, the expansion has a finite number of terms.

Lemma 7.

- (1) For any $X \in \Omega^*(M; op^*(\mathcal{H}))$, we have that $Xe^{-\mathbb{A}^2}$ and $e^{-\mathbb{A}^2}X$ lie in $\Omega^*(M; \mathcal{L}^1(\mathcal{H}))$, and

$$d\text{Str}\left(Xe^{-\mathbb{A}^2}\right) = \text{Str}\left([\mathbb{A}, X]e^{-\mathbb{A}^2}\right). \tag{3.3}$$

- (2) $\text{Ch}(\mathbb{A}) = \text{Str} e^{-\mathbb{A}^2}$ is a closed form on M .

Proof. Expanding $e^{-\mathbb{A}^2} \in \Omega^*(M; op^*)$ around $e^{-\mathbb{A}_{[0]}^2}$ in a Duhamel expansion shows that the component in $\Omega^i(M; op^*)$ is a finite sum of terms of the form

$$\int_{\Delta_k} e^{-t_0 \mathbb{A}_{[0]}^2} F_1 e^{-t_1 \mathbb{A}_{[0]}^2} F_2 \dots F_k e^{-t_k \mathbb{A}_{[0]}^2}, \tag{3.4}$$

where

$$\Delta_k = \{(t_0, \dots, t_k) \in \mathbb{R}^{k+1} : \sum_{j=0}^k t_j = 1\} \tag{3.5}$$

and each F_j lies in $\Omega^{\geq 1}(M; op^*)$. For any $(t_0, \dots, t_k) \in \Delta_k$, we have $t_j \geq \frac{1}{k+1}$ for some j . Thus the integral in (3.4) can be written as a finite sum of integrals where in each integral, $t_j \geq \frac{1}{k+1}$ for some j . The fact that $Xe^{-\mathbb{A}^2}$ and $e^{-\mathbb{A}^2}X$ lie in $\Omega^*(M; \mathcal{L}^1(\mathcal{H}))$ now follows from [Lemma 2.\(3\)](#). Using (2.13), equation (3.3) can be proved along the same lines as the proof of [[3, Lemma 9.15](#)]. Finally, as in [[3, Theorem 9.17\(1\)](#)], part (2) of the lemma is an immediate consequence of (1). \square

Lemma 8. Let $\{\mathbb{A}(t)\}_{t \in [0,1]}$ and $\{\widehat{\mathbb{A}}(t)\}_{t \in [0,1]}$ be two smooth 1-parameter families of superconnections on \mathcal{H} with $\mathbb{A}(0) = \widehat{\mathbb{A}}(0)$ and $\mathbb{A}(1) = \widehat{\mathbb{A}}(1)$. Suppose that the two 1-parameter families are homotopic relative to the endpoints, in sense that there is a smooth 2-parameter family of superconnections $\{\widetilde{\mathbb{A}}(s, t)\}_{s,t \in [0,1]}$ on \mathcal{H} with $\widetilde{\mathbb{A}}(0, t) = \mathbb{A}(t)$, $\widetilde{\mathbb{A}}(1, t) = \widehat{\mathbb{A}}(t)$, $\widetilde{\mathbb{A}}(s, 0) = \mathbb{A}(0) = \widehat{\mathbb{A}}(0)$ and $\widetilde{\mathbb{A}}(s, 1) = \mathbb{A}(1) = \widehat{\mathbb{A}}(1)$. Then

$$\int_0^1 \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\mathbb{A}^2(t)} \right) dt = \int_0^1 \text{Str} \left(\frac{d\widehat{\mathbb{A}}(t)}{dt} e^{-\widehat{\mathbb{A}}^2(t)} \right) dt \tag{3.6}$$

in $\Omega^{\text{odd}}(M)/\text{Im}(d)$.

Proof. Define a superconnection \mathbb{B} on $[0, 1] \times [0, 1] \times M$ by

$$\mathbb{B} = ds \wedge \partial_s + dt \wedge \partial_t + \widetilde{\mathbb{A}}(s, t). \tag{3.7}$$

Then $\text{Ch}(\mathbb{B})$ is given by

$$\begin{aligned} \text{Ch}(\mathbb{B}) &= \text{Ch}(\widetilde{\mathbb{A}}(s, t)) - ds \wedge \text{Str} \left(\frac{d\widetilde{\mathbb{A}}(s, t)}{ds} e^{-\widetilde{\mathbb{A}}^2(s, t)} \right) - \\ &dt \wedge \text{Str} \left(\frac{d\widetilde{\mathbb{A}}(s, t)}{dt} e^{-\widetilde{\mathbb{A}}^2(s, t)} \right) + O(ds \wedge dt). \end{aligned} \tag{3.8}$$

From Lemma 7, $\text{Ch}(\mathbb{B})$ is closed on $[0, 1] \times [0, 1] \times M$. Modulo $\text{Im}(d_M)$, we have

$$\begin{aligned} \int_{\partial([0,1] \times [0,1])} \text{Ch}(\mathbb{B}) &= \int_{[0,1] \times [0,1]} d_{[0,1] \times [0,1]} \text{Ch}(\mathbb{B}) = - \int_{[0,1] \times [0,1]} d_M \text{Ch}(\mathbb{B}) \\ &= - d_M \int_{[0,1] \times [0,1]} \text{Ch}(\mathbb{B}) = 0. \end{aligned} \tag{3.9}$$

The lemma follows. \square

Remark 1. We can weaken the homotopy hypothesis in Lemma 8 to just assume that there is a smooth 2-parameter family $\{\mathcal{O}(s, t)\}_{s,t \in [0,1]}$ in $\Omega^0(M; \mathcal{P})$ with $\mathcal{O}(0, t) = \mathbb{A}(t)_{[0]}$, $\mathcal{O}(1, t) = \widehat{\mathbb{A}}(t)_{[0]}$, $\mathcal{O}(s, 0) = \mathbb{A}(0)_{[0]} = \widehat{\mathbb{A}}(0)_{[0]}$ and $\mathcal{O}(s, 1) = \mathbb{A}(1)_{[0]} = \widehat{\mathbb{A}}(1)_{[0]}$. Then we can construct a 2-parameter family $\{\widetilde{\mathbb{A}}(s, t)\}_{0 \leq s, t \leq 1}$ as in the lemma.

Let \mathbb{A}_0 and \mathbb{A}_1 be two superconnections on \mathcal{H} . Let $\mathbb{A}_{0,[0]}$ and $\mathbb{A}_{1,[0]}$ denote their order-zero parts. Suppose that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; \text{op}^0(\mathcal{H}))$. For $t \in [0, 1]$, put $\mathbb{A}(t) = (1-t)\mathbb{A}_0 + t\mathbb{A}_1$. By Corollary 1, for any $t \in [0, 1]$, we have $\mathbb{A}(t)_{[0]} \in \mathcal{P}$. Define $\eta(\mathbb{A}_0, \mathbb{A}_1) \in \Omega^{\text{odd}}(M)/\text{Im}(d)$ by

$$\eta(\mathbb{A}_0, \mathbb{A}_1) = \int_0^1 \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\mathbb{A}^2(t)} \right) dt. \tag{3.10}$$

Lemma 9.

$$\text{Ch}(\mathbb{A}_1) - \text{Ch}(\mathbb{A}_0) = -d\eta(\mathbb{A}_0, \mathbb{A}_1). \tag{3.11}$$

Proof. Consider the superconnection on $[0, 1] \times M$ given by

$$\mathbb{B} = dt \wedge \partial_t + \mathbb{A}(t). \tag{3.12}$$

Then $\text{Ch}(\mathbb{B}) \in \Omega^*([0, 1] \times M)$ is given by

$$\text{Ch}(\mathbb{B}) = \text{Ch}(\mathbb{A}(t)) - dt \wedge \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\mathbb{A}^2(t)} \right). \tag{3.13}$$

From Lemma 7, $\text{Ch}(\mathbb{B})$ is closed on $[0, 1] \times M$. This implies that

$$\frac{d}{dt} \text{Ch}(\mathbb{A}(t)) = -d \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\mathbb{A}^2(t)} \right) \tag{3.14}$$

in $\Omega^*(M)$. The lemma follows by integrating over $[0, 1]$. \square

Lemma 10. Let $\mathbb{A}_0, \mathbb{A}_1$ and \mathbb{A}_2 be three superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0(\mathcal{H}))$ and $\mathbb{A}_{1,[0]} - \mathbb{A}_{2,[0]} \in \Omega^0(M; op^0(\mathcal{H}))$. Then

$$\eta(\mathbb{A}_0, \mathbb{A}_1) + \eta(\mathbb{A}_1, \mathbb{A}_2) = \eta(\mathbb{A}_0, \mathbb{A}_2). \tag{3.15}$$

Proof. For $s, t \in [0, 1]$, put

$$\begin{aligned} \tilde{\mathbb{A}}(s, t) = & \tag{3.16} \\ & \begin{cases} (1 - t - st)\mathbb{A}_0 + 2st\mathbb{A}_1 + (1 - s)t\mathbb{A}_2 & \text{if } 0 \leq t \leq \frac{1}{2}, \\ (1 - s)(1 - t)\mathbb{A}_0 + 2s(1 - t)\mathbb{A}_1 + (-s + t + ts)\mathbb{A}_2 & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases} \end{aligned}$$

Then $\{\tilde{\mathbb{A}}(0, t)\}_{t \in [0,1]}$ is the linear path from \mathbb{A}_0 to \mathbb{A}_2 , while $\{\tilde{\mathbb{A}}(1, t)\}_{t \in [0,1]}$ is the concatenation of the linear path from \mathbb{A}_0 to \mathbb{A}_1 , with the linear path from \mathbb{A}_1 to \mathbb{A}_2 . After reparametrizing $[0, 1] \times [0, 1]$ to make $\tilde{\mathbb{A}}(s, t)$ smooth in s and t , we can apply Lemma 8. \square

Given a superconnection \mathbb{A} on \mathcal{H} and $t > 0$, define a new superconnection by

$$\mathbb{A}_t = t\mathbb{A}_{[0]} + \mathbb{A}_{[1]} + t^{-1}\mathbb{A}_{[2]} + \dots \tag{3.17}$$

Lemma 11. *Suppose that there is some $c > 0$ so that $\mathbb{A}_{[0]}^2 \geq c^2 \text{Id}$ fiberwise on \mathcal{H} . Put*

$$\eta(\mathbb{A}, \infty) = \int_1^\infty \text{Str} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) dt. \tag{3.18}$$

Then

$$\text{Ch}(\mathbb{A}) = d\eta(\mathbb{A}, \infty) \tag{3.19}$$

Proof. Given $m \in M$, let $\{\lambda_i\}$ denote the eigenvalues of $\mathbb{A}_{[0]}^2$ on the fiber \mathcal{H}_m . By assumption, $\lambda_i \geq c^2$ for each i . Then for $t \geq 1$,

$$\sum_i e^{-t^2 \lambda_i} = \sum_i e^{-(t^2-1)\lambda_i} e^{-\lambda_i} \leq e^{-c^2(t^2-1)} \sum_i e^{-\lambda_i}. \tag{3.20}$$

Hence

$$\left\| e^{-t^2 \mathbb{A}_{[0]}^2} \right\|_1 \leq e^{-c^2(t^2-1)} \left\| e^{-\mathbb{A}_{[0]}^2} \right\|_1 \tag{3.21}$$

and it follows from the proof of [Lemma 2](#) that on any compact subset of M ,

$$\text{Str} \left(e^{-\mathbb{A}_t^2} \right) = O \left(e^{-c^2 t^2 / 2} \right) \tag{3.22}$$

and

$$\text{Str} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right) = O \left(e^{-c^2 t^2 / 2} \right). \tag{3.23}$$

In particular, the integrand in [\(3.18\)](#) is integrable.

As in the proof of [Lemma 9](#),

$$\frac{d}{dt} \text{Ch}(\mathbb{A}_t) = -d \text{Str} \left(\frac{d\mathbb{A}_t}{dt} e^{-\mathbb{A}_t^2} \right). \tag{3.24}$$

Using [\(3.22\)](#) and [\(3.23\)](#), we can integrate [\(3.24\)](#) over $[1, \infty)$, from which [\(3.19\)](#) follows. \square

Lemma 12. *Let \mathbb{E} be a superconnection on a Hilbert bundle over $[0, 1] \times M$. Let $\mathbb{E}(s)$ be the superconnection on the restriction of the Hilbert bundle to $\{s\} \times M$. If $\mathbb{E}(s)_{[0]}$ is invertible for all $s \in [0, 1]$ then*

$$\eta(\mathbb{E}(0), \infty) - \eta(\mathbb{E}(1), \infty) = - \int_0^1 \text{Ch}(\mathbb{E}) \tag{3.25}$$

in $\Omega^{odd}(M)/\text{Im}(d)$. In the case of a product Hilbert bundle, meaning that it pulls back from a Hilbert bundle on M , (3.25) becomes

$$\eta(\mathbb{E}(0), \infty) - \eta(\mathbb{E}(1), \infty) = \eta(\mathbb{E}(0), \mathbb{E}(1)). \tag{3.26}$$

Proof. Consider the superconnection \mathbb{F} on $[1, \infty) \times [0, 1] \times M$ given by

$$dt \wedge \partial_t + \mathbb{E}_t, \tag{3.27}$$

where t is the coordinate on $[1, \infty)$. Given $L > 1$, integrating $\text{Ch}(\mathbb{F})$ over $[1, L] \times [0, 1]$ (cf. (3.9)) gives

$$\begin{aligned} & \int_1^L \text{Str} \left(\frac{d\mathbb{E}(0)_t}{dt} e^{-\mathbb{E}(0)_t^2} \right) dt - \int_1^L \text{Str} \left(\frac{d\mathbb{E}(1)_t}{dt} e^{-\mathbb{E}(1)_t^2} \right) dt = \\ & - \int_0^1 \text{Ch}(\mathbb{E}) + \int_0^1 \text{Ch}(\mathbb{E}_L). \end{aligned} \tag{3.28}$$

From (3.22),

$$\lim_{L \rightarrow \infty} \int_0^1 \text{Ch}(\mathbb{E}_L) = 0. \tag{3.29}$$

This proves (3.25). Equation (3.26) follows as in (3.13). \square

In Appendix A we prove a generalization of (3.26) where we no longer assume the invertibility of $\{\mathbb{E}(s)_{[0]}\}_{s \in [0,1]}$.

4. Differential K -theory

This section contains the main results of the paper. In Subsection 4.1 we define the differential K -group $\check{K}^0(M)$ in terms of superconnections on Hilbert bundles over M . In Subsection 4.2 we construct a map q from $\check{K}^0(M)$ to the standard differential K -group $\check{K}_{stan}^0(M)$. Subsection 4.3 makes the map more explicit when the degree-0 part of the superconnection, $\mathbb{A}_{[0]}$, has vector bundle kernel. In Subsection 4.4 we show that q is an isomorphism, thereby proving Theorem 1. Subsection 4.5 provides a multiplication on $\check{K}^0(M)$.

4.1. Definitions

Let M be a smooth manifold.

Definition 4. A cocycle for $\check{K}^0(M)$ is a triple $[\mathcal{H}, \mathbb{A}, \omega]$ where

- (1) \mathcal{H} is a \mathbb{Z}_2 -graded Hilbert bundle over M ,
- (2) \mathbb{A} is a superconnection on \mathcal{H} and
- (3) $\omega \in \Omega^{odd}(M)/\text{Im}(d)$.

Definition 5. Two cocycles $[\mathcal{H}, \mathbb{A}, \omega]$ and $[\mathcal{H}', \mathbb{A}', \omega']$ for $\check{K}^0(M)$ are isomorphic if there is a smooth isometric isomorphism $i : \mathcal{H} \rightarrow \mathcal{H}'$ so that $[\mathcal{H}, \mathbb{A}, \omega] = [i^*\mathcal{H}', i^*\mathbb{A}', \omega']$.

Definition 6. The group $\check{K}^0(M)$ is the quotient of the free abelian group generated by the isomorphism classes of cocycles, by the subgroup generated by the following relations:

- (1) If $[\mathcal{H}, \mathbb{A}, \omega]$ and $[\mathcal{H}', \mathbb{A}', \omega']$ are cocycles then

$$[\mathcal{H}, \mathbb{A}, \omega] + [\mathcal{H}', \mathbb{A}', \omega'] = [\mathcal{H} \oplus \mathcal{H}', \mathbb{A} \oplus \mathbb{A}', \omega + \omega']. \tag{4.1}$$

- (2) If $\mathbb{A}_{[0]}$ is invertible then

$$[\mathcal{H}, \mathbb{A}, \omega] = [0, 0, \omega + \eta(\mathbb{A}, \infty)]. \tag{4.2}$$

- (3) Suppose that \mathbb{A}_0 and \mathbb{A}_1 are superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0)$. Then

$$[\mathcal{H}, \mathbb{A}_0, \omega] = [\mathcal{H}, \mathbb{A}_1, \omega + \eta(\mathbb{A}_0, \mathbb{A}_1)]. \tag{4.3}$$

Example 6. In the setting of [Example 5](#), given $\omega \in \Omega^{odd}(M)/\text{Im}(d)$, the triple $[\mathcal{H}, \mathbb{A}_{Bismut}, \omega]$ gives an element of $\check{K}^0(M)$. Compare with the “geometric family” cocycles of [\[12\]](#).

It follows from the relations that there is a map $\check{K}^0(M) \rightarrow \Omega^{even}(M)$ that sends a cocycle $[\mathcal{H}, \mathbb{A}, \omega]$ to $\text{Ch}(\mathbb{A}) + d\omega$.

Lemma 13. Let $[\mathcal{H}, \mathbb{A}, \omega]$ be a cocycle for $\check{K}^0(M)$. Let E be a finite dimensional Hermitian vector bundle on M with compatible connection ∇^E . Put $\tilde{E} = E \oplus E$ with \mathbb{Z}_2 -grading $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and connection $\nabla^{\tilde{E}} = \nabla^E \oplus \nabla^E$. Then

$$[\mathcal{H}, \mathbb{A}, \omega] = [\mathcal{H} \oplus \tilde{E}, \mathbb{A} \oplus \nabla^{\tilde{E}}, \omega]. \tag{4.4}$$

Proof. From relation (1) of [Definition 6](#), it suffices to show that $[\tilde{E}, \nabla^{\tilde{E}}, 0]$ vanishes in $\check{K}^0(M)$. Define a superconnection \mathbb{B} on \tilde{E} by

$$\mathbb{B} = \begin{pmatrix} \nabla^E & I \\ I & \nabla^E \end{pmatrix}. \tag{4.5}$$

From relations (2) and (3) of Definition 6, in $\check{K}^0(M)$ we have

$$\begin{aligned} [\tilde{E}, \nabla^{\tilde{E}}, 0] &= [\tilde{E}, \mathbb{B}, 0] + \eta(\nabla^{\tilde{E}}, \mathbb{B}) \\ &= [0, 0, \eta(\nabla^{\tilde{E}}, \mathbb{B}) + \eta(\mathbb{B}, \infty)]. \end{aligned} \tag{4.6}$$

For $t \in [0, 1]$, put

$$\mathbb{A}(t) = (1 - t)\nabla^{\tilde{E}} + t\mathbb{B} = \begin{pmatrix} \nabla^E & tI \\ tI & \nabla^E \end{pmatrix}. \tag{4.7}$$

Then

$$\mathbb{A}(t)^2 = \begin{pmatrix} t^2 + (\nabla^E)^2 & 0 \\ 0 & t^2 + (\nabla^E)^2 \end{pmatrix} \tag{4.8}$$

and

$$\begin{aligned} \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\mathbb{A}^2(t)} \right) &= \\ \text{Str} \left(\begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix} \begin{pmatrix} e^{-t^2 - (\nabla^E)^2} & 0 \\ 0 & e^{-t^2 - (\nabla^E)^2} \end{pmatrix} \right) &= 0. \end{aligned} \tag{4.9}$$

It follows from (3.10) that $\eta(\nabla^{\tilde{E}}, \mathbb{B}) = 0$. By a similar argument, $\eta(\mathbb{B}, \infty) = 0$. This proves the lemma. \square

4.2. Map to the standard finite dimensional version

Define $\check{K}_{stan}^0(M)$ as in the previous subsection, except using finite dimensional vector bundles rather than Hilbert bundles, connections instead of superconnections, removing relation (2) and adding a stabilization relation

$$[\mathcal{H}, \nabla, \omega] = [\mathcal{H} \oplus \tilde{E}, \nabla \oplus \nabla^{\tilde{E}}, \omega] \tag{4.10}$$

as in the conclusion of Lemma 13.

Then $\check{K}_{stan}^0(M)$ is isomorphic to the standard differential K -theory group as defined, for example, in [18].

Suppose now that M is compact. Given a cocycle $c = [\mathcal{H}, \mathbb{A}, \omega]$ for $\check{K}^0(M)$, we construct an equivalent finite dimensional cocycle as follows.

As in [3, Section 9.5], one can find a finite dimensional vector bundle E on M (in fact, a trivial one) and a linear map $s : E \rightarrow \mathcal{H}^-$ so that $\mathbb{A}_{[0]}^+ + s : \mathcal{H}^+ \oplus E \rightarrow \mathcal{H}^-$ is

surjective. Let ∇^E be a connection on E . First, from [Lemma 13](#), the cocycle $[\mathcal{H}, \mathbb{A}, \omega]$ is equivalent to $[\mathcal{H} \oplus \tilde{E}, \mathbb{A} \oplus \nabla^{\tilde{E}}, \omega]$. Next, define $\Delta^+ : \mathcal{H}^+ \oplus \tilde{E}^+ \rightarrow \mathcal{H}^- \oplus \tilde{E}^-$ by

$$\Delta^+ = \begin{pmatrix} 0 & s \\ 0 & 0 \end{pmatrix} \tag{4.11}$$

and define $\Delta^- : \mathcal{H}^- \oplus \tilde{E}^- \rightarrow \mathcal{H}^+ \oplus \tilde{E}^+$ by

$$\Delta^- = \begin{pmatrix} 0 & 0 \\ s^* & 0 \end{pmatrix}. \tag{4.12}$$

Put

$$\hat{\mathbb{A}} = (\mathbb{A} \oplus \nabla^{\tilde{E}}) + \begin{pmatrix} 0 & \Delta^- \\ \Delta^+ & 0 \end{pmatrix}, \tag{4.13}$$

a superconnection on $\hat{\mathcal{H}} = \mathcal{H} \oplus \tilde{E}$. From relation (3) in [Definition 6](#), the cocycle $[\mathcal{H}, \mathbb{A}, \omega]$ is equivalent to $[\hat{\mathcal{H}}, \hat{\mathbb{A}}, \omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \hat{\mathbb{A}})]$. As a map from $\mathcal{H}^+ \oplus \tilde{E}^+$ to $\mathcal{H}^- \oplus \tilde{E}^-$, we have

$$\hat{\mathbb{A}}_{[0]}^+ = \begin{pmatrix} \mathbb{A}_{[0]}^+ & s \\ 0 & 0 \end{pmatrix}. \tag{4.14}$$

Since $\mathbb{A}_{[0]}^+ + s$ is surjective, its kernel is a finite dimensional vector bundle; cf. [\[3, Section 9.5\]](#). Then $\text{Ker}(\hat{\mathbb{A}}_{[0]}^+) = \text{Ker}(\mathbb{A}_{[0]}^+ + s)$ and $\text{Ker}(\hat{\mathbb{A}}_{[0]}^-) \cong \text{Coker}(\hat{\mathbb{A}}_{[0]}^+) = \tilde{E}^-$. This shows that $\hat{\mathbb{A}}_{[0]}$ has $(\mathbb{Z}_2$ -graded) vector bundle kernel.

Let P be orthogonal projection onto $\text{Ker}(\hat{\mathbb{A}}_{[0]})$. Put

$$\hat{\mathbb{A}}' = (I - P)\hat{\mathbb{A}}(I - P) + P\hat{\mathbb{A}}_{[1]}P. \tag{4.15}$$

Then in $\check{K}^0(M)$, $[\mathcal{H}, \mathbb{A}, \omega]$ equals

$$\begin{aligned} & [\mathcal{H} \oplus \tilde{E}, \mathbb{A} \oplus \nabla^{\tilde{E}}, \omega] = \\ & [\hat{\mathcal{H}}, \hat{\mathbb{A}}, \omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \hat{\mathbb{A}})] = \\ & [\hat{\mathcal{H}}, \hat{\mathbb{A}}', \omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \hat{\mathbb{A}}) + \eta(\hat{\mathbb{A}}, \hat{\mathbb{A}}')] = \\ & [\hat{\mathcal{H}}, \hat{\mathbb{A}}', \omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \hat{\mathbb{A}}')] = \\ & [(I - P)\hat{\mathcal{H}}, (I - P)\hat{\mathbb{A}}(I - P), 0] + \\ & [\text{Ker}(\hat{\mathbb{A}}_{[0]}), P\hat{\mathbb{A}}_{[1]}P, \omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \hat{\mathbb{A}}')] = \\ & [\text{Ker}(\hat{\mathbb{A}}_{[0]}), P\hat{\mathbb{A}}_{[1]}P, \end{aligned} \tag{4.16}$$

$$\omega + \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \widehat{\mathbb{A}}') + \eta((I - P)\widehat{\mathbb{A}}(I - P), \infty) \Big].$$

We define the last expression to be $q(c)$.

Thus given a cocycle c for $\check{K}^0(M)$, we have constructed an equivalent cocycle $q(c)$ for $\check{K}^0(M)$, and $q(c)$ is also a cocycle for $\check{K}_{stan}^0(M)$.

Proposition 3. *The class in $\check{K}_{stan}^0(M)$ represented by $q(c)$ is independent of the choices of E , s and ∇^E .*

Proof. We first claim that given E and s , the class is independent of the choice of ∇^E . Let $\underline{\nabla}^E$ be another choice of connection on E . Let $\widehat{\underline{\mathbb{A}}}$ and $\widehat{\underline{\mathbb{A}}}'$ be the corresponding superconnections constructed using $\underline{\nabla}^E$ instead of ∇^E . Note that $\text{Ker}(\widehat{\underline{\mathbb{A}}}_{[0]}) = \text{Ker}(\widehat{\mathbb{A}}_{[0]})$, so the projection operator P doesn't change. Then the change in the class of $q(c)$ in $\check{K}_{stan}^0(M)$ is $[0, 0, \delta\omega]$, where

$$\begin{aligned} \delta\omega = & \eta(P\widehat{\underline{\mathbb{A}}}_{[1]}P, P\widehat{\mathbb{A}}_{[1]}P) + \eta(\mathbb{A} \oplus \underline{\nabla}^{\tilde{E}}, \widehat{\underline{\mathbb{A}}}') + \eta((I - P)\widehat{\underline{\mathbb{A}}}(I - P), \infty) - \\ & \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \widehat{\mathbb{A}}') - \eta((I - P)\widehat{\mathbb{A}}(I - P), \infty). \end{aligned} \tag{4.17}$$

Using Lemma 10 and (3.26), this equals

$$\begin{aligned} & \eta(P\widehat{\underline{\mathbb{A}}}_{[1]}P, P\widehat{\mathbb{A}}_{[1]}P) + \eta(\mathbb{A} \oplus \underline{\nabla}^{\tilde{E}}, \widehat{\underline{\mathbb{A}}}') + \\ & \eta((I - P)\widehat{\underline{\mathbb{A}}}(I - P), (I - P)\widehat{\mathbb{A}}(I - P)) - \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \widehat{\mathbb{A}}') = \\ & \eta(\mathbb{A} \oplus \underline{\nabla}^{\tilde{E}}, \widehat{\underline{\mathbb{A}}}') + \eta(\widehat{\underline{\mathbb{A}}}', \widehat{\mathbb{A}}') - \eta(\mathbb{A} \oplus \nabla^{\tilde{E}}, \widehat{\mathbb{A}}') = \\ & \eta(\mathbb{A} \oplus \underline{\nabla}^{\tilde{E}}, \mathbb{A} \oplus \nabla^{\tilde{E}}) = \eta(\underline{\nabla}^{\tilde{E}}, \nabla^{\tilde{E}}) = 0. \end{aligned} \tag{4.18}$$

This proves the claim.

Now suppose that E_1 , s_1 , E_2 and s_2 are two different choices for E and s . Let $q_1(c)$ and $q_2(c)$ be the ensuing cocycles for $\check{K}_{stan}^0(M)$. Put $F = E_1 \oplus E_2$, with connection $\nabla^{E_1} \oplus \nabla^{E_2}$. Put $S_1 = s_1 \oplus 0$ and $S_2 = 0 \oplus s_2$, both being maps from F to \mathcal{H}^- . Let C_1 and C_2 be the ensuing cocycles for $\check{K}^0(M)$. Then $q_1(c)$ is equivalent to $q(C_1)$, and $q_2(c)$ is equivalent to $q(C_2)$. For $t \in [0, 1]$, put $S(t) = tS_2 + (1 - t)S_1$. Then for all $t \in [0, 1]$, the map $\mathbb{A}_{[0]}^+ + S(t)$ is surjective from $\mathcal{H}^+ \oplus F$ to \mathcal{H}^- . Hence we can reduce to the case when $E_1 = E_2$, which we will again call E , but there are two maps $s_1, s_2 : E \rightarrow \mathcal{H}^-$ that are joined by a 1-parameter family of maps $s(t) : E \rightarrow \mathcal{H}^-$, so that $\mathbb{A}_{[0]}^+ + s(t)$ is surjective for all $t \in [0, 1]$. If $q_1(c)$ is the cocycle for $\check{K}_{stan}^0(M)$ constructed using $s(0)$, and $q_2(c)$ is the cocycle for $\check{K}_{stan}^0(M)$ constructed using $s(1)$, then we want to show that $q_1(c) = q_2(c)$.

Define $\widehat{\mathbb{A}}(t)$, $P(t)$, and $\widehat{\mathbb{A}}'(t)$ accordingly. The family of Hilbert bundles $\{\widehat{\mathcal{H}}(t)\}_{t \in [0, 1]}$ forms a Hilbert bundle \mathcal{L} over $[0, 1] \times M$. There are subbundles \mathcal{M} and \mathcal{M}' of \mathcal{L} formed by

$\{P(t)\widehat{\mathcal{H}}(t)\}_{t \in [0,1]}$ and $\{(I - P(t))\widehat{\mathcal{H}}(t)\}_{t \in [0,1]}$, respectively. The Hilbert bundle \mathcal{L} acquires a superconnection

$$\begin{aligned} \widehat{\mathcal{C}}' = & (I - P(t)) \left(dt \wedge \partial_t + \widehat{\mathbb{A}}(t) \right) (I - P(t)) + \\ & P(t) \left(dt \wedge \partial_t + \widehat{\mathbb{A}}_{[1]}(t) \right) P(t). \end{aligned} \tag{4.19}$$

The finite dimensional bundle \mathcal{M} acquires a connection

$$\mathbb{B} = P(t) \left(dt \wedge \partial_t + \widehat{\mathbb{A}}_{[1]}(t) \right) P(t). \tag{4.20}$$

There is a superconnection on \mathcal{M}' given by

$$\widehat{\mathbb{B}}' = (I - P(t)) \left(dt \wedge \partial_t + \widehat{\mathbb{A}}(t) \right) (I - P(t)). \tag{4.21}$$

In $\check{K}_{stan}^0(M)$, we have

$$\begin{aligned} & \left[\text{Ker}(\widehat{\mathbb{A}}_{[0]}(1)), P(1)\widehat{\mathbb{A}}_{[1]}(1)P(1), 0 \right] = \\ & \left[\text{Ker}(\widehat{\mathbb{A}}_{[0]}(0)), P(0)\widehat{\mathbb{A}}_{[1]}(0)P(0), 0 \right] + \left[0, 0, \int_0^1 \text{Ch}(\mathbb{B}) \right], \end{aligned} \tag{4.22}$$

as can be seen from trivializing \mathcal{M} with respect to $[0, 1]$ and then applying (3.13) and (4.3).

From Lemma 10,

$$\begin{aligned} \eta(\mathbb{A} \oplus \nabla^{\widehat{E}}, \widehat{\mathbb{A}}'(1)) - \eta(\mathbb{A} \oplus \nabla^{\widehat{E}}, \widehat{\mathbb{A}}'(0)) &= \eta(\widehat{\mathbb{A}}'(0), \widehat{\mathbb{A}}'(1)) \\ &= - \int_0^1 \text{Ch}(\widehat{\mathcal{C}}'). \end{aligned} \tag{4.23}$$

From Lemma 12,

$$\begin{aligned} \eta((I - P(1))\widehat{\mathbb{A}}(1)(I - P(1)), \infty) &= \\ \eta((I - P(0))\widehat{\mathbb{A}}(0)(I - P(0)), \infty) &+ \int_0^1 \text{Ch}(\mathbb{B}'). \end{aligned} \tag{4.24}$$

Now

$$\int_0^1 \text{Ch}(\widehat{\mathcal{C}}') = \int_0^1 \text{Ch}(\mathbb{B}) + \int_0^1 \text{Ch}(\mathbb{B}'). \tag{4.25}$$

Equations (4.22), (4.23), (4.24) and (4.25) imply that $q_1(c) = q_2(c)$. \square

Proposition 4. q passes to a map from $\check{K}^0(M)$ to $\check{K}_{stan}^0(M)$.

Proof. We have to show that q vanishes on relations for $\check{K}^0(M)$. This is evident for relations (1) and (2) in Definition 6. For relation (3), suppose that \mathbb{A}_0 and \mathbb{A}_1 are two superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0)$. For $t \in [0, 1]$, put $\mathbb{A}(t) = t\mathbb{A}_1 + (1-t)\mathbb{A}_0$. From Corollary 1, we know that $\mathbb{A}(t) \in \mathcal{P}$. Let \mathcal{K} be the product Hilbert bundle $[0, 1] \times \mathcal{H}$ over $[0, 1] \times M$. We can find a finite dimensional vector bundle E on $[0, 1] \times M$ (in fact a trivial one) and a map $s : E \rightarrow \mathcal{K}^-$ such that for each $t \in [0, 1]$, if $E(t) \rightarrow \{t\} \times M$ is the restricted bundle and $s(t) : E(t) \rightarrow \mathcal{H}^-$ is the restricted map then $\mathbb{A}_{[0]}(t) + s(t) : \mathcal{H}^+ \oplus E(t) \rightarrow \mathcal{H}^-$ is surjective. Define $\widehat{\mathbb{A}}(t)$, $P(t)$, and $\widehat{\mathbb{A}}'(t)$ accordingly. The family of Hilbert bundles $\{\widehat{\mathcal{H}}(t)\}_{t \in [0,1]}$ forms a Hilbert bundle \mathcal{L} over $[0, 1] \times M$. There are subbundles \mathcal{M} and \mathcal{M}' of \mathcal{L} formed by $\{P(t)\widehat{\mathcal{H}}(t)\}_{t \in [0,1]}$ and $\{(I - P(t))\widehat{\mathcal{H}}(t)\}_{t \in [0,1]}$, respectively. Define $\widehat{\mathbb{C}}'$, \mathbb{B} and $\widehat{\mathbb{B}}'$ as in (4.19), (4.20) and (4.21). As in the proof of Proposition 3, we conclude that

$$\begin{aligned} & \left[\text{Ker}(\widehat{\mathbb{A}}_{[0]}(1)), P(1)\widehat{\mathbb{A}}_{[1]}P(1), 0 \right] - \left[\text{Ker}(\widehat{\mathbb{A}}_{[0]}(0)), P(0)\widehat{\mathbb{A}}_{[1]}P(0), 0 \right] = \tag{4.26} \\ & \left[0, 0, \int_0^1 \text{Ch}(\mathbb{B}) \right] = \left[0, 0, \int_0^1 \text{Ch}(\widehat{\mathbb{C}}') \right] - \left[0, 0, \int_0^1 \text{Ch}(\mathbb{B}') \right] = \\ & \left[0, 0, -\eta(\mathbb{A} \oplus \nabla^{\widehat{E}}, \widehat{\mathbb{A}}'(1)) + \eta(\mathbb{A} \oplus \nabla^{\widehat{E}}, \widehat{\mathbb{A}}'(0)) \right] - \\ & \left[0, 0, \eta((I - P(1))\widehat{\mathbb{A}}(1)(I - P(1)), \infty) - \right. \\ & \left. \eta((I - P(0))\widehat{\mathbb{A}}(0)(I - P(0)), \infty) \right]. \end{aligned}$$

The proposition follows. \square

4.3. Vector bundle kernel

In this subsection we simplify the formula for $q(c)$ in the special case when the family $\text{Ker}(\mathbb{A}_{[0]})$ of vector spaces actually form a vector bundle on M .

Proposition 5. Suppose that $\text{Ker}(\mathbb{A}_{[0]})$ is a $(\mathbb{Z}_2\text{-graded})$ vector bundle. Let Q denote orthogonal projection onto $\text{Ker}(\mathbb{A}_{[0]})$. Put

$$\mathbb{B} = (I - Q)\mathbb{A}(I - Q) + Q\mathbb{A}_{[1]}Q. \tag{4.27}$$

If an element $c \in \check{K}^0(M)$ is represented by $c = [\mathcal{H}, \mathbb{A}, \omega]$ then $q(c) \in \check{K}^0(M)_{stan}$ is represented by

$$\left[\text{Ker}(\mathbb{A}_{[0]}), Q\mathbb{A}_{[1]}Q, \omega + \eta(\mathbb{A}, \mathbb{B}) + \eta((I - Q)\mathbb{A}(I - Q), \infty) \right]. \tag{4.28}$$

Proof. Put $c = [\mathcal{H}, \mathbb{A}, \omega]$. Then c is equivalent to $c' = [\mathcal{H}, \mathbb{B}, \omega + \eta(\mathbb{A}, \mathbb{B})]$, so it suffices to look at $q(c')$. In the construction of $q(c')$, put $E = \text{Ker}(\mathbb{A}_{[0]}^-)$, with connection $\nabla^E = Q\mathbb{A}_{[1]}^-Q$. Let $s : E \rightarrow \mathcal{H}^-$ be inclusion. Define $\widehat{\mathbb{B}}$ as in (4.13). In terms of the orthogonal decompositions

$$\widehat{\mathcal{H}}^+ = (I - Q)\mathcal{H}^+ \oplus \text{Ker}(\mathbb{A}_{[0]}^+) \oplus \widetilde{E}^+ \tag{4.29}$$

and

$$\widehat{\mathcal{H}}^- = (I - Q)\mathcal{H}^- \oplus \text{Ker}(\mathbb{A}_{[0]}^-) \oplus \widetilde{E}^-, \tag{4.30}$$

we can write

$$\widehat{\mathbb{B}}_{[0]}^+ = \begin{pmatrix} (I - Q)\mathbb{A}_{[0]}^+(I - Q) & 0 & 0 \\ 0 & 0 & I \\ 0 & 0 & 0 \end{pmatrix} \tag{4.31}$$

and

$$\widehat{\mathbb{B}}_{[0]}^- = \begin{pmatrix} (I - Q)\mathbb{A}_{[0]}^-(I - Q) & 0 & 0 \\ 0 & 0 & 0 \\ 0 & I & 0 \end{pmatrix}. \tag{4.32}$$

Then

$$\text{Ker}(\widehat{\mathbb{B}}_{[0]}^+) \cong 0 \oplus \text{Ker}(\mathbb{A}_{[0]}^+) \oplus 0 \tag{4.33}$$

and

$$\text{Ker}(\widehat{\mathbb{B}}_{[0]}^-) \cong 0 \oplus 0 \oplus \widetilde{E}^-. \tag{4.34}$$

Let P be projection onto $\text{Ker}(\widehat{\mathbb{B}}_{[0]})$. Put

$$\widehat{\mathbb{B}}' = (I - P)\widehat{\mathbb{B}}(I - P) + P\widehat{\mathbb{B}}_{[1]}P. \tag{4.35}$$

Then the cocycle $q(c')$ is represented by

$$\left[\text{Ker}(\widehat{\mathbb{B}}_{[0]}), P\widehat{\mathbb{B}}_{[1]}P, \omega + \eta(\mathbb{A}, \mathbb{B}) + \eta(\mathbb{B} \oplus \nabla^{\widetilde{E}}, \widehat{\mathbb{B}}') + \eta((I - P)\widehat{\mathbb{B}}(I - P), \infty) \right]. \tag{4.36}$$

As $P(Q \oplus 0) = P$, there is an orthogonal decomposition

$$\widehat{\mathcal{H}} = ((I - Q) \oplus 0)\widehat{\mathcal{H}} \oplus ((Q \oplus 0) - P)\widehat{\mathcal{H}} \oplus P\widehat{\mathcal{H}}. \tag{4.37}$$

Put

$$\mathcal{H}'' = (((Q \oplus 0) - P)\widehat{\mathcal{H}} = \widetilde{E}^+ \oplus \text{Ker}(\mathbb{A}_{[0]}^-), \tag{4.38}$$

with the \mathbb{Z}_2 -grading operator $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Define a connection $\nabla^{\mathcal{H}''}$ by $\nabla^{\mathcal{H}''} = \begin{pmatrix} \nabla^E & 0 \\ 0 & \nabla^E \end{pmatrix}$ and a superconnection \mathbb{E} on \mathcal{H}'' by $\mathbb{E} = \begin{pmatrix} \nabla^E & I \\ I & \nabla^E \end{pmatrix}$. Due to cancellations, $\eta(\mathbb{B} \oplus \nabla^{\widetilde{E}}, \widehat{\mathbb{B}}') = \eta(\nabla^{\mathcal{H}''}, \mathbb{E})$.

Next,

$$\eta((I - P)\widehat{\mathbb{B}}(I - P), \infty) = \eta((I - Q)\mathbb{A}(I - Q), \infty) + \eta(\mathbb{E}, \infty). \tag{4.39}$$

As in the proof of [Lemma 13](#),

$$\eta(\nabla^{\mathcal{H}''}, \mathbb{E}) + \eta(\mathbb{E}, \infty) = 0. \tag{4.40}$$

The proposition follows. \square

4.4. Proof of Theorem 1

We now give the proof of [Theorem 1](#), in the following precise form.

Proposition 6. *The map $q : \check{K}^0(M) \rightarrow \check{K}_{stan}^0(M)$ is an isomorphism.*

Proof. Let $e = [F, \nabla^F, \omega]$ be a cocycle for $\check{K}_{stan}^0(M)$, where F is a finite dimensional \mathbb{Z}_2 -graded Hermitian vector bundle on M and ∇^F is a compatible connection on F . We can also consider e to be a cocycle for $\check{K}^0(M)$. Let $r(e)$ denote this cocycle for $\check{K}^0(M)$. Then $r : \check{K}_{stan}^0(M) \rightarrow \check{K}^0(M)$ is well defined, because of [Definition 5](#) and [Lemma 13](#).

Given a cocycle $c = [\mathcal{H}, \mathbb{A}, \omega]$ for $\check{K}^0(M)$, the cocycle $r(q(c))$ amounts to considering $q(c)$ as a cocycle for $\check{K}^0(M)$. Equation (4.16) shows that $r(q(c))$ is equivalent to c in $\check{K}^0(M)$.

Given a cocycle e for $\check{K}_{stan}^0(M)$, applying [Proposition 5](#) to $r(e)$, with $Q = \text{Id}$, shows that $q(r(e))$ is equivalent to e in $\check{K}_{stan}^0(M)$. (Note that in this application, all of the vector bundles in the proof of [Proposition 5](#) are finite dimensional.) This proves the proposition. \square

Remark 2. The isomorphism in [Proposition 6](#) is compatible with the structures in differential cohomology theory.

4.5. Multiplicative structure

Let $[\mathcal{H}_1, \mathbb{A}_1, \omega_1]$ and $[\mathcal{H}_2, \mathbb{A}_2, \omega_2]$ be two cocycles for $\check{K}^0(M)$. Let γ_1 be the \mathbb{Z}_2 -grading operator for \mathcal{H}_1 . Put

$$\begin{aligned} \mathcal{H} &= \mathcal{H}_1 \otimes \mathcal{H}_2, \\ \mathbb{A} &= (\mathbb{A}_1 \otimes I) + (\gamma_1 \otimes A_2), \\ \omega &= (\text{Ch}(\mathbb{A}_1) \wedge \omega_2) + (\omega_1 \wedge \text{Ch}(A_2)) + (\omega_1 \wedge d\omega_2). \end{aligned} \tag{4.41}$$

Proposition 7. *The map that sends $[\mathcal{H}_1, \mathbb{A}_1, \omega_1], [\mathcal{H}_2, \mathbb{A}_2, \omega_2]$ to $[\mathcal{H}, \mathbb{A}, \omega]$ passes to a map $m : \check{K}^0(M) \times \check{K}^0(M) \rightarrow \check{K}^0(M)$.*

Proof. First, the isomorphism class of $[\mathcal{H}, \mathbb{A}, \omega]$ only depends on the isomorphism classes of $[\mathcal{H}_1, \mathbb{A}_1, \omega_1]$ and $[\mathcal{H}_2, \mathbb{A}_2, \omega_2]$. We will check that the relations on the first factor $[\mathcal{H}_1, \mathbb{A}_1, \omega_1]$ pass to relations on $[\mathcal{H}, \mathbb{A}, \omega]$. The argument for the second factor is similar.

Relation (1) of [Definition 6](#) is clearly compatible. For relation (2) of [Definition 6](#), suppose that $\mathbb{A}_{1,[0]}$ is invertible. Since

$$\mathbb{A}_{[0]}^2 = \mathbb{A}_{1,[0]}^2 + \mathbb{A}_{2,[0]}^2, \tag{4.42}$$

it follows that $\mathbb{A}_{[0]}$ is invertible. It suffices to show that

$$\begin{aligned} \eta(\mathbb{A}, \infty) + (\text{Ch}(\mathbb{A}_1) \wedge \omega_2) + (\omega_1 \wedge \text{Ch}(A_2)) + (\omega_1 \wedge d\omega_2) = \\ (\omega_1 + \eta(\mathbb{A}_1, \infty)) \wedge (\text{Ch}(A_2) + d\omega_2) \end{aligned} \tag{4.43}$$

in $\Omega^{odd}(M)/\text{Im}(d)$ or, equivalently, that

$$\eta(\mathbb{A}, \infty) = \eta(\mathbb{A}_1, \infty) \wedge \text{Ch}(A_2) \tag{4.44}$$

in $\Omega^{odd}(M)/\text{Im}(d)$. Put

$$\widehat{\mathbb{A}}(t) = ((\mathbb{A}_1)_t \otimes \text{Id}) + (\gamma_1 \otimes A_2). \tag{4.45}$$

Then

$$\text{Str} \left(\frac{d\widehat{\mathbb{A}}(t)}{dt} e^{-\widehat{\mathbb{A}}^2(t)} \right) = \text{Str} \left(\frac{d(\mathbb{A}_1)_t}{dt} e^{-(\mathbb{A}_1)_t^2} \right) \wedge \text{Ch}(A_2). \tag{4.46}$$

For $s \in [0, 1]$, put

$$\mathbb{B}(s, t) = s\mathbb{A}_t + (1 - s)\widehat{\mathbb{A}}(t). \tag{4.47}$$

Then

$$\mathbb{B}(s, t)_{[0]} = t(\mathbb{A}_{1,[0]} \otimes \text{Id}) + (st + 1 - s)(\gamma_1 \otimes A_{2,[0]}) \tag{4.48}$$

and

$$\mathbb{B}(s, t)_{[0]}^2 = (t^2 \mathbb{A}_{1,[0]}^2 \otimes \text{Id}) + (st + 1 - s)^2 (\text{Id} \otimes \mathbb{A}_{2,[0]}^2) \geq t^2 \mathbb{A}_{1,[0]}^2 \otimes \text{Id}. \tag{4.49}$$

Using the strict positivity of $\mathbb{A}_{1,[0]}^2$, we can apply a homotopy argument as in the proof of Lemma 12. Since $\widehat{\mathbb{A}}(1) = \mathbb{A}$, it follows that $\mathbb{B}(s, 1) = \mathbb{A}$ for all $s \in [0, 1]$. Then the analog of the term $\int_0^1 \text{Ch}(\mathbb{E})$ in Lemma 12 vanishes, so

$$\eta(\mathbb{A}, \infty) = \int_1^\infty \text{Tr}_s \left(\frac{d\widehat{\mathbb{A}}(t)}{dt} e^{-\widehat{\mathbb{A}}^2(t)} \right) dt = \eta(\mathbb{A}_1, \infty) \wedge \text{Ch}(\mathbb{A}_2) \tag{4.50}$$

in $\Omega^{odd}(M)/\text{Im}(d)$. This proves (4.44).

For relation (3) of Definition 6, suppose that \mathbb{A}'_1 is another superconnection on \mathcal{H}_1 with $\mathbb{A}'_{1,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0(\mathcal{H}_1))$. The new product superconnection \mathbb{A}' on \mathcal{H} satisfies

$$\mathbb{A}' - \mathbb{A} = (\mathbb{A}'_1 - \mathbb{A}_1) \otimes \text{Id} \tag{4.51}$$

and hence

$$\mathbb{A}'_{[0]} - \mathbb{A}_{[0]} = (\mathbb{A}'_{1,[0]} - \mathbb{A}_{1,[0]}) \otimes \text{Id} \in op^0(\mathcal{H}). \tag{4.52}$$

It suffices to show that

$$\begin{aligned} \text{Ch}(\mathbb{A}_1) \wedge \omega_2 &= (\text{Ch}(\mathbb{A}'_1) \wedge \omega_2) + (\eta(\mathbb{A}_1, \mathbb{A}'_1) \wedge \text{Ch}(\mathbb{A}_2)) + \\ &\quad (\eta(\mathbb{A}_1, \mathbb{A}'_1) \wedge d\omega_2) + \eta(\mathbb{A}', \mathbb{A}) \end{aligned} \tag{4.53}$$

in $\Omega^{odd}(M)/\text{Im}(d)$ or, equivalently,

$$\eta(\mathbb{A}, \mathbb{A}') = \eta(\mathbb{A}_1, \mathbb{A}'_1) \wedge \text{Ch}(\mathbb{A}_2) \tag{4.54}$$

in $\Omega^{odd}(M)/\text{Im}(d)$. For $t \in [0, 1]$, put

$$\mathbb{A}(t) = (1 - t)\mathbb{A} + t\mathbb{A}' \tag{4.55}$$

and

$$\mathbb{A}_1(t) = (1 - t)\mathbb{A}_1 + t\mathbb{A}'_1. \tag{4.56}$$

Then

$$\mathbb{A}^2(t) = (\mathbb{A}_1^2(t) \otimes \text{Id}) + (\text{Id} \otimes \mathbb{A}_2^2) \tag{4.57}$$

and

$$\frac{d\mathbb{A}(t)}{dt} = \frac{d\mathbb{A}_1(t)}{dt} \otimes \text{Id}, \tag{4.58}$$

from which (4.54) follows. \square

Remark 3. It is straightforward to see that m is unital and associative. That it is commutative can be shown using the graded isomorphism $\mathcal{H}_2 \otimes \mathcal{H}_1 \rightarrow \mathcal{H}_1 \otimes \mathcal{H}_2$ that sends $v_2 \otimes v_1$ to $(-1)^{\text{sign}(v_1) \text{sign}(v_2)} v_1 \otimes v_2$.

5. Pushforward

Given a fiber bundle $\pi : M \rightarrow B$ with even dimensional compact fibers and a Riemannian structure, in this section we construct a pushforward $\pi_* : \check{K}_{fin}^0(M) \rightarrow \check{K}^0(B)$. Here $\check{K}_{fin}^0(M)$ denotes the differential K -theory constructed using finite dimensional Hermitian vector bundles with compatible superconnections. As $\check{K}_{fin}^0(M)$ is isomorphic to $\check{K}^0(M)$, to define the pushforward on $\check{K}^0(M)$ it suffices to just define the pushforward on $\check{K}_{fin}^0(M)$.

In Subsection 5.1 we define the pushforward on cocycles. In Subsection 5.2 we show that it passes to a pushforward on the differential K -groups. Subsection 5.3 has the proof that π_* coincides with the analytic index of [18].

One could consider defining the pushforward directly on cocycles for $\check{K}^0(M)$, instead of just on cocycles for $\check{K}_{fin}^0(M)$. This may be possible but there are some technical issues; see Remark 4.

5.1. Pushforward on cocycles

Let $\check{K}_{fin}^0(M)$ denote the group formed by only allowing finite dimensional Hilbert bundles in the generators and relations of Definition 6. The proof of Proposition 6, when restricted to finite dimensional Hilbert bundles, shows that $\check{K}_{fin}^0(M)$ is isomorphic to $\check{K}_{stan}^0(M)$. (The distinction between $\check{K}_{fin}^0(M)$ and $\check{K}_{stan}^0(M)$ is that cocycles for the former involve superconnections, whereas cocycles for the latter involve connections.) From (4.16), any cocycle for $\check{K}^0(M)$ is equivalent to a cocycle for $\check{K}_{fin}^0(M)$.

Let $\pi : M \rightarrow B$ be a fiber bundle with even dimensional compact fibers. We assume that π has a Riemannian structure in the sense of [18, Section 3.1]. This means that we are given an inner product $g^{T^V M}$ on the vertical tangent bundle $T^V M = \text{Ker}(d\pi)$ and a horizontal distribution $T^H M$ on M . Let $\mathcal{D}^V M$ denote the vertical density bundle on M . We assume that π is spin^c -oriented in the sense that the vertical tangent bundle $T^V M$ on M has a spin^c -structure, with characteristic Hermitian line bundle $L^V M \rightarrow M$. We also assume that π has a differential spin^c -structure in the sense of [18, Section 3.1], meaning that $L^V M$ is equipped with a unitary connection. Let $S^V M$ denote the vertical spinor bundle on M . The connections on $T^V M$ and $L^V M$ induce a connection $\widehat{\nabla}^{T^V M}$ on $S^V M$.

There is a degenerate Clifford module $C_0(M)$ generated by T^*M , with a Clifford action m on $\pi^*T^*B \otimes S^V M$ [3, Section 10.2]. Note that we can identify $C_0(M)$ and $\Lambda^*(T^*M)$ as vector bundles on M . Let \mathbb{B} denote the Bismut superconnection acting on $C^\infty(M; S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}})$.

Let $[E, \mathbb{A}, \omega]$ be a cocycle for $\check{K}_{fin}^0(M)$. We define the push-forward $\pi_*[E, \mathbb{A}, \omega]$ to be the cocycle $[\mathcal{H}, \pi_*\mathbb{A}, \omega']$ where the definition of the terms is as follows. First, \mathcal{H} is the Hilbert bundle whose fiber over $b \in B$ is

$$\mathcal{H}_b = L^2 \left(\pi^{-1}(b); (E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}}) \Big|_{\pi^{-1}(b)} \right). \tag{5.1}$$

The operator D_b on \mathcal{H}_b is the Dirac-type operator. Then

$$\mathcal{H}_b^\infty = C^\infty \left(\pi^{-1}(b); (E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}}) \Big|_{\pi^{-1}(b)} \right) \tag{5.2}$$

and

$$C^\infty(B; \mathcal{H}^\infty) = C^\infty(M; E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}}). \tag{5.3}$$

We use the identification

$$\begin{aligned} C^\infty(M; E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}}) = \\ C^\infty(M; E) \otimes_{C^\infty(M)} C^\infty(M; S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}}). \end{aligned} \tag{5.4}$$

Given $\xi \in C^\infty(M; E)$, write

$$\mathbb{A}\xi = \sum_i \xi_i \otimes \omega_i \in C^\infty(M; E \otimes \Lambda^*(T^*M)), \tag{5.5}$$

a locally finite sum on M , where $\xi_i \in C^\infty(M; E)$ and $\omega_i \in C^\infty(M; \Lambda^*(T^*M))$. With $s \in C^\infty(M; S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}})$, let $m(\mathbb{A} \otimes 1)$ denote the operator that sends $\xi \otimes s$ to

$$\sum_i \xi_i \otimes m(\omega_i)s \in C^\infty(M; E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}} \otimes \pi^*T^*B). \tag{5.6}$$

Acting on $C^\infty(M; E \otimes S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}})$ and using (5.4), we define

$$\pi_*\mathbb{A} = m(\mathbb{A} \otimes \text{Id}) + \text{Id} \otimes \mathbb{B}. \tag{5.7}$$

This is well-defined since one can check, for example, that if $\xi \in C^\infty(M; E)$, $s \in C^\infty(M; S^V M \otimes (\mathcal{D}^V M)^{\frac{1}{2}})$ and $f \in C^\infty(M)$ then

$$(\pi_*\mathbb{A})(\xi f \otimes s) = (\pi_*\mathbb{A})(\xi \otimes fs). \tag{5.8}$$

Let \mathcal{R}_t denote the rescaling operator on superconnections, i.e. $\mathcal{R}_t\mathbb{A} = \mathbb{A}_t$ where the right-hand side is defined in (3.17). Put

$$(\pi_*\mathbb{A})_u = \mathcal{R}_u(m(\mathcal{R}_{u^{-1}}\mathbb{A} \otimes \text{Id}) + \text{Id} \otimes \mathbb{B}). \tag{5.9}$$

From [23, Theorem 5.41], the limit $\lim_{u \rightarrow 0} \eta((\pi_* \mathbb{A})_u, \pi_* \mathbb{A}) \in \Omega^{odd}(B)/\text{Im}(d)$ exists; see also [5, Appendix 1]. Denote the limit by $\eta((\pi_* \mathbb{A})_0, \pi_* \mathbb{A})$. With the characteristic form $\text{Todd}(\widehat{\nabla}^{TV} M)$ from [18, Section 2.1], we put

$$\omega' = \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV} M) \wedge \omega + \eta((\pi_* \mathbb{A})_0, \pi_* \mathbb{A}). \tag{5.10}$$

Definition 7.

$$\pi_* [E, \mathbb{A}, \omega] = [\mathcal{H}, \pi_* \mathbb{A}, \omega']. \tag{5.11}$$

5.2. *Pushforward on differential K-theory*

In this subsection we show that the pushforward π_* is well defined on $\check{K}_{fin}^0(M)$. We begin with a lemma.

Lemma 14. *Suppose that $\{\mathbb{A}(s)\}_{s \in [0,1]}$ is a smooth 1-parameter family of finite dimensional superconnections on E . Then*

$$\lim_{u \rightarrow 0} \eta((\pi_* \mathbb{A}(1))_u, (\pi_* \mathbb{A}(0))_u) = \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV} M) \wedge \eta(\mathbb{A}(1), \mathbb{A}(0)). \tag{5.12}$$

Proof. Put $\mathbb{E} = ds \wedge \partial_s + \mathbb{A}(s)$ and $\pi_* \mathbb{E} = ds \wedge \partial_s + \pi_* \mathbb{A}(s)$. Then

$$\lim_{u \rightarrow 0} \eta((\pi_* \mathbb{A}(1))_u, (\pi_* \mathbb{A}(0))_u) = \lim_{u \rightarrow 0} \int_0^1 \text{Ch}((\pi_* \mathbb{E})_u) \tag{5.13}$$

and

$$\begin{aligned} & \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV} M) \wedge \eta(\mathbb{A}(1), \mathbb{A}(0)) = \\ & \int_0^1 \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV} M) \wedge \text{Ch}(\mathbb{E}). \end{aligned} \tag{5.14}$$

From [23, Theorem 5.33],

$$\lim_{u \rightarrow 0} \text{Ch}((\pi_* \mathbb{E})_u) = \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV} M) \wedge \text{Ch}(\mathbb{E}) \tag{5.15}$$

uniformly on $[0, 1] \times M$; see also [6, (2.127)] and [7, Proposition 11.7]. The lemma follows. \square

Proposition 8. *The map π_* , as defined on cocycles for $\check{K}_{fin}^0(M)$, passes to a map $\pi_* : \check{K}_{fin}^0(M) \rightarrow \check{K}^0(B)$.*

Proof. It is clear that relation (1), in Definition 6 for $\check{K}_{fin}^0(M)$, passes through π_* .

For relation (3), suppose that $[E, \mathbb{A}, \omega]$ is a cocycle for $\check{K}_{fin}^0(M)$ with $\mathbb{A}_{[0]}$ invertible. We must show that $\pi_* [E, \mathbb{A}, \omega]$ is equivalent to

$$\pi_* [0, 0, \omega + \eta(\mathbb{A}, \infty)] = \left[0, 0, \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge (\omega + \eta(\mathbb{A}, \infty)) \right]. \tag{5.16}$$

Equivalently, letting $F(\mathbb{A}) \in \check{K}^0(B)$ be the class represented by the cocycle

$$\left[\mathcal{H}, \pi_* \mathbb{A}, \eta((\pi_* \mathbb{A})_0, \pi_* \mathbb{A}) - \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \eta(\mathbb{A}, \infty) \right], \tag{5.17}$$

we must show that $F(\mathbb{A})$ vanishes.

Lemma 15. *Suppose that $\{\mathbb{A}(s)\}_{s \in [0,1]}$ is a smooth 1-parameter family of finite dimensional superconnections on M , with $\mathbb{A}(0)_{[0]}$ and $\mathbb{A}(1)_{[0]}$ invertible. Then $F(\mathbb{A}(0)) = F(\mathbb{A}(1))$.*

Proof. Using Lemmas 10 and 12, one finds that

$$F(\mathbb{A}(1)) - F(\mathbb{A}(0)) = \left[0, 0, \lim_{u \rightarrow 0} \eta((\pi_* \mathbb{A}(1))_u, (\pi_* \mathbb{A}(0))_u) - \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \eta(\mathbb{A}(1), \mathbb{A}(0)) \right]. \tag{5.18}$$

The lemma now follows from Lemma 14. \square

Lemma 16. *For $v > 0$, put $\mathbb{A}(v) = (v - 1)\mathbb{A}_{[0]} + \mathbb{A}$. Then for all $b \in B$, for sufficiently large v the operator $(\pi_* \mathbb{A}(v))_{[0]}$ is invertible on \mathcal{H}_b .*

Proof. Without loss of generality, suppose that B is a point. Writing

$$\mathbb{A} = \mathbb{A}_{[0]} + \mathbb{A}_{[1]} + X, \tag{5.19}$$

with $X \in C^\infty(M; \text{End}(E) \otimes \Lambda^{\geq 2}(T^*M))$, we have

$$(\pi_* \mathbb{A}(v))_{[0]} = v(\mathbb{A}_{[0]} \otimes \text{Id}) + D_{\mathbb{A}_{[1]}} + m(X \otimes \text{Id}), \tag{5.20}$$

where $D_{\mathbb{A}_{[1]}}$ denotes the Dirac operator on M coupled to the connection $A_{[1]}$ on E . Using the fact that $\mathbb{A}_{[0]}$ anticommutes with γ_E , while $D_{\mathbb{A}_{[1]}}$ commutes with γ_E , it follows that

$$\begin{aligned}
 (\pi_*\mathbb{A}(v))_{[0]}^2 &= v^2(\mathbb{A}_{[0]}^2 \otimes \text{Id}) + vm([\mathbb{A}_{[1]}, \mathbb{A}_{[0]}] \otimes \text{Id}) + \\
 &vm([\mathbb{A}_{[0]}, X] \otimes \text{Id}) + (D_{\mathbb{A}_{[1]}} + m(X \otimes \text{Id}))^2.
 \end{aligned}
 \tag{5.21}$$

Since $\mathbb{A}_{[0]}^2$ is strictly positive, if v is sufficiently large then $(\pi_*\mathbb{A}(v))_{[0]}^2$ is strictly positive. This proves the lemma. \square

By [Lemmas 15 and 16](#), without loss of generality we can assume that $(\pi_*\mathbb{A})_{[0]}$ is invertible. Then the cocycle in [\(5.17\)](#) is equivalent to

$$\left[0, 0, \eta((\pi_*\mathbb{A})_0, \pi_*\mathbb{A}) + \eta(\pi_*\mathbb{A}, \infty) - \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \eta(\mathbb{A}, \infty) \right].
 \tag{5.22}$$

By adiabatic techniques for eta forms, [\(5.22\)](#) vanishes. The proof of this is similar to the proofs in [\[10, Theorem 5.10 and Section 6\]](#) and [\[26\]](#), which deal with adiabatic limits in the more difficult case of a double fibration, i.e. when E is itself the pushforward of a vector bundle with connection. We omit the details.

Finally, to show that relation [\(3\)](#) passes through π_* , suppose that \mathbb{A}_0 and \mathbb{A}_1 are two superconnections on E . Then

$$\begin{aligned}
 \pi_* [E, \mathbb{A}_1, \omega + \eta(\mathbb{A}_0, \mathbb{A}_1)] - \pi_* [E, \mathbb{A}_0, \omega] &= \\
 \left[\mathcal{H}, \pi_*\mathbb{A}_1, \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge (\omega + \eta(\mathbb{A}_0, \mathbb{A}_1)) + \eta((\pi_*\mathbb{A}_1)_0, \pi_*\mathbb{A}_1) \right] - \\
 \left[\mathcal{H}, \pi_*\mathbb{A}_0, \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \omega + \eta((\pi_*\mathbb{A}_0)_0, \pi_*\mathbb{A}_0) \right] &= \\
 \left[0, 0, \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \eta(\mathbb{A}_0, \mathbb{A}_1) - \lim_{u \rightarrow 0} \eta((\pi_*\mathbb{A}_0)_u, (\pi_*\mathbb{A}_1)_u) \right].
 \end{aligned}
 \tag{5.23}$$

This vanishes from [Lemma 14](#). \square

Remark 4. If we start with a cocycle $[E, \mathbb{A}, \omega]$ for $\check{K}^0(M)$, i.e. an infinite dimensional cocycle, then we could consider defining its pushforward $[\mathcal{H}, \pi_*\mathbb{A}, \omega]$ as in [Definition 7](#). The definition of \mathcal{H} , as [\(5.1\)](#), still makes perfect sense. The formal definition of $\pi_*\mathbb{A}$ is the same as [\(5.7\)](#). However, there is the technical point that we want $(\pi_*\mathbb{A})_{[0]}$ to be

θ -summable for all $\theta > 0$. To analyze this requirement, let us assume that B is a point. Then

$$\pi_*\mathbb{A} = \mathbb{A}_{[0]} + D_{\mathbb{A}_{[1]}} + \sum_{i \geq 2} c(\mathbb{A}_{[i]}), \tag{5.24}$$

where $D_{\mathbb{A}_{[1]}}$ is the Dirac-type operator on $C^\infty\left(M; E \otimes SM \otimes (DM)^{\frac{1}{2}}\right)$ as constructed using the connection $\mathbb{A}_{[1]}$ on \mathbb{E} , and c denotes the Clifford action on SM . As

$$\begin{aligned} (\pi_*\mathbb{A})^2 = & \mathbb{A}_{[0]}^2 + c(\nabla\mathbb{A}_{[0]}) + D_{\mathbb{A}_{[1]}}^2 + \\ & \sum_{i \geq 2} c([\mathbb{A}_{[0]}, \mathbb{A}_{[i]}]) + \sum_{i \geq 2} c(\nabla\mathbb{A}_{[i]}) + \left(\sum_{i \geq 2} c(\mathbb{A}_{[i]}) \right)^2, \end{aligned} \tag{5.25}$$

in order for $\pi_*\mathbb{A}$ to be θ -summable it is reasonable to assume that $\nabla\mathbb{A}_{[0]}$ is 1st-order, and if $i \geq 2$ then $\mathbb{A}_{[i]}$ and $\nabla\mathbb{A}_{[i]}$ are 0th-order. (That is, to require that $\nabla\mathbb{A}_{[0]} \in C^\infty(M; T^*M \otimes op^1)$, and for $i \geq 2$ that $\mathbb{A}_{[i]} \in C^\infty(M; \Lambda^i(T^*M) \otimes op^0)$ and $\nabla\mathbb{A}_{[i]} \in C^\infty(M; T^*M \otimes \Lambda^i(T^*M) \otimes op^0)$.)

Returning to the case of general B , even with such additional assumptions on \mathbb{A} , it remains to show that $\eta((\pi_*\mathbb{A})_0, \pi_*\mathbb{A})$ is well defined, i.e. that the analog of [23, Theorem 5.41] holds for infinite dimensional bundles. One can do all this for superconnections \mathbb{A} arising from geometric families in the sense of [12], which gives reason to believe that it can be done more generally.

5.3. Relation with the analytic index

Proposition 9. *Under the isomorphisms $\check{K}^0 \cong \check{K}_{fin}^0 \cong \check{K}_{stan}^0$, the map $\pi_* : \check{K}_{fin}^0(M) \rightarrow \check{K}^0(B)$ coincides with the analytic index $\text{Ind}^{an} : \check{K}_{stan}^0(M) \rightarrow \check{K}_{stan}^0(B)$ of [18, Definition 3.12].*

Proof. Suppose first that $\text{Ker}((\pi_*\mathbb{A})_{[0]})$ is a $(\mathbb{Z}_2$ -graded) vector bundle on B . Let Q be orthogonal projection onto $\text{Ker}((\pi_*\mathbb{A})_{[0]})$. For $T > 0$, put

$$\mathbb{E}_T = (I - Q)(\pi_*\mathbb{A})_T(I - Q) + Q(\pi_*\mathbb{A})_{[1]}Q. \tag{5.26}$$

By Proposition 5 and Definition 7,

$$\begin{aligned} \pi_* [E, \mathbb{A}, \omega] = & \left[\text{Ker}(\mathbb{A}_{[0]}), Q(\pi_*\mathbb{A})_{[1]}Q, \int_{M/B} \text{Todd}(\widehat{\nabla}^{T^V M}) \wedge \omega + \right. \\ & \eta((\pi_*\mathbb{A})_0, \pi_*\mathbb{A}) + \eta(\pi_*\mathbb{A}, \mathbb{E}_1) + \\ & \left. \eta((I - Q)\pi_*\mathbb{A}(I - Q), \infty) \right] \end{aligned} \tag{5.27}$$

$$\begin{aligned}
 &= \left[\text{Ker}(\mathbb{A}_{[0]}), Q(\pi_*\mathbb{A})_{[1]}Q, \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV}M) \wedge \omega + \right. \\
 &\quad \left. \eta((\pi_*\mathbb{A})_0, (\pi_*\mathbb{A})_T) + \eta((\pi_*\mathbb{A})_T, \mathbb{E}_T) + \right. \\
 &\quad \left. \eta((I - Q)(\pi_*\mathbb{A})_T(I - Q), \infty) \right].
 \end{aligned}$$

From Lemma 8,

$$\eta((\pi_*\mathbb{A})_0, (\pi_*\mathbb{A})_T) = \lim_{u \rightarrow 0} \int_u^T \text{Str} \left(\frac{d(\pi_*\mathbb{A})_t}{dt} e^{- (\pi_*\mathbb{A})_t^2} \right) dt. \tag{5.28}$$

From [3, Theorem 9.23], the limit $\lim_{T \rightarrow \infty} \eta((\pi_*\mathbb{A})_0, (\pi_*\mathbb{A})_T)$ exists; it is called $\tilde{\eta}$ in [18], after the Bismut–Cheeger eta form [5]. Using the estimates in [3, Section 9.3], we have $\lim_{T \rightarrow \infty} \eta((\pi_*\mathbb{A})_T, \mathbb{E}_T) = 0$ and $\lim_{T \rightarrow \infty} \eta((I - Q)(\pi_*\mathbb{A})_T(I - Q), \infty) = 0$. Thus

$$\pi_* [E, \mathbb{A}, \omega] = \left[\text{Ker}(\mathbb{A}_{[0]}), Q(\pi_*\mathbb{A})_{[1]}Q, \int_{M/B} \text{Todd}(\widehat{\nabla}^{TV}M) \wedge \omega + \tilde{\eta} \right], \tag{5.29}$$

which is the same as [18, Definition 3.12]. If $\text{Ker}((\pi_*\mathbb{A})_{[0]})$ is not a vector bundle then we can effectively deform to that case, as in Subsection 4.2 and [18, Section 7.12]. \square

6. Odd differential K-groups

In this section we indicate how the results of the previous sections extend to the odd differential K -group $\check{K}^1(\cdot)$. Some of the arguments are similar to those of the previous sections, so we do not write them out in detail. For this reason, we label the results of this section as “claims”.

We use the odd Chern characters of Quillen [32, Section 5]. Let M be a smooth manifold. Let \mathcal{H} be a Hilbert bundle over M as in Subsection 3.1, except ungraded. A superconnection \mathbb{A} on \mathcal{H} is defined as in Definition 3, except removing the oddness condition on $\mathbb{A}_{[0]}$ and the parity condition on $\mathbb{A}_{[i]}$. Note that the only grading on $\Omega^*(M; op^*)$ is the one coming from $\Omega^*(M)$. Let σ be a new formal odd variable with $\sigma^2 = 1$. Put

$$\mathbb{A}_\sigma = \sigma\mathbb{A}_{[0]} + \mathbb{A}_{[1]} + \sigma\mathbb{A}_{[2]} + \dots, \tag{6.1}$$

so that \mathbb{A}_σ has odd total parity. Define $\text{Tr}_\sigma(A + B\sigma) = \text{Tr}(B)$ and

$$\text{Ch}(\mathbb{A}) = \text{Tr}_\sigma \left(e^{-\mathbb{A}_\sigma^2} \right) \in \Omega^{odd}(M). \tag{6.2}$$

Define the eta forms similarly to (3.10) and (3.18), as elements of $\Omega^{even}(M)/\text{Im}(d)$, using Tr_σ instead of Tr_s .

Definition 8. A cocycle for $\check{K}^1(M)$ is a triple $[\mathcal{H}, \mathbb{A}, \omega]$ where

- (1) \mathcal{H} is an ungraded Hilbert bundle over M ,
- (2) \mathbb{A} is a superconnection on \mathcal{H} and
- (3) $\omega \in \Omega^{even}(M)/\text{Im}(d)$.

Definition 9. The group $\check{K}^1(M)$ is the quotient of the free abelian group generated by the isomorphism classes of cocycles, by the subgroup generated by the following relations:

- (1) If $[\mathcal{H}, \mathbb{A}, \omega]$ and $[\mathcal{H}', \mathbb{A}', \omega']$ are cocycles then

$$[\mathcal{H}, \mathbb{A}, \omega] + [\mathcal{H}', \mathbb{A}', \omega'] = [\mathcal{H} \oplus \mathcal{H}', \mathbb{A} \oplus \mathbb{A}', \omega + \omega']. \tag{6.3}$$

- (2) If $\mathbb{A}_{[0]}$ is invertible then

$$[\mathcal{H}, \mathbb{A}, \omega] = [0, 0, \omega + \eta(\mathbb{A}, \infty)]. \tag{6.4}$$

- (3) Suppose that \mathbb{A}_0 and \mathbb{A}_1 are superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0)$. Then

$$[\mathcal{H}, \mathbb{A}_0, \omega] = [\mathcal{H}, \mathbb{A}_1, \omega + \eta(\mathbb{A}_0, \mathbb{A}_1)]. \tag{6.5}$$

It follows from the definitions that there is a map $\check{K}^1(M) \rightarrow \Omega^{odd}(M)$ that sends a cocycle $[\mathcal{H}, \mathbb{A}, \omega]$ to $\text{Ch}(\mathbb{A}) + d\omega$.

Let $\pi : M \rightarrow B$ be a fiber bundle with odd dimensional compact fibers. As in Section 5, we assume that π has a Riemannian structure and a differential $spin^c$ -structure. Given a cocycle $[E, \mathbb{A}, \omega]$ for $\check{K}_{fin}^0(M)$, we define the pushforward $\pi_*[E, \mathbb{A}, \omega] = [\mathcal{H}, \pi_*\mathbb{A}, \omega']$ as in Definition 7, where \mathcal{H} is now ungraded.

Claim 1. *The map π_* on cocycles passes to a map $\pi_* : \check{K}_{fin}^0(M) \rightarrow \check{K}^1(B)$.*

Proof. The proof is similar to that of Proposition 8. \square

Let $p_1 : S^1 \times M \rightarrow S^1$ and $p : S^1 \times M \rightarrow M$ be the projection maps. We define a suspension map p^1 on cocycles for $\check{K}^1(M)$ as follows. Consider the product bundle $q : S^1 \times S^1 \rightarrow S^1$. Give it a product Riemannian structure with circle fibers of constant length. Let (V, ∇^V) be the Hermitian line bundle on $S^1 \times S^1$ with connection of constant curvature, whose restriction to $q^{-1}(e^{i\theta})$ is the flat bundle on S^1 with holonomy $e^{i\theta}$. The sections of V form a Hilbert bundle \mathcal{K} over S^1 , on whose sections the Bismut superconnection acts. Trivializing V over $(0, 2\pi) \times S^1$ and giving coordinates (θ, ϕ) to the latter, we can write this Bismut superconnection on $\mathcal{K} \Big|_{(0, 2\pi)}$ in the form $(\partial_\phi - \frac{i}{2\pi}\theta) + d\theta \wedge \partial_\theta$. Let $[\mathcal{H}, \mathbb{A}, \omega]$ be a cocycle for $\check{K}^1(M)$. Put

$$\begin{aligned}
 \mathcal{H}' &= p^* \mathcal{H} \otimes (p_1^* \mathcal{K} \oplus p_1^* \mathcal{K}), \\
 \gamma_{\mathcal{H}'} &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \\
 \sigma &= \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \\
 \mathbb{A}' &= \begin{pmatrix} d\theta \wedge \partial_\theta & \partial_\phi - \frac{i}{2\pi} \theta \\ -\partial_\phi + \frac{i}{2\pi} \theta & d\theta \wedge \partial_\theta \end{pmatrix} + \mathbb{A}_\sigma, \\
 \omega' &= p_1^*(d\theta) \wedge p^* \omega.
 \end{aligned}
 \tag{6.6}$$

Then $p^![\mathcal{H}, \mathbb{A}, \omega] = [\mathcal{H}', \mathbb{A}', \omega']$, a cocycle for $\check{K}^0(S^1 \times M)$.

Claim 2. *The map $p^!$ on cocycles passes to a map $p^! : \check{K}^1(M) \rightarrow \check{K}^0(S^1 \times M)$.*

Let $K^1(M)$ denote the group defined by similar generators and relations as our definition of $\check{K}^1(M)$, except leaving out the differential form components and the parts of the superconnection beyond $\mathbb{A}_{[0]}$. As in our constructions on differential K-groups, there are maps $p_* : K^0(S^1 \times M) \rightarrow K^1(M)$ and $p^! : K^1(M) \rightarrow K^0(S^1 \times M)$. As in [Proposition 6](#), $K^0(\cdot) \cong K_{stan}^0(\cdot)$ and so

$$\begin{aligned}
 K^0(S^1 \times M) &\cong K_{stan}^0(S^1 \times M) \cong K_{stan}^0(M) \oplus K_{stan}^1(M) \cong \\
 &K^0(M) \oplus K_{stan}^1(M).
 \end{aligned}
 \tag{6.7}$$

Choosing a point $\star \in S^1$ and letting $i : \{\star\} \times M \rightarrow S^1 \times M$ be inclusion, it follows that the group $K^0(S^1 \times M)$ is a direct sum of $K^0(M)$ and $\text{Ker}(i^* : K^0(S^1 \times M) \rightarrow K^0(M))$.

Claim 3. *The composite map $p_* \circ p^!$ is the identity on $K^1(M)$, and $p^! \circ p_*$ is projection from $K^0(S^1 \times M)$ onto $\text{Ker}(i^* : K^0(S^1 \times M) \rightarrow K^0(M))$.*

Claim 4. *The group $K^1(M)$ is isomorphic to the standard K-group $K_{stan}^1(M)$.*

Proof. Using [Claim 3](#),

$$\begin{aligned}
 K^1(M) &\cong \text{Ker}(i^* : K^0(S^1 \times M) \rightarrow K^0(M)) \cong \\
 &\text{Ker}(i^* : K_{stan}^0(S^1 \times M) \rightarrow K_{stan}^0(M)) \cong K_{stan}^1(M).
 \end{aligned}
 \tag{6.8}$$

The claim follows. \square

Let $\Omega^{even}(M)_K$ denote the union of affine subspaces of closed forms whose de Rham cohomology class lies in the image of $\text{Ch} : K^0(M) \rightarrow \Omega^{even}(M)$.

Claim 5. *With our definition of $\check{K}^1(M)$, there is a short exact sequence*

$$0 \rightarrow \frac{\Omega^{even}(M)}{\Omega^{even}(M)_K} \rightarrow \check{K}^1(M) \rightarrow K^1(M) \rightarrow 0. \tag{6.9}$$

Claim 6. *The differential K -group $\check{K}^1(M)$ is isomorphic to the standard differential K -group $\check{K}_{stan}^1(M)$ as defined in [18, Section 9] and [35].*

Proof. There is a short exact sequence

$$0 \rightarrow \frac{\Omega^{even}(M)}{\Omega^{even}(M)_K} \rightarrow \check{K}_{stan}^1(M) \rightarrow K_{stan}^1(M) \rightarrow 0. \tag{6.10}$$

Also, using the desuspension map $D : \check{K}_{stan}^0(S^1 \times M) \rightarrow \check{K}_{stan}^1(M)$ from [18, (9.21)], there is a map

$$\check{K}^1(M) \xrightarrow{p^!} \check{K}^0(S^1 \times M) \cong \check{K}_{stan}^0(S^1 \times M) \xrightarrow{D} \check{K}_{stan}^1(M). \tag{6.11}$$

The claim follows from applying the 5-lemma to the diagram

$$\begin{array}{ccccccc} 0 & \rightarrow & \frac{\Omega^{even}(M)}{\Omega^{even}(M)_K} & \rightarrow & \check{K}^1(M) & \rightarrow & K^1(M) \rightarrow 0 \\ & & \downarrow & & \downarrow & & \downarrow \\ 0 & \rightarrow & \frac{\Omega^{even}(M)}{\Omega^{even}(M)_K} & \rightarrow & \check{K}_{stan}^1(M) & \rightarrow & K_{stan}^1(M) \rightarrow 0, \end{array} \tag{6.12}$$

where the rows are exact and the outer vertical arrows are isomorphisms. \square

Remark 5. One can define multiplications $\check{K}^0(M) \times \check{K}^1(M) \rightarrow \check{K}^1(M)$ and $\check{K}^1(M) \times \check{K}^1(M) \rightarrow \check{K}^0(M)$ in analogy to the multiplication map of Section 4.5.

Remark 6. Definition 9.15 of [18] is missing a homotopy relation; we thank Scott Wilson for pointing this out. Relation (2) on [18, p. 955] should be replaced by a statement that two homotopy equivalent unitary automorphisms are equivalent. Then the corresponding relation on [18, p. 957] will involve the transgressing form CS of [18, p. 956] when applied to homotopy equivalent automorphisms; cf. [35, Appendix A].

7. Twisted differential K -theory

In this section we give the basic definitions for a Hilbert bundle model of twisted differential K -theory, i.e. when differential K -theory is twisted by an element of $H^3(M; \mathbb{Z})$. Subsection 7.1 has a review of abelian gerbes. Subsection 7.2 discusses connective structures on gerbes. Subsection 7.3 recalls twisted de Rham cohomology. In Subsection 7.4 we define superconnections on projective Hilbert bundles. Finally, Subsection 7.5 contains the definition of twisted differential K -theory.

7.1. Gerbes

We first give a summary of facts about abelian gerbes, referring the reader [9, Chapters 4 and 5], [20] and [29] for more details. We will describe abelian gerbes in terms of their descent data.

Let M be a smooth manifold. Let $\{U_\alpha\}_{\alpha \in \Lambda}$ be an open cover of M . Put $U_{\alpha\beta} = U_\alpha \cap U_\beta$, $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$, etc. Any statement about $U_{\alpha\beta}$ will only refer to the case when $U_{\alpha\beta} \neq \emptyset$, and similarly for $U_{\alpha\beta\gamma}$ and $U_{\alpha\beta\gamma\delta}$.

A unitary gerbe \mathcal{L} on M is described by the following data:

- A Hermitian line bundle $\mathcal{L}_{\alpha\beta}$ over each $U_{\alpha\beta}$ and
- An isometric isomorphism $\mu_{\alpha\beta\gamma}: \mathcal{L}_{\alpha\beta}|_{U_{\alpha\beta\gamma}} \otimes \mathcal{L}_{\beta\gamma}|_{U_{\alpha\beta\gamma}} \rightarrow \mathcal{L}_{\alpha\gamma}|_{U_{\alpha\beta\gamma}}$ over each $U_{\alpha\beta\gamma}$ such that
- Over each $U_{\alpha\beta\gamma\delta}$, the following diagram commutes:

$$\begin{array}{ccc}
 \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} \otimes \mathcal{L}_{\gamma\delta} & \xrightarrow{\mu_{\alpha\beta\gamma} \otimes \text{id}} & \mathcal{L}_{\alpha\gamma} \otimes \mathcal{L}_{\gamma\delta} \\
 \text{id} \otimes \mu_{\beta\gamma\delta} \downarrow & & \downarrow \mu_{\alpha\gamma\delta} \\
 \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\delta} & \xrightarrow{\mu_{\alpha\beta\delta}} & \mathcal{L}_{\alpha\delta}.
 \end{array} \tag{7.1}$$

If the cover $\{U_\alpha\}_{\alpha \in \Lambda}$ is good then the bundles $\mathcal{L}_{\alpha\beta}$ are trivializable. After choices of trivializations, the collection $(\mu_{\alpha\beta\gamma})$ can be viewed as a Čech 2-cocycle with coefficients in the sheaf \mathbb{T} of smooth functions with values in $\mathbb{T} = \{z \in \mathbb{C} \mid |z| = 1\}$. This defines a cohomology class $[\mu] \in H^2(M; \mathbb{T}) \cong H^3(M, \mathbb{Z})$.

Given the open cover, there is an evident notion of two gerbes being isomorphic. There is also an evident notion of the tensor product of two gerbes. A *trivial* unitary gerbe has defining line bundle $\mathcal{L}_{\alpha\beta}$ on $U_{\alpha\beta}$ given by $\widehat{\mathcal{L}}_\alpha \otimes \widehat{\mathcal{L}}_\beta^{-1}$, where $\widehat{\mathcal{L}}_\alpha$ is a Hermitian line bundle on U_α . *Stable isomorphism* of unitary gerbes is the equivalence relation generated by isomorphism along with saying that if \mathcal{L} is a unitary gerbe, and \mathcal{L}' is a trivial unitary gerbe, then \mathcal{L} is equivalent to $\mathcal{L} \otimes \mathcal{L}'$. The class $[\mu]$ determines the stable isomorphism class of the gerbe.

7.2. Connective structures on gerbes

Definition 10. A connective structure on \mathcal{L} is given by a collection $(\nabla_{\alpha\beta})$ of Hermitian connections on $(\mathcal{L}_{\alpha\beta})$ such that on $U_{\alpha\beta\gamma}$, we have

$$\mu_{\alpha\beta\gamma}^* \nabla_{\alpha\gamma} = \nabla_{\alpha\beta} \otimes \text{id} + \text{id} \otimes \nabla_{\beta\gamma}. \tag{7.2}$$

Fact 1. A connective structure exists on any gerbe.

Let $\nabla = (\nabla_{\alpha\beta})$ be a connective structure on \mathcal{L} . Put $\kappa_{\alpha\beta} = \nabla_{\alpha\beta}^2$, the curvature of the connection $\nabla_{\alpha\beta}$.

Definition 11. A curving of ∇ is given by a collection K of 2-forms $\kappa_\alpha \in \Omega^2(U_\alpha)$ such that on $U_{\alpha\beta}$, we have

$$\kappa_{\alpha\beta} = \kappa_\alpha - \kappa_\beta. \tag{7.3}$$

Fact 2. Given a gerbe with connective structure ∇ , there is a curving of ∇ .

Given $\tau \in \Omega^2(M)$ and a curving $K = \{\kappa_\alpha\}$ on a gerbe \mathcal{L} with connective structure ∇ , we can define a new curving $K + \tau = \{\kappa_\alpha + \tau\}$. This defines a free transitive action of $\Omega^2(M)$ on the set of all curvings (on a given connective structure).

There is a closed form $c(K) \in \Omega^3(M)$ such that $c(K)|_{U_\alpha} = d\kappa_\alpha$. Note that $c(K + \tau) = c(K) + d\tau$.

If \mathcal{L} is a trivial unitary gerbe with $\mathcal{L}_{\alpha\beta} = \widehat{\mathcal{L}}_\alpha \otimes \widehat{\mathcal{L}}_\beta^{-1}$, and $\widehat{\nabla}_\alpha$ is a Hermitian connection on $\widehat{\mathcal{L}}_\alpha$, then there is a connective structure ∇ on \mathcal{L} given by $\nabla_{\alpha\beta} = \widehat{\nabla}_\alpha \otimes \widehat{\nabla}_\beta^{-1}$. There is a curving K of ∇ coming from $\kappa_\alpha = \nabla_\alpha^2$. It has $c(K) = 0$.

7.3. Twisted cohomology

Let $H \in \Omega^3(M)$ be a closed 3-form. The periodic twisted de Rham complex is the \mathbb{Z}_2 -graded complex $(\Omega^*(M), d_H)$

$$\dots \xrightarrow{d_H} \Omega^{even}(M) \xrightarrow{d_H} \Omega^{odd}(M) \xrightarrow{d_H} \Omega^{even}(M) \xrightarrow{d_H} \dots \tag{7.4}$$

with the differential $d_H = d + H \wedge \cdot$. Its cohomology is called the twisted de Rham cohomology. If $H' = H + d\tau$ then there is an isomorphism of complexes $I_\tau: (\Omega^*(M), d_H) \rightarrow (\Omega^*(M), d_{H'})$ where

$$I_\tau(\xi) = e^{-\tau} \wedge \xi. \tag{7.5}$$

7.4. Projective Hilbert bundles

Definition 12. An \mathcal{L} -projective Hilbert bundle \mathcal{H} is given by

- A Hilbert bundle (in the sense of Subsection 3.1) \mathcal{H}_α over each U_α , and
- A collection of Hilbert bundle isomorphisms $\varphi_{\alpha\beta}: \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \cong \mathcal{H}_\beta$ such that
- On $U_{\alpha\beta\gamma}$, the following diagram commutes:

$$\begin{array}{ccc} \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta} \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\text{id} \otimes \mu_{\alpha\beta\gamma}} & \mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\gamma} \\ \varphi_{\alpha\beta} \otimes \text{id} \downarrow & & \downarrow \varphi_{\alpha\gamma} \\ \mathcal{H}_\beta \otimes \mathcal{L}_{\beta\gamma} & \xrightarrow{\varphi_{\beta\gamma}} & \mathcal{H}_\gamma. \end{array} \tag{7.6}$$

Example 7. To a unitary gerbe \mathcal{L} on M , one can associate a projective space bundle on M [9, Section 4.1]. Using this bundle, there is a natural \mathcal{L} -projective Hilbert bundle on M formed by fermionic Fock spaces; cf. [27].

Note that there is a canonical vector bundle isomorphism over $U_{\alpha\beta}$:

$$op^k(\mathcal{H}_\alpha \otimes \mathcal{L}_{\alpha\beta}) \cong op^k(\mathcal{H}_\alpha). \tag{7.7}$$

Then the bundle isomorphism $\varphi_{\alpha\beta}$ induces an isomorphism of algebra bundles over $U_{\alpha\beta}$:

$$\varphi_{\alpha\beta}^* : op^k(\mathcal{H}_\beta) \rightarrow op^k(\mathcal{H}_\alpha). \tag{7.8}$$

Over $U_{\alpha\beta\gamma}$, we have

$$\varphi_{\alpha\beta}^* \circ \varphi_{\beta\gamma}^* = \varphi_{\alpha\gamma}^*. \tag{7.9}$$

Hence the collection $\{op^k(\mathcal{H}_\alpha)\}_{\alpha \in \Lambda}$ defines a bundle of algebras over M , which we denote by $op^k(\mathcal{H})$. Similarly, there is a bundle $\mathcal{L}^1(\mathcal{H})$ of trace ideals. For every $\alpha \in \Lambda$, we can apply the fiberwise trace to smooth sections of $\mathcal{L}^1(\mathcal{H}_\alpha)$, to obtain a map

$$\text{Tr}_\alpha : C^\infty(U_\alpha; \mathcal{L}^1(\mathcal{H}_\alpha)) \rightarrow C^\infty(U_\alpha). \tag{7.10}$$

For $b \in C^\infty(U_\beta; \mathcal{L}^1(\mathcal{H}_\beta))$, we have $\text{Tr}_\alpha(\varphi_{\alpha\beta}^*(b)) = \text{Tr}_\beta(b)$ on $U_{\alpha\beta}$. Hence we can define $\text{Tr} : C^\infty(M; \mathcal{L}^1(\mathcal{H})) \rightarrow C^\infty(M)$ by saying that

$$\text{Tr}(a) \Big|_{U_\alpha} = \text{Tr}_\alpha \left(a \Big|_{U_\alpha} \right). \tag{7.11}$$

If the bundle \mathcal{H} is \mathbb{Z}_2 -graded then there is a supertrace

$$\text{Str} : C^\infty(M; \mathcal{L}^1(\mathcal{H})) \rightarrow C^\infty(M). \tag{7.12}$$

Assume now that the gerbe \mathcal{L} is equipped with a connective structure ∇ . A superconnection on \mathcal{H} is a choice of superconnection \mathbb{A}_α on each \mathcal{H}_α so that on $U_{\alpha\beta}$, we have

$$\varphi_{\alpha\beta}^* \mathbb{A}_\beta = \mathbb{A}_\alpha \otimes \text{id} + \text{id} \otimes \nabla_{\alpha\beta}. \tag{7.13}$$

Let K be a curving of ∇ . Put $H = c(K)$.

Lemma 17. *There is a $\theta_{\mathbb{A},K} \in \Omega^*(M, op^*(\mathcal{H}))$ such that for all $\alpha \in \Lambda$, we have*

$$(\theta_{\mathbb{A},K}) \Big|_{U_\alpha} = \mathbb{A}_\alpha^2 + \kappa_\alpha. \tag{7.14}$$

Proof. Equation (7.13) implies that on $U_{\alpha\beta}$, we have

$$\varphi_{\alpha\beta}^* \mathbb{A}_\beta^2 = \mathbb{A}_\alpha^2 + \kappa_{\alpha\beta}. \tag{7.15}$$

Hence the collection $\{\mathbb{A}_\alpha^2 + \kappa_\alpha\}_{\alpha \in \Lambda}$ satisfies the relations

$$\varphi_{\alpha\beta}^*(\mathbb{A}_\beta^2 + \kappa_\beta) = \mathbb{A}_\alpha^2 + \kappa_\alpha. \tag{7.16}$$

The lemma follows. \square

For $\xi \in \Omega^*(M, op^*(\mathcal{H}))$ we can define $[\mathbb{A}, \xi] \in \Omega^*(M, op^*(\mathcal{H}))$ by

$$[\mathbb{A}, \xi] \Big|_{U_\alpha} = \left[\mathbb{A}_\alpha, \xi \Big|_{U_\alpha} \right]. \tag{7.17}$$

One can check that this is well defined.

Lemma 18. *We have $[\mathbb{A}, \theta_{\mathbb{A}, K}] = H$.*

Proof. On U_α , we know that $[\mathbb{A}_\alpha, \mathbb{A}_\alpha^2 + \kappa_\alpha] = d\kappa_\alpha = H \Big|_{U_\alpha}$. The lemma follows. \square

Definition 13. The Chern character of \mathbb{A} is given by

$$\text{Ch}(\mathbb{A}, K) = \text{Str } e^{-\theta_{\mathbb{A}, K}} \in \Omega^{\text{even}}(M). \tag{7.18}$$

Lemma 19.

- (1) *We have $d_H \text{Ch}(\mathbb{A}, K) = 0$.*
- (2) *Let $K' = K + \tau$ be another curving. Put $H' = c(K') = H + d\tau$. Then*

$$I_\tau(\text{Ch}(\mathbb{A}, K)) = \text{Ch}(\mathbb{A}, K'). \tag{7.19}$$

Proof. (1). We have

$$\begin{aligned} d \text{Str } e^{-\theta_{\mathbb{A}, K}} &= \text{Str}[\mathbb{A}, e^{-\theta_{\mathbb{A}, K}}] = -\text{Str}[\mathbb{A}, \theta_{\mathbb{A}, K}] e^{-\theta_{\mathbb{A}, K}} \\ &= -\text{Str } H e^{-\theta_{\mathbb{A}, K}} = -H \text{Str } e^{-\theta_{\mathbb{A}, K}}. \end{aligned} \tag{7.20}$$

Hence $d_H \text{Ch}(\mathbb{A}, K) = 0$.

- (2). As $\theta_{\mathbb{A}, K'} = \theta_{\mathbb{A}, K} + \tau$, the lemma follows. \square

The proofs of the next four lemmas are similar to those in Subsection 3.2.

Lemma 20. Let $\{\mathbb{A}(t)\}_{t \in [0,1]}$ and $\{\widehat{\mathbb{A}}(t)\}_{t \in [0,1]}$ be two smooth 1-parameter families of superconnections on \mathcal{H} with $\mathbb{A}(0) = \widehat{\mathbb{A}}(0)$ and $\mathbb{A}(1) = \widehat{\mathbb{A}}(1)$. Suppose that the two 1-parameter families are homotopic relative to the endpoints, in sense that there is a smooth 2-parameter family of superconnections $\{\widetilde{\mathbb{A}}(s, t)\}_{s, t \in [0,1]}$ on \mathcal{H} with $\widetilde{\mathbb{A}}(0, t) = \mathbb{A}(t)$, $\widetilde{\mathbb{A}}(1, t) = \widehat{\mathbb{A}}(t)$, $\widetilde{\mathbb{A}}(s, 0) = \mathbb{A}(0) = \widehat{\mathbb{A}}(0)$ and $\widetilde{\mathbb{A}}(s, 1) = \mathbb{A}(1) = \widehat{\mathbb{A}}(1)$.

Then

$$\int_0^1 \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\theta_{\mathbb{A}(t), \kappa}} \right) dt = \int_0^1 \text{Str} \left(\frac{d\widehat{\mathbb{A}}(t)}{dt} e^{-\theta_{\widehat{\mathbb{A}}(t), \kappa}} \right) dt \tag{7.21}$$

in $\Omega^{odd}(M)/\text{Im}(d_H)$.

Let \mathbb{A}_0 and \mathbb{A}_1 be two superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0(\mathcal{H}))$. For $t \in [0, 1]$, put $\mathbb{A}(t) = (1 - t)\mathbb{A}_0 + t\mathbb{A}_1$. Define $\eta(\mathbb{A}_0, \mathbb{A}_1) \in \Omega^{odd}(M)/\text{Im}(d_H)$ by

$$\eta(\mathbb{A}_0, \mathbb{A}_1) = \int_0^1 \text{Str} \left(\frac{d\mathbb{A}(t)}{dt} e^{-\theta_{\mathbb{A}(t), \kappa}} \right) dt. \tag{7.22}$$

Lemma 21.

$$\text{Ch}(\mathbb{A}_1) - \text{Ch}(\mathbb{A}_0) = -d_H \eta(\mathbb{A}_0, \mathbb{A}_1). \tag{7.23}$$

Lemma 22. Let $\mathbb{A}_0, \mathbb{A}_1$ and \mathbb{A}_2 be three superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0(\mathcal{H}))$ and $\mathbb{A}_{1,[0]} - \mathbb{A}_{2,[0]} \in \Omega^0(M; op^0(\mathcal{H}))$. Then

$$\eta(\mathbb{A}_0, \mathbb{A}_1) + \eta(\mathbb{A}_1, \mathbb{A}_2) = \eta(\mathbb{A}_0, \mathbb{A}_2). \tag{7.24}$$

Lemma 23. Suppose that there is some $c > 0$ so that $\mathbb{A}_{[0]}^2 \geq c^2 \text{Id}$ fiberwise on \mathcal{H} . Put

$$\eta(\mathbb{A}, \infty) = \int_1^\infty \text{Str} \left(\frac{d\mathbb{A}_t}{dt} e^{-\theta_{\mathbb{A}_t, \kappa}} \right) dt. \tag{7.25}$$

Then

$$\text{Ch}(\mathbb{A}) = d_H \eta(\mathbb{A}, \infty) \tag{7.26}$$

7.5. Definition of twisted differential K-theory

Let M be a smooth manifold. Let \mathcal{L} be a gerbe on M with a connective structure ∇ . Let K be a curving of ∇ . Put $H = c(K)$.

Definition 14. A cocycle for $\check{K}_{\mathcal{L}}^0(M)$ is a triple $[\mathcal{H}, \mathbb{A}, \omega]$ where

- (1) \mathcal{H} is a \mathbb{Z}_2 -graded \mathcal{L} -projective Hilbert bundle over M ,
- (2) \mathbb{A} is a superconnection on \mathcal{H} and
- (3) $\omega \in \Omega^{odd}(M)/\text{Im}(d_H)$.

Definition 15. The twisted differential K -theory group $\check{K}_{\mathcal{L}}^0(M)$ is the quotient of the free abelian group generated by the isomorphism classes of cocycles, by the subgroup generated by the following relations:

- (1) If $[\mathcal{H}, \mathbb{A}, \omega]$ and $[\mathcal{H}', \mathbb{A}', \omega']$ are cocycles then

$$[\mathcal{H}, \mathbb{A}, \omega] + [\mathcal{H}', \mathbb{A}', \omega'] = [\mathcal{H} \oplus \mathcal{H}', \mathbb{A} \oplus \mathbb{A}', \omega + \omega']. \tag{7.27}$$

- (2) If $\mathbb{A}_{[0]}$ is invertible then

$$[\mathcal{H}, \mathbb{A}, \omega] = [0, 0, \omega + \eta(\mathbb{A}, \infty)]. \tag{7.28}$$

- (3) Suppose that \mathbb{A}_0 and \mathbb{A}_1 are superconnections on \mathcal{H} such that $\mathbb{A}_{0,[0]} - \mathbb{A}_{1,[0]} \in \Omega^0(M; op^0)$. Then

$$[\mathcal{H}, \mathbb{A}_0, \omega] = [\mathcal{H}, \mathbb{A}_1, \omega + \eta(\mathbb{A}_0, \mathbb{A}_1)]. \tag{7.29}$$

The map $[\mathcal{H}, \mathbb{A}, \omega] \rightarrow \text{Ch}(\mathbb{A}) + d_H\omega$ passes to a map $\check{K}_{\mathcal{L}}^0(M) \rightarrow \Omega^{even}(M)$ whose image is d_H -closed.

Proposition 10. *Up to isomorphism, $\check{K}_{\mathcal{L}}^0(M)$ only depends on \mathcal{L} through its stable isomorphism class; it is independent of the choices of connective structure and curving.*

Proof. Fixing the connective structure ∇ on the gerbe \mathcal{L} , if $K' = K + \tau$ is another curving of ∇ then the map $[\mathcal{H}, \mathbb{A}, \omega] \rightarrow [\mathcal{H}, \mathbb{A}, e^{-\tau}\omega]$ induces an isomorphism of the corresponding $\check{K}_{\mathcal{L}}^0(M)$ -groups.

To see what happens when the connective structure on \mathcal{L} varies, suppose that $\nabla' = (\nabla'_{\alpha\beta})$ is another connective structure. One can find a collection $\phi_\alpha \in \Omega^1(U_\alpha)$ such that $\nabla'_{\alpha\beta} - \nabla_{\alpha\beta} = \phi_\alpha - \phi_\beta$. If $K = (\kappa_\alpha)$ is a curving of ∇ then we obtain a curving $K' = (\kappa'_\alpha)$ of ∇' by putting $\kappa'_\alpha = \kappa_\alpha + d\phi_\alpha$. The corresponding 3-form $H' = d\kappa'_\alpha$ is the same as $H = d\kappa_\alpha$. Suppose now that \mathbb{A} is a superconnection on an \mathcal{L} -projective Hilbert bundle, as defined using the connective structure ∇ . Put $\mathbb{A}'_\alpha = \mathbb{A}_\alpha - \phi_\alpha$. This defines a superconnection \mathbb{A}' on \mathcal{H} that is compatible with ∇' . Note that $\theta_{\mathbb{A}', K'} = \theta_{\mathbb{A}, K}$. It follows that the map $[\mathcal{H}, \mathbb{A}, \omega] \rightarrow [\mathcal{H}, \mathbb{A}', \omega]$ induces an isomorphism of the corresponding $\check{K}_{\mathcal{L}}^0(M)$ -groups. This isomorphism depends on the choice of the ϕ_α 's.

To check what happens under stable isomorphism of the gerbe, suppose that \mathcal{L}' is given by $\mathcal{L}'_{\alpha\beta} = \widehat{\mathcal{L}}_\alpha \otimes \mathcal{L}_{\alpha\beta} \otimes \widehat{\mathcal{L}}_\beta^{-1}$. Choose Hermitian connections $\widehat{\nabla}_\alpha$ on the \mathcal{L}_α 's. Put $\nabla'_{\alpha\beta} = \widehat{\nabla}_\alpha \otimes \nabla_{\alpha\beta} \otimes \widehat{\nabla}_\beta^{-1}$. Given a cocycle $[\mathcal{H}, \mathbb{A}, \omega]$ for $\check{K}_{\mathcal{L}}^0(M)$, we obtain a cocycle

$[\mathcal{H}', \mathbb{A}', \omega']$ for $\check{K}_{\mathcal{L}'}^0(M)$ by $\mathcal{H}'_{\alpha} = \mathcal{H}_{\alpha} \otimes \widehat{L}_{\alpha}^{-1}$, $\mathbb{A}'_{\alpha} = \mathbb{A}_{\alpha} \otimes \widehat{V}_{\alpha}^{-1}$ and $\omega' = \omega$. This gives an isomorphism between $\check{K}_{\mathcal{L}}^0(M)$ and $\check{K}_{\mathcal{L}'}^0(M)$. It depends on the choice of the \widehat{V}_{α} 's. \square

Appendix A. Chern character in relative cohomology

In [32], Quillen stated without proof that if \mathbb{A} is a superconnection on a finite dimensional vector bundle E over M , and the degree-0 component $\mathbb{A}_{[0]}$ is invertible on an open subset $U \subset M$, then the pair $(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty))$ represents the Chern character of E in relative cohomology. In this appendix we provide a proof of the statement in the more general setting of superconnections on Hilbert bundles. We give an application to a difference formula for eta forms.

Remark 7. As mentioned by Paradan and Vergne [30], in the finite dimensional case one can also prove Quillen’s claim by showing that the pair satisfies Schneiders’ axiomatic characterization of the relative Chern character [33, Proposition 4.5.2].

Let \mathbb{A} be a superconnection, in the sense of Definition 3, on a \mathbb{Z}_2 -graded Hilbert bundle \mathcal{H} over a manifold M . Let $U \subset M$ be an open set such that $\mathbb{A}_{[0]}^2 \geq c^2 \text{Id} > 0$ on U , for some $c > 0$. Then $\eta(\mathbb{A}, \infty)$ is defined on U .

Recall that $H^*(M, U; \mathbb{R})$ is the cohomology of the complex

$$\Omega^*(M, U; \mathbb{R}) = \Omega^*(M) \oplus \Omega^{*-1}(U) \tag{A.1}$$

with differential

$$d(\omega, \sigma) = (d\omega, i^*\omega - d\sigma), \tag{A.2}$$

where $i : U \rightarrow M$ is the inclusion map.

Lemma 24. *The pair $(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty))$ defines a class in $H^*(M, U; \mathbb{R})$.*

Proof. This follows from Lemmas 7 and 11. \square

Lemma 25. *Let \mathbb{A} and \mathbb{A}' be two superconnections as above such that $\mathbb{A}_{[0]} - \mathbb{A}'_{[0]} \in \Omega^0(M; \text{op}^0(\mathcal{H}))$ and $\mathbb{A}_{[0]} = \mathbb{A}'_{[0]}$ on U . Then*

$$[(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty))] = [(\text{Ch}(\mathbb{A}'), \eta(\mathbb{A}', \infty))] \in H^*(M, U; \mathbb{R}). \tag{A.3}$$

Proof. On the product space $[0, 1] \times M$, consider the superconnection

$$\mathbb{B} = dt \wedge \partial_t + t\mathbb{A} + (1 - t)\mathbb{A}'. \tag{A.4}$$

Note that $\mathbb{B}_{[0]}^2 \geq c^2 \text{Id}$ on $[0, 1] \times U$. Since $d\text{Ch}(\mathbb{B}) = 0$ in $\Omega^*([0, 1] \times M)$, one finds that in $\Omega^*(M, U; \mathbb{R})$, we have

$$d \left(\int_0^1 \text{Ch}(\mathbb{B}), \int_0^1 \eta(\mathbb{B}, \infty) \right) = \tag{A.5}$$

$$(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty)) - (\text{Ch}(\mathbb{A}'), \eta(\mathbb{A}', \infty)).$$

The lemma follows. \square

Since $\mathbb{A}_{[0]}$ is invertible on U , we can define a class $\text{Ind}(\mathbb{A}_{[0]}^\pm) \in K^0(M, U)$ as follows.

Let γ denote the \mathbb{Z}_2 -grading operator on \mathcal{H} . Put $\tilde{\mathcal{H}} = \mathcal{H} \oplus \mathcal{H}$ with the \mathbb{Z}_2 -grading operator $\tilde{\gamma} = \begin{bmatrix} \gamma & 0 \\ 0 & -\gamma \end{bmatrix}$. Let $f \in C^\infty(\mathbb{R})$ be a real-valued odd function such that $xf(x) = 1 - \xi(x)$, with $\xi(0) = 1$ and $\text{supp } \xi(x) \subset [-c, c]$. Put $Q = f(\mathbb{A}_{[0]})$. Then Q is a parametrix for $\mathbb{A}_{[0]}$ which is odd with respect to γ , self-adjoint and is the inverse of $\mathbb{A}_{[0]}$ over U .

Put $S_0 = 1 - Q\mathbb{A}_{[0]}$ and $S_1 = 1 - \mathbb{A}_{[0]}Q$. Note that S_0 and S_1 vanish over U . Put $L = \begin{bmatrix} S_0 & -(1 + S_0)Q \\ \mathbb{A}_{[0]} & S_1 \end{bmatrix}$, with $L^{-1} = \begin{bmatrix} S_0 & (1 + S_0)Q \\ -\mathbb{A}_{[0]} & S_1 \end{bmatrix}$. These are operators on $\tilde{\mathcal{H}}$ that are even with respect to $\tilde{\gamma}$.

Put $P = L^{-1} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} L$ and $P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. They are both even projection operators. Then

$$P - P_0 = \begin{pmatrix} S_0^2 & -S_0(1 + S_0)Q \\ -\mathbb{A}_{[0]}S_0 & -S_1^2 \end{pmatrix} \tag{A.6}$$

is a smooth family of finite rank operators, i.e. $P - P_0 \in \Omega^0(M; \mathcal{L}^{fr}(H))$; the smoothness can be seen by repeated differentiation. Also, $P - P_0$ vanishes on U .

Put $P^\pm = P|_{\tilde{\mathcal{H}}^\pm}$ and $P_0^\pm = P_0|_{\tilde{\mathcal{H}}^\pm}$.

Definition 16. The index $\text{Ind}(\mathbb{A}_{[0]}^\pm) \in K^0(M, U)$ is represented by the virtual projection $[P^\pm - P_0^\pm]$.

Remark 8. As in [15, Chapter II.9.α], $[P^\pm - P_0^\pm]$ is the index class in $K^0(M)$. As $P^\pm - P_0^\pm$ vanishes on U , we can consider $[P^\pm - P_0^\pm]$ to give a class in $K^0(M, U)$. To justify this statement, suppose that there is a closed subset $A \subset M$ which is a strong deformation retract of U (as is often the case). Then there are canonical isomorphisms $K^0(M, U) \cong K^0(M, A) \cong \tilde{K}^0(M/A)$. The virtual projection $[P^\pm - P_0^\pm]$ descends to a reduced K -theory class of M/A .

Note that $\text{Ind}(\mathbb{A}_{[0]}^-) = -\text{Ind}(\mathbb{A}_{[0]}^+)$.

The real-valued Chern character $\text{Ch}(\text{Ind}(\mathbb{A}_{[0]}^+))$ in $H^{\text{even}}(M, U; \mathbb{R})$ can be represented as follows. Put $\tilde{\mathbb{A}} = \mathbb{A} \oplus \mathbb{A}$, a superconnection on \tilde{H} . Then $\tilde{\mathbb{A}}_{[1]}$ is a connection on \tilde{H} .

Definition 17. The Chern character of $\text{Ind} \left(\mathbb{A}_{[0]}^+ \right)$ in $\tilde{H}^{\text{even}}(M/A; \mathbb{R})$ is represented by

$$\begin{aligned} \text{Ch} \left(\text{Ind} \left(\mathbb{A}_{[0]}^+ \right) \right) = & \tag{A.7} \\ \left[\left(\frac{1}{2} \text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}}_{[1]} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}_{[1]} \circ P_0)^2} P_0 \right), 0 \right) \right]. \end{aligned}$$

Remark 9. To justify this definition, we first note that the closed form

$$\frac{1}{2} \text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}}_{[1]} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}_{[1]} \circ P_0)^2} P_0 \right) \tag{A.8}$$

represents the Chern character of the index in $H^{\text{even}}(M; \mathbb{R})$. It has support in $M \setminus U$ and so, in the setting of Remark 8, extends to a compactly supported form in $(M/A) - (A/A)$. Under the isomorphisms $H_c^{\text{even}}((M/A) - (A/A); \mathbb{R}) \cong H^{\text{even}}(M/A, A/A; \mathbb{R}) \cong H^{\text{even}}(M, A; \mathbb{R}) \cong H^{\text{even}}(M, U; \mathbb{R})$, the form gets mapped to (A.7).

Proposition 11.

$$\text{Ch} \left(\text{Ind} \left(\mathbb{A}_{[0]}^+ \right) \right) = [(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty))]. \tag{A.9}$$

Proof. We claim that

$$\begin{aligned} & \left[\left(\text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}}_{[1]} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}_{[1]} \circ P_0)^2} P_0 \right), 0 \right) \right] = & \tag{A.10} \\ & \left[\left(\text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2} P_0 \right), 0 \right) \right] \end{aligned}$$

in $H^{\text{even}}(M, U; \mathbb{R})$. To see this, for $t \in [0, 1]$ put $\tilde{\mathbb{A}}(t) = t\tilde{\mathbb{A}} + (1-t)\tilde{\mathbb{A}}_{[1]}$, a superconnection on $\tilde{\mathcal{H}}$. Now

$$P e^{-(P \circ \tilde{\mathbb{A}}(t) \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}(t) \circ P_0)^2} P_0 \tag{A.11}$$

is a (2×2) -matrix with entries in $\Omega^*(M; \mathcal{L}^{fr}(\mathcal{H}))$, that are smooth in t . Using the finite rank property, it is easy to justify that

$$\begin{aligned} & \frac{d}{dt} \text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}}(t) \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}(t) \circ P_0)^2} P_0 \right) = & \tag{A.12} \\ & d \text{Tr}_s \left(P \frac{d\tilde{\mathbb{A}}(t)}{dt} P e^{-(P \circ \tilde{\mathbb{A}}(t) \circ P)^2} P - P_0 \frac{d\tilde{\mathbb{A}}(t)}{dt} P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}(t) \circ P_0)^2} P_0 \right), \end{aligned}$$

so

$$\begin{aligned} & \text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}}_{[1]} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}_{[1]} \circ P_0)^2} P_0 \right) = & \tag{A.13} \\ & \text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2} P_0 \right) + \end{aligned}$$

$$d \int_0^1 \text{Tr}_s \left(P \frac{d\tilde{\mathbb{A}}(t)}{dt} P e^{-(P \circ \tilde{\mathbb{A}}(t) \circ P)^2} P - P_0 \frac{d\tilde{\mathbb{A}}(t)}{dt} P_0 e^{-(P_0 \circ \tilde{\mathbb{A}}(t) \circ P_0)^2} P_0 \right) dt.$$

This proves (A.10).

Considering $P \circ \tilde{\mathbb{A}} \circ P$ to be a superconnection on $\text{Im}(P)$, and $P_0 \circ \tilde{\mathbb{A}} \circ P_0$ to be a superconnection on $\text{Im}(P_0)$, Lemma 24 implies that

$$\left(\text{Tr}_s \left(e^{-(P \circ \tilde{\mathbb{A}} \circ P)^2} \right), \eta(P \circ \tilde{\mathbb{A}} \circ P, \infty) \right) \tag{A.14}$$

and

$$\left(\text{Tr}_s \left(e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2} \right), \eta(P_0 \circ \tilde{\mathbb{A}} \circ P_0, \infty) \right) \tag{A.15}$$

are closed elements of $\Omega^{even}(M, U)$. Since $P = P_0$ on U , we have

$$\begin{aligned} & \left[\left(\text{Tr}_s \left(P e^{-(P \circ \tilde{\mathbb{A}} \circ P)^2} P - P_0 e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2} P_0 \right), 0 \right) \right] = \tag{A.16} \\ & \left[\left(\text{Tr}_s \left(e^{-(P \circ \tilde{\mathbb{A}} \circ P)^2} \right), \eta(P \circ \tilde{\mathbb{A}} \circ P, \infty) \right) \right] - \\ & \left[\left(\text{Tr}_s \left(e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2} \right), \eta(P_0 \circ \tilde{\mathbb{A}} \circ P_0, \infty) \right) \right]. \end{aligned}$$

Next, since $P_0 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ and the \mathbb{Z}_2 -grading operator $\tilde{\gamma}$ is $-\gamma$ on $\text{Im}(P_0)$, we have

$$\left(\text{Tr}_s e^{-(P_0 \circ \tilde{\mathbb{A}} \circ P_0)^2}, \eta(P_0 \circ \tilde{\mathbb{A}} \circ P_0, \infty) \right) = (-\text{Ch}(\mathbb{A}), -\eta(\mathbb{A}, \infty)). \tag{A.17}$$

Finally, put $P_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. Then

$$P \circ \tilde{\mathbb{A}} \circ P = L^{-1} P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1 L. \tag{A.18}$$

Hence

$$\begin{aligned} & (\text{Ch}(P \circ \tilde{\mathbb{A}} \circ P), \eta(P \circ \tilde{\mathbb{A}} \circ P, \infty)) = \tag{A.19} \\ & (\text{Ch}(P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1), \eta(P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1, \infty)). \end{aligned}$$

The superconnections \mathbb{A} and $P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1$ on $\text{Im}(P_1) = \mathcal{H}$ satisfy the conditions of Lemma 25. Hence

$$\begin{aligned} & [(\text{Ch}(P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1), \eta(P_1 L \circ \tilde{\mathbb{A}} \circ L^{-1} P_1, \infty))] = \tag{A.20} \\ & [(\text{Ch}(\mathbb{A}), \eta(\mathbb{A}, \infty))]. \end{aligned}$$

The proposition now follows from combining (A.7), (A.10), (A.16), (A.17), (A.19) and (A.20). \square

Corollary 2. *Suppose that $\{\mathbb{E}(t)\}_{t \in [0,1]}$ is a smooth 1-parameter family of superconnections on \mathcal{H} . Put $\mathbb{A} = \mathbb{E}(0)$ and $\mathbb{A}' = \mathbb{E}(1)$. If $\mathbb{A}_{[0]}$ and $\mathbb{A}'_{[0]}$ are invertible then*

$$\eta(\mathbb{A}, \mathbb{A}') - \eta(\mathbb{A}, \infty) + \eta(\mathbb{A}', \infty) \in \text{Im}(\text{Ch} : K^1(M) \rightarrow H^{\text{odd}}(M; \mathbb{R})). \tag{A.21}$$

If $\mathbb{E}_{[0]}(t)$ is invertible for all $t \in [0, 1]$ then the expression in (A.21) vanishes in $\Omega^{\text{odd}}(M)/\text{Im}(d)$.

Proof. Let V be a codimension-zero submanifold of M with compact closure. (If M is compact then we just take $V = M$.) Put $M' = V \times \mathbb{R}$ and $U = V \times ((-\infty, 0) \cup (1, \infty))$. There are isomorphisms $i_K : K^0(M', U) \rightarrow K^1(V)$ and $i_H : H^{\text{even}}(M', U) \rightarrow H^{\text{odd}}(V)$, where i_H is represented by

$$i_H(\omega, \eta) = \left(\int_0^1 \omega \right) - \eta(1) + \eta(0). \tag{A.22}$$

The maps i_K and i_H are consistent in the sense that

$$\text{Ch} \circ i_K = i_H \circ \text{Ch}. \tag{A.23}$$

Extend $\mathbb{E}(t)$ to be constant on $(-\infty, 0)$ and constant on $(1, \infty)$. By reparametrizing \mathbb{R} if necessary, we can assume that \mathbb{E} is smooth in t . Put $\mathbb{B} = dt \wedge \partial_t + \mathbb{E}(t)$, a superconnection on M' . The family $\mathbb{B}_{[0]}^+$ of operators defines an element $[\mathbb{B}_{[0]}^+] \in K^0(M', U)$, since the operators are invertible on U .

Equations (3.10) and (A.22) along with Proposition 11, when applied to \mathbb{B} , M' and U , give

$$\text{Ch}(i_K([\mathbb{B}_{[0]}^+])) = -\eta(\mathbb{A}, \mathbb{A}') - \eta(\mathbb{A}', \infty) + \eta(\mathbb{A}, \infty). \tag{A.24}$$

After exhausting M by such V 's, this proves the first part of the corollary. If each $\mathbb{E}(t)$ is invertible then $[\mathbb{B}_{[0]}^+]$ vanishes in $K^0(M', U)$, which implies the second part of the corollary. \square

References

[1] M. Atiyah, *K-Theory*, Adv. Book Class., Westview Press, 1994.
 [2] S. Baaĵ, P. Julg, Théorie bivariante de Kasparov et opérateurs non bornés dans les C^* -modules, *C. R. Math. Acad. Sci. Paris* 296 (21) (1983) 875–878.
 [3] N. Berline, E. Getzler, M. Vergne, *Heat Kernels and Dirac Operators*, Springer, New York, 1996.
 [4] J.-M. Bismut, The Atiyah–Singer index theorem for families of Dirac operators: two heat equation proofs, *Invent. Math.* 83 (1986) 91–151.

- [5] J.-M. Bismut, J. Cheeger, η -invariants and their adiabatic limits, *J. Amer. Math. Soc.* 2 (1989) 33–70.
- [6] J.-M. Bismut, H. Gillet, C. Soulé, Analytic torsion and holomorphic determinant bundles II, *Comm. Math. Phys.* 115 (1988) 79–126.
- [7] J.-M. Bismut, G. Lebeau, Complex immersions and Quillen metrics, *Publ. Math. Inst. Hautes Études Sci.* 74 (1991) 1–298.
- [8] C. Blohmann, M. Fernandes, A. Weinstein, Groupoid symmetry and constraints in general relativity, *Commun. Contemp. Math.* 15 (2013).
- [9] J.-L. Brylinski, *Loop Spaces, Characteristic Classes and Geometric Quantization*, *Progr. Math.*, vol. 107, Birkhäuser, Boston, 1993.
- [10] U. Bunke, X. Ma, Index and secondary index theory for flat bundles with duality, in: *Aspects of Boundary Problems in Analysis and Geometry*, in: *Oper. Theory Adv. Appl.*, vol. 151, 2004, pp. 265–341.
- [11] U. Bunke, T. Nikolaus, Twisted differential cohomology, preprint, <http://arxiv.org/abs/1406.3231>, 2014.
- [12] U. Bunke, T. Schick, Smooth K-theory, *Astérisque* 328 (2009) 45–135.
- [13] U. Bunke, T. Schick, Differential K-theory. A survey, in: C. Bär, J. Lohkamp, M. Schwarz (Eds.), *Global Differential Geometry*, in: *Springer Proc. Math.*, vol. 17, Springer, New York, 2012, pp. 303–357.
- [14] A. Carey, J. Mickelsson, B. Wang, Differential twisted K-theory and applications, *J. Geom. Phys.* 59 (2009) 632–653.
- [15] A. Connes, *Noncommutative Geometry*, Academic Press, San Diego, 1994.
- [16] A. Connes, H. Moscovici, The local index formula in noncommutative geometry, *Geom. Funct. Anal.* 5 (1995) 174–243.
- [17] A. Connes, G. Skandalis, The longitudinal index theorem for foliations, *Publ. Res. Inst. Math. Sci.* 20 (1984) 1139–1183.
- [18] D. Freed, J. Lott, An index theorem in differential K-theory, *Geom. Topol.* 14 (2010) 903–966.
- [19] E. Getzler, A. Szenes, On the Chern character of a theta-summable Fredholm module, *J. Funct. Anal.* 84 (1989) 343–357.
- [20] N. Hitchin, Lectures on special Lagrangian submanifolds, in: *Winter Summer School on Mirror Symmetry, Vector Bundles and Lagrangian Submanifolds*, in: *AMS/IP Stud. Adv. Math.*, vol. 21, Amer. Math. Soc., Providence, 2001, pp. 151–182.
- [21] M. Hopkins, I. Singer, Quadratic functions in geometry, topology and M-theory, *J. Differential Geom.* 70 (2005) 329–452.
- [22] P. Iglesias-Zemmour, *Diffeology*, Amer. Math. Soc., Providence, 2013.
- [23] A. Kahle, Superconnections and index theory, *J. Geom. Phys.* 61 (2011) 1601–1624.
- [24] A. Kahle, A. Valentino, T-duality and differential K-theory, *Commun. Contemp. Math.* 16 (2014) 1350014.
- [25] G. Kasparov, Equivariant KK-theory and the Novikov conjecture, *Invent. Math.* 91 (1988) 147–201.
- [26] B. Liu, Functoriality of equivariant eta forms, preprint, <http://arxiv.org/abs/1505.04454>, 2015.
- [27] J. Mickelsson, Gerbes and quantum field theory, in: J.-P. Francoise, G. Naber, T.-S. Tsun (Eds.), *Encyclopedia of Mathematical Physics*, Elsevier, Amsterdam, 2006.
- [28] A. Mishchenko, A. Fomenko, The index of elliptic operators over C^* -algebras, *Izv. Akad. Nauk SSSR* 43 (1979) 831–859.
- [29] M. Murray, D. Stevenson, Bundle gerbes: stable isomorphism and local theory, *J. London Math. Soc.* 62 (2000) 925–937.
- [30] P.-E. Paradan, M. Vergne, Quillen’s relative Chern character, in: *Algebraic Analysis and Around*, in: *Adv. Stud. Pure Math.*, vol. 54, 2009, pp. 255–291.
- [31] B. Park, Geometric models of twisted differential K-theory I, preprint, <http://arxiv.org/abs/1602.02292>, 2016.
- [32] D. Quillen, Superconnections and the Chern character, *Topology* 24 (1985) 89–95.
- [33] J.-P. Schneiders, *Introduction to Characteristic Classes and Index Theory*, *Textos Math.*, vol. 13, Universidade de Lisboa, 2000, <http://www.analg.ulg.ac.be/jps/rec/icc.pdf>.
- [34] J. Simons, D. Sullivan, Structured vector bundles define differential K-theory, in: *Quanta of Maths*, in: *Clay Math. Proc.*, vol. 11, Amer. Math. Soc., Providence, 2010, pp. 579–599.
- [35] T. Tradler, S. Wilson, M. Zeinalian, An elementary differential extension of odd K-theory, *J. K-Theory* 12 (2013) 331–361.