

# Evolution of three-dimensional Ricci flow

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The question

Two-dimensional Ricci flow

Three-dimensional Ricci flows with symmetry

Main results

Equilibrium configurations of Ricci flow

Locally homogeneous Ricci flows

Proofs of main results

# Question

Take a compact orientable three-dimensional manifold.

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Run the Ricci flow.

What happens?

# The goal

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2. Prove convergence of the Ricci flow to an equilibrium metric.

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# Warmup case

Say  $M$  is a compact orientable surface.

The Ricci flow equation :

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Here

1.  $g \equiv g_{ij}$  is a Riemannian metric on  $M$ .
2.  $K$  is the Gaussian curvature of  $g$ .

# Some explicit solutions

1. The shrinking two-sphere :

$$g(t) = r^2(t) g_{\text{round}},$$

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2. The static two-torus :

$$g(t) = g_{flat}.$$

3. The expanding higher-genus surface :

$$g(t) = r^2(t) g_{hyp},$$

$$r^2(t) = r^2(0) + 2t.$$

# Conformal invariance

From the 2-D Ricci flow equation

$$\frac{dg}{dt} = -2 K g,$$

we get

$$g(t) = \Phi(t) g(0)$$

for some positive function  $\Phi(t)$ .

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for some positive function  $\Phi(t)$ .

Here  $\Phi$  satisfies the **logarithmic fast diffusion equation**

$$\frac{\partial \Phi}{\partial t} = \Delta_{g(0)}(\ln \Phi) - 2K_0.$$

## Results of Hamilton (1988), Chow (1991)

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1. If  $M$  is a two-sphere then there is a finite extinction time  $T < \infty$ . Also,

$$\lim_{t \rightarrow T^-} \frac{g(t)}{T - t} = 2 g_{\text{round}}.$$

2. If  $M$  is a two-torus then the flow exists for  $t \in [0, \infty)$  and

$$\lim_{t \rightarrow \infty} g(t) = g_{\text{flat}}$$

for some flat metric  $g_{\text{flat}}$ .

3. If  $M$  is a higher genus surface then the flow exists for  $t \in [0, \infty)$ . Putting

$$\widehat{g}(t) = \frac{g(t)}{t},$$

we have

$$\lim_{t \rightarrow \infty} \widehat{g}(t) = 2 g_{hyp}$$

for some metric  $g_{hyp}$  of constant curvature  $-1$ .

# Uniformization theorem

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Not quite : there's a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof **uses** the uniformization theorem.

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Does this give a Ricci flow proof of the uniformization theorem?

Not quite : there's a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof **uses** the uniformization theorem.

However, one can get around this (Chen-Lu-Tian 2006).

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The “de Turck” trick gives an equivalent parabolic equation.

If  $M$  is compact then for any initial Riemannian metric  $g(0)$  on  $M$ , there is a smooth Ricci flow solution on some maximal time interval  $[0, T)$  with  $0 < T \leq \infty$ .

# Constant curvature solutions

Some explicit solutions :

1. Round shrinking sphere
2. Static flat metric
3. Expanding hyperbolic metric

If  $(M, g(\cdot))$  is a Ricci flow solution on a surface  $M$  then we get a Ricci flow solution on  $M \times S^1$  :

$$h(t) = g(t) + d\theta^2.$$

Here the  $S^1$ -factor is static.

## First nontrivial case

Take the manifold  $M \times S^1$  again, but now allow the length of the circle fiber over  $m \in M$  to depend on  $m$ .

This gives a **warped product** metric on  $M \times S^1$  :

$$h = g + e^{2u} d\theta^2,$$

where  $u$  is a function on the surface  $M$ .

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Fact : if the initial metric  $h(0)$  is a warped product metric then so is the Ricci flow metric  $h(t)$ .

As the metric evolves, what happens? An old question in Ricci flow.

# Warped product Ricci flow

The Ricci flow equation for the metric

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on  $M \times S^1$  becomes the coupled equations on the surface  $M$  :

$$\begin{aligned}\frac{\partial u}{\partial t} &= \Delta_{g(t)} u, \\ \frac{\partial g_{ij}}{\partial t} &= -2K g_{ij} + 2(\partial_i u)(\partial_j u).\end{aligned}$$

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**Problem :** this flow is no longer conformal.

## $U(1) \times U(1)$ symmetry

If  $U(1) \times U(1)$  acts freely on a compact orientable three-dimensional manifold then it is topologically a 3-torus. It fibers over a circle, with the 2-torus fibers being the orbits of the  $U(1) \times U(1)$  action.

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A  $U(1) \times U(1)$ -invariant metric takes the form

$$g = \sum_{i,j=1}^2 G_{ij}(y) dx^i dx^j + g_{yy}(y) dy^2,$$

where

- ▶  $y$  is a coordinate for the base circle,
- ▶  $x^1$  and  $x^2$  are linear coordinates for the two-torus, and
- ▶  $(G_{ij}(y))$  is a positive-definite symmetric  $2 \times 2$  matrix.

# Flow equation

The Ricci flow equation becomes

$$\begin{aligned}\frac{\partial G}{\partial t} &= g^{yy} \left( G'' - G' G^{-1} G' \right), \\ \frac{\partial g_{yy}}{\partial t} &= \frac{1}{2} \operatorname{Tr} \left( \left( G^{-1} G' \right)^2 \right).\end{aligned}$$

Periodic boundary condition :

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Periodic boundary condition :

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Hamilton (1995) : Under some additional assumptions, as  $t \rightarrow \infty$ , the metric approaches a flat metric on  $T^3$ .

# Twisted torus bundles

Twisted boundary conditions : require

$$G(y + 1) = H^T G(y) H,$$

where  $H \in \text{SL}(2, \mathbb{Z})$ . This gives a metric on the 2-torus bundle over a circle with holonomy  $H$ .



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Hamilton-Isenberg (1993) : If  $H$  has distinct real eigenvalues then under some additional assumptions, as  $t \rightarrow \infty$ , the metric approaches a locally homogeneous metric of Sol-type.

From Perelman's first Ricci flow paper :

“The natural questions that remain open are whether the normalized curvatures must stay bounded as  $t \rightarrow \infty$ , and whether reducible manifolds and manifolds with finite fundamental group can have metrics which evolve smoothly by the Ricci flow on the infinite interval.”

# Natural questions

From Perelman's first Ricci flow paper :

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The second question was answered in the negative in Perelman's third Ricci flow paper.

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The second question was answered in the negative in Perelman's third Ricci flow paper.

We'll answer the first question in the positive in the special cases of warped products and 2-torus bundles.

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Joint work with Natasa Sesum (Rutgers)

# Warped products

## Theorem

*Suppose that  $h(\cdot)$  is a Ricci flow on  $M \times S^1$  whose initial metric  $h(0)$  is a warped product metric over a two-dimensional compact base  $M$ .*

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- 1. If  $M$  is a 2-sphere then there is a finite extinction time  $T < \infty$ . As  $t \rightarrow T^-$ , the metric on  $M \times S^1$  is asymptotic to the product of a shrinking round metric on  $M$  with a static metric on  $S^1$ .*



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- 2. If  $M$  is a 2-torus then the flow exists for  $t \in [0, \infty)$ . As  $t \rightarrow \infty$ , the metric on  $M \times S^1$  approaches a flat metric exponentially fast.*

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- 2. If  $M$  is a 2-torus then the flow exists for  $t \in [0, \infty)$ . As  $t \rightarrow \infty$ , the metric on  $M \times S^1$  approaches a flat metric exponentially fast.*
- 3. If  $M$  is a higher genus surface then the flow exists for  $t \in [0, \infty)$ . The sectional curvatures on  $M \times S^1$  are  $O(t^{-1})$ . For large  $t$ , there is a large region of  $M$  on which the curvature of  $\frac{g(t)}{t}$  is close to  $-\frac{1}{2}$ , and over which the circle fibers have almost constant length.*

## Theorem

*Suppose that  $N$  is the total space of a 2-torus bundle over the circle, with holonomy  $H \in \mathrm{SL}(2, \mathbb{Z})$ . Let  $h(\cdot)$  be a Ricci flow solution on  $N$  which is locally  $U(1) \times U(1)$  invariant. Then the sectional curvatures of  $(N, h(t))$  are  $O(t^{-1})$  in magnitude.*

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- 1. If  $H$  has finite order then as  $t \rightarrow \infty$ , the metric  $h(t)$  approaches a flat metric exponentially fast.*
- 2. If  $H$  is hyperbolic then as  $t \rightarrow \infty$ , the manifold  $(N, \frac{g(t)}{t})$  approaches a circle in the Gromov-Hausdorff sense. When pulled back to the universal cover, as  $t \rightarrow \infty$ , the Ricci flow solution approaches the homogeneous solution*

$$\left( \mathbb{R}^3, e^{-2z} dx^2 + e^{2z} dy^2 + 4t dz^2 \right).$$

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# Parabolic rescaling

If  $f(x, t)$  is a solution of the **heat equation**

$$\frac{\partial f}{\partial t} = \Delta f$$

on  $\mathbb{R}^n$  then so is

$$f_s(x, t) = f(\sqrt{s}x, st).$$

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$$f_s(x, t) = f(\sqrt{s}x, st).$$

If  $g(t)$  is a solution to the **Ricci flow equation**

$$\frac{dg}{dt} = -2 \text{ Ric}$$

then so is

$$g_s(t) = \frac{1}{s}g(st).$$



# Diffeomorphism invariance

If  $\phi : N \rightarrow N$  is a diffeomorphism and  $g(t)$  is a Ricci flow solution on  $N$  then so is  $\phi^*g(t)$ .

# Static solutions

The **static** solutions to the Ricci flow equation

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Allowing uniform expansion or contraction, we also get the **Einstein** metrics

$$\operatorname{Ric} = cg, \quad c \in \left\{ -\frac{1}{2}, 0, \frac{1}{2} \right\}.$$

# Soliton solutions

If a metric satisfies the **steady soliton equation**

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = 0$$

then we get a Ricci flow solution

$$g(t) = \phi_t^* g,$$

where  $\{\phi_t\}$  is the flow generated by the vector field  $V$ .

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where  $\{\phi_t\}$  is the flow generated by the vector field  $V$ .

Similarly, there are the **shrinking soliton equation**

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = \frac{1}{2} g$$

and the **expanding soliton equation**

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = -\frac{1}{2} g.$$

**Algebraic fact** : if  $N$  is a three-dimensional manifold then any Einstein metric on  $N$  has constant curvature.

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**Analytic fact** : if  $N$  is a compact three-dimensional manifold then any Ricci soliton on  $N$  has constant curvature.

# Hyperbolizable manifolds

The **only** general result about the long-time asymptotics of three-dimensional Ricci flow :



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## Theorem

*(Perelman 2003) Suppose that  $N$  is a compact three-dimensional manifold that admits a hyperbolic metric  $g_{hyp}$ . Let  $g(0)$  be any Riemannian metric on  $N$ . Run the Ricci flow starting with  $g(0)$ . Then*

- ▶ *There is a finite number of surgeries.*
- ▶ *As  $t \rightarrow \infty$ , the rescaled metric  $\frac{g(t)}{t}$  approaches  $4g_{hyp}$ .*

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Suppose that we run the Ricci flow on a compact three-dimensional manifold that **doesn't** admit a constant curvature metric.

**What happens?**

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# Locally homogeneous manifolds

A Riemannian manifold is **locally homogeneous** if any two points have isometric neighborhoods.

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**Thurston's geometrization conjecture = Perelman's theorem :**

Any compact three-dimensional manifold  $N$  can be decomposed into pieces that admit finite-volume locally homogeneous metrics.



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**Thurston's geometrization conjecture = Perelman's theorem :**

Any compact three-dimensional manifold  $N$  can be decomposed into pieces that admit finite-volume locally homogeneous metrics.

Suppose that  $N$  has a locally homogeneous metric. Run the Ricci flow. What happens?

# Locally homogeneous Ricci flow

If  $N$  is locally homogeneous then its universal cover  $\tilde{N}$  is a globally homogeneous space  $G/K$ .

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If  $N$  is locally homogeneous then its universal cover  $\tilde{N}$  is a globally homogeneous space  $G/K$ .

The Ricci flow on  $G/K$  reduces to a system of ODE's. In three dimensions, the system can either be solved explicitly or its asymptotics can be computed (Isenberg-Jackson 1992).

# Gromov-Hausdorff limits

Thurston type      GH limit of  $(N, \frac{g(t)}{t})$

$H^3$	3-manifold of constant curvature $-\frac{1}{4}$
$H^2 \times \mathbb{R}$ or $\widetilde{\text{SL}(2, \mathbb{R})}$	2-orbifold of constant curvature $-\frac{1}{2}$
Sol	circle or interval
Nil or $\mathbb{R}^3$	point

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In general, the compact 3-manifold  $N$  **collapses** to a lower-dimensional space.

Let's pass to the universal cover  $\tilde{N}$ .

## Proposition

*(L. 2007) Suppose that  $N$  is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for  $t \in [0, \infty)$ . On the universal cover  $\tilde{N}$ , the rescaled pullback metric approaches a homogeneous expanding soliton.*

# Expanding solitons

## Proposition

(L. 2007) Suppose that  $N$  is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for  $t \in [0, \infty)$ . On the universal cover  $\tilde{N}$ , the rescaled pullback metric approaches a homogeneous expanding soliton.

<u>Thurston type</u>	<u>Expanding soliton</u>
$H^3$	$4 t g_{H^3}$
$H^2 \times \mathbb{R}$ or $\widetilde{\text{SL}(2, \mathbb{R})}$	$2 t g_{H^2} + g_{\mathbb{R}}$
Sol	$e^{-2z} dx^2 + e^{2z} dy^2 + 4 t dz^2$
Nil	$\frac{1}{3t^{\frac{1}{3}}} \left( dx + \frac{1}{2} y dz - \frac{1}{2} z dy \right)^2 + t^{\frac{1}{3}} (dy^2 + dz^2)$
$\mathbb{R}^3$	$g_{\mathbb{R}^3}$



The meaning of the limit in the Proposition :

Let  $g(\cdot)$  be a Ricci flow solution on  $N$  that exists for  $t \in [0, \infty)$ .  
For  $s > 0$ , define a blowdown solution by

$$g_s(t) = \frac{1}{s} g(st).$$

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For  $s > 0$ , define a blowdown solution by

$$g_s(t) = \frac{1}{s} g(st).$$

Then on the universal cover  $\tilde{N}$ , there are diffeomorphisms  $\{\phi_s\}$   
so that

$$\lim_{s \rightarrow \infty} \phi_s^* \tilde{g}_s = g_{\text{expander}},$$

with smooth convergence on compact subsets.

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## Theorem

*(L. 2010) Suppose that  $(N, g(\cdot))$  is a Ricci flow on a compact three-dimensional manifold, that exists for  $t \in [0, \infty)$ . Suppose that the sectional curvatures are  $O(t^{-1})$  in magnitude, and the diameter is  $O(\sqrt{t})$ . Then the pullback of the Ricci flow to  $\tilde{N}$  approaches a homogeneous expanding soliton.*

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## Remarks :

- ▶ The hypotheses are invariant under parabolic rescaling.
- ▶ The hypotheses imply that there is only one piece in the Thurston decomposition of  $N$ , so  $N$  admits a locally homogeneous metric.
- ▶ One can also describe the Gromov-Hausdorff limit of  $(N, g(\cdot))$ .

1. Monotonic quantities : modifications of Perelman's  $W$ -functional.

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2. Compactness theorem : an extension of Hamilton's compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.

# Ingredients of the proof

1. Monotonic quantities : modifications of Perelman's  $W$ -functional.
2. Compactness theorem : an extension of Hamilton's compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.
3. Contradiction arguments, using blowdown limits.



# Warped products and torus bundles

In the case of warped products and 2-torus bundles, the main work is to show that the sectional curvatures are  $O(t^{-1})$  in magnitude, and that the diameter is  $O(\sqrt{t})$ .

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In the warped product case, the sectional curvature bound comes from a contradiction argument, using blowdown limits and the Gauss-Bonnet theorem.

In the case of torus bundles, the diameter bound comes from the evolution formula for lengths, along with the maximum principle.

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3. What about the general case of a free  $U(1)$  action?