Evolution of three-dimensional Ricci flow

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The question

Two-dimensional Ricci flow

Three-dimensional Ricci flows with symmetry

Main results

Equilibrium configurations of Ricci flow

Locally homogeneous Ricci flows

Proofs of main results
Question

Take a compact orientable three-dimensional manifold.

Put a Riemannian metric on it.
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Run the Ricci flow.
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Take a compact orientable three-dimensional manifold.

Put a Riemannian metric on it.

Run the Ricci flow.

What happens?
The goal

1. Identify the equilibrium metrics.
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1. Identify the equilibrium metrics.
2. Prove convergence of the Ricci flow to an equilibrium metric.
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Proofs of main results
Say $M$ is a compact orientable surface.

The Ricci flow equation:

$$\frac{dg}{dt} = -2 \, K \, g.$$
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The Ricci flow equation:

$$\frac{dg}{dt} = -2Kg.$$ 

Here

1. $g \equiv g_{ij}$ is a Riemannian metric on $M$.
2. $K$ is the Gaussian curvature of $g$. 
1. The shrinking two-sphere:

\[ g(t) = r^2(t) \, g_{\text{round}}, \]
\[ r^2(t) = r^2(0) - 2t. \]
Some explicit solutions

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2. The static two-torus:

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2. The static two-torus:

\[ g(t) = g_{\flat}. \]

3. The expanding higher-genus surface:

\[ g(t) = r^2(t) g_{\text{hyp}}, \]
\[ r^2(t) = r^2(0) + 2t. \]
Conformal invariance

From the 2-D Ricci flow equation

$$\frac{dg}{dt} = -2Kg,$$

we get

$$g(t) = \Phi(t)g(0)$$

for some positive function \(\Phi(t)\).
Conformal invariance

From the 2-D Ricci flow equation

\[ \frac{dg}{dt} = -2 K g, \]

we get

\[ g(t) = \Phi(t) g(0) \]

for some positive function \( \Phi(t) \).

Here \( \Phi \) satisfies the logarithmic fast diffusion equation

\[ \frac{\partial \Phi}{\partial t} = \triangle g(0)(\ln \Phi) - 2K_0. \]
Say $g(0)$ is an arbitrary Riemannian metric on the surface $M$. 

1. If $M$ is a two-sphere then there is a finite extinction time $T < \infty$. Also, 
   \[ \lim_{t \to T^-} g(t) = 2g \text{ round}. \]

2. If $M$ is a two-torus then the flow exists for $t \in [0, \infty)$ and 
   \[ \lim_{t \to \infty} g(t) = g_{\text{flat}} \] for some flat metric $g_{\text{flat}}$. 

Results of Hamilton (1988), Chow (1991)
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$$\lim_{t \to \infty} g(t) = g_{\text{flat}}$$

for some flat metric $g_{\text{flat}}$. 
3. If $M$ is a higher genus surface then the flow exists for $t \in [0, \infty)$. Putting

$$\hat{g}(t) = \frac{g(t)}{t},$$

we have

$$\lim_{t \to \infty} \hat{g}(t) = 2g_{hyp}$$

for some metric $g_{hyp}$ of constant curvature $-1$. 
Uniformization theorem

The uniformization theorem says that any Riemannian metric on a compact orientable surface is conformally equivalent to a constant curvature metric.
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The **uniformization theorem** says that any Riemannian metric on a compact orientable surface is conformally equivalent to a constant curvature metric.

Does this give a Ricci flow proof of the uniformization theorem?

Not quite: there’s a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof uses the uniformization theorem.
The uniformization theorem says that any Riemannian metric on a compact orientable surface is conformally equivalent to a constant curvature metric.

Does this give a Ricci flow proof of the uniformization theorem?

Not quite: there's a circularity in the argument.

In the two-sphere case, the Hamilton/Chow convergence proof uses the uniformization theorem.

However, one can get around this (Chen-Lu-Tian 2006).
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The Ricci flow equation

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The “de Turck” trick gives an equivalent parabolic equation.
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The “de Turck” trick gives an equivalent parabolic equation.

If \( M \) is compact then for any initial Riemannian metric \( g(0) \) on \( M \), there is a smooth Ricci flow solution on some maximal time interval \( [0, T) \) with \( 0 < T \leq \infty \).
Constant curvature solutions

Some explicit solutions:

1. Round shrinking sphere
2. Static flat metric
3. Expanding hyperbolic metric
If \((M, g(\cdot))\) is a Ricci flow solution on a surface \(M\) then we get a Ricci flow solution on \(M \times S^1\):

\[ h(t) = g(t) + d\theta^2. \]

Here the \(S^1\)-factor is static.
First nontrivial case

Take the manifold $M \times S^1$ again, but now allow the length of the circle fiber over $m \in M$ to depend on $m$.

This gives a warped product metric on $M \times S^1$:

$$h = g + e^{2u} d\theta^2,$$

where $u$ is a function on the surface $M$. 
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Fact: if the initial metric $h(0)$ is a warped product metric then so is the Ricci flow metric $h(t)$. 
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Fact: if the initial metric $h(0)$ is a warped product metric then so is the Ricci flow metric $h(t)$.

As the metric evolves, what happens? An old question in Ricci flow.
The Ricci flow equation for the metric

\[ h = g + e^{2u} d\theta^2 \]

on \( M \times S^1 \) becomes the coupled equations on the surface \( M \):

\[
\frac{\partial u}{\partial t} = \triangle_{g(t)} u, \\
\frac{\partial g_{ij}}{\partial t} = -2K g_{ij} + 2(\partial_i u)(\partial_j u).
\]
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on \( M \times S^1 \) becomes the coupled equations on the surface \( M \) :

\[
\frac{\partial u}{\partial t} = \triangle g(t) u,
\]

\[
\frac{\partial g_{ij}}{\partial t} = -2K g_{ij} + 2(\partial_i u)(\partial_j u).
\]

**Problem**: this flow is no longer conformal.
If $U(1) \times U(1)$ acts freely on a compact orientable three-dimensional manifold then it is topologically a 3-torus. It fibers over a circle, with the 2-torus fibers being the orbits of the $U(1) \times U(1)$ action.
U(1) x U(1) symmetry

If $U(1) \times U(1)$ acts freely on a compact orientable three-dimensional manifold then it is topologically a 3-torus. It fibers over a circle, with the 2-torus fibers being the orbits of the $U(1) \times U(1)$ action.

A $U(1) \times U(1)$-invariant metric takes the form

$$g = \sum_{i,j=1}^{2} G_{ij}(y) dx^i dx^j + g_{yy}(y) dy^2,$$

where

- $y$ is a coordinate for the base circle,
- $x^1$ and $x^2$ are linear coordinates for the two-torus, and
- $(G_{ij}(y))$ is a positive-definite symmetric $2 \times 2$ matrix.
The Ricci flow equation becomes

\[ \frac{\partial G}{\partial t} = g^{yy} \left( G'' - G' G^{-1} G' \right), \]

\[ \frac{\partial g_{yy}}{\partial t} = \frac{1}{2} \text{Tr} \left( \left( G^{-1} G' \right)^2 \right). \]

Periodic boundary condition:

\[ G(y + 1) = G(y). \]
The Ricci flow equation becomes

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Periodic boundary condition:

\[ G(y + 1) = G(y). \]

Hamilton (1995) : Under some additional assumptions, as \( t \to \infty \), the metric approaches a flat metric on \( T^3 \).
Twisted boundary conditions: require

\[ G(y + 1) = H^T G(y) H, \]

where \( H \in \text{SL}(2, \mathbb{Z}) \). This gives a metric on the 2-torus bundle over a circle with holonomy \( H \).
Twisted boundary conditions: require

\[ G(y + 1) = H^T G(y) H, \]

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Hamilton-Isenberg (1993): If \( H \) has distinct real eigenvalues then under some additional assumptions, as \( t \to \infty \), the metric approaches a locally homogeneous metric of Sol-type.
From Perelman’s first Ricci flow paper:

“The natural questions that remain open are whether the normalized curvatures must stay bounded as $t \to \infty$, and whether reducible manifolds and manifolds with finite fundamental group can have metrics which evolve smoothly by the Ricci flow on the infinite interval.”
From Perelman’s first Ricci flow paper:

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The second question was answered in the negative in Perelman’s third Ricci flow paper.
From Perelman’s first Ricci flow paper:

“The natural questions that remain open are whether the normalized curvatures must stay bounded as $t \to \infty$, and whether reducible manifolds and manifolds with finite fundamental group can have metrics which evolve smoothly by the Ricci flow on the infinite interval.”

The second question was answered in the negative in Perelman’s third Ricci flow paper.

We’ll answer the first question in the positive in the special cases of warped products and 2-torus bundles.
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The question

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Proofs of main results
Joint work with Natasa Sesum (Rutgers)
Theorem
Suppose that $h(\cdot)$ is a Ricci flow on $M \times S^1$ whose initial metric $h(0)$ is a warped product metric over a two-dimensional compact base $M$.

1. If $M$ is a 2-sphere then there is a finite extinction time $T < \infty$. As $t \to T^-$, the metric on $M \times S^1$ is asymptotic to the product of a shrinking round metric on $M$ with a static metric on $S^1$.

2. If $M$ is a 2-torus then the flow exists for $t \in [0, \infty)$. As $t \to \infty$, the metric on $M \times S^1$ approaches a flat metric exponentially fast.

3. If $M$ is a higher genus surface then the flow exists for $t \in [0, \infty)$. The sectional curvatures on $M \times S^1$ are $O(t^{-1})$. For large $t$, there is a large region of $M$ on which the curvature of $g(t)$ is close to $-\frac{1}{2}$, and over which the circle fibers have almost constant length.
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2. If \( M \) is a 2-torus then the flow exists for \( t \in [0, \infty) \). As \( t \to \infty \), the metric on \( M \times S^1 \) approaches a flat metric exponentially fast.
Warped products

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3. If \( M \) is a higher genus surface then the flow exists for \( t \in [0, \infty) \). The sectional curvatures on \( M \times S^1 \) are \( O(t^{-1}) \). For large \( t \), there is a large region of \( M \) on which the curvature of \( \frac{g(t)}{t} \) is close to \(-\frac{1}{2}\), and over which the circle fibers have almost constant length.
Theorem
Suppose that $N$ is the total space of a 2-torus bundle over the circle, with holonomy $H \in \text{SL}(2, \mathbb{Z})$. Let $h(\cdot)$ be a Ricci flow solution on $N$ which is locally $U(1) \times U(1)$ invariant. Then the sectional curvatures of $(N, h(t))$ are $O(t^{-1})$ in magnitude.
Theorem

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1. If $H$ has finite order then as $t \to \infty$, the metric $h(t)$ approaches a flat metric exponentially fast.
Theorem
Suppose that $N$ is the total space of a $2$-torus bundle over the circle, with holonomy $H \in \text{SL}(2, \mathbb{Z})$. Let $h(\cdot)$ be a Ricci flow solution on $N$ which is locally $U(1) \times U(1)$ invariant. Then the sectional curvatures of $(N, h(t))$ are $O \left( t^{-1} \right)$ in magnitude.

1. If $H$ has finite order then as $t \to \infty$, the metric $h(t)$ approaches a flat metric exponentially fast.

2. If $H$ is hyperbolic then as $t \to \infty$, the manifold $\left( N, \frac{g(t)}{t} \right)$ approaches a circle in the Gromov-Hausdorff sense. When pulled back to the universal cover, as $t \to \infty$, the Ricci flow solution approaches the homogeneous solution

\[
\left( \mathbb{R}^3, e^{-2z} \, dx^2 + e^{2z} \, dy^2 + 4 \, t \, dz^2 \right).
\]
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Proofs of main results
If $f(x, t)$ is a solution of the heat equation

$$\frac{\partial f}{\partial t} = \triangle f$$

on $\mathbb{R}^n$ then so is

$$f_s(x, t) = f(\sqrt{s}x, st).$$
Parabolic rescaling

If \( f(x, t) \) is a solution of the heat equation

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f_s(x, t) = f(\sqrt{s}x, st).
\]

If \( g(t) \) is a solution to the Ricci flow equation

\[
\frac{dg}{dt} = -2 \text{ Ric}
\]
then so is

\[
g_s(t) = \frac{1}{s}g(st).
\]
If $\phi : N \rightarrow N$ is a diffeomorphism and $g(t)$ is a Ricci flow solution on $N$ then so is $\phi^* g(t)$. 
The static solutions to the Ricci flow equation

$$\frac{dg}{dt} = -2 \text{ Ric}$$

are the Ricci-flat metrics.
The **static** solutions to the Ricci flow equation

\[ \frac{dg}{dt} = -2 \text{ Ric} \]

are the **Ricci-flat** metrics.

Allowing uniform expansion or contraction, we also get the **Einstein** metrics

\[ \text{Ric} = cg, \quad c \in \left\{ -\frac{1}{2}, 0, \frac{1}{2} \right\}. \]
Soliton solutions

If a metric satisfies the steady soliton equation

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = 0$$

then we get a Ricci flow solution

$$g(t) = \phi_t^* g,$$

where \( \{\phi_t\} \) is the flow generated by the vector field \( V \).
Soliton solutions

If a metric satisfies the steady soliton equation

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Similarly, there are the shrinking soliton equation

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = \frac{1}{2} g$$

and the expanding soliton equation

$$\text{Ric} + \frac{1}{2} \mathcal{L}_V g = -\frac{1}{2} g.$$
Algebraic fact: if $N$ is a three-dimensional manifold then any Einstein metric on $N$ has constant curvature.
3D solitons

**Algebraic fact**: if $N$ is a three-dimensional manifold then any Einstein metric on $N$ has constant curvature.

**Analytic fact**: if $N$ is a compact three-dimensional manifold then any Ricci soliton on $N$ has constant curvature.
The only general result about the long-time asymptotics of three-dimensional Ricci flow:

Theorem (Perelman 2003) Suppose that $N$ is a compact three-dimensional manifold that admits a hyperbolic metric $g_{hyp}$. Let $g(0)$ be any Riemannian metric on $N$. Run the Ricci flow starting with $g(0)$. Then:

- There is a finite number of surgeries.
- As $t \to \infty$, the rescaled metric $g(t)/t$ approaches $4g_{hyp}$. 
The only general result about the long-time asymptotics of three-dimensional Ricci flow:

**Theorem**

*(Perelman 2003)* Suppose that $N$ is a compact three-dimensional manifold that admits a hyperbolic metric $g_{hyp}$. Let $g(0)$ be any Riemannian metric on $N$. Run the Ricci flow starting with $g(0)$. Then

- There is a finite number of surgeries.
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Most compact three-dimensional manifolds do not admit a constant curvature metric.
Apparent paradox

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But the only quasistatic Ricci flow solutions on compact three-dimensional manifolds have constant curvature.
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Suppose that we run the Ricci flow on a compact three-dimensional manifold that doesn’t admit a constant curvature metric.
Apparent paradox

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A Riemannian manifold is \textit{locally homogeneous} if any two points have isometric neighborhoods.
A Riemannian manifold is **locally homogeneous** if any two points have isometric neighborhoods.

**Thurston’s geometrization conjecture = Perelman’s theorem:**

Any compact three-dimensional manifold $N$ can be decomposed into pieces that admit finite-volume locally homogeneous metrics.
Locally homogeneous manifolds

A Riemannian manifold is **locally homogeneous** if any two points have isometric neighborhoods.

Thurston’s geometrization conjecture = Perelman’s theorem :

Any compact three-dimensional manifold $N$ can be decomposed into pieces that admit finite-volume locally homogeneous metrics.

Suppose that $N$ has a locally homogeneous metric. Run the Ricci flow. What happens?
If $N$ is locally homogeneous then its universal cover $\tilde{N}$ is a globally homogeneous space $G/K$. 
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The Ricci flow on $G/K$ reduces to a system of ODE’s. In three dimensions, the system can either be solved explicitly or its asymptotics can be computed (Isenberg-Jackson 1992).


<table>
<thead>
<tr>
<th>Thurston type</th>
<th>GH limit of ( (N, \frac{g(t)}{t}) )</th>
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<tbody>
<tr>
<td>( H^3 )</td>
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In general, the compact 3-manifold $N$ collapses to a lower-dimensional space.
Gromov-Hausdorff limits

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In general, the compact 3-manifold $N$ collapses to a lower-dimensional space.

Let’s pass to the universal cover $\tilde{N}$. 
Proposition

(L. 2007) Suppose that $N$ is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for $t \in [0, \infty)$. On the universal cover $\tilde{N}$, the rescaled pullback metric approaches a homogeneous expanding soliton.
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(L. 2007) Suppose that \( N \) is a compact locally homogeneous three-dimensional manifold whose Ricci flow exists for \( t \in [0, \infty) \). On the universal cover \( \tilde{N} \), the rescaled pullback metric approaches a homogeneous expanding soliton.

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<td>( H^3 )</td>
<td>( 4 , t , g_{H^3} )</td>
</tr>
<tr>
<td>( H^2 \times \mathbb{R} ) or ( \text{SL}(2, \mathbb{R}) )</td>
<td>( 2 , t , g_{H^2} + g_{\mathbb{R}} )</td>
</tr>
<tr>
<td>Sol</td>
<td>( e^{-2z} , dx^2 + e^{2z} , dy^2 + 4 , t , dz^2 )</td>
</tr>
<tr>
<td>Nil</td>
<td>( \frac{1}{3t^{\frac{1}{3}}} \left( dx + \frac{1}{2} ydz - \frac{1}{2} zdy \right)^2 + t^{\frac{1}{3}} \left( dy^2 + dz^2 \right) )</td>
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<td>( \mathbb{R}^3 )</td>
<td>( g_{\mathbb{R}^3} )</td>
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The meaning of the limit in the Proposition:

Let $g(\cdot)$ be a Ricci flow solution on $N$ that exists for $t \in [0, \infty)$. For $s > 0$, define a blowdown solution by

$$g_s(t) = \frac{1}{s} g(st).$$
Blowdown limits

The meaning of the limit in the Proposition:

Let \( g(\cdot) \) be a Ricci flow solution on \( N \) that exists for \( t \in [0, \infty) \). For \( s > 0 \), define a blowdown solution by

\[
g_s(t) = \frac{1}{s} g(st).
\]

Then on the universal cover \( \tilde{N} \), there are diffeomorphisms \( \{ \phi_s \} \) so that

\[
\lim_{s \to \infty} \phi_s^* \tilde{g}_s = g_{\text{expander}},
\]

with smooth convergence on compact subsets.
Evolution of three-dimensional Ricci flow

The question

Two-dimensional Ricci flow

Three-dimensional Ricci flows with symmetry

Main results

Equilibrium configurations of Ricci flow

Locally homogeneous Ricci flows

Proofs of main results
Theorem
(L. 2010) Suppose that \((N, g(\cdot))\) is a Ricci flow on a compact three-dimensional manifold, that exists for \(t \in [0, \infty)\). Suppose that the sectional curvatures are \(O\left(t^{-1}\right)\) in magnitude, and the diameter is \(O(\sqrt{t})\). Then the pullback of the Ricci flow to \(\tilde{N}\) approaches a homogeneous expanding soliton.
**Main tool: a convergence criterion**

**Theorem**

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**Remarks:**

- The hypotheses are invariant under parabolic rescaling.
- The hypotheses imply that there is only one piece in the Thurston decomposition of \(N\), so \(N\) admits a locally homogeneous metric.
- One can also describe the Gromov-Hausdorff limit of \((N, g(\cdot))\).
Ingredients of the proof


2. Compactness theorem: an extension of Hamilton’s compactness theorem for Ricci flows, to remove the assumption of a lower bound on the injectivity radius.
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In the case of torus bundles, the diameter bound comes from the evolution formula for lengths, along with the maximum principle.
1. For warped products over a higher genus surface, the bound \( \text{diam}(M, g(t)) = O(\sqrt{t}) \) is missing.
What’s missing so far

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3. What about the general case of a free \( U(1) \) action?