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Eigenvalue estimates and differential form Laplacians on Alexandrov spaces

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Abstract We give upper bounds on the eigenvalues of the differential form Laplacian on a compact Riemannian manifold. The proof uses Alexandrov spaces with curvature bounded below. We also construct differential form Laplacians on Alexandrov spaces. Under a local biLipschitz assumption on the Alexandrov space, which is conjecturally always satisfied, we show that the differential form Laplacian has a compact resolvent. We identify its kernel with an intersection homology group.

1 Introduction

For a closed connected Riemannian manifold, let λ_k denote the kth positive eigenvalue of the (nonnegative) function Laplacian, counted with multiplicity. In 1975, S.-Y. Cheng proved the following upper bound on the eigenvalues.

Theorem 1.1 (Cheng [7]) *There is a function $c : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}^+$ with the following property. Given $n \in \mathbb{Z}^+$, $D \in \mathbb{R}^+$ and $K \in \mathbb{R}$, if M is an n-dimensional closed connected Riemannian manifold with diameter D, and* $\mathrm{Ric}_M \geq K g_M$, *then*

$$\lambda_k \leq c(n, K D^2) \frac{k^2}{D^2}. \tag{1.2}$$

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We give extensions of Cheng's result to the differential form Laplacian. Let $\lambda_{k,p}$ denote the kth positive eigenvalue of the Hodge Laplacian $dd^* + d^*d$ on p-forms, counted with multiplicity. First, we assume a lower volume bound.

Theorem 1.3 *There is a function $C_1 : \mathbb{Z}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}^+$ with the following property. Given $n \in \mathbb{Z}^+$, $D \in \mathbb{R}^+$, $K \in \mathbb{R}$ and $v \in \mathbb{R}^+$, if M is an n-dimensional closed connected Riemannian manifold with diameter D, sectional curvatures bounded below by K and volume bounded below by v, then for all $p \in [0, n]$ we have*

$$\lambda_{k,p} \leq C_1(n, KD^2, vD^{-n}) \frac{k^{\frac{2}{n}}}{D^2}. \tag{1.4}$$

To remove the lower volume bound, we use the notion of a strainer [3, §5]. Given a complete Riemannian manifold (or Alexandrov space) with curvature bounded below by $K \in \mathbb{R}$, and an integer $s > 0$, an s-strainer of quality δ and size S at a point m consists of points $\{a_i, b_i\}_{i=1}^s$ with $d(m, a_i) = d(m, b_i) = S$ so that

$$\begin{aligned} \tilde{\measuredangle} a_i m b_i &> \pi - \delta, \quad \tilde{\measuredangle} a_i m a_j > \frac{\pi}{2} - \delta, \\ \tilde{\measuredangle} a_i m b_j &> \frac{\pi}{2} - \delta, \quad \tilde{\measuredangle} b_i m b_j > \frac{\pi}{2} - \delta, \end{aligned} \tag{1.5}$$

whenever $i \neq j$. Here $\tilde{\measuredangle}$ is the comparison angle at m, relative to the model space of constant sectional curvature K.

Theorem 1.6 *There is a function $C_2 : \mathbb{Z}^+ \times \mathbb{R} \to \mathbb{R}^+$ with the following property. Given $n \in \mathbb{Z}^+$ and $K \in \mathbb{R}$, let M be an n-dimensional closed connected Riemannian manifold with sectional curvatures bounded below by K. Suppose that there is some $m \in M$ with an s-strainer of quality $\frac{1}{10}$ and size S, where $1 \leq s \leq n$. Then for all $p \in [0, s]$, we have*

$$\lambda_{k,p} \leq C_2(n, KS^2) \frac{k^{\frac{2}{s}}}{S^2}. \tag{1.7}$$

Corollary 1.8 *Given $n \in \mathbb{Z}^+$, there is some $\kappa_n < \infty$ with the following property. Let M be an n-dimensional closed connected Riemannian manifold with nonnegative sectional curvature. Suppose that there is some $m \in M$ with an s-strainer of quality $\frac{1}{10}$ and size S, where $1 \leq s \leq n$. Then for all $p \in [0, s]$, we have*

$$\lambda_{k,p} \leq \kappa_n \frac{k^{\frac{2}{s}}}{S^2}. \tag{1.9}$$

Remark 1.10 The choice of $\frac{1}{10}$ for the quality of the strainer is arbitrary. In the proof of Theorem 1.6 we actually get upper eigenvalue bounds for the Laplacian on $\Omega^p(M)/\operatorname{Ker}(d)$ when $p < s$, and for the Laplacian on $\operatorname{Im}(d) \subset \Omega^s(M)$.

Theorem 1.6 implies Theorem 1.1 under the stronger assumption of a lower sectional curvature bound, by taking $s = 1$ and $S = \frac{D}{2}$. One would not expect to be able to control eigenvalues of the p-form Laplacian from a lower Ricci curvature bound if $p \notin \{0, 1, n-1, n\}$.

Remark 1.11 Theorem 1.3 actually follows from Theorem 1.6. Under the hypotheses of Theorem 1.3, after rescaling the diameter to be 1, convergence theory implies that there is some point with an n-strainer of quality $\frac{1}{10}$ and a certain size.

Remark 1.12 To get upper eigenvalue bounds for the p-form Laplacian, Theorem 1.6 has an assumption about the existence of a p-strainer at some point. The need for some such assumption can be seen by taking $M = X \times S^N$, where X is a closed Riemannian manifold, and shrinking the S^N-factor. If $\dim(X) < p$ and $N > p$ then the eigenvalues of the p-form Laplacian on M go to infinity. As the sphere shrinks, there is clearly no p-strainer on M of quality $\frac{1}{10}$ whose size is uniformly bounded below.

Taking $s = \dim(X)$, this example also shows the sharpness of the exponent $\frac{2}{s}$ in (1.7), which corresponds to Weyl-type asymptotics on an s-dimensional manifold.

The constant c in Theorem 1.1 can be made explicit. The constants C_1 and C_2 in Theorems 1.3 and 1.6 are not explicit. The reason is that the proofs of Theorems 1.3 and 1.6 are by a contradiction argument. One common feature of all the proofs is the use of a minmax argument on an appropriate class of test functions or test forms. To prove Theorem 1.1, Cheng transplanted functions from a model space, using the exponential map from a point. Dodziuk extended Cheng's result, by transplanting differential forms from a model space, to prove an analog of Theorem 1.3 under the stronger assumptions of a double sided curvature bound and a lower bound on the injectivity radius [9]. We instead pullback differential forms from an Alexandrov space (with curvature bounded below) that arises in the contradiction argument.

This leads to the question of whether differential form Laplacians make sense for Alexandrov spaces. (The function Laplacian on an Alexandrov space was studied in [21,31].) To see some of the issues involved, note that on a smooth Riemannian manifold, when written in local coordinates, the differential form Laplacian $dd^* + d^*d$ involves two derivatives of the metric tensor. An Alexandrov space has a dense open set with the structure of a Riemannian Lipschitz manifold, meaning in particular that there is a Riemannian metric whose components, in local coordinates, are in L^∞_{loc} [26]. Hence defining $dd^* + d^*d$ directly on an Alexandrov space does not look promising.

Instead of trying to directly define the differential form Laplacian as an operator, one could try to define the putative spectrum. On a smooth closed Riemannian manifold M, the minmax formula says that

$$\lambda_{k,p} = \inf_V \sup_{\omega \in V, \omega \neq 0} \frac{|d\omega|^2_{L^2} + |d^*\omega|^2_{L^2}}{|\omega|^2_{L^2}}, \tag{1.13}$$

where V ranges over k-dimensional subspaces of $\Omega^p(M)$. In local coordinates, $d^*\omega$ involves first derivatives of the metric tensor. On an Alexandrov space, one knows that the first derivatives of the metric components exist as measures [26], but this is not enough to make sense of (1.13).

To gain another derivative, we use the observation, essentially due to Cheeger and Dodziuk [9], that the minmax equation (1.13) takes a nicer form if we look instead at the Laplacian $\triangle_*$ on $\Omega^*(M)/\overline{\mathrm{Im}(d)}$. For this Laplacian, the minmax equation becomes

$$\lambda_{k,p} = \inf_V \sup_{\omega \in V, \omega \neq 0} \frac{|d\omega|^2_{L^2}}{|\omega|^2_{L^2}}, \tag{1.14}$$

where V now ranges over k-dimensional subspaces of $\Omega^p(M)/\overline{\mathrm{Im}(d)}$. The right-hand side of (1.14) does not involve any derivatives of the metric tensor. Using the Hodge decomposition and the isomorphism $\Omega^*(M)/\overline{\mathrm{Im}(d)} \cong \mathrm{Ker}(dd^* + d^*d) \oplus \overline{\mathrm{Im}(d^*)}$, the spectrum of $\triangle_*$ is the same as the spectrum of $dd^* + d^*d$, with the multiplicities related by a factor of at most two.

For this reason, in making sense of a differential form Laplacian on an Alexandrov space, we only consider an analog of the Laplacian $\triangle_*$ on $\Omega^*(M)/\overline{\mathrm{Im}(d)}$.

Theorem 1.15 *If X is a compact Alexandrov space then there is a well-defined non-negative self-adjoint differential form Laplacian $\triangle_*$. When X is a smooth Riemannian manifold M with (possibly empty) convex boundary, the operator $\triangle_*$ becomes the usual Hodge Laplacian on $\Omega^*(M)/\overline{\mathrm{Im}(d)}$ with relative (Dirichlet) boundary conditions.*

We prove Theorem 1.3 in the generality of compact Alexandrov spaces. The use of $\Omega^*/\mathrm{Ker}(d)$ is key in proving Theorems 1.3 and 1.6. Their proofs do not need the existence of the differential form Laplacian on the limit space, but rather the existence of differential forms.

We construct $\triangle_*$ more generally for compact metric spaces X that have an open subset, of full Hausdorff measure, with the structure of a Riemannian Lipschitz manifold. The basic analytic property of $\triangle_*$ that one would like to show is that $(I + \triangle_*)^{-1}$ is compact; this implies discreteness of the spectrum of $\triangle_*$. In order to show that $(I + \triangle_*)^{-1}$ is compact, it is necessary to make an additional assumption about X. To motivate this assumption, we recall that in a finite dimensional Alexandrov space X, any $x \in X$ has a neighborhood that is homeomorphic to the truncated tangent cone $T^1_x X$ [18,27]. It seems likely that any $x \in X$ has a neighborhood that is biLipschitz homeomorphic to $T^1_x X$; this has been claimed, although no proof is available. Based on this, we consider a class $\mathcal{C}_*$ of compact metric spaces that are Lipschitz analogs of the topological multiconical spaces (MCS) introduced in [32] and used in [18,27]. First, $\mathcal{C}_0$ consists of finite metric spaces. Inductively, if $X \in \mathcal{C}_n$ with $n \geq 1$ then any point in X has a neighborhood that is biLipschitz homeomorphic to the truncated open metric cone over some element of $\mathcal{C}_{n-1}$ with diameter at most π. Conjecturally, any n-dimensional compact Alexandrov space is an element of $\mathcal{C}_n$. (If one is just interested in Alexandrov spaces then one can just start with elements of $\mathcal{C}_0$ consisting of one or two points. For boundaryless Alexandrov spaces, one can just start with elements of $\mathcal{C}_0$ consisting of two points.) Examples of elements of $\mathcal{C}_*$ come from quotients of smooth closed Riemannian manifolds by compact groups of isometries. Other examples come from compact stratified spaces with iterated cone-edge Riemannian metrics.

Theorem 1.16 (1) *If $X \in \mathcal{C}_n$ then $\mathrm{Ker}(\triangle_*)$ is isomorphic to $\mathrm{IH}^{GM}_{n-*}(X; \mathcal{O})$, the Goresky-MacPherson intersection homology of X as defined using the upper middle perversity.* (2) *If $X \in \mathcal{C}_n$ then $(I + \triangle_*)^{-1}$ is compact.*

Here $\mathcal{O}$ is the orientation line bundle of the codimension-zero stratum of X. The upper middle perversity is the function $\overline{p} : \mathbb{Z}^{\geq 0} \to \mathbb{Z}$ given by $\overline{p}(0) = 0$ and $\overline{p}(j) = [\frac{j-1}{2}]$ for $j \geq 1$. If X is a conically stratified pseudomanifold (i.e. has no codimension-one strata) then it is well known that the L^2-cohomology of X is related to the intersection (co)homology of X with middle perversity. If X is allowed to have codimension-one strata, as in our case, then there are various notions of intersection (co)homology [11]. It is not immediately clear which one is the right one to describe $\mathrm{Ker}(\triangle_*)$. It turns out that the right one is the original Goresky-MacPherson intersection homology, extended to spaces with codimension-one strata, after an appropriate change of degree.

Remark 1.17 A finite dimensional Alexandrov space is locally Lipschitz contractible [24]. Unfortunately, this does not help in proving Theorem 1.16 for Alexandrov spaces that are not *a priori* in $\mathcal{C}_*$, due to boundedness issues.

To summarize, we construct self-adjoint differential form Laplacians for a class of compact metric spaces, that includes compact Alexandrov spaces. For a more restricted class of compact metric spaces, that conjecturally includes compact Alexandrov spaces, we show that the differential form Laplacian has a compact resolvent.

The structure of the paper is as follows. In Sect. 2 we construct the differential form Laplacian $\triangle_*$ on a class of compact metric spaces. Section 3 has the construction of a sheaf of certain locally-L^2 differential forms. The eigenvalue bounds of Theorems 1.3 and 1.6 are proven in Sect. 4. In Sect. 5 we consider the Lipschitz multiconical spaces $\mathcal{C}_*$ and prove Theorem 1.16.

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2 Differential form Laplacian on an Alexandrov space

In this section we define differential form Laplacians on a class of metric spaces that includes Alexandrov spaces. In Sect. 2.1 we consider a certain class of test forms built out of Lipschitz functions. Using them, in Sect. 2.2 we define a complex of L^2-forms. Section 2.3 has the construction of the differential form Laplacian.

For background material on Alexandrov spaces, we refer to [2, Chapter 10].

Let (X, d_X) be a compact metric space with Hausdorff dimension n and finite n-dimensional Hausdorff mass. If X is disconnected then we assume that the distance between points in distinct connected components is infinity. Suppose that there is an open subset $X^* \subset X$, with full Hausdorff n-measure, having the structure of an n-dimensional Riemannian Lipschitz manifold. This means that X^* has a manifold structure with locally Lipschitz transition maps, and that it is equipped with a Riemannian metric g so that in coordinate charts, g and g^{-1} are in L^∞_{loc}. In addition, d_X is compatible with the metric d_{X^*} on X^* coming from g [8], in the sense that d_X and d_{X^*} coincide on some neighborhood of the diagonal in $X^* \times X^*$. In particular, if F is a function with compact support in a coordinate neighborhood of X^*, and F is Lipschitz in terms of the coordinates, then F is a Lipschitz function on X.

Example 2.1 Let X be a compact Alexandrov space with curvature bounded below, of Hausdorff dimension n. There is some $\delta_0 > 0$ with the following property. Given $\delta \in (0, \delta_0)$, let X^*_δ be the set of points $x \in X$ such that the space of directions Σ_x has $(n-1)$-dimensional Hausdorff mass more than $(1-\delta)$ times that of S^{n-1}. Then X^*_δ is an open convex subset of X of full Hausdorff measure, with the structure of a Riemannian Lipschitz manifold [26]. In fact, there is a stronger DC-structure, but this doesn't seem to matter for the considerations of this paper.

2.1 Test forms

For a smooth compact Riemannian manifold, we can define an operator d, on a dense subset of L^2-forms, by saying that $\omega \in \mathrm{Dom}(d)$ if the distributional differential $d\omega$ is L^2. Here the notion of distributional differential uses smooth test forms. On our space X, it doesn't make sense to talk about smooth forms. We will instead use "test forms" made from Lipschitz functions. (Not to be confused with the test forms mentioned in the introduction.) Let $\Omega^*_{\mathrm{Lip}}(X)$ be the graded-commutative differential graded algebra generated by $\{f_0 df_1 \dots df_k\}$, where $f_i \in \mathrm{Lip}(X)$. In particular, an element of $\Omega^k_{\mathrm{Lip}}(X)$ is a finite sum of expressions $f_0 df_1 \dots df_k$, and $d(f_0 df_1 \dots df_k) = 1 \cdot df_0 df_1 \dots df_k$. There is a relation $d(fg) = f dg + g df$ for $f, g \in \mathrm{Lip}(X)$. The elements of $\Omega^*_{\mathrm{Lip}}(X)$ are also known as the Kähler forms of the algebra $\mathrm{Lip}(X)$. There is a homomorphism ρ from $\Omega^*_{\mathrm{Lip}}(X)$ to the locally-L^∞ differential forms on X^*. This homomorphism need not be injective or surjective.

The test forms, or more precisely their image under ρ, will actually be twisted by the flat orientation line bundle $\mathcal{O}$ of X^*. The fiber of $\mathcal{O}$ over $x \in X^*$ is $\mathrm{H}^n(X^*, X^* - x; \mathbb{R})$. If X^* is orientable then with a given orientation c, the homomorphism ρ_c to the $\mathcal{O}$-valued differential forms on X^* can be identified with the ρ of before. If c' is a different orientation then in the applications, $\rho_{c'}(\omega')$ will be equivalent to the result of changing $\rho_c(\omega')$ by a sign on the components of X^* where c' differs from c. We write $\Omega^*_{\mathrm{Lip}}(X; \mathcal{O})$ for the elements of $\Omega^*_{\mathrm{Lip}}(X)$ when we consider them to be twisted by $\mathcal{O}$.

If X^* is not orientable then we only consider the case when X is a boundaryless Alexandrov space. There is a notion of an orientation cover $\widehat{X}$ of X [16]. It is also an Alexandrov space and is equipped with a $\mathbb{Z}_2$-action whose quotient is X. Choose an orientation on the connected space $\widehat{X}^*$. Then $\Omega^*_{\mathrm{Lip}}(X; \mathcal{O}) \cong \Omega^*_{\mathrm{Lip}}(\widehat{X}) \otimes_{\mathbb{Z}_2} \mathbb{R}$, where $\mathbb{R}$ has the nontrivial representation of $\mathbb{Z}_2$. (The papers [16,23] discuss various equivalent notions of orientability for Alexandrov spaces.)

In the rest of the paper we will only discuss the case when X^* is oriented, as the nonorientable case can be handled by working $\mathbb{Z}_2$-equivariantly on $\widehat{X}$.

Lemma 2.2 *If $\omega' \in \Omega^{n-1}_{\mathrm{Lip}}(X; \mathcal{O})$ is such that $\rho(\omega')$ and $\rho(d\omega')$ have compact support in X^* then $\int_{X^*} \rho(d\omega') = 0$.*

Proof Put $K = \mathrm{supp}(\rho(\omega')) \cup \mathrm{supp}(\rho(d\omega'))$. Let $\{U_i\}_{i=1}^N$ be relatively compact coordinate neighborhoods of X^* that cover K. Let $\{\phi_i\}_{i=1}^N$ be nonnegative subordinate Lipschitz functions whose sum is one on K. Write ω' as a finite sum

$\sum_j f_0^j df_1^j \dots df_{n-1}^j$. Then

$$\rho(\omega') = \sum_{i=1}^{N} \phi_i \rho(\omega') = \sum_{i=1}^{N} \sum_j \rho\left(\phi_i f_0^j df_1^j \dots df_{n-1}^j\right) \tag{2.3}$$

and

$$\rho(d\omega') = \sum_{i=1}^{N} (d\phi_i \wedge \rho(\omega') + \phi_i \rho(d\omega')) = \sum_{i=1}^{N} \sum_j \rho\left(d(\phi_i f_0^j) df_1^j \dots df_{n-1}^j\right). \tag{2.4}$$

Hence it suffices to prove the lemma with f_0^j replaced by $\phi_i f_0^j$, for some fixed i. Choose a Lipschitz function η_i with compact support in U_i so that $\eta_i \phi_i = \phi_i$, i.e. η_i is one on $\mathrm{supp}(\phi_i)$. Then

$$\begin{aligned} &d(\phi_i f_0^j) d(\eta_i f_1^j) \dots d(\eta_i f_{n-1}^j) \\ &\quad = ((d\phi_i) f_0^j + \phi_i df_0^j) \cdot \left((d\eta_i) f_1^j + \eta_i df_1^j\right) \dots \left((d\eta_i) f_{n-1}^j + \eta_i df_{n-1}^j\right) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} &\rho\left(d(\phi_i f_0^j) d(\eta_i f_1^j) \dots d(\eta_i f_{n-1}^j)\right) \\ &\quad = ((d\phi_i)\rho(f_0^j) + \phi_i \rho(df_0^j)) \wedge ((d\eta_i)\rho(f_1^j) \\ &\qquad + \eta_i \rho(df_1^j)) \wedge \dots \wedge (d\eta_i \wedge \rho(f_{n-1}^j) + \eta_i \rho(df_{n-1}^j)) \\ &\quad = ((d\phi_i)\rho(f_0^j) + \phi_i \rho(df_0^j)) \wedge \rho(df_1^j) \wedge \dots \wedge \rho(df_{n-1}^j) \\ &\quad = \rho(d(\phi_i f_0^j) df_1^j \dots df_{n-1}^j)). \end{aligned}$$

Hence we can reduce the lemma to the case when each of $f_0^j, f_1^j, \dots, f_{n-1}^j$ has compact support in U_i. Using Euclidean coordinates on U_i, we can mollify the functions by convolution and take the mollification parameter to zero, to reduce to the case when $f_0^j, f_1^j, \dots, f_{n-1}^j$ are smooth functions of the coordinates, in which case the lemma is evident. □

It is not immediately clear that $\rho(d\omega')$ is determined by $\rho(\omega')$, but this turns out to be the case.

Lemma 2.7 *Given $\omega'_1, \omega'_2 \in \Omega^p_{\mathrm{Lip}}(X)$, if $\rho(\omega'_1) = \rho(\omega'_2)$ then $\rho(d\omega'_1) = \rho(d\omega'_2)$*

Proof It is equivalent to show that if $\rho(\omega') = 0$ then $\rho(d\omega') = 0$. Suppose that $\rho(\omega') = 0$. For any $\omega \in \Omega^{n-p-1}_{\mathrm{Lip}}(X; \mathcal{O})$ such that $\rho(\omega)$ and $\rho(d\omega)$ have compact support in X^*, Lemma 2.2 implies that

$$0 = \int_{X^*} \rho(d(\omega' \wedge \omega)) = \int_{X^*} \rho(d\omega') \wedge \rho(\omega). \tag{2.8}$$

Let U be a relatively compact coordinate neighborhood for X^*. Let F be a Lipschitz function with support in U. Let ϕ be a Lipschitz function with support in U that is identically one on $\mathrm{supp}(F)$. Put $\omega = F d(\phi x^{i_1}) \dots d(\phi x^{i_p})$. Then $\rho(\omega) = F dx^{i_1} \wedge \dots \wedge dx^{i_p}$. Letting ω vary over such choices, the lemma follows. □

Lemma 2.9 *If* $\omega' \in \Omega^p_{\mathrm{Lip}}(X)$ *then* $\mathrm{supp}(\rho(d\omega')) \subset \mathrm{supp}(\rho(\omega'))$.

Proof Suppose that $\mathrm{supp}(\rho(d\omega'))$ is not contained in $\mathrm{supp}(\rho(\omega'))$. Let $\omega \in \Omega^{n-p-1}(X; \mathcal{O})$ be such that $\rho(\omega)$ and $\rho(d\omega)$ have support in a relatively compact coordinate neighborhood of $X^* - \mathrm{supp}(\rho(\omega'))$; such ω can be constructed as in the proof of Lemma 2.7. Then

$$\begin{aligned} 0 &= \int_{X^*} \rho(d(\omega' \wedge \omega)) = \int_{X^*} \big(\rho(d\omega') \wedge \rho(\omega) + (-1)^p \rho(\omega') \wedge \rho(d\omega)\big) \\ &= \int_{X^*} \rho(d\omega') \wedge \rho(\omega). \end{aligned} \tag{2.10}$$

Letting ω vary over such choices gives a contradiction. □

For brevity, we will write ω' for $\rho(\omega')$ on X^*, and $d\omega'$ for $\rho(d\omega')$ on X^*. In what follows, this should not cause confusion.

2.2 L^2-complex

Let $\Omega^*_{L^2}(X)$ be the L^2-differential forms on X^*. There is a well-defined integration $\int_{X*} : \Omega^n_{L^2}(X; \mathcal{O}) \to \mathbb{R}$.

The map ρ sends $\Omega^*_{\mathrm{Lip}}(X)$ to $\Omega^*_{L^2}(X)$.

Lemma 2.11 *The image of* ρ *is dense in* $\Omega^*_{L^2}(X)$.

Proof As in the proof of Lemma 2.7, let U be a relatively compact coordinate neighborhood and let F be a Lipschitz function with support in U. Then $F dx^{i_1} \wedge \dots \wedge dx^{i_p}$ is in the image of ρ, from which the lemma follows. □

Let $\Omega^p_{L^2,d}(X)$ be the elements $\omega \in \Omega^p_{L^2}(X)$ for which there is some $\eta \in \Omega^{p+1}_{L^2}(X)$ so that for all $\omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}(X; \mathcal{O})$, we have

$$\int_{X^*} \Big(d\omega' \wedge \omega + (-1)^{n-p-1} \omega' \wedge \eta\Big) = 0. \tag{2.12}$$

If such an η exists then it is unique, and we put $d\omega = \eta$. This defines a map $d : \Omega^p_{L^2,d}(X) \to \Omega^{p+1}_{L^2}(X)$. (In the case when X is a smooth closed Riemannian manifold, the definition of d is similar to how one defines the maximal closed extension of the exterior derivative on smooth forms.)

Example 2.13 If $X = [0, 1]$ then $\Omega^0_{L^2,d}([0, 1]) = \{f \in H^1([0, 1]) : f(0) = f(1) = 0\}$ and $\Omega^1_{L^2,d}([0, 1]) = \Omega^1_{L^2}([0, 1])$. More generally, if X is a smooth compact Riemannian manifold-with-boundary, with boundary inclusion $i : \partial X \to X$, then an element ω of $\Omega^*_{L^2,d}(X)$ has a well-defined restriction $i^*\omega$ in $\Omega^*(\partial X)/\overline{\mathrm{Im}(d)}$ that vanishes. These are relative (Dirichlet) boundary conditions.

Remark 2.14 If we replace X^* by an open subset of X^* with full measure then $\Omega^p_{L^2}(X)$ and $\Omega^p_{L^2,d}(X)$ do not change. (This would not be the case if we required ω' to have support in a compact subset of X^*.) As a consequence, $\Omega^p_{L^2}(X)$ and $\Omega^p_{L^2,d}(X)$ are independent of the choice of X^*. Namely, if X^*_1 and X^*_2 are two different choices then in each case, the ensuing spaces $\Omega^p_{L^2}(X)$ and $\Omega^p_{L^2,d}(X)$ are the same as those coming from $X^*_3 = X^*_1 \cap X^*_2$.

In particular, if X is a compact n-dimensional Alexandrov space, let X^*_δ be the subspace of Example 2.1. If $\delta' < \delta$ then $X^*_{\delta'}$ is an open subset of X^*_δ with full measure. Hence the notions of $\Omega^p_{L^2}(X)$ and $\Omega^p_{L^2,d}(X)$ are independent of δ.

Lemma 2.15 *The subspace $\Omega^p_{L^2,d}(X)$ is dense in $\Omega^p_{L^2}(X)$.*

Proof It suffices to prove the corresponding statement for the elements of $\Omega^p_{L^2}(X)$ with support in a fixed but arbitrary compact set $K \subset X^*$. If $\{\sigma_i\}_{i=1}^N$ are Lipschitz functions on X and $\{\omega_i\}_{i=1}^N$ are elements of $\Omega^p_{L^2,d}(X)$ then one can check that $\sum_{i=1}^N \sigma_i\omega_i \in \Omega^p_{L^2,d}(X)$, with $d\sum_{i=1}^N \sigma_i\omega_i = \sum_{i=1}^N (d\sigma_i \wedge \omega_i + \sigma_i d\omega_i)$. Using a covering of K by relatively compact coordinate neighborhoods of X^*, and nonnegative subordinate Lipschitz functions whose sum is one on K, we can reduce to the case when K is a closed ball in a fixed coordinate neighborhood. Considering forms with support in K that are smooth with respect to the given coordinates, the lemma follows. □

Lemma 2.16 *The operator $d : \Omega^p_{L^2,d}(X) \to \Omega^{p+1}_{L^2}(X)$ is closed.*

Proof Suppose that $\{\omega_i\}_{i=1}^\infty$ is a sequence in $\Omega^p_{L^2,d}(X)$ so that there is a limit of pairs $\lim_{i\to\infty}(\omega_i, d\omega_i) = (\omega_\infty, \eta_\infty)$ for some $(\omega_\infty, \eta_\infty) \in \Omega^p_{L^2}(X) \oplus \Omega^{p+1}_{L^2}(X)$. Replacing ω in (2.12) by ω_i and passing to the limit shows that $\omega_\infty \in \Omega^p_{L^2,d}(X)$ and $\eta_\infty = d\omega_\infty$. This proves the lemma. □

Lemma 2.17 *The image of $d : \Omega^p_{L^2,d}(X) \to \Omega^{p+1}_{L^2}(X)$ lies in $\Omega^{p+1}_{L^2,d}(X)$, and $\overline{\mathrm{Im}(d)} \subset \mathrm{Ker}(d)$.*

Proof Given $\omega \in \Omega^p_{L^2,d}(X)$, replacing ω' in (2.12) with $d\omega'$ gives

$$\int_{X^*} d\omega' \wedge d\omega = 0 \tag{2.18}$$

for all $\omega' \in \Omega^{n-p-2}_{\mathrm{Lip}}(X; \mathcal{O})$. It follows that $\mathrm{Im}(d) \subset \Omega^{p+1}_{L^2,d}(X)$ and $d^2 = 0$. Since d is a closed operator, $\mathrm{Ker}(d)$ is a closed subset of $\Omega^{p+1}_{L^2}(X)$. Hence $\overline{\mathrm{Im}(d)} \subset \mathrm{Ker}(d)$. □

2.3 Differential form Laplacian

Define a quadratic form on $\Omega^p_{L^2,d}(X)/\overline{\mathrm{Im}(d)} \subset \Omega^p_{L^2}(X)/\overline{\mathrm{Im}(d)}$ by

$$Q(\omega) = \int_{X^*} \langle d\omega, d\omega \rangle \ \mathrm{dvol}_{X^*}. \tag{2.19}$$

As d is closed, it follows that Q is a closed quadratic form in the sense of [28, Section VIII.6]. There is a corresponding self-adjoint operator $\triangle_p = d^* d$, densely defined on $\Omega^p_{L^2}(X)/\overline{\mathrm{Im}(d)}$ [28, Theorem VIII.15], [29, Theorem X.25]. Its domain is

$$\mathrm{Dom}(\triangle_p) = \left\{ \omega \in \Omega^p_{L^2,d}(X)/\overline{\mathrm{Im}(d)} \ : \ d\omega \in \mathrm{Dom}(d^*) \right\}. \tag{2.20}$$

We have isometric isomorphisms

$$\Omega^*_{L^2}(X) \cong \left(\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)} \right) \oplus \overline{\mathrm{Im}(d)} \tag{2.21}$$

and

$$\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)} \cong \left(\Omega^*_{L^2}(X)/\mathrm{Ker}(d) \right) \oplus \left(\mathrm{Ker}(d)/\overline{\mathrm{Im}(d)} \right). \tag{2.22}$$

Using the isomorphisms

$$\Omega^*_{L^2}(X)/\mathrm{Ker}(d) \cong (\mathrm{Ker}(d))^\perp \tag{2.23}$$

and

$$\mathrm{Ker}(d)/\overline{\mathrm{Im}(d)} \cong \mathrm{Ker}(\triangle), \tag{2.24}$$

we obtain an orthogonal decomposition

$$\Omega^*_{L^2}(X) \cong (\mathrm{Ker}(d))^\perp \oplus \mathrm{Ker}(\triangle_*) \oplus \overline{\mathrm{Im}(d)}. \tag{2.25}$$

We have already defined $\triangle_p$ on

$$\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)} \cong (\mathrm{Ker}(d))^\perp \oplus \mathrm{Ker}(\triangle_*). \tag{2.26}$$

We can define the Laplacian on $\overline{\mathrm{Im}(d)} \subset \Omega^*_{L^2}(X)$ by using the isomorphism d : $\Omega^{*-1}_{L^2}(X)/\mathrm{Ker}(d) \to \mathrm{Im}(d)$ to transfer the Laplacian from $\Omega^{*-1}_{L^2}(X)/\mathrm{Ker}(d)$. In this way, there is a Laplacian $\triangle_*^{Hodge}$ on $\Omega^*_{L^2}(X)$ with a full Hodge decomposition. Using this Hodge decomposition, spectral questions for $\triangle_*^{Hodge}$ reduce to spectral questions about the Laplacian $\triangle_*$ on $\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)}$. In the rest of the paper, we mostly concentrate on the latter Laplacian.

Example 2.27 If X is a compact Riemannian manifold-with-boundary then $\triangle_p$ is the densely-defined Laplacian on the Hilbert space $\Omega^p_{L^2}(X)/\overline{\mathrm{Im}(d)}$, with relative (Dirichlet) boundary conditions, e.g. $\mathrm{Ker}(\triangle_p) \cong \mathrm{H}^p(X, \partial X; \mathbb{R})$.

Example 2.28 We give an example in which $\triangle_p$ has an infinite dimensional kernel. Start with the cone $(0, 1] \times S^3$, equipped with the metric $g = dr^2 + r^2 g_{S^3}$. Glue a 4-ball onto the S^3-boundary. Choose some point $m \in S^3$ and for each $i > 1$, perform a connected sum with a copy of $\mathbb{C}P^2$, with size comparable to 100^{-i}, at the point (i^{-1}, m) in the conical region. Call the result X^* and let X be its 1-point compactification. Now $\mathrm{Im}\left(\mathrm{H}^p_c(X^*) \to \mathrm{H}^p(X^*)\right)$ injects into $\mathrm{Ker}(d : \Omega^p_{L^2,d}(X) \to \Omega^{p+1}_{L^2,d}(X))/\overline{\mathrm{Im}(d : \Omega^{p-1}_{L^2,d}(X) \to \Omega^p_{L^2,d}(X))}$; see [22, Proposition 4], whose proof does not need completeness of X^*. As $\mathrm{Im}\left(\mathrm{H}^2_c(X^*) \to \mathrm{H}^2(X^*)\right)$ is infinite dimensional, it follows that $\mathrm{Ker}(\triangle_2)$ is infinite dimensional.

Remark 2.29 An alternative differential form Laplacian can be defined using the closure of the differential on $\Omega^*_{\mathrm{Lip}}(X)$ (sometimes called the minimal closed extension). Namely, say that an element $\omega \in \Omega^*_{L^2}(X)$ lies in $\mathrm{Dom}(d)$ if there is a sequence $\omega_i \in \Omega^*_{\mathrm{Lip}}(X)$ such that $\lim_{i\to\infty} \omega_i = \omega$ in $\Omega^*_{L^2}(X)$, and $\lim_{i\to\infty} d\omega_i$ exists in $\Omega^{*+1}_{L^2}(X)$. If this is the case, put $d\omega = \lim_{i\to\infty} d\omega_i$; it is independent of the particular choice of $\{\omega_i\}_{i=1}^\infty$. Then d is a closed operator and one can consider d^*d, acting on $\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)}$.

If X is a compact Riemannian manifold-with-boundary then one recovers the differential form Laplacian on $\Omega^*_{L^2}(X)/\overline{\mathrm{Im}(d)}$, with absolute (Neumann) boundary conditions, this way. The differential form Laplacian $\triangle_*$, as defined following (2.19), is more convenient for the purposes of this paper, as will be seen in the proof of Theorem 1.6.

Remark 2.30 If X is a smooth closed Riemannian manifold then there is a Hodge Laplacian $\triangle_p^{Hodge} = dd^* + d^*d$ acting on the H^2-regular p-forms. The space of H^2-regular p-forms is independent of the particular Riemannian metric.

If X is a closed Riemannian Lipschitz manifold then a Hodge Laplacian $\triangle_*^{Hodge}$, with dense domain in $\Omega^*_{L^2}(X)$, was defined in [33]. However, there are some subtleties. For example, the corresponding quadratic form

$$Q^{Hodge}(\omega) = \int_X \left(\langle d\omega, d\omega\rangle + \langle d^*\omega, d^*\omega\rangle\right) \mathrm{dvol}_X \tag{2.31}$$

has domain $\mathrm{Dom}(Q^{Hodge}) = \{\omega \in \Omega^*_{L^2}(X) : d\omega \in \Omega^{*+1}_{L^2}(X), d * \omega \in \Omega^{n-*+1}_{L^2}(X)\}$. Due to the appearance of the Hodge duality operator $*$ in $d * \omega$, the domain of Q^{Hodge} definitely depends on the precise L^∞_{loc}-Riemannian metric used on X [33, p. 46].

In contrast, the quadratic form Q of (2.19) has domain $\Omega^*_{L^2,d}(X)/\overline{\mathrm{Im}(d)}$ which, in the case of a closed Riemannian Lipschitz manifold, is independent of the precise Riemannian metric. Hence the domain of $\triangle_*$ is also independent of the precise Riemannian metric. This is one manifestation of the fact that $\triangle_*$ has better biLipschitz properties than $\triangle_*^{Hodge}$.

3 L^2-sheaf

In this section we define a sheaf $\Omega^*_{L^2_{loc},d}$ of differential graded complexes, constructed from certain locally-L^2 differential forms. The result of this section will be used in Sect. 5.

We continue with the setup of Sect. 2. In particular, X is a compact metric space and $X^* \subset X$ is an open subset of full Hausdorff measure, with the structure of a Riemannian Lipschitz manifold.

Given an open set $U \subset X$, let $\Omega^*_{L^2_{loc}}(U)$ be the locally-L^2 differential forms on $U^* = X^* \cap U$. Let $\Omega^p_{L^2_{loc},d}(U)$ be the elements $\omega \in \Omega^p_{L^2_{loc}}(U)$ for which there is some $\eta \in \Omega^{p+1}_{L^2_{loc}}(U)$ so that for all compact subsets $K \subset U$ and all $\omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}(X; \mathcal{O})$ with support in K, we have

$$\int_{U^*} \left(d\omega' \wedge \omega + (-1)^{n-p-1} \omega' \wedge \eta \right) = 0. \tag{3.1}$$

If such an η exists then it is unique, and we put $d\omega = \eta$. Note that an element of $\Omega^p_{L^2_{loc},d}(U)$ may not satisfy relative boundary conditions in any sense.

Lemma 3.2 *The assignment $U \to \Omega^p_{L^2_{loc},d}(U)$ defines a sheaf $\Omega^p_{L^2_{loc},d}$ on X.*

Proof Given $V \subset U$, there is clearly a restriction map $r_{V,U} : \Omega^p_{L^2_{loc},d}(U) \to \Omega^p_{L^2_{loc},d}(V)$ that defines a presheaf. Let $\{U_\alpha\}$ be an open covering of U. Given elements $\omega_1, \omega_2 \in \Omega^p_{L^2_{loc},d}(U)$, if $r_{U_\alpha,U}(\omega_1) = r_{U_\alpha,U}(\omega_2)$ for all α, then $\omega_1 = \omega_2$.

Now suppose that $\omega_\alpha \in \Omega^p_{L^2_{loc},d}(U_\alpha)$ are such that $r_{U_\alpha \cap U_\beta, U_\alpha}(\omega_\alpha) = r_{U_\alpha \cap U_\beta, U_\beta}(\omega_\beta)$ for all α and β. There is a unique $\omega \in \Omega^p_{L^2_{loc}}(U)$ so that $\omega_\alpha = r_{U_\alpha,U}(\omega)$ for all α. We have to show that $\omega \in \Omega^p_{L^2_{loc},d}(U)$. Let K be a compact subset of U. Then K is covered by a finite subset $\{U_i\}_{i=1}^N$ of the U_α's. Let $\{\sigma_i\}_{i=1}^N$ be nonnegative subordinate Lipschitz functions whose sum is one on K. In particular, the support of σ_i is a compact set $K_i \subset U_i$. For any $\omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}(X; \mathcal{O})$ with support in K, since $\sigma_i \omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}(X; \mathcal{O})$ has support in U_i, and $\omega_i \in \Omega^p_{L^2_{loc},d}(U_i)$, we have

$$\int_{U_i^*} \left(d(\sigma_i \omega') \wedge \omega + (-1)^{n-p-1} \sigma_i \omega' \wedge d\omega_i \right) = 0. \tag{3.3}$$

Summing over i gives

$$\int_{U^*} \left(d\omega' \wedge \omega + (-1)^{n-p-1} \omega' \wedge \sum_{i=1}^N \sigma_i d\omega_i \right) = 0. \tag{3.4}$$

Thus $\omega \in \Omega^p_{L^2_{loc},d}(U)$, with $d\omega = \sum_{i=1}^N \sigma_i d\omega_i$. This proves the lemma. □

4 Eigenvalue estimates

In this section we prove the eigenvalue estimates of Theorems 1.3 and 1.6. In both cases, the proof is by a contradiction argument.

As mentioned in Remark 1.11, Theorem 1.3 actually follows from Theorem 1.6. As the proof of Theorem 1.3 makes the strategy clearer, we give it first, in Sect. 4.1. To get the upper eigenvalue bound, we pullback forms from a nice subset of a limiting Alexandrov space that has the same dimension n as the manifolds. In Sect. 4.2 we prove Theorem 1.6. The proof is more involved, in that we need to pullback forms from an Alexandrov space that may be of a lower dimension. We show that there is a region in the limiting Alexandrov space that is almost Euclidean, and above which the approximating manifold has a fibration structure with controlled geometry. We then pullback forms as in the proof of Theorem 1.3.

4.1 Noncollapsing case

Let X be a compact metric space of the type considered in Sect. 2. Given $k \in \mathbb{Z}^+$ and $p \in [0, n]$, put

$$\lambda_{k,p}(X) = \inf_{V_{k,p}} \sup_{\psi \in V_{k,p}, \psi \neq 0} \frac{\|d\psi\|^2}{\|\psi\|^2}, \tag{4.1}$$

where $V_{k,p}$ ranges over k-dimensional subspaces of $\Omega^p_{L^2,d}(X)/\operatorname{Ker}(d)$. If $\triangle_p$ has discrete spectrum with finite multiplicities then $\lambda_{k,p}(X)$ is the k^{th} eigenvalue of $\triangle_p$ on $\Omega^p_{L^2}(X)/\operatorname{Ker}(d)$, counted with multiplicity [30, Theorem XIII.2].

Lemma 4.2 *If X_1 and X_2 are C-biLipschitz then*

$$C^{-2n-4p-2}\lambda_{k,p}(X_1) \leq \lambda_{k,p}(X_2) \leq C^{2n+4p+2}\lambda_{k,p}(X_1) \tag{4.3}$$

Proof By reducing X_1^* and X_2^* if necessary, we can assume that the biLipschitz map $h : X_1 \to X_2$ restricts to a biLipschitz map between X_1^* and X_2^*. Then on X_1^*, we have $C^{-2}g_1 \leq h^*g_2 \leq C^2 g_1$. There is a bounded pullback map $h^* : \Omega^*_{L^2}(X_2) \to \Omega^*_{L^2}(X_1)$. One can check that h^* sends $\Omega^*_{L^2,d}(X_2)$ to $\Omega^*_{L^2,d}(X_1)$, with $h^*d = dh^*$. Hence there is also a pullback $h^* : \Omega^p_{L^2,d}(X_2)/\operatorname{Ker}(d_{X_2}) \to \Omega^p_{L^2,d}(X_1)/\operatorname{Ker}(d_{X_1})$.

Given a k-dimensional subspace $V_{k,p}$ of $\Omega^p_{L^2,d}(X_2)/\operatorname{Ker}(d_{X_2})$ and a nonzero element $\widetilde{\psi} \in V_{k,p}$, let $\psi \in \Omega^p_{L^2,d}(X_2)$ be a representative for $\widetilde{\psi}$. Then

$$\begin{aligned}
\|dh^*\widetilde{\psi}\|^2 &= \int_{X_1^*} \langle dh^*\psi, dh^*\psi\rangle_{g_1} \,\mathrm{dvol}_{X_1^*} = \int_{X_1^*} \langle h^*d\psi, h^*d\psi\rangle_{g_1} \,\mathrm{dvol}_{X_1^*} \\
&\le C^{2(p+1)} C^n \int_{X_1^*} \langle h^*d\psi, h^*d\psi\rangle_{h^*g_2} h^* \,\mathrm{dvol}_{X_2^*} \\
&= C^{2(p+1)+n} \int_{X_2^*} \langle d\psi, d\psi\rangle_{g_2} \,\mathrm{dvol}_{X_2^*} \\
&= C^{2(p+1)+n} \|d\widetilde{\psi}\|^2
\end{aligned} \tag{4.4}$$

and

$$\begin{aligned}
\|h^*\widetilde{\psi}\|^2 &= \inf_{\tau\in\operatorname{Ker}(d_{X_1})} \|h^*\psi+\tau\|^2_{L^2} = \inf_{\sigma\in\operatorname{Ker}(d_{X_2})} \|h^*(\psi+\sigma)\|^2_{L^2} \\
&= \inf_{\sigma\in\operatorname{Ker}(d_{X_2})} \int_{X_1^*} \langle h^*(\psi+\sigma), h^*(\psi+\sigma)\rangle_{g_1} \,\mathrm{dvol}_{X_1^*} \\
&\ge C^{-2p} C^{-n} \inf_{\sigma\in\operatorname{Ker}(d_{X_2})} \int_{X_1^*} \langle h^*(\psi+\sigma), h^*(\psi+\sigma)\rangle_{h^*g_2} h^* \,\mathrm{dvol}_{X_2^*} \\
&= C^{-2p-n} \inf_{\sigma\in\operatorname{Ker}(d_{X_2})} \int_{X_2^*} \langle \psi+\sigma, \psi+\sigma\rangle_{g_2} \,\mathrm{dvol}_{X_2^*} = C^{-2p-n} \|\widetilde{\psi}\|^2.
\end{aligned} \tag{4.5}$$

It follows that

$$C^{-2n-4p-2}\lambda_{k,p}(X_1) \le \lambda_{k,p}(X_2). \tag{4.6}$$

Reversing the roles of X_1 and X_2 gives (4.3). This proves the lemma. □

We now prove a version of Theorem 1.3 for compact Alexandrov spaces.

Proposition 4.7 *Given $n \in \mathbb{Z}^+$, $K \in \mathbb{R}$ and $v > 0$, there is some $L = L(n, K, v) < \infty$ so that for any n-dimensional compact connected Alexandrov space X for which*

1. *the curvature of X is bounded below by K,*
2. *$\operatorname{diam}(X) \le 1$ and*
3. *$\operatorname{vol}(X) \ge v$,*

and any $p \in [0, n-1]$ and $k \in \mathbb{Z}^+$, we have $\lambda_{k,p}(X) \le Lk^{\frac{2}{n}}$. Here $\lambda_{k,p}(X)$ is defined by (4.1).

Proof Suppose that the claim about $\lambda_{k,p}$ is not true. Then there is a sequence $\{X_i\}_{i=1}^\infty$ of n-dimensional Alexandrov spaces and some $p \in [0, n-1]$ so that

1. the curvature of X_i is bounded below by K,
2. $\operatorname{diam}(X_i) \le 1$, and
3. $\operatorname{vol}(X_i) \ge v$, but

4. $\lambda_{k_i,p}(X_i) \geq i k_i^{\frac{2}{n}}$ for some $k_i \in \mathbb{Z}^+$.

Let c_i be the smallest integer such that $k_i \leq 2^{nc_i}$. Then

$$\lambda_{2^{nc_i},p}(X_i) \geq \lambda_{k_i,p}(X_i) \geq i k_i^{\frac{2}{n}} \geq i \left(2^{n(c_i-1)}\right)^{\frac{2}{n}} = \frac{i}{4} 4^{c_i}. \tag{4.8}$$

Putting $\overline{k}_i = 2^{nc_i}$, we have $\lambda_{\overline{k}_i,p}(X_i) \geq \frac{1}{4} i \overline{k}_i^{\frac{2}{n}}$.

After passing to a subsequence, we can assume that $\lim_{i\to\infty} X_i = X_\infty$ in the Gromov–Hausdorff topology, where X_∞ is also an n-dimensional Alexandrov space. Let x_∞ be a regular point of X_∞. Then there is a neighborhood U_∞ of x_∞, that is biLipschitz homeomorphic to $(0, 1)^n \subset \mathbb{R}^n$, such that for large i, there are

- points $x_i \in X_i$,
- neighborhoods U_i of x_i, and
- bijective maps $\sigma_i : U_i \to U_\infty$

that are uniformly biLipschitz [3, Theorem 9.8].

Letting h_i be the composition of σ_i with the biLipschitz homeomorphism from U_∞ to $(0, 1)^n$, the maps $h_i : U_i \to (0, 1)^n$ are Λ-biLipschitz for some $\Lambda < \infty$ independent of i.

Let ψ be a smooth compactly supported p-form on $(0, 1)^n$ with $\int_{(0,1)^n} |\psi|^2 d^n x = 1$ and $d\psi \neq 0$. Put $E_\psi = \int_{(0,1)^n} |d\psi|^2 \, d^n x$.

Let $\mathrm{Ker}(d_{(0,1)^n})$ be the forms $\mu \in \Omega^p_{L^2}((0, 1)^n)$ such that

$$\int_{(0,1)^n} d\omega' \wedge \mu = 0 \tag{4.9}$$

for all compactly supported $\omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}((0, 1)^n)$. Then $\mathrm{Ker}(d_{(0,1)^n})$ is closed in $\Omega^p_{L^2}((0, 1)^n)$.

Lemma 4.10 *ψ does not lie in* $\mathrm{Ker}(d_{(0,1)^n})$.

Proof By assumption, $d\psi \neq 0$ as a smooth form. Hence we can find some $\omega' \in \Omega^{n-p-1}_{\mathrm{Lip}}((0, 1)^n)$ so that $\int_{(0,1)^n} d\omega' \wedge \psi = (-1)^{n-p} \int_{(0,1)^n} \omega' \wedge d\psi \neq 0$. □

With reference to Lemma 4.10, let N_ψ be the square of the norm of the image of ψ in $\Omega^p_{L^2}((0, 1)^n)/\mathrm{Ker}(d_{(0,1)^n})$.

Let $R_i : (0, 2^{-c_i})^n \to (0, 1)^n$ be multiplication by 2^{c_i}. Under rescaling, $\int_{(0,2^{-c_i})^n} |dR_i^*\psi|^2 \, dx^n = 2^{c_i(2p+2-n)} E_\psi$ and the square norm of $R_i^*\psi$ in $\Omega^p_{L^2}((0, 2^{-c_i})^n)/\mathrm{Ker}(d)$ is $2^{c_i(2p-n)} N_\psi$.

There are $\overline{k}_i = 2^{nc_i}$ disjoint boxes $\{B_j\}_{j=1}^{\overline{k}_i}$ in $(0, 1)^n$, each congruent to $(0, 2^{-c_i})^n$. Let ψ_j be the translate of $R_i^*\psi$ to B_j. Let V_i^∞ be the span of $\{\psi_j\}_{j=1}^{\overline{k}_i}$ in $\Omega^p_{L^2}((0, 1)^n)$, let $h_i^* V_i^\infty$ denote the extension by zero from $\Omega^p_{L^2,d}(U_i)$ to $\Omega^p_{L^2,d}(X_i)$ of the pullback, and let $\widetilde{h_i^* V_i^\infty}$ denote the image of $h_i^* V_i^\infty$ in $\Omega^p_{L^2,d}(X_i)/\mathrm{Ker}(d_{X_i})$. We claim that

$\widetilde{h_i^* V_i^\infty}$ is $\overline{k}_i$-dimensional. To see this, if there is a relation $\sum_{j=1}^{\overline{k}_i} \alpha_j h_i^* \psi_j \in \mathrm{Ker}(d_{X_i})$ then

$$h_i^* \left(\sum_{j=1}^{\overline{k}_i} \alpha_j d\psi_j \right) = d \left(\sum_{j=1}^{\overline{k}_i} \alpha_j h_i^* \psi_j \right) = 0, \tag{4.11}$$

and hence $\sum_{j=1}^{\overline{k}_i} \alpha_j d\psi_j = 0$, which implies that each α_j vanishes.

If $\eta_i = \sum_{j=1}^{\overline{k}_i} \beta_j h_i^* \psi_j$ is a nonzero element of $h_i^* V_i^\infty$ then as in the proof of (4.4), for large i, we have

$$\begin{aligned} \int_{X_i^*} \langle d\eta_i, d\eta_i \rangle_{g_i} \, \mathrm{dvol}_{X_i^*} &\le \Lambda^{2p+2+n} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2 \int_{(0,1)^n} \langle d\psi_j, d\psi_j \rangle_{(0,1)^n} \, \mathrm{dvol}_{(0,1)^n} \\ &= 2^{c_i(2p+2-n)} E_\psi \Lambda^{2p+2+n} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \end{aligned} \tag{4.12}$$

Let $\widetilde{\eta}_i$ be the class of η_i in $\widetilde{h_i^* V_i^\infty}$. We claim that

$$\|\widetilde{\eta}_i\|^2 \ge \frac{1}{4} 2^{c_i(2p-n)} N_\psi \Lambda^{-2p-n} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \tag{4.13}$$

To see this, suppose that $\|\widetilde{\eta}_i\|^2 < \frac{1}{4} 2^{c_i(2p-n)} N_\psi \Lambda^{-2p-n} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2$. Then there is some $\sigma \in \mathrm{Ker}(d_{X_i})$ so that

$$\int_{X_i^*} \langle \eta_i + \sigma, \eta_i + \sigma \rangle_{g_i} \, \mathrm{dvol}_{X_i^*} < \frac{1}{2} 2^{c_i(2p-n)} N_\psi \Lambda^{-2p-n} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2 \tag{4.14}$$

As in (4.5), it follows that

$$\begin{aligned} &\int_{(0,1)^n} \left\langle \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + (h_i^{-1})^* \sigma, \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + (h_i^{-1})^* \sigma \right\rangle \mathrm{dvol}_{(0,1)^n} \\ &\quad < \frac{1}{2} 2^{c_i(2p-n)} N_\psi \sum_{j=1}^{\overline{k}_i} |\beta_j|^2 \end{aligned} \tag{4.15}$$

However,

$$\int_{(0,1)^n} \left\langle \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + (h_i^{-1})^* \sigma, \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + (h_i^{-1})^* \sigma \right\rangle \mathrm{dvol}_{(0,1)^n}$$

$$= \sum_{j=1}^{\overline{k}_i} \int_{B_j} \left\langle \beta_j \psi_j + (h_i^{-1})^* \sigma, \beta_j \psi_j + (h_i^{-1})^* \sigma \right\rangle \mathrm{dvol}_{B_j}$$

$$\geq 2^{c_i(2p-n)} N_\psi \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \tag{4.16}$$

This is a contradiction.

Combining (4.12) and (4.13) gives

$$\lambda_{\overline{k}_i,p}(X_i) \leq 4 \cdot 2^{2c_i} \Lambda^{4p+2n+2} E_\psi N_\psi^{-1} = 4\Lambda^{4p+2n+2} E_\psi N_\psi^{-1} \overline{k}_i^{\frac{2}{n}}. \tag{4.17}$$

For large i, this contradicts the fact that $\lambda_{\overline{k}_i,p}(X_i) \geq \frac{1}{4} i \overline{k}_i^{\frac{2}{n}}$. This proves the proposition. ◻

Corollary 4.18 *Given $n \in \mathbb{Z}^+$, $K \in \mathbb{R}$ and $v > 0$, there is some $L = L(n, K, v) < \infty$ with the following property. Let M be an n-dimensional compact connected Riemannian manifold-with-boundary whose boundary, if nonempty, is convex. Suppose that*

1. *the sectional curvature of M is bounded below by K,*
2. $\mathrm{diam}(M) \leq 1$ *and,*
3. $\mathrm{vol}(M) \geq v$.

Then for any $p \in [0, n]$ and any $k \in \mathbb{Z}^+$, the k^{th} positive eigenvalue of the p-form Laplacian on M, as defined with relative boundary conditions, satisfies $\lambda_{k,p}(M) \leq Lk^{\frac{2}{n}}$.

Proof Proposition 4.7 gives upper bounds on the eigenvalues of the Laplacian on $\Omega^p_{L^2}(M)/\operatorname{Ker}(d)$ for all $p \in [0, n-1]$. From the Hodge decomposition, we also get an upper bound on the eigenvalues of the Laplacian on $\operatorname{Im}(d) \subset \Omega^n_{L^2}(M)$. The corollary follows. ◻

Theorem 1.3 follows from Corollary 4.18.

4.2 General case

We now prove Theorem 1.6.

Proposition 4.19 *Given $n \in \mathbb{Z}^+$ and $K \in \mathbb{R}$, there is some $A = A(n, K) < \infty$ with the following property. Let M be an n-dimensional closed connected Riemannian manifold for which*

1. *the sectional curvature of M is bounded below by K and,*
2. *there is some point $m \in M$ with an s-strainer of quality $\frac{1}{10}$ and size 1, where $1 \leq s \leq n$.*

Then for any $p \in [0, s-1]$ and $k \in \mathbb{Z}^+$, we have $\lambda_{k,p}(M) \leq Ak^{\frac{2}{s}}$. Here $\lambda_{k,p}(M)$ is the k^{th} eigenvalue of the Laplacian on $\Omega^p_{L^2}(M)/\operatorname{Ker}(d)$.

Proof Suppose that the claim about $\lambda_{k,p}$ is not true. Then for some $n \in \mathbb{Z}^+$, some $s \in [1, n]$ and some $p \in [0, s-1]$, there is a sequence $\{M_i\}_{i=1}^{\infty}$ of n-dimensional closed Riemannian manifolds and numbers $k_i \in \mathbb{Z}^+$ so that □

1. the curvature of M_i is bounded below by K and,
2. there is some point $m_i \in M_i$ with an s-strainer of quality $\frac{1}{10}$ and size 1, but
3. $\lambda_{k_i,p}(M_i) \geq ik_i^{\frac{2}{s}}$.

Let c_i be the smallest integer such that $k_i \leq 2^{n!c_i}$. Then

$$\lambda_{2^{n!c_i},p}(M_i) \geq \lambda_{k_i,p}(M_i) \geq ik_i^{\frac{2}{s}} \geq i\left(2^{n!(c_i-1)}\right)^{\frac{2}{s}}. \tag{4.20}$$

Putting $\overline{k}_i = 2^{n!c_i}$, we have $\lambda_{\overline{k}_i,p}(M_i) \geq \frac{i}{4^{\frac{n!}{s}}}\overline{k}_i^{\frac{2}{s}}$.

After passing to a subsequence, we can assume that $\lim_{i\to\infty}(M_i, m_i) = (X_\infty, x)$ in the pointed Gromov-Hausdorff topology, where X_∞ is a complete Alexandrov space, say of dimension n_∞. From the strainer condition, $n_\infty \geq s$. Let x_∞ be a regular point of X_∞, say within distance $\frac{1}{1000}$ from x.

Given $\delta > 0$, we can find a n_∞-strainer $\{a_l, b_l\}_{l=1}^{n_\infty}$ of quality δ around x_∞, say of size r_δ, with $\lim_{\delta\to 0} r_\delta = 0$.

I thank Vitali Kapovitch for the proof of the next lemma.

Lemma 4.21 *If δ is sufficiently small then for large i, the following holds. There are an open subset U_i of M_i, a closed Lipschitz manifold Z_i, a Lipschitz surjection $\tau_i : U_i \to B(0, \delta r_\delta)$ and a commutative diagram* □

$$\begin{array}{ccc} U_i & \xrightarrow{\alpha_i} & Z_i \times B(0, \delta r_\delta) \\ \tau_i \downarrow & & p_2 \downarrow \\ B(0, \delta r_\delta) & \xrightarrow{\mathrm{Id}} & B(0, \delta r_\delta), \end{array} \tag{4.22}$$

where α_i is a biLipschitz homeomorphism. Furthermore, τ_i is an almost metric submersion in the sense of [35, p. 318].

Proof Let $\widetilde{m}_i \in M_i$ be such that $\{\widetilde{m}_i\}_{l=1}^{n_\infty}$ converges to x_∞. Given $\delta > 0$, for large i, let $\{\widetilde{a}_{i,l}, \widetilde{b}_{i,l}\}_{l=1}^{n_\infty}$ be an n_∞-strainer of quality 2δ in M_i such that as $i \to \infty$, $\{\widetilde{a}_{i,l}\}_{l=1}^{n_\infty}$ converges to $\{a_l\}_{l=1}^{n_\infty}$ and $\{\widetilde{b}_{i,l}\}_{l=1}^{n_\infty}$ converges to $\{b_l\}_{l=1}^{n_\infty}$. Define $\gamma_i : B(\widetilde{m}_i, r_\delta) \to \mathbb{R}^{n_\infty}$ by $\gamma_i(p_i) = \{d(\widetilde{m}_i, \widetilde{a}_{i,l}) - d(p_i, \widetilde{a}_{i,l})\}_{l=1}^{n_\infty}$.

Consider the pointed Riemannian manifolds $\{(M_i, r_\delta^{-2}g_i, \widetilde{m}_i)\}_{i=1}^{\infty}$. The rescaled limit space $r_\delta^{-1}X_\infty$ has a strainer of size 1 and quality δ, centered at x_∞. There is some $\rho > 0$ so that as $\delta \to 0$, the ρ-balls around x_∞ in $r_\delta^{-1} X_\infty$ converge in the pointed Gromov-Hausdorff topology to $B(0, \rho) \subset \mathbb{R}^{n_\infty}$. It follows that for any $\epsilon > 0$, there is a $\delta_0 > 0$ so that if $\delta < \delta_0$ then for all large i, the map $r_\delta^{-1} \gamma_i : B(\widetilde{m}_i, r_\delta) \to \mathbb{R}^{n_\infty}$ defines a pointed ϵ-Gromov-Hausdorff approximation between $(\gamma_i^{-1}(B(0, \rho r_\delta)), \widetilde{m}_i)$ and $(B(0, \rho), 0)$, where $\gamma_i^{-1}(B(0, \rho r_\delta))$ has the restricted metric from $(M_i, r_\delta^{-2} g_i)$. For an appropriate choice of ϵ and taking δ small enough, we can apply [35] in the pointed setting to get the Lipschitz fibration, along with the almost metric submersion property. (In fact, the fibration in [35] is C^1.) As $B(0, \delta r_\delta)$ is contractible, the fibration structure is a product structure. □

Since τ_i is Lipschitz and surjective, for almost all $u \in B(0, \delta r_\delta)$ and almost all $u_i \in \tau_i^{-1}(u)$, the differential $d\tau_i : T_{u_i} M_i \to \mathbb{R}^{n_\infty}$ is defined. Given such a point $u_i \in U_i$, put $V_{u_i} = \mathrm{Ker}((d\tau_i)_{u_i})$, an $(n - n_\infty)$-dimensional subspace of $T_{u_i} U_i$. Put $H_{u_i} = V_{u_i}^\perp$. The “almost metric submersion” property implies that if δ is small enough then for all large i, we have

$$\frac{1}{2}|v| \leq |(d\tau_i)_{u_i} v| \leq 2|v|. \tag{4.23}$$

for all $v \in H_{u_i}$. It follows that for $\eta \in \Lambda^p(T_u^* B(0, \delta r_\delta))$, we have

$$2^{-p}|\eta| \leq |(d\tau_i)_{u_i}^* \eta| \leq 2^p |\eta|. \tag{4.24}$$

Lemma 4.25 *There is some $\widehat{C} < \infty$ so that for all sufficiently large i and all $u, u' \in B(0, \delta r_\delta)$, we have*

$$\widehat{C}^{-1} \operatorname{vol}\left(\tau_i^{-1}(u)\right) \leq \operatorname{vol}\left(\tau_i^{-1}(u')\right) \leq \widehat{C} \operatorname{vol}\left(\tau_i^{-1}(u)\right). \tag{4.26}$$

Proof The proof uses gradient flow. We sketch the argument, which is similar to [18, Pf. of Lemma 6.15]. Write $\tau_i = (\xi_{i,1}, \ldots, \xi_{i,n_\infty})$ and $u = (a_1, \ldots, a_{n_\infty})$. From the construction of τ_i using distance functions from strainer points, the functions $\xi_{i,\cdot}$ are quantitatively semiconcave, independent of i. Put $H_-(u) = \{\widehat{u} = (b_1, \ldots, b_{n_\infty}) \in \mathbb{R}^{n_\infty} : b_l \leq a_l \text{ for } 1 \leq l \leq n_\infty\}$. Given $m_i \in \tau_i^{-1}(u')$, we can perform a gradient flow starting from m_i with respect to the gradient of the distance function from an appropriate point in M_i, for a controlled amount of time, to ensure that the result lies in $\tau_i^{-1}(H_-(u))$. Then we perform a gradient flow with respect to the gradient of $F = \min(0, \xi_{i,1} - a_1, \ldots, \xi_{i,n_\infty} - a_{n_\infty})$. After a controlled amount of time, the result of the flow lies in $\tau_i^{-1}(u)$. This gives a map $L_{u',u} : \tau_i^{-1}(u') \to \tau_i^{-1}(u)$. Using the control on the semiconcavity of the distance functions and of F, along with the ensuing distortion bounds for gradient flow, we obtain a bound on the Lipschitz constant of $L_{u',u}$ that is independent of u, u' and i. Replacing u' by a point u'' moving along a line from u' to u, and performing the same construction, shows that if the fibers are orientable then $L_{u',u}$ has degree one; if the fibers are not orientable then we pass to orientable double covers of the fibers and apply the same arguments. In all, from the

Lipschitz bound on $L_{u',u}$, we obtain a bound $\frac{\mathrm{vol}(\tau_i^{-1}(u))}{\mathrm{vol}(\tau_i^{-1}(u'))} \leq \widehat{C}$ with $\widehat{C}$ independent of u, u' and i, thereby giving the first inequality in (4.26). Reversing the roles of u and u' gives the second inequality in (4.26). □

We continue with the proof of Proposition 4.19. Let $\mu_\delta : U_\delta \to (0,1)^{n_\infty}$ be linear coordinates for a neighborhood U_δ of 0 in $B(0, \delta r_\delta)$. We redefine U_i to be $\tau_i^{-1}(U_\delta)$ and put $h_i = \mu_\delta \circ \tau_i : U_i \to (0,1)^{n_\infty}$.

Lemma 4.27 *There exist $\Delta < \infty$ and, for sufficiently large i, a closed element $\chi_i \in \Omega^{n-n_\infty}_{\mathrm{Lip}}(U_i; \mathcal{O}_{U_i})$ so that for all $u \in (0,1)^{n_\infty}$, we have $\int_{h_i^{-1}(u)} \chi_i = 1$, and $\int_{h_i^{-1}(u)} |\chi|^2 \, \mathrm{dvol}_{h_i^{-1}(u)} \leq \Delta \left(\inf_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \right)^{-1}$.*

Proof We can assume that h_i is similarly defined on a slightly larger open set containing U_i, so that the fiber $h_i^{-1}(1, \ldots, 1)$ is well defined. Write $h_i = (h_{i,1} \ldots, h_{i,n_\infty})$. Similarly to the proof of Lemma 4.25, we perform a gradient flow on $\overline{U_i}$ with respect to the gradient of $F = \min(0, h_{i,1} - 1, \ldots, h_{i,n_\infty} - 1)$. After a controlled amount of time, the result of the flow lies in $h_i^{-1}(1, \ldots, 1)$. This gives a deformation retraction $L : \overline{U_i} \to h_i^{-1}(1, \ldots, 1)$. Using the control on the semiconcavity of F, along with the ensuing distortion bounds for gradient flow, we obtain a bound on the Lipschitz constant of F. Let χ_i be the pullback under L of the normalized volume density

$$\mathrm{dvol}_{h_i^{-1}(1,\ldots,1)} / \mathrm{vol}(h_i^{-1}(1, \ldots, 1)) \in \Omega^{n-n_\infty}_{\mathrm{Lip}} \left(h_i^{-1}(1, \ldots, 1); \mathcal{O}_{h_i^{-1}(1,\ldots,1)} \right). \tag{4.28}$$

Then $\chi_i \in \Omega^{n-n_\infty}_{\mathrm{Lip}}(U_i; \mathcal{O}_{U_i})$. The bound on the Lipschitz constant of F, which is independent of i, gives a pointwise bound on χ_i of the form $|\chi_i| \leq \mathrm{const.} \left(\mathrm{vol}(h_i^{-1}(1, \ldots, 1)) \right)^{-1}$. As

$$\begin{aligned} \int_{h_i^{-1}(u)} |\chi_i|^2 \, \mathrm{dvol}_{h_i^{-1}(u)} &\leq \mathrm{const.} \left(\mathrm{vol}(h_i^{-1}(1, \ldots, 1)) \right)^{-2} \sup_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \\ &\leq \mathrm{const.} \, \widehat{C} \left(\inf_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \right)^{-1}, \end{aligned}$$

the lemma follows. □

Define ψ, E_ψ, $\mathrm{Ker}(d_{(0,1)^{n_\infty}})$ and N_ψ as in the proof of Proposition 4.7, with n replaced by n_∞.

Let $R_i : (0, 2^{-\frac{n!}{n_\infty} c_i})^{n_\infty} \to (0,1)^{n_\infty}$ be multiplication by $2^{\frac{n!}{n_\infty} c_i}$. Under rescaling,

$$\int_{(0, 2^{-\frac{n!}{n_\infty} c_i})^{n_\infty}} |d R_i^* \psi|^2 \, dx^{n_\infty} = 2^{\frac{n!}{n_\infty} c_i (2p + 2 - n_\infty)} E_\psi \tag{4.29}$$

and the square norm of $R_i^* \psi$ in $\Omega^p_{L^2}((0, 2^{-\frac{n!}{n_\infty} c_i})^{n_\infty}) / \mathrm{Ker}(d)$ is $2^{\frac{n!}{n_\infty} c_i (2p - n_\infty)} N_\psi$.

There are $\overline{k}_i = 2^{n!c_i}$ disjoint boxes $\{B_j\}_{j=1}^{\overline{k}_i}$ in $(0,1)^{n_\infty}$, each congruent to $(0, 2^{-\frac{n!}{n_\infty}c_i})^{n_\infty}$. Let ψ_j be the translate of $R_i^*\psi$ to B_j. Let V_i^∞ be the span of $\{\psi_j\}_{j=1}^{\overline{k}_i}$ in $\Omega^p_{L^2}((0,1)^{n_\infty})$. Let $h_i^* V_i^\infty$ denote the extension by zero from $\Omega^p_{L^2}(U_i)$ to $\Omega^p_{L^2}(M_i)$ of the pullback.

Lemma 4.30 *For large i, $h_i^* V_i^\infty$ is a subspace of $\Omega^p_{L^2,d}(M_i)$ that projects to a $\overline{k}_i$-dimensional subspace $\widetilde{h_i^* V_i^\infty}$ of $\Omega^p_{L^2,d}(M_i)/\operatorname{Ker}(d_{M_i})$.*

Proof We claim that $h_i^*\psi_j$ is a well-defined element of $\Omega^p_{L^2,d}(M_i)$, with $dh_i^*\psi_j = h_i^* d\psi_j$. Let $\Omega^*_{\mathrm{Lip},c}((0,1)^{n_\infty})$ denote the differential graded algebra constructed from compactly supported Lipschitz functions on $(0,1)^{n_\infty}$. There is a pullback $h_i^* : \Omega^*_{\mathrm{Lip},c}((0,1)^{n_\infty}) \to \Omega^*_{\mathrm{Lip}}(M_i)$ of differential graded algebras. Since ψ_j is a smooth compactly supported p-form on $(0,1)^{n_\infty}$, it follows as in the proof of Lemma 2.7 that there is a $\psi'_j \in \Omega^*_{\mathrm{Lip},c}((0,1)^{n_\infty})$ so that $\rho(\psi'_j) = \psi_j$. Then $h_i^*\psi_j = \rho(h_i^*\psi'_j)$ in $\Omega^p_{L^2}(M_i)$. Using Lemma 2.2, one shows that $h_i^*\psi_j$ lies in $\Omega^p_{L^2,d}(M_i)$, with differential given by $dh_i^*\psi_j = \rho(dh_i^*\psi'_j) = \rho(h_i^* d\psi'_j) = h_i^* d\psi_j$.

The lemma now follows as in the proof of Proposition 4.7. $\square$

For sufficiently small δ and all large i, the ratio $\frac{(h_i)_* \operatorname{dvol}_{M_i}}{\operatorname{dvol}_{(0,1)^{n_\infty}}}$ is bounded above by twice the function on $(0,1)^{n_\infty}$ which, to a point $u \in (0,1)^{n_\infty}$, assigns the volume of the fiber $h_i^{-1}(u)$. If $\eta_i = \sum_{j=1}^{\overline{k}_i} \beta_j h_i^*\psi_j$ is a nonzero element of $h_i^* V_i^\infty$ then using (4.24), there is some $\Lambda < \infty$ such that for large i, we have

$$\begin{aligned} &\int_{M_i} \langle d\eta_i, d\eta_i\rangle_{g_i} \operatorname{dvol}_{M_i} \\ &\quad \le \Lambda^{p+1} \sum_{j=1}^{\overline{k}_i} |\beta_j|^2 \int_{(0,1)^n} \langle d\psi_j, d\psi_j\rangle_{(0,1)^n} \frac{(h_i)_* \operatorname{dvol}_{M_i}}{\operatorname{dvol}_{(0,1)^{n_\infty}}} \operatorname{dvol}_{(0,1)^n} \\ &\quad \le 2\Lambda^{p+1} 2^{\frac{n!}{n_\infty} c_i (2p+2-n_\infty)} E_\psi \cdot \sup_{u\in(0,1)^{n_\infty}} \operatorname{vol}(h_i^{-1}(u)) \cdot \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \end{aligned} \tag{4.31}$$

Let $\widetilde{\eta}_i$ be the class of η_i in $\widetilde{h_i^* V_i^\infty}$. We claim that

$$\|\widetilde{\eta}_i\|^2 \ge \frac{1}{4} 2^{\frac{n!}{n_\infty} c_i (2p-n_\infty)} \Delta^{-1} N_\psi \left(\inf_{u\in(0,1)^{n_\infty}} \operatorname{vol}(h_i^{-1}(u)) \right) \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \tag{4.32}$$

To see this, suppose that $\|\widetilde{\eta}_i\|^2 < \frac{1}{4} 2^{\frac{n!}{n_\infty} c_i (2p - n_\infty)} \Delta^{-1} N_\psi \left(\inf_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \right)$ $\sum_{j=1}^{\overline{k}_i} |\beta_j|^2$. Then there is some $\sigma_i \in \mathrm{Ker}(d_{M_i})$ so that

$$\int_{M_i} \langle \eta_i + \sigma_i, \eta_i + \sigma_i \rangle_{g_i} \ \mathrm{dvol}_{M_i} < \frac{1}{2} 2^{\frac{n!}{n_\infty} c_i (2p - n_\infty)} \Delta^{-1} N_\psi \left(\inf_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \right) \sum_{j=1}^{\overline{k}_i} |\beta_j|^2. \tag{4.33}$$

Let $\int_{Z_i} : \Omega^*_{L^2}(U_i; \mathcal{O}_i) \to \Omega^{*-(n-n_\infty)}_{L^2}((0,1)^{n_\infty})$ denote fiberwise integration.

Lemma 4.34 *The form* $\int_{Z_i} \chi_i \wedge \sigma_i \in \Omega^p_{L^2}((0,1)^{n_\infty})$ *lies in* $\mathrm{Ker}(d_{(0,1)^{n_\infty}})$.

Proof Given a compactly supported $\omega' \in \Omega^{n_\infty - p - 1}_{\mathrm{Lip}}((0,1)^{n_\infty})$, we have

$$\int_{(0,1)^{n_\infty}} d\omega' \wedge \int_{Z_i} \chi_i \wedge \sigma_i = \int_{U_i} dh_i^* \omega' \wedge \chi_i \wedge \sigma_i. \tag{4.35}$$

Smoothing σ_i by applying the heat operator on M_i, and using Lemma 2.2, gives

$$\int_{U_i} dh_i^* \omega' \wedge \chi_i \wedge \sigma_i = \int_{U_i} d\left(h_i^* \omega' \wedge \chi_i \wedge \sigma_i \right) = 0. \tag{4.36}$$

The lemma follows. □

Put $\nu_i = \int_{Z_i} \chi_i \wedge \sigma_i$. As

$$\int_{Z_i} \chi_i \wedge \eta_i = \int_{Z_i} \chi_i \wedge h_i^* \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j = \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j, \tag{4.37}$$

it follows that

$$\sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + \nu_i = \int_{Z_i} \chi_i \wedge (\eta_i + \sigma_i). \tag{4.38}$$

Using Lemma 4.27 and (4.33),

$$\begin{aligned}
&\int_{(0,1)^n} \left\langle \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + \nu_i, \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + \nu_i \right\rangle \mathrm{dvol}_{(0,1)^n} \\
&= \int_{(0,1)^n} \left\langle \int_{Z_i} \chi_i \wedge (\eta_i + \sigma_i), \int_{Z_i} \chi_i \wedge (\eta_i + \sigma_i) \right\rangle \mathrm{dvol}_{(0,1)^n} \\
&\leq \Delta \left(\inf_{u \in (0,1)^{n_\infty}} \mathrm{vol}(h_i^{-1}(u)) \right)^{-1} \\
&\quad \times \int_{(0,1)^n} \int_{h_i^{-1}(u)} \langle \eta_i + \sigma_i, \eta_i + \sigma_i \rangle \, \mathrm{dvol}_{h_i^{-1}(u)} \, \mathrm{dvol}_{(0,1)^{n_\infty}}(u) \\
&< \frac{1}{2} 2^{\frac{n!}{n_\infty} c_i (2p - n_\infty)} N_\psi \sum_{j=1}^{\overline{k}_i} |\beta_j|^2.
\end{aligned} \tag{4.39}$$

However,

$$\begin{aligned}
&\int_{(0,1)^n} \left\langle \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + \nu_i, \sum_{j=1}^{\overline{k}_i} \beta_j \psi_j + \nu_i \right\rangle \mathrm{dvol}_{(0,1)^n} \\
&= \sum_{j=1}^{\overline{k}_i} \int_{B_j} \left\langle \beta_j \psi_j + \nu_i, \beta_j \psi_j + \nu_i \right\rangle \mathrm{dvol}_{B_j} \geq 2^{\frac{n!}{n_\infty} c_i (2p - n_\infty)} N_\psi \sum_{j=1}^{\overline{k}_i} |\beta_j|^2.
\end{aligned} \tag{4.40}$$

This is a contradiction.

To finish the proposition, combining Lemma 4.25, (4.31) and (4.32) gives

$$\begin{aligned}
\lambda_{\overline{k}_i, p} &\leq 8 \cdot 2^{\frac{2(n!)}{n_\infty} c_i} \Lambda^{p+1} \Delta \widehat{C} E_\psi N_\psi^{-1} \\
&= 8 \Lambda^{p+1} \Delta \widehat{C} E_\psi N_\psi^{-1} \overline{k}_i^{\frac{2}{n_\infty}} \leq 8 \Lambda^{p+1} \Delta \widehat{C} E_\psi N_\psi^{-1} \overline{k}_i^{\frac{2}{s}}
\end{aligned} \tag{4.41}$$

For large i, this contradicts the fact that $\lambda_{\overline{k}_i, p}(M_i) \geq \frac{i}{4^{\frac{n!}{s}}} \overline{k}_i^{\frac{2}{s}}$. □

Theorem 1.6 follows from Proposition 4.19 and the Hodge decomposition.

Remark 4.42 In addition to upper bounds on the eigenvalues, one could ask about lower bounds. Under the assumption of a lower bound on the curvature operator, there are lower bounds on $\lambda_{k,p}$ in terms of the diameter, coming from heat trace estimates, with the bound proportionate to $k^{\frac{2}{n}}$ as $k \to \infty$ [1]. (For some finite number of k's, the lower bound may be zero.) If we only assume a lower bound on the sectional curvature then it is not so clear if there are eigenvalue bounds from below, in terms of the diameter. As a consistency check, we note that eigenvalue bounds from below, going to infinity as $k \to \infty$, imply upper bounds on Betti numbers. Such upper bounds on Betti numbers exist if we assume a lower bound on the curvature operator or, more generally a lower bound on sectional curvatures. However, the proofs are very different in the two cases. With a lower bound on the curvature operator, a Betti number bound (in terms of the diameter) follows easily from heat trace estimates. In comparison, if we assume a lower bound on sectional curvatures then there is again a Betti number bound, but the proof uses completely different methods [13].

One can also ask about spectral convergence. That is, suppose that $\{(M_i, g_i)\}_{i=1}^{\infty}$ is a sequence of Riemannian manifolds satisfying the assumptions of Theorem 1.3, with a Gromov–Hausdorff limit X. The question is whether $\lim_{i\to\infty} \lambda_{k,p}(M_i) = \lambda_{k,p}(X)$. This is known for functions when the lower sectional curvature bound is replaced by a lower Ricci curvature bound [5], and for 1-forms when the lower sectional curvature bound is replaced by a double sided Ricci bound [17]. It may be necessary to assume that the Riemannian manifolds $\{(M_i, g_i)\}_{i=1}^{\infty}$ have a uniform lower bound on the curvature operator.

5 Hodge theorem and compact resolvent

In this section we introduce the class $\mathcal{C}_*$ of Lipschitz multiconical spaces. If $X \in \mathcal{C}_*$ then in Sect. 5.1 we prove a Hodge theorem, in the sense that we identify $\mathrm{Ker}(\triangle_*)$ with a certain intersection homology group. In Sect. 5.2 we show that $(I + \triangle_*)^{-1}$ is compact.

To begin, if Y is a metric space of diameter at most π, and $\epsilon > 0$, then the truncated open metric cone $CY(\epsilon)$ over Y is homeomorphic to the topological space $([0,\epsilon) \times Y)/\sim$, where $(0, y_1) \sim (0, y_2)$ for all $y_1, y_2 \in Y$. The vertex of the cone, i.e. the equivalence class $\{(0, y)\}_{y\in Y}$, is denoted by $\star$. The metric on CY comes from

$$d_{CY}((t_1, y_1), (t_2, y_2)) = t_1^2 + t_2^2 - 2t_1t_2 \cos(d_Y(y_1, y_2)). \tag{5.1}$$

We define a class $\mathcal{C}_*$ of compact metric spaces inductively. An element of $\mathcal{C}_0$ is a finite metric space. A compact metric space X lies in $\mathcal{C}_n$ if every point $x \in X$ has a neighborhood U so that there is a pointed biLipschitz homeomorphism $h : (U, x) \to (CY(\epsilon), \star)$ for some $\epsilon > 0$ and some $Y \in \mathcal{C}_{n-1}$ with $\mathrm{diam}(Y) \le \pi$.

Note that for any $\epsilon > 0$, the cone $CY(\epsilon)$ is biLipschitz homeomorphic to $CY(1)$, so the parameter ϵ is not really needed in the definition.

Example 5.2 If X is an n-dimensional closed Riemannian Lipschitz manifold then $X \in \mathcal{C}_n$.

By induction, if $X \in \mathcal{C}_n$ then there is an open dense subset X^* of full Hausdorff n-measure that has the structure of a Riemannian Lipschitz n-manifold. Namely if $n = 0$ then $X^* = X$. If $n > 0$, cover X by a finite number of neighborhoods $\{U_i\}_{i=1}^N$ of points $\{x_i\}_{i=1}^N$ so that there are pointed biLipschitz homeomorphisms $\phi_i : (U_i, x_i) \to (CY_i(\epsilon_i), \star_i)$, with $Y_i \in \mathcal{C}_{n-1}$. Then we can take $X^* = \bigcup_{i=1}^N \phi_i^{-1}(CY_i^*(\epsilon_i) - \star_i)$.

We can apply the setup of Sect. 2.

Example 5.3 An element X of $\mathcal{C}_1$ is a finite metric graph G. The subset X^* is the union of the open edges. For convenience, suppose that X^* is oriented. Given $f \in \Omega^0_{L^2}(X)$

and an edge e of length L_e, write $f\Big|_e = f_e$, where $f_e \in L^2(0, L_e)$. Then $f \in \mathrm{Dom}(d)$ if

1. Each $f_e \in H^1(0, L_e)$, and
2. For each vertex v of G, the sum of the limiting values of f_e along edges e incoming to v equals the sum of the limiting values of f_e along edges outgoing from v.

Note that $\Omega^0_{L^2}(X)/\overline{\mathrm{Im}(d)} = \Omega^0_{L^2}(X)$. A 1-form $\omega \in \Omega^1_{L^2,d}(X) = \Omega^1_{L^2}(X)$ consists of a union of L^2-regular 1-forms on the open edges. The L^2-cohomology of X is given in degree zero by $\mathrm{Ker}(d : \Omega^0_{L^2,d}(X) \to \Omega^1_{L^2,d}(X)) \cong \mathbb{R}^{b_1(G)}$, and in degree one by $\Omega^1_{L^2,d}(X)/\mathrm{Im}(d) \cong \mathbb{R}^{b_0(G)}$.

Given $\omega \in \Omega^1_{L^2}(X)$ and an edge e of length L_e, write $\omega\Big|_e = \omega_e ds$, where $\omega_e \in L^2(0, L_e)$ and s is the oriented length parameter along e. Then $\omega \in \mathrm{Dom}(d^*)$ if

1. Each ω_e lies in $H^1(0, L_e)$, and
2. For each vertex $v \in G$, there is a number F_v so that for each edge e adjoining v, the limiting value of ω_e on e, toward v, is $\pm F_v$, depending on whether e is incoming or outgoing.

If $\omega \in \mathrm{Dom}(d^*)$ then the restriction of $d^*\omega \in \Omega^0_{L^2}(X)$ to an edge e is $-\frac{d\omega_e}{ds}$.

Then $\mathrm{Dom}(\triangle_0) = \{f \in \Omega^0_{L^2}(X) : df \in \mathrm{Dom}(d^*)\}$. The restriction of $\triangle_0 f$ to an edge e is $-\frac{d^2 f_e}{ds^2}$. The operator $\triangle_1$ on $\Omega^1_{L^2}(X)/\overline{\mathrm{Im}(d)}$ vanishes.

To see how the orientation of G affects the calculations, suppose that G'_e is the oriented graph obtained by starting with G and reversing the orientation of a particular edge e. Given f in the domain of d for G, define f' by

$$f'\Big|_{e'} = \begin{cases} -f\Big|_{e'}, & \text{if } e' = e \\ f\Big|_{e'}, & \text{if } e' \neq e. \end{cases} \tag{5.4}$$

Then f' is in the domain of d for G', and has the same energy as f. Hence if f is in the domain of $\triangle_0$ for G, then f' is in the domain of $\triangle_0$ for G'. It follows that choosing different orientations of the graph gives unitarily equivalent representations of $\triangle_0$.

5.1 Hodge theorem

For background information on intersection homology, we refer to [20].

Proposition 5.5 *If $X \in \mathcal{C}_n$ then for all $p \in [0, n]$, the unreduced L^2-cohomology*

$$\mathcal{H}^p_{L^2}(X) = \mathrm{Ker}\left(d : \Omega^p_{L^2,d}(X) \to \Omega^{p+1}_{L^2,d}(X)\right) \Big/ \mathrm{Im}\left(d : \Omega^{p-1}_{L^2,d}(X) \to \Omega^p_{L^2,d}(X)\right) \tag{5.6}$$

is isomorphic to $\mathrm{IH}^{GM}_{n-p}(X; \mathcal{O})$. Here IH^{GM}_* denotes the Goresky–MacPherson intersection homology with real coefficients, computed with the perversity $\overline{p}$ given by $\overline{p}(0) = 0$ and $\overline{p}(j) = [\frac{j-1}{2}]$ for $j \geq 1$, and $\mathcal{O}$ is the orientation line bundle of the codimension-zero stratum of X.

Proof We first note that IH is a topological invariant of X [19, Theorem 9] but can be computed using a topological stratification. In our case there is a natural stratification $X = X_n \supset X_{n-1} \supset \cdots \supset X_0 \supset X_{-1} = \emptyset$ given by saying that $x \in X_j$ if and only if there is no neighborhood of x that is biLipschitz equivalent to $B^{j+1} \times CY^{n-j-2}$ for any $Y^{n-j-2} \in \mathcal{C}_{n-j-2}$ of diameter at most π. Here B^{j+1} is the open unit ball in $\mathbb{R}^{j+1}$. The associated codimension-k stratum is $X_{n-k} - X_{n-k-1}$, a manifold of dimension $n-k$. A point in the codimension-k stratum has a neighborhood that splits off a B^k-factor, but there is no neighborhood that splits off a B^{k+1}-factor.

If $\mathcal{O}$ is the orientation line bundle of the codimension-zero stratum then since $\overline{p}(1) = 0$, it follows as in [12, Section 2.2] that perversity-$\overline{p}$ intersection homology with values in $\mathcal{O}$ is well-defined.

If X is a pseudomanifold, i.e. if $X_{n-1} = X_{n-2}$, then the result of Proposition 5.5 is well known and goes back to work of Cheeger [4] and Cheeger–Goresky–MacPherson [6]. The proofs that are relevant for us are sheaf-theoretic in nature, and appear in [25] and [36].

There are two relevant sheaves of differential graded complexes on X. The first one is the sheaf $\Omega^*_{L^2_{loc},d}$ defined in Sect. 3. We will use the fact that it only depends on X through the biLipschitz homeomorphism class of X. The second relevant sheaf, as pointed out to me by Greg Friedman, is the sheaf $\mathrm{IC}_{n-\star}$ coming from the presheaf whose sections, over an open set $U \subset X$, are the singular intersection chains $C_{n-\star}(X, X - \overline{U}; \mathcal{O})$ relative to the perversity $\overline{p}$ [10, Section 3]. (If X is a PL-stratified space then $IC_{n-\star}$ is derived isomorphic to the sheaf $IC^{\infty}_{n-\star}$ whose sections, over an open set U, are the locally finite $\mathcal{O}$-valued simplicial intersection chains relative to $\overline{p}$.) The hypercohomology groups of the two sheaves are isomorphic to $\mathcal{H}^*_{L^2}(X)$ and $\mathrm{IH}^{GM}_{n-*}(X; \mathcal{O})$, respectively. Hence it suffices to show that the two sheaves are isomorphic in the derived category of differential graded sheaves on X.

When X is a pseudomanifold, the strategy of [25] was to use the unique extension result of [12]. On the codimension-zero stratum, each sheaf was quasi-isomorphic to the constant $\mathbb{R}$-sheaf in degree 0. As each sheaf satisfied the axioms of [12, Section 3.4], the stratum-by-stratum argument of [12, Section 3.5] showed that the two extensions from the codimension-zero stratum to all of X are isomorphic in the derived category.

When the codimension-one stratum is nonempty, this strategy has to be slightly modified. The relevant unique extension result for us is in [14, Section 4], with $c_p = c = 2$. We have to know that the restrictions of the two sheaves, on the union of the codimension-zero and codimension-one strata, are equivalent coefficient systems in the sense of [14, Definition 5.1]. We also have to know that the conditions of [14, Proposition 5.2(1)] are satisfied. Then [14, Proposition 4.5] implies that the two sheaves are isomorphic in the derived category of differential graded sheaves on X.

As the steps are similar to those in the pseudomanifold case, we just give the main points. We let $\mathrm{H}^*_{L^2,d}(\cdot)$ denote hypercohomology of the sheaf $\Omega^*_{L^2_{loc},d}$ and we let $\mathrm{H}^*_{L^2,d,c}(\cdot)$ denote compactly-supported hypercohomology. Similarly, we let $\mathrm{IC}^{GM}_{n-\star}(\cdot)$ denote hypercohomology of the sheaf $\mathrm{IC}_{n-\star}$ and we let $\mathrm{IC}^{GM}_{n-\star,c}(\cdot)$ denote compactly supported hypercohomology, i.e the usual $\mathcal{O}$-twisted Goresky–MacPherson intersection homology in degree $n - \star$.

First, we give the relevant cohomology of the truncated open metric cone $CY = CY(1)$ over some $Y \in \mathcal{C}_{k-1}$. If $k = 1$ then

$$
\begin{aligned}
\mathrm{H}^0_{L^2,d}(CY) &= \mathrm{IH}^{GM}_1(CY) = \widetilde{\mathrm{H}}_0(Y),\\
\mathrm{H}^1_{L^2,d}(CY) &= \mathrm{IH}^{GM}_0(CY) = 0,
\end{aligned} \tag{5.7}
$$

where $\widetilde{\mathrm{H}}$ denotes reduced homology, and

$$
\begin{aligned}
\mathrm{H}^0_{L^2,d,c}(CY) &= \mathrm{IH}^{GM}_{1,c}(CY) = 0,\\
\mathrm{H}^1_{L^2,d,c}(CY) &= \mathrm{IH}^{GM}_{0,c}(CY) = \mathbb{R}.
\end{aligned} \tag{5.8}
$$

If $k > 1$ then

$$
\mathrm{H}^i_{L^2,d}(CY) \cong \begin{cases} \mathrm{H}^i_{L^2,d}(Y) & \text{if } i < \frac{k}{2},\\ 0 & \text{if } i \geq \frac{k}{2}, \end{cases} \tag{5.9}
$$

$$
\mathrm{H}^i_{L^2,d,c}(CY) \cong \begin{cases} \mathrm{H}^{i-1}_{L^2,d}(Y) & \text{if } i \geq \frac{k}{2}+1,\\ 0 & \text{if } i < \frac{k}{2}+1, \end{cases} \tag{5.10}
$$

$$
\mathrm{IH}^{GM}_{k-i}(CY) \cong \begin{cases} \mathrm{IH}^{GM}_{k-1-i}(Y) & \text{if } i < \frac{k}{2},\\ 0 & \text{if } i \geq \frac{k}{2} \end{cases} \tag{5.11}
$$

and

$$
\mathrm{IH}^{GM}_{k-i,c}(CY) \cong \begin{cases} \mathrm{IH}^{GM}_{k-i,c}(Y) & \text{if } i \geq \frac{k}{2}+1,\\ 0 & \text{if } i < \frac{k}{2}+1. \end{cases} \tag{5.12}
$$

Equations (5.9) and (5.10) can be proved using separation of variables as in [36]. Equation (5.9) can be understood as saying that the L^2-cohomology of CY comes from pulling back harmonic forms from Y with respect to the projection map $(0, 1) \times Y \to Y$, provided that the pullback is square integrable. Equation (5.10) can be understood as saying that the compactly supported L^2-cohomology of CY is generated by forms of the type $\phi' dr \wedge \omega$, where $\phi \in C^\infty(0, 1)$ is a nonincreasing function that is identically one on $(0, 1/3)$ and identically zero on $(2/3, 1)$, and ω is a harmonic $(i-1)$-form on Y. Then when $i \geq \frac{k}{2} + 1$, the putative primitive $\phi\omega$ fails to be square integrable on CY. Equations (5.11) and (5.12) follow from [10, Proposition 2.18].

Next, for both sheaf cohomology theories there are Künneth formulas :

$$
\begin{aligned}
\mathrm{H}^i_{L^2,d}(B^{n-k} \times CY) &\cong \mathrm{H}^i_{L^2,d}(CY),\\
\mathrm{H}^i_{L^2,d,c}(B^{n-k} \times CY) &\cong \mathrm{H}^{i-n+k}_{L^2,d,c}(CY),\\
\mathrm{IH}^{GM}_{n-i}(B^{n-k} \times CY) &\cong \mathrm{IH}^{GM}_{k-i}(CY),\\
\mathrm{IH}^{GM}_{n-i,c}(B^{n-k} \times CY) &\cong \mathrm{IH}^{GM}_{n-i,c}(CY).
\end{aligned} \tag{5.13}
$$

The third isomorphism comes from [10, Proposition 2.20].

To check the conditions of [14], let A denote one of the two above differential graded sheaves. They are both cohomologically constructible with respect to the stratification. As in [12, Section 1.12], if v is the vertex of a cone CY, and $f : (0, v) \to B^{n-k} \times CY$ is inclusion, then $\mathrm{H}^i(f^* A) \cong \mathrm{H}^i(B^{n-k} \times CY; A)$ and $\mathrm{H}^i(f^! A) \cong \mathrm{H}^i_c(B^{n-k} \times CY; A)$. We will use the fact that the cohomology of $\Omega^*_{L^2_{loc},d}$ is biLipschitz invariant, in order to compute the L^2-cohomology of a neighborhood of a point $x \in X$ that is biLipschitz to $B^{n-k} \times CY$, using the conical metric on CY and the product metric on $B^{n-k} \times CY$.

The restrictions of $\Omega^*_{L^2_{loc},d}$ and $\mathrm{IC}_{n-\star}$ to $X_n - X_{n-2}$ are quasi-isomorphic; c.f. (5.7) and (5.8). Let $\mathcal{E}$ denote their common class in the derived category of differential graded sheaves on $X_n - X_{n-2}$. From (5.7), (5.8) and (5.13), if $x \in X_n - X_{n-2}$ then $\mathrm{H}^i(\mathcal{E}_x) = 0$ for $i > 0$ and $\mathrm{H}^i(f_x^! \mathcal{E}) = 0$ for $i < n$. It follows from [14, Proposition 5.2] that the coefficient system satisfies [14, Definition 5.1].

Finally, from (5.9), (5.11) and (5.13), we have

$$\mathrm{H}^i_{L^2,d}(B^{n-k} \times CY) = \mathrm{H}^{GM}_{n-i}(B^{n-k} \times CY) = 0 \tag{5.14}$$

if $Y \in \mathcal{C}_{k-1}$ and $i > \overline{p}(k)$. From (5.10), (5.12) and (5.13), we have

$$\mathrm{H}^i_{L^2,d,c}(B^{n-k} \times CY) = \mathrm{H}^{GM}_{n-i,c}(B^{n-k} \times CY) = 0 \tag{5.15}$$

if $Y \in \mathcal{C}_{k-1}$ and $i < n - \max(n - 2 - \overline{p}(k), 0)$. Using [14, Lemma 4.6], the sheaves $\Omega^*_{L^2_{loc},d}$ and $\mathrm{IC}_{n-\star}$ satisfy the axioms $A_{\overline{p}}$ in the sense of [14, Definition 4.3]. From [14, Remark 4.2] and [14, Proposition 4.5], it follows that the two sheaves are derived isomorphic on X. This proves the proposition. □

Corollary 5.16 *If $X \in \mathcal{C}_n$ then for all $p \in [0, n]$, we have* $\dim(\mathrm{Ker}(\triangle_p)) < \infty$.

Corollary 5.17 *If $X \in \mathcal{C}_n$ then for all $p \in [0, n]$, we have* $\overline{\mathrm{Im}(d)} = \mathrm{Im}(d) \subset \Omega^p_{L^2}(X)$.

In particular, $\mathrm{Ker}(\triangle_*) \cong \mathcal{H}^*_{L^2}(X)$.

5.2 Compactness of the resolvent

Proposition 5.18 *For any $X \in \mathcal{C}_n$, and for any $p \in [0, n]$, the operator $(I + \triangle_p)^{-1}$ is a compact operator on $\Omega^p_{L^2}(X)/\overline{\mathrm{Im}(d)}$.*

Proof We will prove the proposition by induction on n. It is true if $n = 0$.

Using Corollary 5.17, we have

$$\Omega^p_{L^2,d}(X)/\mathrm{Im}(d) = \Omega^p_{L^2,d}(X)/\overline{\mathrm{Im}(d)} \cong \mathcal{H}^p_{L^2}(X) \oplus \left(\Omega^p_{L^2,d}(X)/\operatorname{Ker}(d)\right). \tag{5.19}$$

The restriction of $I + \triangle_p$ to $\Omega^p_{L^2,d}(X)/\operatorname{Ker}(d)$ equals $I + d^*d$. Let $\Omega^p_{H^1,d}(X)/\operatorname{Ker}(d) \subset \Omega^p_{L^2}(X)/\operatorname{Ker}(d)$ denote $\Omega^p_{L^2,d}(X)/\operatorname{Ker}(d)$ with the Hilbert space norm $\|\omega\|^2_{H^1} = \|d\omega\|^2_{L^2} + \|\omega\|^2_{L^2}$. The map $d : \Omega^p_{H^1,d}(X)/\operatorname{Ker}(d) \to \mathrm{Im}(d)$ is invertible. Since

$\mathrm{Im}(d)$ is closed in $\Omega^{p+1}_{L^2}(X)$, the open mapping theorem implies that the inverse $d^{-1} : \mathrm{Im}(d) \to \Omega^p_{H^1,d}(X)/\mathrm{Ker}(d) \subset \Omega^p_{L^2}(X)/\mathrm{Ker}(d)$ is L^2-bounded. As $(d^*d)^{-1} = d^{-1}\left(d^{-1}\right)^*$, it suffices to show that d^{-1} is compact.

We will construct an approximate inverse of d. Given $x \in X$, let V_x be a neighborhood of x for which there is a biLipschitz homeomorphism $\phi_x : (V_x, x) \to (CY_x(2), \star_x)$ with $Y_x \in \mathcal{C}_{n-1}$ of diameter at most π. (Note that the parameter 2 in $CY_x(2)$ can always be chosen.) Put $U_x = \phi_x^{-1}(CY_x(1))$. As $\{U_x\}_{x\in X}$ is an open covering of X, we can choose a finite subcovering $\{U_i\}_{i=1}^N$, where we write $U_i = U_{x_i}$. We write the corresponding biLipschitz homeomorphisms as $\phi_i : (V_i, x_i) \to (CY_i(2), \star_i)$.

Let $\Omega^*_{L^2}(V_i)$ be the square-integrable forms of V_i. Let $\Omega^*_{L^2,d,abs}(V_i)$ be the square-integrable elements of $\Omega^*_{L^2_{loc},d}(V_i)$, the latter being defined in Sect. 3. In terms of the conical coordinate r on V_i, an element of $\Omega^*_{L^2}(V_i)$ can be written as $\omega = \omega_0(r) + dr \wedge \omega_1(r)$, with $\omega_0(r) \in \Omega^*_{L^2}(Y_i)$ and $\omega_1(r) \in \Omega^{*-1}_{L^2}(Y_i)$. If $\omega \in \Omega^*_{L^2,d,abs}(V_i)$ then $\omega_0(r) \in \Omega^*_{L^2,d}(Y_i)$, $\omega_1(r) \in \Omega^{*-1}_{L^2,d}(Y_i)$, and $d_{Y_i}\omega_0$ and $\partial_r\omega_0 - d_{Y_i}\omega_1$ are square-integrable on V_i. Let $d_i : \Omega^*_{L^2,d,abs}(V_i) \to \Omega^{*+1}_{L^2}(V_i)$ be the differential on V_i. From (5.9), the unreduced L^2-cohomology of V_i is finite dimensional and so $\mathrm{Im}(d_i)$ is closed in $\Omega^{p+1}_{L^2}(V_i)$. As before, the map $d_i : \Omega^p_{L^2,d,abs}(V_i)/\mathrm{Ker}(d_i) \to \mathrm{Im}(d_i)$ has an inverse that extends to a bounded map $d_i^{-1} : \mathrm{Im}(d_i) \to \Omega^p_{L^2}(V_i)/\mathrm{Ker}(d_i)$.

Lemma 5.20 *For each i, the operator d_i^{-1} is compact.*

Proof Under a biLipschitz change of metric on V_i, one obtains equivalent norms on $\mathrm{Im}(d_i)$ and $\Omega^p_{L^2}(V_i)/\mathrm{Ker}(d_i)$. Hence to prove the lemma, we can replace V_i by $CY_i(2)$ with its conical metric.

The quadratic form Q_i on $\Omega^p_{L^2,d_i,abs}(CY_i(2))/\mathrm{Ker}(d_i)$ is defined as in (2.19). The corresponding operator $d_i^*d_i$, densely defined on $\Omega^p_{L^2}(CY_i(2))/\mathrm{Ker}(d_i)$, has a domain whose elements satisfy absolute (Neumann) boundary conditions on $Y_i = \partial\overline{CY_i(2)}$. The spectrum of $d_i^*d_i$ can be explicitly computed using separation of variables. If Y_i is a smooth manifold then this was done in [15,34]. (Since we are not interested in enforcing Hodge duality, subtleties about ideal boundary conditions do not arise.) In our case, by the induction assumption, the differential form Laplacian on Y_i has a discrete spectrum with finite multiplicities. Then by the same separation of variable argument, this is also true for $d_i^*d_i$ on $CY_i(2)$. From the explicit spectral decomposition, one sees that $d_i^{-1} = (d_i^*d_i)^{-1}d_i^*$ is a compact operator on $\mathrm{Im}(d_i)$. □

There is a restriction map from $\mathrm{Im}(d) \subset \Omega^*_{L^2}(X)$ to $\mathrm{Im}(d_i) \subset \Omega^*_{L^2}(V_i)$. Let $L_i : \Omega^*_{L^2}(V_i)/\mathrm{Ker}(d_i) \to \mathrm{Ker}(d_i)^\perp$ be the lifting isomorphism. Let $q : \Omega^*_{L^2}(X) \to \Omega^*_{L^2}(X)/\mathrm{Ker}(d)$ be the quotient map. Let $\{\sigma_i\}_{i=1}^N$ be a Lipschitz partition of unity subordinate to $\{U_i\}_{i=1}^N$. Define a compact operator $A : \mathrm{Im}(d) \to \Omega^p_{L^2}(X)/\mathrm{Ker}(d)$ by

$$A\omega = q\sum_{i=1}^{N} \sigma_i L_i d_i^{-1}\left(\omega\Big|_{V_i}\right). \tag{5.21}$$

Then

$$dA\omega = \sum_{i=1}^{N} d\sigma_i \wedge L_i d_i^{-1} \left(\omega \Big|_{V_i} \right) + \omega. \tag{5.22}$$

The operator K given by

$$K\omega = \sum_{i=1}^{N} d\sigma_i \wedge L_i d_i^{-1} \left(\omega \Big|_{V_i} \right) \tag{5.23}$$

is compact. As $dA = I + K$ and d^{-1} is bounded, we see that $d^{-1} = A - d^{-1}K$ is compact. □

This proves Theorem 1.16.

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