DIFFERENTIAL FORMS, SPINORS AND BOUNDED CURVATURE COLLAPSE

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From preprints

“Collapsing and the Differential Form Laplacian”

“On the Spectrum of a Finite-Volume Negatively-Curved Manifold”

“Collapsing and Dirac-Type Operators”

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WHAT I WILL (NOT) TALK ABOUT

Fukaya (1987) : Studied the function Laplacian, under a bounded curvature collapse.

Cheeger-Colding (preprint) : Studied the function Laplacian, under a collapse with Ricci curvature bounded below.

Today : The differential form Laplacian and geometric Dirac-type operators, under a bounded curvature collapse.
MOTIVATION: CHEEGER’S INEQUALITY

Let $M$ be a connected compact Riemannian manifold.

Spectrum of $\triangle^M$:

$$0 = \lambda_1(M) < \lambda_2(M) \leq \lambda_3(M) \leq \ldots$$

Theorem 1. (Cheeger 1969)

$$\lambda_2(M) \geq \frac{h^2}{4}.$$

Definition 1.

$$h = \inf_A \frac{\text{Area}(A)}{\min(\text{vol}(M_1), \text{vol}(M_2))},$$

where $A$ ranges over separating hypersurfaces.

Question (Cheeger): Is there a similar inequality for the $p$-form Laplacian? (Problem # 79 on Yau’s list.)
THE $p$-FORM LAPLACIAN

The $p$-form Laplacian is

$$\Delta^M_p = dd^* + d^* d : \Omega^p(M) \to \Omega^p(M).$$

Spectrum of $\Delta^M_p$:

$$0 \leq \lambda_{p,1}(M) \leq \lambda_{p,2}(M) \leq \lambda_{p,3}(M) \leq ...$$

By Hodge theory,

$$\text{Ker}(\Delta^M_p) \cong H^p(M; \mathbb{R}),$$

so

$$0 = \lambda_{p,1}(M) = ... = \lambda_{p,b_p(M)}(M) < \lambda_{p,b_p(M)+1}(M) \leq ... .$$
Notation:

\[ R^M = \text{Riemann sectional curvatures.} \]
\[ \text{diam}(M) = \sup_{p,q \in M} d_M(p,q). \]

Fix a number \( K \geq 0 \). Consider

\[ \{ \text{connected Riem. manifolds } M^n : \| R^M \|_\infty \leq K \text{ and diam}(M) \leq 1 \}. \]

There is a number \( v_0(n, K) > 0 \) such that one has the following dichotomy:

I. Noncollapsing case: \( \text{vol}(M) \geq v_0 \).

Finite number of topological types, \( C^{1,\alpha} \)-metric rigidity. In particular, uniform bounds on eigenvalues of \( \triangle_p \).

or

II. Collapsing case: \( \text{vol}(M) < v_0 \).

Special structure. Need to analyze \( \triangle_p \) in this case.
BERGER EXAMPLE OF COLLAPSING

Hopf fibration \( \pi : S^3 \to S^2 \)

Shrink the circles to radius \( \epsilon \). Look at the 1-form Laplacian \( \Delta_1 \). Since \( H^1(S^3; \mathbb{R}) = 0 \), the first eigenvalue \( \lambda_{1,1} \) of \( \Delta_1 \) is positive.

**Fact** (Colbois-Courtois 1990)

\[
\lim_{\epsilon \to 0} \lambda_{1,1} = 0.
\]

**New phenomenon**: (uncontrollably) small eigenvalues.

When does this happen?
FUKAYA’S WORK ON THE FUNCTION LAPLACIAN

Suppose that \( \{M_i\}_{i=1}^\infty \) are connected \( n \)-dimensional Riemannian manifolds with \( \| R^{M_i} \|_\infty \leq K \) and \( \text{diam}(M_i) \leq 1 \).

Suppose that \( M_i \overset{\text{GH}}{\rightarrow} X \).

Consider the function Laplacian on \( M_i \), with eigenvalues \( \{\lambda_j(M_i)\}_{j=1}^\infty \).

**Question**: Suppose that \( X \) is a smooth Riemannian manifold. Is it true that \( \lim_{i \to \infty} \lambda_j(M_i) = \lambda_j(X) \)?

**Answer**: In general, no. Need to add a probability measure \( \mu \) to \( X \).

Laplacian on weighted \( L^2 \)-space:

\[
< f, \triangle^X \mu f > \quad \frac{< f, f >}{\int_X f^2 \, d\mu} = \frac{\int_X |\nabla f|^2 \, d\mu}{\int_X f^2 \, d\mu}.
\]

**Theorem 2.** (Fukaya 1987) If

\[
\lim_{i \to \infty} \left( M_i, \frac{dvol_{M_i}}{vol(M_i)} \right) = (X, \mu)
\]

in the measured Gromov-Hausdorff topology then

\[
\lim_{i \to \infty} \lambda_j(M_i) = \lambda_j(X, \mu).
\]
GOAL

Want a “p-form Laplacian” on the limit space $X$ so that after taking a subsequence,

$$\lambda_{p,j}(M_i) \longrightarrow \lambda_{p,j}(X).$$

**Question**: What kind of structure do we need on $X$?
SUPERCONNECTIONS (Quillen 1985, Bismut-L. 1995)

Input:

$B$ a smooth manifold,

$E = \bigoplus_{j=0}^{m} E^j$ a $\mathbb{Z}$-graded real vector bundle on $B$.

The (degree-1) superconnections $A'$ that we need will be formal sums of the form

$$A' = A'[0] + A'[1] + A'[2]$$

where

- $A'[0] \in C^\infty(B; \text{Hom}(E^*, E^{*-1}))$,
- $A'[1]$ is a grading-preserving connection $\nabla^E$ on $E$ and
- $A'[2] \in \Omega^2(B; \text{Hom}(E^*, E^{*-1}))$.

Then $A' : C^\infty(B; E) \to \Omega(B; E)$ extends by Leibniz’ rule to an operator $A' : \Omega(B; E) \to \Omega(B; E)$.

Flatness condition: $(A')^2 = 0$,

i.e.

- $(A'[0])^2 = (A'[2])^2 = 0$,
- $\nabla^E A'[0] = \nabla^E A'[2] = 0$ and

Note: $A'[0]$ gives a differential complex on each fiber of $E$. 
THE NEEDED STRUCTURE ON THE LIMIT SPACE $X$

A triple $(E, A', h^E)$, where

1. $E$ is a $\mathbb{Z}$-graded real vector bundle on $X$,
2. $A'$ is a flat degree-1 superconnection on $E$ and
3. $h^E$ is a Euclidean inner product on $E$.

We have

$$A' : \Omega(X; E) \rightarrow \Omega(X; E).$$

Using $g^{TX}$ and $h^E$, we get

$$(A')^* : \Omega(X; E) \rightarrow \Omega(X; E).$$

Put

$$\Delta^E = A'(A')^* + (A')^* A',$$

the superconnection Laplacian.

Example: If $E$ is the trivial $\mathbb{R}$-bundle on $X$, $A'$ is the trivial connection and $h^E$ is the standard inner product on $E$ then $\Delta^E$ is the Hodge Laplacian.
ANALYTIC COMPACTNESS

Theorem 3. If $M_i \rightarrow^GH X$ with bounded sectional curvature then after taking a subsequence, there is a certain triple $(E,A',h^E)$ on $X$ such that

$$\lim_{i \to \infty} \sigma(\triangle_{p}^{M_i}) = \sigma(\triangle_{p}^{E}).$$

Remark 1: This is a pointwise convergence statement, i.e. for each $j$, the $j$-th eigenvalue converges.

Remark 2: Here the limit space $X$ is assumed to be a Riemannian manifold. There is an extension to singular limit spaces (see later).

Remark 3: The relation to Fukaya’s work on functions: For functions, only $\Omega^0(X;E^0)$ is relevant. Here $E^0$ is a trivial $\mathbb{R}$-bundle on $X$ with a trivial connection. But its metric $h^{E^0}$ may be nontrivial and corresponds to Fukaya’s measure $\mu$. 
IDEA OF PROOF

1. The individual eigenvalues $\lambda_{p,j}$ are continuous with respect to the metric on $M$, in the $C^0$-topology (Cheeger-Dodziuk).

2. By Cheeger-Fukaya-Gromov, if $M$ is Gromov-Hausdorff close to $X$ then we can slightly perturb the metric to get a Riemannian affine fiber bundle. That is,

affine fiber bundle : $M$ is the total space of a fiber bundle $M \to X$ with infranil fiber $Z$, whose holonomy can be reduced from $\text{Diff}(Z)$ to $\text{Aff}(Z)$.

Riemannian affine fiber bundle : In addition, one has
a. A horizontal distribution $T^HM$ on $M$ with holonomy in $\text{Aff}(Z)$, and
b. Fiber metrics $g^{TZ}$ which are fiberwise affine-parallel.

So it’s enough to just consider Riemannian affine fiber bundles.

3. If $M$ is a Riemannian affine fiber bundle then $\sigma(\triangle_p^M)$ equals $\sigma(\triangle_p^E)$ up to a high level, which is on the order of $d_{GH}(M,X)^{-2}$. Here $E$ is the vector bundle on $X$ whose fiber over $x \in X$ is

$$E_x = \{\text{affine-parallel forms on } Z_x\}.$$

4. Show that the ensuing triples $\{(E_i, A'_i, h^{E_i})\}_{i=1}^\infty$ have a convergent subsequence (modulo gauge transformation).
APPLICATION TO SMALL EIGENVALUES

Fix $M$ and $K \geq 0$. Consider
$$
\{ g : \| R^M(g) \|_\infty \leq K \text{ and } \text{diam}(M, g) \leq 1 \}.
$$

**Question**: Among these metrics, are there more than $b_p(M)$ small eigenvalues of $\Delta_p^M$?

Suppose so, i.e. that for some $j > b_p(M)$, there are metrics $\{g_i\}_{i=1}^\infty$ so that
$$
\lambda_{p,j}(M, g_i) \to 0.
$$

**Step 1.** Using Gromov precompactness, take a convergent subsequence
$$
(M, g_i) \to X.
$$

Since there are small positive eigenvalues, we must be in the collapsing situation.

**Step 2.** Using the analytic compactness theorem, take a further subsequence to get a triple $(E, A', h^E)$ on $X$. Then
$$
\lambda_{p,j}(\Delta^E) = 0.
$$

In the limit, we’ve turned the small eigenvalues into **extra zero eigenvalues**.

Recall that $\Delta^E$ has the Hodge form $A'(A')^* + (A')^* A'$. Then from Hodge theory,
$$
\dim(H^p(A')) \geq j.
$$

**Analysis** → **Topology**

**Fact**: There is a spectral sequence to compute $H^p(A')$. Analyze the spectral sequence.
RESULTS ABOUT SMALL EIGENVALUES

**Theorem 4.** Given $M$, there are no more than $b_1(M) + \dim(M)$ small eigenvalues of the 1-form Laplacian.

More precisely, if there are $j$ small eigenvalues and $j > b_1(M)$ then in terms of the limit space $X$,

$$j \leq b_1(X) + \dim(M) - \dim(X).$$

(Sharp in the case of the Berger sphere.)

More generally, where do small eigenvalues come from?

**Theorem 5.** Let $M$ be the total space of an affine fiber bundle $M \to X$, which collapses to $X$. Suppose that there are small positive eigenvalues of $\triangle_p$ in the collapse. Then there are exactly three possibilities:

1. The infranil fiber $Z$ has small eigenvalues of its $q$-form Laplacian for some $0 \leq q \leq p$. That is, $b_q(Z) < \dim\{\text{affine-parallel } q\text{-forms on } Z\}$.

   **OR**

2. The “direct image” cohomology bundle $H^q$ on $X$ has a holonomy representation $\pi_1(X) \to \text{Aut}(H^q(Z; \mathbb{R}))$ which fails to be semisimple, for some $0 \leq q \leq p$.

   **OR**

3. The Leray spectral sequence to compute $H^p(M; \mathbb{R})$ does not degenerate at the $E_2$ term.

Each of these cases occurs in examples.
UPPER EIGENVALUE BOUNDS

**Theorem 6.** Fix $M$. If there is not a uniform upper bound on $\lambda_{p,j}$ (among metrics with $\| R^M \|_\infty \leq K$ and $\text{diam}(M) = 1$) then $M$ collapses to a limit space $X$ with $1 \leq \text{dim}(X) \leq p - 1$.

In addition, the generic fiber $Z$ of the fiber bundle $M \to X$ is an infranilmanifold which does not admit nonzero affine-parallel $k$-forms for $p - \text{dim}(X) \leq k \leq p$.

**Example:** Given $M$, if there is *not* a uniform upper bound on the $j$-th eigenvalue of the 2-form Laplacian then $M$ collapses with bounded curvature to a 1-dimensional limit space. We know what such $M$ look like.
SINGULAR LIMIT SPACES

Technical problem: in general, a limit space of a bounded-curvature collapse is not a manifold.

Theorem 7. (Fukaya 1986): A limit space $X$ is of the form $\check{X}/G$, where $\check{X}$ is a Riemannian manifold and $G \subset \text{Isom}(\check{X})$.

What should the “forms on $X$” be? Answer: the basic forms on $\check{X}$. $\Omega_{\text{basic}}^*(\check{X}) = \{ \omega \in \Omega^*(\check{X}) : \omega \text{ is } G\text{-invariant and for all } x \in g, i_x \omega = 0 \}$.

Fact: One can do analysis on the singular space $X$ by working $G$-equivariantly on $\check{X}$, i.e. construct superconnection Laplacians, etc. The preceding results extend to this setting.
Theorem 8. Let $M^n$ be a complete connected Riemannian manifold with $\text{vol}(M) < \infty$ and $-b^2 \leq R^M \leq -a^2$, with $0 < a \leq b$. Then the space of square-integrable harmonic $p$-forms on $M$ is finite-dimensional.

Previously known to be true if $p \neq \frac{n-1}{2}$ and $\frac{b}{a}$ is close enough to one (Donnelly-Xavier).

The result is also true if $M$ just has bounded curvature and asymptotically-cylindrical ends, as long as the cross-sections of the ends are not too big.

Theorem 9. There is a number $\delta(n) > 0$ such that if
1. $M^n$ is a complete connected Riemannian manifold,
2. $\|R^M\|_{\infty} \leq b^2$ and
3. The ends of $M$ are $\delta(n) b^{-1}$-Gromov-Hausdorff close to rays
then the space of square-integrable harmonic $p$-forms on $M$ is finite-dimensional.

Theorem 10. If $M$ is a finite-volume negatively-curved manifold as above then one can write down an explicit ordinary differential operator whose essential spectrum coincides with that of the $p$-form Laplacian on $M$. 
GEOMETRIC DIRAC-TYPE OPERATORS

Spinor modules $V$:

Say $G$ is $SO(n)$ or $Spin(n)$, and $V$ is a Hermitian $G$-module. Suppose that there is a $G$-equivariant map $\gamma : \mathbb{R}^n \to \text{End}(V)$ such that

$$\gamma(v)^2 = |v|^2 \text{Id}.$$ 

Geometric Dirac-type operators:

Let $M^n$ be a closed Riemannian manifold which is oriented or spin. Let $V$ be a spinor module and let $D^M$ be the corresponding Dirac-type operator. (Special cases: signature operator, pure Dirac operator.)

Theorem 11. Suppose that $M_i \overset{GH}{\longrightarrow} X$ with bounded curvature, with $X$ smooth. Then after taking a subsequence, there are a Clifford-module $E$ on $X$ and a certain first-order elliptic operator $D^E$ on $C^\infty(X; E)$ such that

$$\lim_{i \to \infty} \sigma(D^{M_i}) = \sigma(D^E).$$
Suppose now that $M_i \xrightarrow{GH} X$ with bounded curvature, but with $X$ singular. To describe the limit of $\sigma(D^M_i)$, we need a Dirac-type operator on the singular space $X$. How to do this?

Let $P_i$ be the principal $G$-bundle on $M_i$. Following Fukaya, we can assume that $P_i \xrightarrow{GH} \tilde{X}$, with $\tilde{X}$ a $G$-manifold. We want to define a Dirac-type operator on $X = \tilde{X}/G$.

**Fundamental Problem** : There is no notion of a “$G$-basic spinor”.

**Resolution** : Observe that a spinor field on $M_i$ is a $G$-invariant element of $C^\infty(P_i) \otimes V$. Take $P_i \longrightarrow \tilde{X}$.

**Definition 2.** A “spinor field on $X$” is a $G$-invariant element of $C^\infty(\tilde{X}) \otimes V$.

**Fact** : There’s a certain first-order transversally elliptic operator $\tilde{D}$ on $C^\infty(\tilde{X}) \otimes V$.

**Definition 3.** The Dirac-type operator $D$ on $X$ is the restriction of $\tilde{D}$ to the $G$-invariant subspace of $C^\infty(\tilde{X}) \otimes V$. 
APPLICATIONS TO SPECTRAL ANALYSIS OF DIRAC-TYPE OPERATORS

With this notion of the Dirac operator on $X$, one can prove a general convergence theorem for $\sigma(D^{M_i})$.

An application to upper eigenvalue bounds:

**Theorem 12.** Fix $M$ and the spinor module $V$. If there is not a uniform upper bound on the $j$-th eigenvalue of $|D^M|$ (among metrics with $\|R^M\|_\infty \leq K$ and $\text{diam}(M) = 1$) then $M$ collapses to a limit space $X$. Furthermore, the generic fiber $Z$ of the map $M \to X$ is an infranilmanifold which does not admit any affine-parallel spinor fields.

Finally, one can characterize the essential spectrum of a geometric Dirac-type operator on a finite-volume negatively curved manifold. That is, one can show that it equals the essential spectrum of a certain first-order ordinary differential operator associated to the ends.