Classical and Quantum Gravity

PAPER

Backreaction in the future behavior of an expanding vacuum spacetime

To cite this article: John Lott 2018 *Class. Quantum Grav.* **35** 035010

IOP Publishing Classical and Quantum Gravity

Class. Quantum Grav. **35** (2018) 035010 (10pp) https://doi.org/10.1088/1361-6382/aaa13e

Backreaction in the future behavior of an expanding vacuum spacetime*

John Lott

Department of Mathematics, University of California-Berkeley, Berkeley, CA 94720-3840, United States of America

E-mail: lott@berkeley.edu

Received 5 October 2017, revised 29 November 2017
Accepted for publication 13 December 2017
Published 8 January 2018

Abstract
We perform a rescaling analysis to analyze the future behavior of a class of T^2-symmetric vacuum spacetimes. We show that on the universal cover, there is C^0-convergence to a spatially homogeneous spacetime that does not satisfy the vacuum Einstein equations.

Keywords: backreaction, expanding, homogeneous

1. Introduction

In this paper we study the future behavior of an expanding vacuum spacetime (M, g) with compact spatial slices. A basic question is whether the gravitational dynamics, in the form of the equation $\mathrm{Ric}(g) = 0$, force the solution to approach a locally spatially homogeneous spacetime in the future; see [17, part I] for discussion. To make the question precise, one must say what sort of limit one is considering.

We take the viewpoint that the relevant notion of convergence is that of a sequence of pointed vacuum spacetimes. Details are in section 2 but to give the idea, let $\{p_i\}_{i=1}^{\infty}$ be a sequence of points in M going to future infinity. Let $\{c_i\}_{i=1}^{\infty}$ be a sequence of positive numbers. Then $\{(M, c_i g, p_i)\}_{i=1}^{\infty}$ is a sequence of pointed vacuum spacetimes and we can ask whether there is a limit $(M_\infty, g_\infty, p_\infty)$ in the pointed sense. The latter roughly means that we compare neighborhoods of p_i of an arbitrary but fixed size, as $i \to \infty$, to the corresponding neighborhood of p_∞. This notion is prevalent in Riemannian geometry and Ricci flow.

One basic issue is that the coordinates used to compute the future asymptotics of g may not be well adapted to describe the geometry around p_i for large i. Hence in the definition of convergence, before taking a limit one allows i-dependent changes of coordinates. One can think of taking normal coordinates around p_i.

There is some freedom in the choice of parameters $\{c_i\}_{i=1}^{\infty}$, which determine the scales at which we are making comparisons. They should have engineering dimension *time*$^{-2}$ or

* Research partially supported by NSF grant DMS-1510192.

distance^{-2}, so that $c_i g_i$ is dimensionless. By a 'type-III rescaling' we mean that c_i is constructed using the proper time of p_i from a fixed hypersurface, or the Hubble time $t = -\frac{3}{H}$ of p_i with respect to a constant mean curvature (CMC) spatial foliation with mean curvature function $H : (T_0, \infty) \to (H_0, \infty)$, where $H_0 < 0$. (The negativity of H is the expanding nature of the spacetime.)

One must also specify the sense in which the metric tensors converge. In [1, 14], it was shown that in the case of a CMC foliation, if the curvature has quadratic decay in the Hubble time then, after passing to a subsequence, there is a limit of the metrics in the pointed weak $W^{2,q}$-topology, for any $q \in [1, \infty)$, with the limit being a vacuum spacetime. In such a case, we can also assume that the metrics converge in the pointed $C^{1,\alpha}$-topology for any $\alpha \in (0, 1)$.

In this paper we look instead at type-III rescalings of vacuum spacetimes that may not satisfy the curvature decay condition. Although a limit is no longer guaranteed, we can still ask whether the pointed spacetimes have a limit $(M_\infty, g_\infty, p_\infty)$, say in the pointed C^0-topology. If (M_∞, g_∞) exists, and is locally spatially homogeneous, then it makes sense to say that the original (M, g) approaches a locally spatially homogeneous spacetime in the C^0-topology, along the sequence $\{p_i\}$. We will actually pass to the universal cover $\widetilde{M}$ and ask whether the lifted spacetime $(\widetilde{M}, \widetilde{g})$ approaches a spatially homogeneous spacetime. It is quite possible that $(\widetilde{M}, \widetilde{g})$ approaches a spatially homogeneous spacetime in the C^0-topology, but not in some stronger topology.

We perform this rescaling analysis for a class of vacuum spacetimes with compact spatial slices diffeomorphic to T^3, and invariance under the action of the group T^2. Such a spacetime is *polarized* if the Killing fields can be taken to be orthogonal, i.e. if the T^2-invariance can be promoted to an $(O(2) \times O(2))$-invariance. A T^2-symmetric spacetime has a *twist constant K*; if $K = 0$ then the spacetime is *Gowdy*. Future asymptotics of T^2-invariant spacetimes were considered by Ringström in the Gowdy case [15, 16] and the nonGowdy case [18]. More precise asymptotics were obtained by LeFloch and Smulevici in the polarized nonGowdy case, for initial data that is sufficiently close to the asymptotic regime [11].

Proposition 1.1. *After passing to the universal cover, any vacuum spacetime of the type considered in [11] has a smooth type-III rescaling limit g_∞, in the pointed C^0-topology. The Lorentzian metric g_∞ is spatially homogeneous but does not satisfy the vacuum Einstein equations.*

The effective stress–energy tensor $T = \mathrm{Ric}_{g_\infty} - \frac{1}{2} R_{g_\infty} g_\infty$ of g_∞ vanishes except for the T_{00} component, which is positive. (We do not claim that T has any physical meaning.) The fact that g_∞ does not satisfy the vacuum Einstein equations implies that the convergence cannot be in the pointed $(C^0 \cap H^1)$-topology, as otherwise the vacuum Einstein equations would make sense weakly and pass to the limit; see [13].

That a limit of vacuum spacetimes can have a nonzero stress–energy tensor is called backreaction [6, 8, 10]. In effect, fluctations of the geometry, with increasing frequency, can average out to zero in some parts of the Einstein equations, but give a nonzero contribution through nonlinearities to other parts. This phenomenon of increasing fluctuations also arose in the analysis of expanding spacetimes [14–16]. In [8], a framework was developed to analyze backreaction, with one of the main conclusions being that the effective stress–energy tensor is trace-free. We see that the framework of [8] does not apply to our rescaling examples.

The fact that T_{00} is positive in proposition 1.1 makes one wonder how generally a limiting stress–energy tensor satisfies some positive energy condition. Motivated by the results of [5, 9], in section 4 we raise a purely Riemannian question about the behavior of scalar curvature when taking a C^0-limit of Riemannian metrics. In proposition 4.3 we show that a positive answer to this question implies that if a sequence of CMC vacuum spacetimes converges in the pointed weak H^1-topology and the pointed C^0-topology, then the limiting spacetime has a nonnegative energy density.

The structure of the paper is the following. In section 2 we discuss rescaling limits of expanding vacuum spacetimes. In section 3 we analyze the polarized T^2-symmetric spacetimes of [11]. Section 4 has the link to questions of scalar curvature in Riemannian geometry. Section 5 has a short discussion of the results of the paper.

I thank Hans Ringström for references to the literature, and the referees for useful comments.

2. Rescaling limits

In this section we discuss notions of pointed convergence and rescaling for spacetimes. These notions are not new, at least in spirit; see [1, 2].

2.1. Rescaling limits of spacetimes

Let $\{(M_i, g_i)\}_{i=1}^{\infty}$ be a sequence of $(n+1)$-dimensional Lorentzian manifolds. For the moment, we do not specify the regularity of the metrics. Let $p_i \in M_i$ be a basepoint. Let (M_∞, g_∞) be another such Lorentzian manifold, with basepoint $p_\infty \in M_\infty$. We say that $\lim_{i\to\infty}(M_i, g_i, p_i) = (M_\infty, g_\infty, p_\infty)$ if there is

- An exhaustion of M_∞ by compact codimension-zero submanifolds-with-boundary $p_\infty \in K_1 \subset K_2 \subset \ldots$, and
- Maps $\phi_{i,j} : K_j \to M_i$, with $\phi_{i,j}(p_\infty) = p_i$, that are diffeomorphisms onto their images for large i, such that
- For all j, we have $\lim_{i\to\infty} \phi_{i,j}^* g_i = g_\infty$ on K_j.

Here the notion of convergence of metrics depends on the topology that we want to consider, e.g. C^0, C^k, C^∞, $W^{q,k}$, etc. If each (M_i, g_i) is a C^2-regular vacuum spacetime, i.e. $\mathrm{Ric}(g_i) = 0$, and g_∞ is C^2-regular, then (M_∞, g_∞) is a vacuum spacetime provided that the metric convergence is $C^0 \cap H^1$, since the Ricci-flat condition then makes sense weakly.

If we start with a single smooth Lorentzian manifold (M, g), and a sequence $\{p_i\}_{i=1}^{\infty}$ in M, then we may want to take $M_i = M$ and $g_i = c_i g$ for some constants $c_i > 0$. The goal is to find constants c_i and maps $\phi_{i,j} : K_j \to M$ so that there is a limit (M_∞, g_∞). We want c_i to have engineering dimension *time*$^{-2}$ or *distance*$^{-2}$, so that g_i is dimensionless. If one takes $\{c_i\}_{i=1}^{\infty}$ increasing sufficiently quickly then one can always get a flat limit in the smooth topology, but this would be considered uninteresting.

2.2. Rescaling limits of CMC spacetimes

Going back to the sequence $\{(M_i, g_i)\}_{i=1}^{\infty}$, suppose that each (M_i, g_i) is smooth and that there is a globally hyperbolic foliation $M_i = (T_i, \infty) \times X_i$ by spatial hypersurfaces. We can generally perform spatial diffeomorphisms to write

$$g_i = -L_i^2(t)\mathrm{d}t^2 + h_i(t), \tag{2.1}$$

where $L_i(t) \in C^\infty(X_i)$ is the lapse function and $h_i(t)$ is a Riemannian metric on X_i; for example, we can always put g_i in this form if X_i is compact. Write $p_i = (t_i, x_i)$.

Let $M_\infty = I_\infty \times Y$ be a putative limit space, for some open interval $I_\infty \subset \mathbb{R}$, with basepoint (u_∞, y_∞). One may wish to consider comparison maps that preserve the structure

(2.1). To do so, let $u_\infty \in C_1 \subset C_2 \ldots$ be an exhaustion of I_∞ by compact intervals. Let $\sigma_{i,j} : C_j \to (T_i, \infty)$ be a map with $\sigma_{i,j}(u_\infty) = t_i$ that is a diffeomorphism to its image for large i. Let $y_\infty \in K'_1 \subset K'_2 \subset \ldots$ be an exhaustion of Y by compact codimension-zero submanifolds-with-boundary. Put $K_j = C_j \times K'_j$. Given maps $\eta_{i,j} : K'_j \to X_i$ with $\eta_{i,j}(y_\infty) = x_i$ that are diffeomorphisms to their images for large i, we define $\phi_{i,j} : K_j \to M_i$ by $\phi_{i,j}(u, y) = (\sigma_{i,j}(u), \eta_{i,j}(y))$. Then $\phi^*_{i,j} g_i$ has the form (2.1). If the convergence $\lim_{i\to\infty} \phi^*_{i,j} g_i = g_\infty\big|_{K_j}$ is C^0 for each j, then g_∞ also has the form (2.1).

A special case is when for each $t \in (T_i, \infty)$, the hypersurface $\{t\} \times X_i$ has constant mean curvature $H_i(t)$. Then $\phi^*_{i,j} g_i$ also has a constant mean curvature (CMC) foliation. If the convergence $\lim_{i\to\infty} \phi^*_{i,j} g_i = g_\infty\big|_{K_j}$ is C^1 for each j then g_∞ acquires a CMC foliation.

We say that (M, g) is an expanding CMC spacetime if $H : (T_0, \infty) \to (H_0, 0)$ is bijective and increasing, where $H_0 < \infty$. Then we can assume that the time parameter t of M satisfies $t = -\frac{n}{H}$, i.e. t is the Hubble time of the CMC foliation.

2.3. Type-III rescalings

Continuing with an expanding CMC spacetime $M = (T_0, \infty) \times X$, parametrized by Hubble time t, and a sequence $p_i = (t_i, x_i)$ with $\lim_{i\to\infty} t_i = \infty$, we say that a type-III rescaling is when $M_i = M$, $X_i = X$, $I_\infty = (0, \infty)$, $u_\infty = 1$, $C_j = \left[\frac{1}{j}, j\right]$, $\sigma_{i,j}(u) = t_i u$ (for all i sufficiently large that $t_i > T_0 j$) and $c_i = t_i^{-2}$. Then u is the Hubble time for the CMC foliation of $\phi^*_{i,j} g$. If the convergence $\lim_{i\to\infty} \phi^*_{i,j} g = g_\infty\big|_{K_j}$ is C^1 then u is also the Hubble time for the CMC foliation of g_∞.

Define a curvature norm for (M, g), at a point $m \in M$, as follows [1, 0.7]. Let $\{e_i\}_{i=0}^n$ be an orthonormal basis for $T_m M$ with $e_0 = (-g(\partial_t, \partial_t))^{-\frac{1}{2}} \partial_t$. Put

$$|\mathrm{Rm}|(m) = \sqrt{\sum_{\alpha,\beta,\gamma,\delta=0}^{n} R(e_\alpha, e_\beta, e_\gamma, e_\delta)^2}. \tag{2.2}$$

Assuming that each $(X, h(t))$ is complete, and $|\mathrm{Rm}| = O\left(t^{-2}\right)$, we can use the type-III scaling to extract a subsequential limit $(M_\infty, g_\infty, p_\infty)$ [14, corollary 3.4]. The limit is in the pointed weak $W^{2,q}$-topology for all $q \in [1, \infty)$, but Y may be an étale groupoid rather than a manifold, if there is 'collapsing'. In some cases, such as if X is compact and aspherical, one can stay in the world of manifolds by lifting g to the universal cover $(T_0, \infty) \times \widetilde{X}$ and taking a pointed limit there.

Example 2.3. Consider a Kasner spacetime $(0, \infty) \times T^n$ with metric

$$g = -\frac{1}{n^2} \mathrm{d}t^2 + \sum_{k=1}^{n} t^{2p_k} (\mathrm{d}x^k)^2, \tag{2.3}$$

where $\sum_{k=1}^n p_k = \sum_{k=1}^n p_k^2 = 1$. Then t is the Hubble time. Passing to the universal cover, we take $x^k \in \mathbb{R}$. Put $M_\infty = (0, \infty) \times \mathbb{R}^n$, with $p_\infty = (1, 0)$. Writing a point $x \in \mathbb{R}^n$ as $x = (x^k)$, define $\eta_{i,j}(y) \in \mathbb{R}^n$ to be the point whose kth-coordinate is $t_i^{1-p_k} y^k + x_i^k$. Then $\phi^*_{i,j} g_i$ is the Kasner metric on M_∞, now with time parameter u. Hence in this case the rescaling limit exists on the universal cover.

If the foliation $M = (T_0, \infty) \times X$ may not be a CMC foliation, an alternative type-III rescaling uses the proper time from a fixed hypersurface [1]. Fixing $T_1 \in (T_0, \infty)$, define

$\tau : [T_1, \infty) \to \mathbb{R}$ by saying that $\tau(t)$ is the maximal length of causal curves from the time-T_1 hypersurface to the time-t hypersurface. We reparametrize M by τ. Putting $\tau_i = \tau(t_i)$, we can rescale using $\sigma_{i,j}(u) = \tau_i u$ and $c_i = \tau_i^{-2}$.

2.4. Type-II rescalings

Again in the case of an expanding CMC spacetime, parametrized by Hubble time t, if $|\mathrm{Rm}|$ is not $O\left(t^{-2}\right)$ then it makes sense to do a type-II rescaling. Put $I_\infty = \mathbb{R}$, $u_\infty = 0$ and $C_j = [-j, j]$. Given a sequence $p_i = (t_i, x_i)$ with $\lim_{i\to\infty} t_i = \infty$, put $\sigma_{i,j}(u) = |\mathrm{Rm}(p_i)|^{-\frac{1}{2}} u + t_i$ (for all i sufficiently large that $t_i - |\mathrm{Rm}(p_i)|^{-\frac{1}{2}} j > T_0$) and $c_i = |\mathrm{Rm}(p_i)|$. By construction, the curvature tensor of g_i at p_i has norm one. With the right choice of $\{p_i\}_{i=1}^\infty$, there is a subsequential limit g_∞ in the pointed weak $W^{2,q}$-topology for any $q \in [1, \infty)$ [14, proposition 2.51]. Two caveats must be made. First, in the collapsing case, Y may be an étale groupoid rather than a manifold. Second, the limiting lapse function L_∞ may vanish. If this happens then g_∞ is a static solution of the constraint equations.

If the second fundamental form K of g satisfies an inequality $|K|^2 \leqslant \mathrm{const.}H^2$ then $L_\infty > 0$ [14, proposition 4.1]. If in addition $n = 3$ then g_∞ turns out to be a flat static spacetime [14, corollary 4.6]. The interpretation is that there are increasing fluctuations of the curvature tensor, at least in neighborhoods of the points p_i, that average out the normalized curvature to become zero in the weak limit.

3. Polarized T^2-symmetric nonGowdy spacetimes

We now take $n = 3$ and $X = T^3$, with linear coordinates $(\theta, x, y) \in (\mathbb{R}/2\pi\mathbb{Z})^3$. We assume that there is an $(O(2) \times O(2))$-symmetry, acting on the (x, y)-factor. Take the time parameter R so that the area of the T^2-orbit is R^2. As in [11, 2–2], the metric can be written

$$g = \mathrm{e}^{2(\eta-U)}(-dR^2 + a^{-2}\mathrm{d}\theta^2) + \mathrm{e}^{2U}(\mathrm{d}x + G\mathrm{d}\theta)^2 + \mathrm{e}^{-2U}R^2(\mathrm{d}y + H\mathrm{d}\theta)^2, \tag{3.1}$$

where η, U, a, G and H are functions of R and θ. Let K be the twist constant. We assume that $K \neq 0$. Let $\langle\eta\rangle(R)$ denote the average value of $\eta(R, \theta)$ with respect to $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, and similarly for $\langle U\rangle(R)$. From [11, theorem 7.1 and (2–8)], if the initial data are close enough to the asymptotic regime then the leading asymptotics of the metric parameters are

$$\begin{aligned}
|K^2\mathrm{e}^{2\eta} - R^2| &= O\left(R^{\frac{7}{4}}\right), \\
|\eta - \langle\eta\rangle| &= O\left(R^{-\frac{1}{2}}\right), \\
|a^{-1} - \frac{2}{\sqrt{5}}C_\infty^{\frac{1}{2}}R^{\frac{1}{2}}\mathcal{L}(\theta)| &= O\left(R^{-1}\right), \\
|U - C_U| &= O\left(R^{-\frac{1}{2}}\right), \\
|G - \mathcal{G}(\theta)| &= 0, \\
|H - \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}R^{\frac{1}{2}}\mathcal{L}(\theta)| &= O\left(R^{\frac{1}{4}}\right),
\end{aligned} \tag{3.2}$$

where C_U, C_∞ are constants with $C_\infty > 0$, and $\mathcal{G}(\theta), \mathcal{L}(\theta)$ are functions with $\mathcal{L}(\theta) > 0$.

The constant-R slices are generally not CMC. The maximal length of causal curves between a constant R_0-slice and a constant R-slice is asymptotic to

$$\int_{R_0}^{R} \mathrm{e}^{\langle\eta\rangle(r)-\langle U\rangle(r)}\,\mathrm{d}r \sim \text{const.}\int_{R_0}^{R} r\,\mathrm{d}r \sim \text{const.}R^2; \tag{3.3}$$

see [16, proof of proposition 3]. Changing variable from R to $t = R^2$, the asymptotic behavior of g is

$$\begin{aligned} g \sim &-\frac{1}{4}K^{-2}\mathrm{e}^{-2C_U}\mathrm{d}t^2 + \frac{4}{5}K^{-2}\mathrm{e}^{-2C_U}C_\infty\mathcal{L}^2 t^{\frac{3}{2}}\mathrm{d}\theta^2 + \mathrm{e}^{2C_U}(\mathrm{d}x + \mathcal{G}\mathrm{d}\theta)^2 \\ &+ \mathrm{e}^{-2C_U}t\left(\mathrm{d}y + \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}\mathcal{L}t^{\frac{1}{4}}\mathrm{d}\theta\right)^2. \end{aligned} \tag{3.4}$$

That is, when restricted to a future time interval $t \in [c, \infty)$, the two sides of (3.4) are $(1 + o(c))$-biLipschitz.

Let $p_i = (t_i, x_i)$ be a sequence with $\lim_{i\to\infty} t_i = \infty$. The choice of points $x_i \in T^3$ will be irrelevant. Putting $t = t_i u$ gives

$$\begin{aligned} t_i^{-2}g \sim &-\frac{1}{4}K^{-2}\mathrm{e}^{-2C_U}\mathrm{d}u^2 + \frac{4}{5}K^{-2}\mathrm{e}^{-2C_U}C_\infty\mathcal{L}^2 t_i^{-\frac{1}{2}}u^{\frac{3}{2}}\mathrm{d}\theta^2 + \mathrm{e}^{2C_U}\left(t_i^{-1}\mathrm{d}x + t_i^{-1}\mathcal{G}\mathrm{d}\theta\right)^2 \\ &+ \mathrm{e}^{-2C_U}u\left(t_i^{-\frac{1}{2}}\mathrm{d}y + \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}\mathcal{L}t_i^{-\frac{1}{4}}u^{\frac{1}{4}}\mathrm{d}\theta\right)^2. \end{aligned} \tag{3.5}$$

Passing to the universal cover, we take $(\theta, x, y) \in \mathbb{R}^3$. For simplicity, we just take the spatial basepoint to be $0 \in \mathbb{R}^3$. We define $\widehat{\theta}$, $\widehat{x}$ and $\widehat{y}$ by $\mathrm{d}\widehat{\theta} = t_i^{-\frac{1}{4}}\mathcal{L}\mathrm{d}\theta$, $\widehat{x} = t_i^{-1}x$ and $\widehat{y} = t_i^{-\frac{1}{2}}y$. Then

$$\begin{aligned} t_i^{-2}g \sim &-\frac{1}{4}K^{-2}\mathrm{e}^{-2C_U}\mathrm{d}u^2 + \frac{4}{5}K^{-2}\mathrm{e}^{-2C_U}C_\infty u^{\frac{3}{2}}\mathrm{d}\widehat{\theta}^2 + \mathrm{e}^{2C_U}(\mathrm{d}\widehat{x} + \mathcal{G}\mathcal{L}^{-1}t_i^{-\frac{3}{4}}\mathrm{d}\widehat{\theta})^2 \\ &+ \mathrm{e}^{-2C_U}u\left(\mathrm{d}\widehat{y} + \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}u^{\frac{1}{4}}\mathrm{d}\widehat{\theta}\right)^2. \end{aligned} \tag{3.6}$$

Since $\lim_{i\to\infty} t_i = \infty$, there is a limit $\lim_{i\to\infty} t_i^{-2}g_i = g_\infty$ in the pointed C^0-topology:

$$\begin{aligned} g_\infty = &-\frac{1}{4}K^{-2}\mathrm{e}^{-2C_U}\mathrm{d}u^2 + \frac{4}{5}K^{-2}\mathrm{e}^{-2C_U}C_\infty u^{\frac{3}{2}}\mathrm{d}\widehat{\theta}^2 + \mathrm{e}^{2C_U}\mathrm{d}\widehat{x}^2 \\ &+ \mathrm{e}^{-2C_U}u\left(\mathrm{d}\widehat{y} + \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}u^{\frac{1}{4}}\mathrm{d}\widehat{\theta}\right)^2. \end{aligned} \tag{3.7}$$

We see that g_∞ has a spatial $\mathbb{R}^3$-symmetry. Redefining $u = \widehat{R}^2$ gives

$$g_\infty = \mathrm{e}^{2(\widehat{\eta}-\widehat{U})}(-\,\mathrm{d}\widehat{R}^2 + \widehat{a}^{-2}\mathrm{d}\widehat{\theta}^2) + \mathrm{e}^{2\widehat{U}}(\mathrm{d}\widehat{x} + \widehat{G}\mathrm{d}\widehat{\theta})^2 + \mathrm{e}^{-2\widehat{U}}\widehat{R}^2(\mathrm{d}\widehat{y} + \widehat{H}\mathrm{d}\widehat{\theta})^2, \tag{3.8}$$

where

$$\begin{aligned} \mathrm{e}^{2\widehat{\eta}} &= K^{-2}\widehat{R}^2, \\ \widehat{a}^{-1} &= \frac{2}{\sqrt{5}}C_\infty^{\frac{1}{2}}\widehat{R}^{\frac{1}{2}}, \\ \widehat{U} &= C_U, \\ \widehat{G} &= 0, \\ \widehat{H} &= \frac{4}{K\sqrt{5}}C_\infty^{\frac{1}{2}}\widehat{R}^{\frac{1}{2}}. \end{aligned} \tag{3.9}$$

To check whether g_∞ satisfies the vacuum Einstein equations, we can plug (3.9) into [11, 2–3–2–8]. One finds that these equations are satisfied except for the constraint equation [11, 2–6], which instead becomes

$$\widehat{\eta}_{\widehat{R}} + \frac{K^2}{4\widehat{R}^3} e^{2\widehat{\eta}} - \widehat{a}\widehat{R}\left(\widehat{a}^{-1}\widehat{U}_{\widehat{R}}^2 + \widehat{a}\widehat{U}_{\widehat{\theta}}^2\right) = \frac{5}{4\widehat{R}}. \tag{3.10}$$

The left-hand side of (3.10) is proportionate to $(\mathrm{Ric}_\infty - \frac{1}{2}R_\infty g_\infty)\left(\partial_{\widehat{R}}, \partial_{\widehat{R}}\right)$. We conclude that g_∞ satisfies the Einstein equations, except for the nonvanishing of $(\mathrm{Ric}_\infty - \frac{1}{2}R_\infty g_\infty)\left(\partial_{\widehat{R}}, \partial_{\widehat{R}}\right)$.

Remark 3.11. One can also consider rescaling limits of Gowdy spacetimes, i.e. T^2-symmetric spacetimes with spatial slices diffeomorphic to T^3, and vanishing twist constant. For such spacetimes, the curvature decays like the inverse square of the proper time function, as measured from a fixed hypersurface [16, theorem 2]. Hence we expect that they have type-III rescaling limits that are vacuum spacetimes. If the metric is independent of the parameter θ of $S^1 = T^3/T^2$ then before rescaling, the solution on the universal cover is a spatially homogeneous Kasner spacetime. In this case the rescaling limit exists and is also a Kasner spacetime; see example 2.3. If the metric is not θ-independent then asymptotics were given in [15, 16]. Some rough calculations indicate that the rescaled metrics should approach a flat metric (in the *weak* $W^{2,q}$-topology). However, it does not seem to be possible to prove this rigorously from the known asymptotics.

Another interesting vacuum spacetime is the Bianchi VIII solution. The so-called non-NUT type does not have curvature that decays like the inverse square of the Hubble time (or the proper time) [16, theorem 3]. Based on some calculations in terms of the coordinates from [7, section 4.3] or [12, section 3.3.5], there does not appear to be a type-III rescaling limit in the C^0-topology.

4. Nonnegativity of induced energy density

We recall that in section 2.3, there was a subsequential rescaling limit of a CMC vacuum spacetime with quadratic curvature decay, that exists in the pointed weak $W^{2,q}$-topology for all $q \in [1, \infty)$. This is related to the fact from Riemannian geometry that a sequence of complete pointed Riemannian manifolds, with uniformly bounded curvature, has a subsequential limit in the pointed weak $W^{2,q}$-topology.

In Riemannian geometry, if one weakens the curvature assumptions to a uniform lower bound on the Ricci curvature, and a uniform lower bound on the injectivity radius, then there is a subsequential limit in the pointed weak $W^{1,q}$-topology for all $q \in [1, \infty)$, and hence also a subsequential limit in the pointed C^α-topology for all $\alpha \in (0, 1)$ [3]. Motivated by this, we consider a sequence of CMC vacuum spacetimes $(T_i, \infty) \times X_i$, as in section 2.2, so that for each $u \in I_\infty$,

- The pullback metrics $h_i(u)$ and the pullback lapse functions $L_i(u)$ converge in the pointed weak H^1-topology and the pointed C^0-topology, and
- The pullback second fundamental forms $K_i(u)$ converge in the pointed weak L^2-topology.

In Riemannian geometry, there is a general principle that curvature can only go up when taking limits. In the case of scalar curvature, a precise statement along these lines is the following result.

Theorem 4.1 ([5, 9]). *Let Y be a smooth manifold. Given $\kappa \in C(Y)$, let $\{\mathfrak{g}_i\}_{i=1}^{\infty}$ be a sequence of C^2-regular Riemannian metrics on Y with scalar curvature function bounded below by κ. If $\{\mathfrak{g}_i\}_{i=1}^{\infty}$ converges on compact subsets in the C^0-topology to a C^2-regular Riemannian metric $\mathfrak{g}_\infty$, then $\mathfrak{g}_\infty$ has scalar curvature function bounded below by κ.*

Question 4.2. Let Y be a smooth manifold. Let $\{\mathfrak{g}_i\}_{i=1}^{\infty}$ be a sequence of C^2-regular Riemannian metrics on Y that C^0-converges on compact subsets to a C^2-regular Riemannian metric $\mathfrak{g}_\infty$ on Y. Suppose that on any compact subset, the scalar curvatures of $\{\mathfrak{g}_i\}_{i=1}^{\infty}$ are uniformly bounded below. Is it true that for every nonnegative compactly supported smooth density ω on Y, the scalar curvature $R_{\mathfrak{g}_\infty}$ satisfies $\int_Y R_{\mathfrak{g}_\infty}\omega \geqslant \liminf_{i\to\infty}\int_X R_{\mathfrak{g}_i}\omega$?

A positive answer to question 4.2 clearly implies theorem 4.1.

Proposition 4.3. *Consider a sequence of expanding CMC vacuum spacetimes $(T_i,\infty)\times X_i$, as in section 2.2, that by assumption converges in the sense of the bullet-points above to a CMC spacetime $I_\infty \times Y$, equipped with a C^2-regular metric g_∞ that is also parametrized by the Hubble time Then a positive answer to question 4.2 implies that $\left(\mathrm{Ric}_{g_\infty} - \frac{1}{2}R_{g_\infty}g_\infty\right)(\partial_u,\partial_u) \geqslant 0$.*

Proof. On a given u-slice of the limit space, the Gauss equation gives

$$(\mathrm{Ric}_{g_\infty} - \frac{1}{2}R_{g_\infty}g_\infty)(\partial_u,\partial_u) = \frac{1}{2}\left(R_{h_\infty} - |K_\infty|^2 + H_\infty^2\right). \tag{4.1}$$

By assumption, $H_\infty = -\frac{n}{u}$. Choose a nonnegative compactly supported smooth density ω on Y. Then

$$\int_Y \left((\mathrm{Ric}_{g_\infty} - \frac{1}{2}R_{g_\infty}g_\infty)(\partial_u,\partial_u)\right)\omega = \frac{1}{2}\int_Y\left(R_{h_\infty} - |K_\infty|^2 + \frac{n^2}{u^2}\right)\omega. \tag{4.2}$$

For large i, we can pullback h_i and K_i to $\mathrm{supp}(\omega)$, so we assume that everything lives on Y. The constraint equations give

$$R_{h_i} - |K_i|^2 + \frac{n^2}{u^2} = R_{h_i} - |K_i|^2 + H_i^2 = 0. \tag{4.3}$$

In particular, R_{h_i} is bounded below in terms of u. A positive answer to question 4.2 implies that

$$\int_Y R_{h_\infty}\omega \geqslant \liminf_{i\to\infty}\int_Y R_{h_i}\omega = \liminf_{i\to\infty}\int_Y\left(|K_i|^2 - \frac{n^2}{u^2}\right)\omega. \tag{4.4}$$

Then

$$\begin{aligned}
&\int_Y \left((\mathrm{Ric}_{g_\infty} - \frac{1}{2}R_{g_\infty}g_\infty)(\partial_u,\partial_u)\right)\omega \\
&\geqslant \frac{1}{2}\liminf_{i\to\infty}\int_Y\left(|K_i|^2 - |K_\infty|^2\right)\omega \\
&= \frac{1}{2}\liminf_{i\to\infty}\int_Y\left(|K_i - K_\infty|^2 + 2\langle K_i - K_\infty, K_\infty\rangle\right)\omega.
\end{aligned} \tag{4.5}$$

As $\lim_{i\to\infty} K_i = K_\infty$ in the weak L^2-topology on $\mathrm{supp}(\omega)$, it follows that

$$\lim_{i\to\infty} \int_Y \langle K_i - K_\infty, K_\infty \rangle \omega = 0. \tag{4.6}$$

From (4.5), we obtain that

$$\int_Y \left((\mathrm{Ric}_{g_\infty} - \frac{1}{2} R_{g_\infty} g_\infty)(\partial_u, \partial_u) \right) \omega \geqslant 0 \tag{4.7}$$

for every nonnegative compactly supported smooth density ω on Y. This implies that $(\mathrm{Ric}_{g_\infty} - \frac{1}{2} R_{g_\infty} g_\infty)(\partial_u, \partial_u) \geqslant 0$. □

5. Discussion

In this paper we described a notion of rescaling limits for Lorentzian spacetimes. For a class of T^2-symmetric vacuum spacetimes, we showed that on the universal cover, there is a rescaling limit in the pointed C^0-topology that is smooth and spatially homogeneous, but does not satisfy the vacuum Einstein equations.

The paper [8] showed that under certain assumptions, a weak limit of a 1-parameter family of vacuum spacetimes has an effective stress–energy tensor that is traceless. The assumptions are about the asymptotics of the metric tensors as the parameter λ goes to zero; we refer to [8] for the details. In our examples, the effective stress–energy tensor is not traceless. Hence the assumed asymptotics of [8] do not hold. One could try to perform a more detailed analysis.

More generally, one could look at rescaling limits of other solutions of the Einstein equations. In this paper, we focused on the future behavior of expanding solutions. One could also consider rescaling limits as a singularity develops or, similarly, as one goes backward in time toward an initial singularity. There is some relation here to the paper [4].

ORCID iDs

John Lott https://orcid.org/0000-0002-5107-8719

References

[1] Anderson M 2001 On long-time evolution in general relativity and geometrization of 3-manifolds *Commun. Math. Phys.* **222** 533–67
[2] Anderson M 2003 Regularity for Lorentz metrics under curvature bounds *J. Math. Phys.* **44** 2994–3012
[3] Anderson M and Cheeger J 1992 C^α-compactness for manifolds with Ricci curvature and injectivity radius bounded below *J. Differ. Geom.* **35** 265–81
[4] Andersson L, van Elst H, Lim W and Uggla C 2005 Asymptotic silence of generic cosmological singularities *Phys. Rev. Lett.* **94** 051101
[5] Bamler R 2016 A Ricci flow proof of a result by Gromov on lower bounds for scalar curvature *Math. Res. Lett.* **23** 325–37
[6] Buchert T and Räsänen S 2012 Backreaction in late time cosmology *Ann. Rev. Nucl. Part. Sci.* **62** 57–79
[7] Glickenstein D 2008 Riemannian groupoids and solitons for three-dimensional homogeneous Ricci and cross curvature flows *Int. Math. Res. Not.* **2008** rnn034
[8] Green S and Wald R 2011 A new framework for analyzing the effects of small scale inhomogeneities in cosmology *Phys. Rev.* D **83** 084020

[9] Gromov M 2014 Dirac and plateau billiards in domains with corners *Cent. Eur. J. Math.* **12** 1109–56
[10] Huneau C and Luk J 2017 High-frequency backreaction for the Einstein equations under polarized $U(1)$ symmetry (arXiv:1706.0951)
[11] LeFloch P and Smulevici J 2016 Future asymptotics and geodesic completeness of polarized T^2-symmetric spacetimes *Anal. PDE* **9** 363–95
[12] Lott J 2007 On the long-time behavior of type-III Ricci flow solutions *Math. Ann.* **339** 627–66
[13] Lott J 2016 Ricci measure for some singular Riemannian metrics *Math. Ann.* **365** 449–71
[14] Lott J 2017 Collapsing in the Einstein flow (arXiv:1701.05150)
[15] Ringström H 2004 On a wave map equation arising in general relativity *Commun. Pure Appl. Math.* **57** 657–703
[16] Ringström H 2006 On curvature decay in expanding cosmological models *Commun. Math. Phys.* **264** 613–30
[17] Ringström H 2013 *On the Topology and Future Stability of the Universe* (Oxford: Oxford University Press)
[18] Ringström H 2015 Instability of spatially homogeneous solutions in the class of T^2-symmetric solutions to Einstein's vacuum equations *Commun. Math. Phys.* **334** 1299–375