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Local index theory over étale groupoids

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Abstract. We give a superconnection proof of Connes' index theorem for proper cocompact actions of étale groupoids. This includes Connes' general foliation index theorem for foliations with Hausdorff holonomy groupoid.

1. Introduction

This paper is concerned with a families index theorem in which the family of operators is parametrized by a noncommutative space which comes from a smooth Hausdorff étale groupoid G. The relevant index theorem was stated by Connes in [8], Section III.7.γ, Theorem 12. We give a superconnection proof of Connes' theorem. The desirability of having such a proof was mentioned in [6]. In the case of a foliation, by taking a complete transversal, one recovers Connes' general foliation index theorem for a foliation whose holonomy groupoid is Hausdorff. For the history and significance of Connes' foliation index theorem we refer to [8], Sections I.5, II.8–9 and III.6–7, along with the references cited therein.

For concreteness, let us first discuss the case when G is the cross-product groupoid coming from the action of a finitely-generated discrete group Γ on a smooth manifold B. In this case the geometric setup for the index theorem consists of a manifold $\hat{M}$ on which Γ acts, and a submersion $\pi : \hat{M} \to B$ which is Γ-equivariant. In addition, we assume that the action of Γ on $\hat{M}$ is free, properly discontinuous and cocompact. Put $M = \hat{M}/\Gamma$, a compact manifold. (A relevant example is when $\hat{M} = \mathbb{R} \times S^1$, $B = S^1$ and $\Gamma = \mathbb{Z}$, with the action of $n \in \mathbb{Z}$ on $(r, e^{i\theta}) \in \mathbb{R} \times S^1$ given by $n \cdot (r, e^{i\theta}) = (r + n, e^{i(\theta + n\alpha)})$ for some $\alpha \in \mathbb{R}$. Then $M = T^2$.)

There is a quotient map $M \to B/\Gamma$, which will be the intuitive setting for our families index theorem. In general B/Γ is highly singular and, following Connes, we will treat it as a "noncommutative space" $\mathfrak{B}$. We will give a superconnection proof of a families index theorem in such a setting.

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Let us consider two special cases. If $\Gamma = \{e\}$ then we just have a submersion $M \to B$ with M compact. In this case, the relevant index theorem is the Atiyah-Singer families index theorem [1], for which a superconnection proof was given by Bismut [3]. At the other extreme, if B is a point then we just have a covering space $\hat{M} \to M$ of a compact manifold M. In this case the relevant index theorem is due to Miscenko-Fomenko [22] and Connes-Moscovici [9], and a superconnection proof was given by the second author [20]. To some degree, the present paper combines the superconnection proofs of these two special cases in order to deal with general Γ and B. However, there are some new features, which we will emphasize.

One motivation for giving a superconnection proof of Connes' theorem is that the superconnection formalism gives a somewhat canonical proof. In particular, it expresses the Chern character of the index as an explicit differential form. This is one of the reasons that the superconnection formalism allows extensions to the case of manifolds with boundary. When $\Gamma = \{e\}$, this is due to Bismut-Cheeger [4]. When B is a point, it is due to the second author [21] and Leichtnam-Piazza [17]. Based on the present paper, it should be possible to extend Connes' index theorem to manifolds-with-boundary. In particular, this would give an index theorem for a foliated manifold-with-boundary if the foliation is transverse to the boundary.

We now describe in some detail Connes' index theorem and the superconnection approach to its proof. Before setting up the superconnection formalism, we must first describe what we mean by functions and differential forms on the noncommutative base space $\mathfrak{B}$. There is a clear choice for a class of "smooth functions" on $\mathfrak{B}$, namely the algebraic cross-product $C_c^\infty(B) \rtimes \Gamma$. The choice of "differential forms" on $\mathfrak{B}$ is dictated by two facts. First, if $\Gamma = \{e\}$, i.e. in the commutative situation, we want to recover the smooth compactly-supported differential forms $\Omega_c^*(B)$. Second, the choice should extend to general smooth Hausdorff étale groupoids G. This dictates that we should take the differential forms to be elements of $\Omega^*(B, \mathbb{C}\Gamma) = \Omega_c^*(B) \hat{\otimes} \Omega^*(\mathbb{C}\Gamma)$, where $\Omega^*(\mathbb{C}\Gamma)$ is the graded differential algebra of noncommutative forms on $\mathbb{C}\Gamma$ ([19], Section 2.6), and the product in $\Omega^*(B, \mathbb{C}\Gamma)$ takes into account that Γ acts on $\Omega_c^*(B)$.

Given these choices, we need to know that the ensuing "homology" of $\mathfrak{B}$ is sufficiently rich. This is shown in the following theorem. Let $GT_{*,\langle e \rangle}$ denote the complex of graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ that are concentrated at the identity conjugacy class in Γ. (Only these graded traces will be relevant for the paper.) We let $C_*(B)$ denote the currents on B and we let $\overline{\mathscr{C}}_*(\Gamma)$ denote a certain complex of differential forms on $E\Gamma$, described in (2.49).

Theorem 1. *The homology of $GT_{*,\langle e \rangle}$ is isomorphic to the homology of the total complex of the double complex $\big((\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)) \otimes C_*(B)\big)^\Gamma$.*

In particular, if η is a closed graded n-trace on $\Omega^*(B, \mathbb{C}\Gamma)$, concentrated at the identity conjugacy class, then we obtain a corresponding cohomology class

$$\Phi_\eta \in \mathrm{H}_\tau^{n+\dim(B)+2\mathbb{Z}}\big((E\Gamma \times B)/\Gamma; \mathbb{R}\big),$$

where the $2\mathbb{Z}$ denotes an even-odd grading and the τ denotes a twisting by the orientation bundle of B. This shows our choice of differential forms gives the "right" answer, as $(E\Gamma \times B)/\Gamma$ is the classifying space BG for the groupoid G; compare the cyclic cohomology calculations in [6], [8], Section III.2.δ, [10], [11].

For analytic reasons arising from finite propagation speed estimates, we introduce a slightly larger space of differential forms. Let $\|\cdot\|$ be a word-length metric on Γ and put

$$\mathscr{B}^{\omega} = \left\{ \sum_{\gamma \in \Gamma} c_\gamma \gamma : |c_\gamma| \text{ decays faster than any exponential in } \|\gamma\| \right\}. \tag{1.1}$$

Put $C^\infty(B, \mathscr{B}^\omega) = \mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} \big(C_c^\infty(B) \rtimes \Gamma\big)$. It contains $C_c^\infty(B) \rtimes \Gamma$ as a dense subset. There is a corresponding space of differential forms $\Omega^*(B, \mathscr{B}^\omega)$, defined in (3.30).

Let us now state Connes' index theorem. Let $\hat{M}$ and M be as above. The Γ-covering $\hat{M} \to M$ is classified by a Γ-equivariant continuous map $\mu : \hat{M} \to E\Gamma$, defined up to Γ-homotopy. Let $\hat{v} : \hat{M} \to E\Gamma \times B$ be (μ, π) and let $v : M \to (E\Gamma \times B)/\Gamma$ be the Γ-quotient of $\hat{v}$, a classifying map for the action of the groupoid G on $\hat{M}$.

Let Z denote a fiber of the submersion $\pi : \hat{M} \to B$ and let TZ denote the vertical tangent bundle, a Γ-invariant vector bundle on $\hat{M}$. There is a foliation $\mathscr{F}$ on M whose leaves are the images of the fibers Z under π. The tangent bundle to the foliation is $T\mathscr{F} = (TZ)/\Gamma$, a vector bundle on M. Choose a smooth Γ-invariant vertical Riemannian metric g^{TZ} on $\hat{M}$. Assume that TZ has a Γ-invariant spin structure, with corresponding spinor bundle S^Z. Let $\hat{V}$ be a Γ-invariant vector bundle on $\hat{M}$, with Γ-invariant Hermitian connection $\nabla^{\hat{V}}$. Put $V = \hat{V}/\Gamma$, a vector bundle on M, and put $\hat{E} = S^Z \otimes \hat{V}$.

There is an ensuing Γ-invariant family D of Dirac-type operators which act fiberwise on $C_c^\infty(\hat{M}; \hat{E})$. Equivalently, the family D is G-invariant. Let $\mathrm{Ind}(D)$ denote the index of this family; we will say more about it later. Let η be a closed graded trace on $\Omega^*(B, \mathscr{B}^\omega)$ which is concentrated at the identity conjugacy class. In particular, η restricts to a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$. In this situation, Connes' index theorem ([8], Section III.7.γ, Theorem 12) becomes the statement that

$$\big\langle \mathrm{ch}\big(\mathrm{Ind}(D)\big), \eta \big\rangle = \int_M \hat{A}(T\mathscr{F}) \, \mathrm{ch}(V) v^* \Phi_\eta. \tag{1.2}$$

As mentioned before, special cases of Connes' index theorem are the Atiyah-Singer families index theorem and the covering-space index theorem.

The goal of the paper is to give a superconnection proof of (1.2). A key ingredient will be the $\mathbb{C}\Gamma$-vector bundle $\mathscr{E} = (\hat{M} \times \mathbb{C}\Gamma)/\Gamma$ on M, where Γ acts diagonally on $\hat{M} \times \mathbb{C}\Gamma$. By construction, $C^\infty(M; \mathscr{E}) \cong C_c^\infty(\hat{M})$. The natural flat connection $\nabla^{1,0}$ on $\mathscr{E}$ sends $f \in C_c^\infty(\hat{M})$ to df. An important part of our proof is a certain differentiation

$$\nabla^{0,1} : C_c^\infty(\hat{M}) \to \Omega^1(\mathbb{C}\Gamma) \otimes_{\mathbb{C}\Gamma} C_c^\infty(\hat{M}) \tag{1.3}$$

in the "noncommutative" directions. The explicit formula for $\nabla^{0,1}$ is given in Section 2. The sum of $\nabla^{1,0}$ and $\nabla^{0,1}$ is a nonflat connection

$$\nabla^{\mathrm{can}} : C_c^\infty(\hat{M}) \to \Omega^1(\hat{M}, \mathbb{C}\Gamma) \otimes_{C_c^\infty(\hat{M}) \rtimes \Gamma} C_c^\infty(\hat{M}). \tag{1.4}$$

Choose a Γ-invariant horizontal distribution $T^H\hat{M}$ on $\hat{M}$. Suppose that Z is even-dimensional. For $s > 0$, there is an ensuing Bismut superconnection A_s^{Bismut} on the submersion $\hat{M} \to B$ ([2], Section 10.3, [3], Section IIIa). Our noncommutative superconnection is simply $A_s = A_s^{\mathrm{Bismut}} + \nabla^{0,1}$. Let $\Omega^*(B, \mathscr{B}^\omega)_{\mathrm{ab}}$ be the quotient of $\Omega^*(B, \mathscr{B}^\omega)$ by the closure of its graded commutator. We show that the Chern character $\mathrm{Tr}_{s,\langle e \rangle}(e^{-A_s^2})$ of the superconnection A_s is well-defined in $\Omega^*(B, \mathscr{B}^\omega)_{\mathrm{ab}}$. It is a closed form and its cohomology class is independent of s. The next result gives its $s \to 0$ limit.

Let $\phi \in C_c^\infty(\hat{M})$ be such that $\sum_{\gamma \in \Gamma} \gamma \cdot \phi = 1$. Let $\mathscr{R}$ be the rescaling operator on $\Omega^{\mathrm{even}}(B, \mathscr{B}^\omega)_{\mathrm{ab}}$ which multiplies an element of $\Omega^{2k}(B, \mathscr{B}^\omega)_{\mathrm{ab}}$ by $(2\pi i)^{-k}$.

Theorem 2.

$$\lim_{s \to 0} \mathscr{R}\, \mathrm{Tr}_{s,\langle e \rangle}(e^{-A_s^2}) = \int_Z \phi \hat{A}(\nabla^{TZ})\, \mathrm{ch}(\nabla^{\hat{V}})\, \mathrm{ch}(\nabla^{\mathrm{can}}) \in \Omega^*(B, \mathscr{B}^\omega)_{\mathrm{ab}}. \tag{1.5}$$

The proof of Theorem 2 is by local index theory techniques.

In order to prove (1.2), it remains to relate $\langle \mathrm{Tr}_{s,\langle e \rangle}(e^{-A_s^2}), \eta \rangle$ to $\langle \mathrm{ch}(\mathrm{Ind}(D)), \eta \rangle$. Let $C_0(B) \rtimes_r \Gamma$ be the reduced cross-product C^*-algebra. A technical problem is that the Dirac-type operator D, when considered as an operator on a $C_0(B) \rtimes_r \Gamma$-Hilbert module, may not have closed range. This problem also arises in the superconnection proof of the families index theorem [3]. Recall that in the families index theorem, a special case is when the kernels and the cokernels of the fiberwise operators form vector bundles on the base. In this case, one defines the analytic index to be the difference of these two vector bundles, as an element of the K-theory of the base. If the kernels and the cokernels do not form vector bundles then one can deform the fiberwise operators in order to reduce oneself to the case in which they do [1].

In order to carry out this deformation argument at the level of superconnection Chern characters requires a pseudodifferential operator calculus ([3], Section 2d). In our context one can set up such a calculus for a class of operators on $C_0(B) \rtimes_r \Gamma$-vector bundles. However, this would not be enough for our purposes, as we would need such a calculus for operators on $C^\infty(B, \mathscr{B}^\omega)$-vector bundles. There seem to be serious problems in constructing such a calculus, as $C^\infty(B, \mathscr{B}^\omega)$ is generally not closed under the holomorphic functional calculus in $C_0(B) \rtimes_r \Gamma$.

To get around this problem, we use a method which seems to be new even in the case of the families index theorem. The idea, which is due to Nistor [25], is to define $\mathrm{Ind}(D)$ to be the K-theory element represented by the difference between the index projection p and a standard projection p_0, and then relate the Chern character of $[p - p_0]$ to the superconnection Chern character. In order to relate the two, Nistor works in a universal setting and shows that a certain cyclic cohomology group is singly generated, which implies that the two expressions are related by a computable constant. Unfortunately, Nistor's assumptions do not hold in our setting and his argument does not seem to be adaptable. Instead, we give a direct proof relating $[p - p_0]$ to the superconnection Chern character. We show that the pairing of η with $[p - p_0]$ can be written as the pairing of η with the Chern character of a certain $\mathbb{Z}_2$-graded connection ∇'. We then homotop between the connection ∇' and the superconnection A_s. Of course one cannot do so in a purely formal way, as the vector bundles involved are infinite-dimensional. However, we show that one can write things so that one has uniformly smoothing operators inside the traces during the homotopy, thereby justifying the formal argument. In this way we prove

Theorem 3. *For all $s > 0$,*

$$\big\langle \mathrm{ch}\big(\mathrm{Ind}(D)\big), \eta \big\rangle = \langle \mathscr{R}\,\mathrm{Tr}_{s,\langle e \rangle}(e^{-A_s^2}), \eta \rangle \in \mathbb{C}. \tag{1.6}$$

Equation (1.2) follows from combining Theorems 2 and 3.

Equation (1.2) deals with closed graded traces η on $\Omega^*(B, \mathscr{B}^\omega)$. On the other hand, the geometric and topological consequences of index theory involve the K-theory of $C_0(B) \rtimes_r \Gamma$. Equation (1.2) is in some ways a stronger result than the analogous $\mathrm{K}_*\big(C_0(B) \rtimes_r \Gamma\big)$-valued index theorem, as the underlying algebra in (1.2) is $C^\infty(B, \mathscr{B}^\omega) \subset C_0(B) \rtimes_r \Gamma$. However, in order to obtain geometric consequences from (1.2), one must consider certain "smooth" algebras $\mathscr{A}$ such that $C^\infty(B, \mathscr{B}^\omega) \subset \mathscr{A} \subset C_0(B) \rtimes_r \Gamma$. The requirements on $\mathscr{A}$ are that it should be closed under the holomorphic functional calculus in $C_0(B) \rtimes_r \Gamma$, and that η should extend to a continuous cyclic cocycle on $\mathscr{A}$. The existence of such subalgebras $\mathscr{A}$ is discussed in [8], Chapter III and we do not have anything new to say about it in this paper.

We give the extensions of Theorems 2 and 3 to the setting of smooth Hausdorff étale groupoids. As mentioned above, our proof applies to foliations, to give a superconnection proof of Connes' general foliation index theorem. There has been some previous work along these lines. Theorem 2 was proven by Heitsch, using A_s^{Bismut}, in the special case when η comes from a holonomy-invariant transverse current to the foliation [14]. A corresponding analog of Theorem 3 was proven by Heitsch and Lazarov when η comes from a holonomy-invariant transverse current and under some additional technical assumptions regarding the spectral densities of the leafwise Dirac-type operators [15]. In [18], Liu and Zhang gave an adiabatic limit proof of a certain case of a vanishing result of Connes for foliations with spin leaves of positive scalar curvature. (Connes' result is a corollary of his general foliation index theorem.) The additional assumptions in [18] were that the foliation is almost-Riemannian and that the pairing object comes from the Pontryagin classes of the normal bundle. The adiabatic limit is closely related to superconnections.

The methods of [14], [15] and [18] have the limitation that the dimension of the pairing objects is at most the codimension of the foliation, as they come from the normal bundle to the foliation. Consequently, one misses the noncommutative features of the foliation, which lead to the phenomenon that the dimension of the leaf space, treated as a noncommutative space, can be greater than the codimension of the foliation. For example, a manifold with a codimension-1 foliation has a Godbillon-Vey class which is a three-dimensional cohomology class. One important aspect of Connes' foliation index theorem is that it allows a pairing between the foliation index and the Godbillon-Vey class ([8], Section III.6). One would miss this pairing if one treated the foliation in a "commutative" way.

Let us also mention the paper [24] which proves (1.2) in the case when $\hat{M} = Z \times S^1$, $B = S^1$, the action of Γ on S^1 preserves orientation, V is trivial and η corresponds to the Godbillon-Vey class. The method of proof of [24] is to represent $\mathrm{Ind}(D)$ by means of a "graph" projection and then compute the pairing of this projection with a cyclic cocycle representing the Godbillon-Vey class.

Not all foliations have a holonomy groupoid that is Hausdorff. We expect that our results can be extended to the nonHausdorff case, but we have not worked this out in detail. Relevant treatments of the cyclic cohomology of an étale groupoid algebra, in the nonHausdorff case, are in [10], [11] and [12].

The results of this paper are valid in the generality of a smooth Hausdorff étale groupoid G with a free, properly discontinuous and cocompact action on a manifold $\hat{M}$. For simplicity of notation, we first present all of the arguments in the case of a cross-product groupoid $B \rtimes \Gamma$. We then explain how to extend the proofs to general G.

The structure of the paper is as follows. In Section 2 we prove Theorem 1. In Section 3 we define a class of fiberwise-smoothing operators on $\hat{M} \to B$ and construct their $\Omega^*(B, \mathscr{B}^\omega)_{\mathrm{ab}}$-valued traces. In Section 4 we define the superconnection A_s and prove Theorem 2. In Section 5 we prove Theorem 3 and give some consequences.

In Section 6 we give the extension of the preceding results to general étale groupoids. The extension is not entirely straightforward and the expressions in Section 6 would be unmotivated if it were not for the results of the preceding sections. In Subsection 6.1 we show how in the case of a cross-product groupoid, the expressions of Section 6 reduce to expressions of the preceding sections. The reader may wish to read Subsection 6.1 simultaneously with the rest of Section 6.

More detailed information is given at the beginnings of the sections.

Background information on superconnections and index theory is in [2]. For background results we sometimes refer to the relevant sections in [2], [8] or [19], where references to the original articles can be found.

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2. Closed graded traces

In this section we compute the homology of $GT_{*,\langle e \rangle}$. We show in Proposition 1 that it is isomorphic to the homology of the total complex of a certain double complex $\mathscr{T}_{*,*}$. We then construct a morphism from this double complex to the double complex $\big((\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)) \otimes C_*(B)\big)^\Gamma$. In order to construct this morphism, we first describe the connection ∇^{can} on $\mathscr{E} = (\hat{M} \times \mathbb{C}\Gamma)/\Gamma$, along with its Chern character. Taking $\hat{M}$ to be $E\Gamma$, we obtain a connection ∇^{univ}, whose Chern character implements the morphism. We show that the morphism induces an isomorphism between the E^1-terms of the two double complexes. It follows that it induces an isomorphism between the homologies of the total complexes.

The material in this section is necessarily of a technical nature. A trusting reader may be willing to take the main result of the section, Corollary 1, on faith. To read the rest of the paper, it is also worthwhile to read the important digression of the section, which is labeled as such.

Let $\mathscr{B}$ be a unital algebra over $\mathbb{C}$. Let $\Omega^*(\mathscr{B})$ denote its universal graded differential algebra (GDA) ([16], Section 2.24). As a vector space,

$$\Omega^k(\mathscr{B}) = \mathscr{B} \otimes \big(\otimes^k(\mathscr{B}/\mathbb{C})\big). \tag{2.1}$$

As a GDA, $\Omega^*(\mathscr{B})$ is generated by $\mathscr{B}$ and $d\mathscr{B}$ with the relations

$$d1 = 0, \quad d^2 = 0, \quad d(\omega_k \omega_l) = (d\omega_k)\omega_l + (-1)^k \omega_k (d\omega_l) \tag{2.2}$$

for $\omega_k \in \Omega^k(\mathscr{B})$, $\omega_l \in \Omega^l(\mathscr{B})$. It will be convenient to write an element ω_k of $\Omega^k(\mathscr{B})$ as a finite sum $\omega_k = \sum b_0\, db_1 \dots db_k$. Let $[\Omega^*(\mathscr{B}), \Omega^*(\mathscr{B})]$ denote the graded commutator of $\Omega^*(\mathscr{B})$ and let $\Omega^*(\mathscr{B})_{\mathrm{ab}} = \dfrac{\Omega^*(\mathscr{B})}{[\Omega^*(\mathscr{B}), \Omega^*(\mathscr{B})]}$ denote the abelianization of $\Omega^*(\mathscr{B})$. Put $\overline{\Omega}^*(\mathscr{B})_{\mathrm{ab}} = \dfrac{\Omega^*(\mathscr{B})}{\mathbb{C} \oplus [\Omega^*(\mathscr{B}), \Omega^*(\mathscr{B})]}$. Let $HDR^*(\mathscr{B})$ and $\overline{H}DR^*(\mathscr{B})$ denote the cohomologies of the differential complexes $\Omega^*(\mathscr{B})_{\mathrm{ab}}$ and $\overline{\Omega}^*(\mathscr{B})_{\mathrm{ab}}$, respectively. Then $HDR^*(\mathscr{B}) \cong \overline{H}DR^*(\mathscr{B})$ if $* \geqq 1$, and there is a short exact sequence $0 \to \mathbb{C} \to HDR^0(\mathscr{B}) \to \overline{H}DR^0(\mathscr{B}) \to 0$. Furthermore, Connes and Karoubi showed that $\overline{H}DR^*(\mathscr{B})$ is expressed in terms of reduced cyclic and Hochschild homology ([19], Theorem 2.6.7) by

$$\overline{H}DR^*(\mathscr{B}) \cong \operatorname{Ker}\bigl(B : \overline{HC}_*(\mathscr{B}) \to \overline{HH}_{*+1}(\mathscr{B})\bigr). \tag{2.3}$$

If Γ is a discrete group and $\mathscr{B}$ is the group algebra $\mathbb{C}\Gamma$, i.e. finite sums $\sum\limits_{\gamma \in \Gamma} c_\gamma \gamma$, let us recall the calculation of $HDR^*(\mathbb{C}\Gamma)$. It breaks up with respect to the conjugacy classes of Γ, as do the Hochschild and cyclic cohomologies of $\mathbb{C}\Gamma$, and we will only be interested in the component $HDR^*_{\langle e \rangle}(\mathbb{C}\Gamma)$ corresponding to the identity conjugacy class. Let $\mathrm{H}_*(\Gamma; \mathbb{C})$ denote the group homology of Γ and let $\overline{\mathrm{H}}_*(\Gamma; \mathbb{C})$ denote the reduced group homology, i.e. $\overline{\mathrm{H}}_*(\Gamma; \mathbb{C}) = \mathrm{H}_*(B\Gamma, *; \mathbb{C})$. Then it follows from Burghelea's work that the reduced Hochschild and cyclic homologies of $\mathbb{C}\Gamma$, when considered at the identity conjugacy class, are

$$\overline{HH}_{*, \langle e \rangle}(\mathbb{C}\Gamma) \cong \overline{\mathrm{H}}_*(\Gamma; \mathbb{C})$$

and

$$\overline{HC}_{*, \langle e \rangle}(\mathbb{C}\Gamma) \cong \bigoplus_{i \geqq 0} \overline{\mathrm{H}}_{*-2i}(\Gamma; \mathbb{C}), \tag{2.4}$$

with the map $B : \overline{HC}_{*, \langle e \rangle}(\mathbb{C}\Gamma) \to \overline{HH}_{*+1, \langle e \rangle}(\mathbb{C}\Gamma)$ vanishing ([19], Section 7.4). Hence

$$HDR^*_{\langle e \rangle}(\mathbb{C}\Gamma) \cong \begin{cases} \mathrm{H}_0(\Gamma; \mathbb{C}) & \text{if } * = 0, \\ \bigoplus\limits_{i \geqq 0} \overline{\mathrm{H}}_{*-2i}(\Gamma; \mathbb{C}) & \text{if } * > 0. \end{cases} \tag{2.5}$$

If $\mathscr{B}$ is a locally convex topological algebra then there is a natural completion of the algebraic GDA $\Omega^*(\mathscr{B})$ to a locally convex GDA ([16], Section 5.1). For simplicity of notation, when the context is clear we will also denote this completion by $\Omega^*(\mathscr{B})$. We will also denote by $\Omega^*(\mathscr{B})_{\mathrm{ab}}$ the quotient of $\Omega^*(\mathscr{B})$ by the closure of $[\Omega^*(\mathscr{B}), \Omega^*(\mathscr{B})]$, where the quotienting by the closure is done in order to obtain a Hausdorff space. In general, we take the tensor product of two locally convex topological vector spaces to be the projective topological tensor product and we let $\hat{\otimes}$ denote a graded (projective) tensor product.

Let B be a smooth manifold on which a finitely-generated discrete group Γ acts on the right, not necessarily freely or properly discontinuously. Given $\gamma \in \Gamma$, let $R_\gamma \in \mathrm{Diff}(B)$ denote the action of γ on B. We let Γ act on $C_c^\infty(B)$ on the left so that $\gamma \cdot f = R_\gamma^* f$, i.e. $(\gamma \cdot f)(b) = f(b\gamma)$. Given $\gamma \in \Gamma$, let B^γ denote the subset of B which is pointwise fixed by γ. Given $b \in B$, let $\Gamma_b \subset \Gamma$ be the isotropy subgroup at b for the action of Γ on B.

Let $C_c^\infty(B) \rtimes \Gamma$ denote the cross product algebra, whose elements are finite sums $\sum_{\gamma \in \Gamma} f_\gamma \gamma$, with $f_\gamma \in C_c^\infty(B)$. We wish to define an appropriate GDA whose zeroth-degree component equals $C_c^\infty(B) \rtimes \Gamma$. One's first choice might be the universal GDA $\Omega^*\big(C_c^\infty(B) \rtimes \Gamma\big)$. The corresponding de Rham cohomology is expressed in (2.3) in terms of the cyclic and Hochschild homologies of $C_c^\infty(B) \rtimes \Gamma$. Now such homologies have been computed in [8], Section III.2.δ, [6], [10] and [11] in cases of increasing generality. In particular, the periodic cyclic cohomology of $C_c^\infty(B) \rtimes \Gamma$ contains a factor consisting of the twisted equivariant cohomology of B. This implies that the space of closed graded traces on $\Omega^*\big(C_c^\infty(B) \rtimes \Gamma\big)$ is sufficiently rich for our purposes.

On the other hand, the choice of $\Omega^*\big(C_c^\infty(B) \rtimes \Gamma\big)$ as the GDA is inconvenient from the point of view of superconnections. In the case when Γ is the trivial group, and one is dealing with a fiber bundle $M \to B$, the usual superconnection formalism uses the graded differential algebra $\Omega_c^*(B)$. It appears that it would be quite cumbersome to redo the superconnection proof of the Atiyah-Singer families index theorem using the noncommutative differential forms $\Omega^*\big(C_c^\infty(B)\big)$ instead of $\Omega_c^*(B)$.

For this reason, instead of using $\Omega^*\big(C_c^\infty(B) \rtimes \Gamma\big)$ as the GDA, we replace it by an appropriate quotient. By universality, there will be a map from the closed graded traces on the quotient GDA to the closed graded traces on $\Omega^*\big(C_c^\infty(B) \rtimes \Gamma\big)$. We want this map to have a sufficiently large image. It turns out that an appropriate GDA is $\Omega^*(B, \mathbb{C}\Gamma) = \Omega_c^*(B) \hat{\otimes} \Omega^*(\mathbb{C}\Gamma)$, the graded algebraic tensor product over $\mathbb{C}$, where the multiplication in $\Omega^*(B, \mathbb{C}\Gamma)$ takes into account that $\mathbb{C}\Gamma$ acts on $\Omega_c^*(B)$.

Then $\Omega^*(B, \mathbb{C}\Gamma)$ is a GDA, with $\Omega^0(B, \mathbb{C}\Gamma) = C_c^\infty(B) \rtimes \Gamma$. We wish to compute the homology of the complex of graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$, or at least the part of the homology which is concentrated at the identity conjugacy class. Let $GT_{n,\langle e \rangle}$ be the graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ which are concentrated on the elements

$$\sum_{\gamma_0, \dots, \gamma_n :\, \gamma_0 \cdot \dots \cdot \gamma_n = e} \omega_{\gamma_0, \dots, \gamma_n} \gamma_0 \, d\gamma_1 \dots d\gamma_n .$$

Let $d^t : GT_{n,\langle e \rangle} \to GT_{n-1,\langle e \rangle}$ be the boundary operator. A closed graded trace is an element of $\mathrm{Ker}(d^t)$.

Consider the space of maps $\tau_k : \Gamma^{k+1} \to \mathbb{C}$. We now define certain operators that arise in the computation of the cyclic homology of $\mathbb{C}\Gamma$ [16], Section 2.21. Namely, put

$$\begin{aligned}
(2.6) \qquad (t\tau)_k(\gamma_0, \gamma_1, \dots, \gamma_k) &= (-1)^k \tau_k(\gamma_1, \dots, \gamma_k, \gamma_0), \\
(\delta_i \tau)_{k+1}(\gamma_0, \dots, \gamma_{k+1}) &= \tau_k(\gamma_0, \dots, \gamma_{i-1}, \hat{\gamma}_i, \gamma_{i+1}, \dots, \gamma_k), \\
(\sigma_i \tau)_{k-1}(\gamma_0, \dots, \gamma_{k-1}) &= \tau_k(\gamma_0, \dots, \gamma_i, \gamma_i, \dots, \gamma_{k-1}), \\
(B_0 \tau)_{k-1}(\gamma_0, \dots, \gamma_{k-1}) &= \tau_k(\gamma_{k-1}, \gamma_0, \dots, \gamma_{k-1}).
\end{aligned}$$

Put

$$(2.7) \qquad (b\tau)_{k+1} = \sum_{i=0}^{k} (-1)^i (\delta_i \tau)_{k+1}.$$

The operator $B_0 b + b B_0$ acts by

$$(2.8)\quad \big((B_0 b + bB_0)\tau\big)_k(\gamma_0,\ldots,\gamma_k) = \tau_k(\gamma_0,\ldots,\gamma_k) - (-1)^k \tau_k(\gamma_k,\gamma_0,\ldots,\gamma_{k-1}) + (-1)^k \tau_k(\gamma_{k-1},\gamma_0,\ldots,\gamma_{k-1}).$$

Let $C_*(B)$ be the currents on B. We denote the pairing of $\omega \in \Omega_c^*(B)$ with $c \in C_*(B)$ by $\langle \omega, c\rangle$. The de Rham boundary $\partial : C_*(B) \to C_{*-1}(B)$ satisfies $\langle d\omega, c\rangle = \langle \omega, \partial c\rangle$. The action of Γ on $C_*(B)$ is such that $\langle \gamma\cdot\omega, \gamma\cdot c\rangle = \langle \omega, c\rangle$.

Let $C^k(\Gamma)$ be the vector space of $\mathbb{C}$-valued functions on Γ^{k+1}. Let $\mathscr{C}_{k,l}$ be the elements $\tau_k \in C^k(\Gamma) \otimes C_l(B)$ which are normalized in the sense that $(\sigma_i\tau)_{k-1} = 0$ for all $0 \leqq i < k$, and which are Γ-invariant in the sense that

$$(2.9)\qquad \tau_k(\gamma\gamma_0,\ldots,\gamma\gamma_k) = \gamma\cdot\tau_k(\gamma_0,\ldots,\gamma_k).$$

Put $\mathscr{C}_n = \bigoplus_{k+l=n} \mathscr{C}_{k,l}$. The operators ∂, b and B_0 can be defined on $\mathscr{C}_{*,*}$ in the natural way. Let $\mathscr{T}_{*,*}$ be the double complex given by

$$(2.10)\qquad \mathscr{T}_{k,l} = \mathrm{Ker}(b) \cap \mathrm{Ker}(bB_0 + B_0 b) \subset \mathscr{C}_{k,l}$$

with boundary operators $(-1)^l B_0$ and ∂.

Proposition 1. *The vector space $GT_{n,\langle e\rangle}$ is isomorphic to $\bigoplus_{k+l=n} \mathscr{T}_{k,l}$. Under this isomorphism, the action of d^t on $GT_{n,\langle e\rangle}$ is equivalent to the action of $\partial + (-1)^l B_0$ on $\bigoplus_{k+l=n} \mathscr{T}_{k,l}$.*

Proof. Given $\tau \in \mathscr{C}_n$, write it as $\tau = \sum_{k=0}^{n} \tau_k$, with $\tau_k \in \mathscr{C}_{k,n-k}$. We obtain a linear functional Ψ_τ on $\Omega^*(B, \mathbb{C}\Gamma)$, which is concentrated at the identity conjugacy class, by the formula

$$(2.11)\quad \Psi_\tau(\omega\gamma_0\,d\gamma_1\ldots d\gamma_k) = \begin{cases} \langle \omega, \tau_k(\gamma_0, \gamma_0\gamma_1, \ldots, \gamma_0\cdots\gamma_{k-1}, e)\rangle & \text{if } \gamma_0\gamma_1\cdots\gamma_k = e,\\ 0 & \text{if } \gamma_0\gamma_1\cdots\gamma_k \neq e.\end{cases}$$

Conversely, all linear functionals on $\Omega^*(B, \mathbb{C}\Gamma)$, which are concentrated at the identity conjugacy class, arise in this way.

Ψ_τ will be a graded trace if and only if it satisfies

$$(2.12)\qquad \Psi_\tau(\gamma\omega\gamma_0\,d\gamma_1\,d\gamma_2\ldots d\gamma_k) = \Psi_\tau(\omega\gamma_0\,d\gamma_1\,d\gamma_2\ldots d\gamma_k\,\gamma)$$

and

$$(2.13)\quad \Psi_\tau(d\gamma_k\,\omega\gamma_0\,d\gamma_1\,d\gamma_2\ldots d\gamma_{k-1}) = (-1)^{k-1+|\omega|}\Psi_\tau(\omega\gamma_0\,d\gamma_1\,d\gamma_2\ldots d\gamma_k).$$

Equation (2.12) is equivalent to

$$\begin{aligned}(2.14)\quad &\Psi_\tau\big((\gamma\cdot\omega)\gamma\gamma_0\,d\gamma_1\,d\gamma_2\ldots d\gamma_k\big)\\ &= \Psi_\tau\big(\omega[\gamma_0\,d\gamma_1\,d\gamma_2\ldots d(\gamma_k\gamma) - \gamma_0\,d\gamma_1\,d\gamma_2\ldots d(\gamma_{k-1}\gamma_k)\,d\gamma\\ &\qquad + \cdots + (-1)^k\gamma_0\gamma_1\,d\gamma_2\ldots d\gamma_k\,d\gamma]\big).\end{aligned}$$

This is equivalent to

$$\begin{aligned}(2.15)\quad &\langle \gamma \cdot \omega, \tau(\gamma\gamma_0, \gamma\gamma_0\gamma_1, \dots, \gamma\gamma_0 \cdots \gamma_{k-1}, e)\rangle \\ &= \langle \omega, \tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, e) - \tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-2}, \gamma_0 \cdots \gamma_k, e) \\ &\quad + \cdots + (-1)^k \tau(\gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_k, e)\rangle,\end{aligned}$$

which is equivalent to

$$\begin{aligned}(2.16)\quad &\tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, \gamma_0 \cdots \gamma_k) \\ &= \tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, e) - \tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-2}, \gamma_0 \cdots \gamma_k, e) \\ &\quad + \cdots + (-1)^k \tau(\gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_k, e),\end{aligned}$$

which is equivalent to

$$\begin{aligned}(2.17)\quad 0 = (-1)^k[&\tau(\gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_k, e) \\ &+ \cdots + (-1)^{k-3}\tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-2}, \gamma_0 \cdots \gamma_k, e) \\ &+ (-1)^{k-2}\tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, e) \\ &+ (-1)^{k-1}\tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, \gamma_0 \cdots \gamma_k)].\end{aligned}$$

After a change of variable and using the Γ-invariance of τ, this is equivalent to $b\tau = 0$.

Next,

$$\begin{aligned}(2.18)\quad &(-1)^{|\omega|}\Psi_\tau(d\gamma_k\, \omega\gamma_0\, d\gamma_1\, d\gamma_2 \dots d\gamma_{k-1}) \\ &= \Psi_\tau\big((\gamma_k \cdot \omega)\, d\gamma_k\, \gamma_0\, d\gamma_1\, d\gamma_2 \dots d\gamma_{k-1}\big) \\ &= \Psi_\tau\big((\gamma_k \cdot \omega)\, d(\gamma_k\gamma_0)\, d\gamma_1\, d\gamma_2 \dots d\gamma_{k-1}\big) \\ &\quad - \Psi_\tau\big((\gamma_k \cdot \omega)\gamma_k\, d\gamma_0\, d\gamma_1\, d\gamma_2 \dots d\gamma_{k-1}\big) \\ &= \langle \gamma_k \cdot \omega, \tau_k(e, \gamma_k\gamma_0, \gamma_k\gamma_0\gamma_1, \dots, \gamma_k\gamma_0 \cdots \gamma_{k-2}, e)\rangle \\ &\quad - \langle \gamma_k \cdot \omega, \tau_k(\gamma_k, \gamma_k\gamma_0, \gamma_k\gamma_0\gamma_1, \dots, \gamma_k\gamma_0 \cdots \gamma_{k-2}, e)\rangle \\ &= \langle \omega, \tau_k(\gamma_0 \cdots \gamma_{k-1}, \gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}) \\ &\qquad - \tau_k(e, \gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1})\rangle.\end{aligned}$$

Thus (2.13) is equivalent to

$$\begin{aligned}(2.19)\quad &\tau_k(\gamma_0 \cdots \gamma_{k-1}, \gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}) - \tau_k(e, \gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}) \\ &= (-1)^{k-1}\tau_k(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \cdots \gamma_{k-1}, e).\end{aligned}$$

By a change of variables, this in turn is equivalent to

$$\begin{aligned}(2.20)\quad &\tau_k(g_{k-1}, g_0, g_1, \dots, g_{k-1}) - \tau_k(e, g_0, \dots, g_{k-1}) \\ &= (-1)^{k-1}\tau_k(g_0, \dots, g_{k-1}, e).\end{aligned}$$

Using the Γ-equivariance, this is equivalent to

$$\tau_k(g_{k-1}, g_0, g_1, \dots, g_{k-1}) - \tau_k(g_k, g_0, \dots, g_{k-1}) = (-1)^{k-1}\tau_k(g_0, \dots, g_{k-1}, g_k), \tag{2.21}$$

which, from (2.8), is equivalent to $(B_0 b + bB_0)\tau = 0$.

Finally, given $\tau \in C^k(\Gamma) \otimes C_l(B)$, let $\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}$ be an element of $\Omega^{l'}(B) \hat{\otimes} \Omega^{k'}(\mathbb{C}\Gamma)$. Then

$$(d^t\Psi_\tau)(\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}) = \Psi_\tau\big(d\omega\,\gamma_0\, d\gamma_1 \dots d\gamma_{k'} + (-1)^{l'}\omega\, d\gamma_0 \dots d\gamma_{k'}\big). \tag{2.22}$$

If $k' = k$ and $l' = l - 1$ then

$$\begin{aligned}(d^t\Psi_\tau)(\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}) &= \langle d\omega, \tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \dots, \gamma_{k-1}, e)\rangle \\ &= \langle \omega, \partial\tau(\gamma_0, \gamma_0\gamma_1, \dots, \gamma_0 \dots, \gamma_{k-1}, e)\rangle \\ &= \Psi_{\partial\tau}(\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}).\end{aligned} \tag{2.23}$$

If $k' = k - 1$ and $l' = l$ then

$$\begin{aligned}(d^t\Psi_\tau)(\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}) &= (-1)^{l'}\langle \omega, \tau(e, \gamma_0, \dots, \gamma_0 \cdots \gamma_{k'-1}, e)\rangle \\ &= (-1)^{l'}\langle \omega, (B_0\tau)(\gamma_0, \dots, \gamma_0 \cdots \gamma_{k'-1}, e)\rangle \\ &= \Psi_{(-1)^l B_0\tau}(\omega\gamma_0\, d\gamma_1 \dots d\gamma_{k'}).\end{aligned} \tag{2.24}$$

This proves the proposition. $\square$

From Proposition 1, (2.7) and (2.8), we see that any antisymmetric group cocycle for Γ, which takes values in the closed currents on B, gives a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$. In this way, we have a map $\mathrm{H}^k\big(\Gamma; Z_l(B)\big) \to \mathrm{H}_{k+l}(GT_{*,\langle e\rangle}, d^t)$. In particular, if $k = 0$ then we obtain closed graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ from Γ-invariant closed currents on B. We now use Proposition 1 to describe all of the homology of the complex $GT_{*,\langle e\rangle}$.

Consider the E^1-term of the double complex $\mathscr{T}_{*,*}$. That is, $E^1_{k,l}$ is the k-th homology group of the complex $\mathscr{T}_{*,l}$ with respect to the differential $(-1)^l B_0$. We first want to compute this homology group. To do so, we follow the general method of proof of [8], Section III.1.β, Theorem 22. We fix l for the moment.

Let us define operators b', A and B on $C^*(\Gamma) \otimes C_l(B)$ by the usual formulas

$$\begin{aligned}(b'\tau)_{k+1} &= \sum_{i=0}^{k-1}(-1)^i(\delta_i\tau)_{k+1}, \\ (A\tau)_k &= \sum_{i=0}^{k}(-1)^i t^i\tau, \\ B &= AB_0.\end{aligned} \tag{2.25}$$

(The "B" in this Connes B-operator should not be confused with the manifold B.)

Lemma 1. *Acting on* $\mathscr{C}_{*,l}$, *we have* $\mathrm{Ker}(b) \cap \mathrm{Ker}(B_0) = \mathrm{Ker}(b) \cap \mathrm{Ker}(1-t)$.

Proof. If $\tau \in \mathscr{C}_{*,l}$ and $(1-t)\tau = 0$ then $B_0\tau = 0$. Thus

$$\mathrm{Ker}(b) \cap \mathrm{Ker}(1-t) \subset \mathrm{Ker}(b) \cap \mathrm{Ker}(B_0).$$

On the other hand, the identity $b'B_0 + B_0 b = 1 - t$ shows that

$$\mathrm{Ker}(b) \cap \mathrm{Ker}(B_0) \subset \mathrm{Ker}(b) \cap \mathrm{Ker}(1-t). \quad \square$$

Lemma 2. *An element* $\tau \in \mathscr{T}_{k,l}$ *lies in the image of* $B_0 : \mathscr{T}_{k+1,l} \to \mathscr{T}_{k,l}$ *if and only if* $\tau = B\phi$ *for some* $\phi \in \mathrm{Ker}(b) \subset \mathscr{C}_{k+1,l}$.

Proof. Suppose that $\tau \in \mathscr{T}_{k,l}$ satisfies $\tau = B_0\phi$ for some $\phi \in \mathscr{T}_{k+1,l}$. Then $B_0\tau = 0$. By the previous lemma $\tau = t\tau$ and so $\tau = \frac{1}{k+1} A\tau = \frac{1}{k+1} B\phi$.

Now suppose that $\tau \in \mathscr{T}_{k,l}$ satisfies $\tau = B\phi$ for some $\phi \in \mathrm{Ker}(b) \subset \mathscr{C}_{k+1,l}$. Put

$$\Theta = B_0\phi - \frac{1}{k+1}\tau. \tag{2.26}$$

Note that $B_0\phi \in \mathrm{Ker}(B_0) \subset \mathscr{C}_{k,l}$. Also, as $\tau = B\phi$, it follows that $\tau = t\tau$ which, along with the fact that $\tau \in \mathscr{T}_{k,l}$, implies that $\tau \in \mathrm{Ker}(B_0)$. Hence $\Theta \in \mathrm{Ker}(B_0) \subset \mathscr{C}_{k,l}$. As $A\Theta = 0$, we can write $\Theta = \psi - t\psi$ where $\psi \in \mathrm{Ker}(B_0) \subset \mathscr{C}_{k,l}$ is given by $\psi = -\frac{1}{k+1}\sum_{i=0}^{k} it^i\Theta$. Then $\Theta = (1-t)\psi = (b'B_0 + B_0 b)\psi = B_0 b\psi$. Put $\phi' = \phi - b\psi$. Then from (2.26), $\tau = (k+1)B_0\phi'$. Furthermore, $b\phi' = b(\phi - b\psi) = b\phi = 0$ and $(k+1)bB_0\phi' = b\tau = 0$. Hence $\phi' \in \mathscr{T}_{k+1,l}$. This proves the lemma. $\square$

Put

$$C_\lambda^k\big(\Gamma; C_l(B)\big) = \mathrm{Ker}(B_0) \cap \mathrm{Ker}(B_0 b + bB_0) \subset \mathscr{C}_{k,l}. \tag{2.27}$$

From (2.6) and (2.8), the elements of $C_\lambda^k\big(\Gamma; C_l(B)\big)$ can be considered to be cyclic cochains which are reduced if $k > 0$. Put

$$\begin{aligned} Z_\lambda^k\big(\Gamma; C_l(B)\big) &= \mathrm{Ker}\big(b : C_\lambda^k\big(\Gamma; C_l(B)\big) \to C_\lambda^{k+1}\big(\Gamma; C_l(B)\big)\big), \\ B_\lambda^k\big(\Gamma; C_l(B)\big) &= \mathrm{Im}\big(b : C_\lambda^{k-1}\big(\Gamma; C_l(B)\big) \to C_\lambda^k\big(\Gamma; C_l(B)\big)\big), \\ H_\lambda^k\big(\Gamma; C_l(B)\big) &= Z_\lambda^k\big(\Gamma; C_l(B)\big) / B_\lambda^k\big(\Gamma; C_l(B)\big). \end{aligned} \tag{2.28}$$

We define the Hochschild objects $CH^k\big(\Gamma; C_l(B)\big)$, $ZH^k\big(\Gamma; C_l(B)\big)$, $BH^k\big(\Gamma; C_l(B)\big)$ and $HH^k\big(\Gamma; C_l(B)\big)$ similarly, but without the cyclic condition. The Connes B-operator induces a map $B : HH^k\big(\Gamma; C_l(B)\big) \to H_\lambda^{k-1}\big(\Gamma; C_l(B)\big)$.

We now prove a result which, when B is a point, amounts to the dual of (2.3), when applied to $\mathscr{B} = \mathbb{C}\Gamma$ and considered at the identity conjugacy class.

Lemma 3. *There is an isomorphism*

$$H^k(\mathscr{T}_{*,l}) \cong H_\lambda^k\big(\Gamma; C_l(B)\big) / \mathrm{Im}\big(B : HH^{k+1}\big(\Gamma; C_l(B)\big) \to H_\lambda^k\big(\Gamma; C_l(B)\big)\big). \tag{2.29}$$

Proof. By Lemma 1, $\mathrm{Ker}(B_0) \subset \mathscr{T}_{k,l}$ is isomorphic to $Z^k_\lambda(\Gamma; C_l(B))$. By Lemma 2,

$$\text{(2.30)} \qquad \mathrm{Im}(B_0 : \mathscr{T}_{k+1,l} \to \mathscr{T}_{k,l}) = \mathrm{Im}\big(B : ZH^{k+1}(\Gamma; C_l(B)) \to \mathscr{T}_{k,l}\big).$$

Also,

$$\text{(2.31)} \qquad B^k_\lambda(\Gamma; C_l(B)) \subset \mathrm{Im}\big(B : ZH^{k+1}(\Gamma; C_l(B)) \to \mathscr{T}_{k,l}\big),$$

as if $\tau = b\psi$ with $\psi \in C^{k-1}_\lambda(\Gamma; C_l(B))$ then $\psi = B\psi'$ for some $\psi' \in CH^k(\Gamma; C_l(B))$ ([8], Section III.1.β, Corollary 20), and so $\tau = bB\psi' = -Bb\psi'$. The lemma follows. □

Let $\mathrm{H}^*(\Gamma; C_l(B))$ denote the group cohomology of Γ with coefficients in the Γ-module $C_l(B)$. Let $\overline{\mathrm{H}}^*(\Gamma; C_l(B))$ denote the reduced group cohomology with coefficients in $C_l(B)$. That is, let $E\Gamma_{(0)}$ be the set of vertices in $E\Gamma$. Then $\overline{\mathrm{H}}^*(\Gamma; C_l(B))$ is the cohomology of the complex $\big(C^*(E\Gamma, E\Gamma_{(0)}) \otimes_{\mathbb{Z}} C_l(B)\big)^\Gamma$.

We now give a result which, when B is a point, amounts to the dual of (2.4).

Lemma 4.

$$\text{(2.32)} \qquad HH^k(\Gamma; C_l(B)) \cong \begin{cases} \mathrm{H}^0(\Gamma; C_l(B)) & \text{if } k = 0, \\ \overline{\mathrm{H}}^k(\Gamma; C_l(B)) & \text{if } k > 0 \end{cases}$$

and

$$\text{(2.33)} \qquad H^k_\lambda(\Gamma; C_l(B)) \cong \begin{cases} \mathrm{H}^0(\Gamma; C_l(B)) & \text{if } k = 0, \\ \bigoplus_{i \geqq 0} \overline{\mathrm{H}}^{k-2i}(\Gamma; C_l(B)) & \text{if } k > 0, \end{cases}$$

with $B : HH^{k+1}(\Gamma; C_l(B)) \to H^k_\lambda(\Gamma; C_l(B))$ *vanishing.*

Proof. The proof of this follows algebraically from the method of proof of (2.4). That is, we have the same sort of cyclic structures. We omit the details. □

Putting together Lemmas 3 and 4, we have shown

Proposition 2. *The* E^1*-term of* $\mathscr{T}_{*,*}$ *is given by*

$$\text{(2.34)} \qquad E^1_{k,l} \cong \begin{cases} \mathrm{H}^0(\Gamma; C_l(B)) & \text{if } k = 0, \\ \bigoplus_{i \geqq 0} \overline{\mathrm{H}}^{k-2i}(\Gamma; C_l(B)) & \text{if } k > 0. \end{cases}$$

Clearly the differential $d^1_{k,l} : E^1_{k,l} \to E^1_{k,l-1}$ is induced from ∂.

Important digression. To digress for a moment, let $\hat{M}$ be a smooth manifold on which Γ acts freely, properly discontinuously and cocompactly. Put $M = \hat{M}/\Gamma$, a closed manifold. We construct a connection

$$\text{(2.35)} \qquad \nabla^{\mathrm{can}} : C^\infty_c(\hat{M}) \to \Omega^1(\hat{M}, \mathbb{C}\Gamma) \otimes_{C^\infty_c(\hat{M}) \rtimes \Gamma} C^\infty_c(\hat{M}).$$

Let us note that

$$\text{(2.36)} \qquad \begin{aligned} &\Omega^1(\hat{M}, \mathbb{C}\Gamma) \otimes_{C^\infty_c(\hat{M}) \rtimes \Gamma} C^\infty_c(\hat{M}) \\ &\quad = \big(\Omega^1_c(\hat{M}) \otimes_{C^\infty_c(\hat{M})} C^\infty_c(\hat{M})\big) \oplus \big(\Omega^1(\mathbb{C}\Gamma) \otimes_{\mathbb{C}\Gamma} C^\infty_c(\hat{M})\big) \end{aligned}$$

is isomorphic to $\Omega^1_c(\hat{M}) \oplus \big(\Omega^1(\mathbb{C}\Gamma) \otimes_{\mathbb{C}\Gamma} C^\infty_c(\hat{M})\big)$.

Lemma 5 ([20], Prop. 9). *Let $h \in C_c^\infty(\hat{M})$ satisfy $\sum_{\gamma \in \Gamma} \gamma \cdot h = 1$. Define ∇^{can} by*

$$\nabla^{\mathrm{can}} f = d^{\hat{M}} f \oplus \sum_{\gamma \in \Gamma} d\gamma \otimes h(\gamma^{-1} \cdot f) \tag{2.37}$$

for $f \in C_c^\infty(\hat{M})$. Then ∇^{can} is a connection on $C_c^\infty(\hat{M})$.

One sees that

$$(\nabla^{\mathrm{can}})^2 \in \mathrm{Hom}_{C_c^\infty(\hat{M}) \rtimes \Gamma}\big(C_c^\infty(\hat{M}), \Omega^2(\hat{M}, \mathbb{C}\Gamma) \otimes_{C_c^\infty(\hat{M}) \rtimes \Gamma} C_c^\infty(\hat{M})\big) \tag{2.38}$$

acts on $C_c^\infty(\hat{M})$ as left multiplication by a 2-form Θ which commutes with $C_c^\infty(\hat{M}) \rtimes \Gamma$. Explicitly,

$$\begin{aligned}\Theta &= \sum_{\gamma \in \Gamma} d^{\hat{M}}(\gamma \cdot h)\, d\gamma\gamma^{-1} - \sum_{\gamma, \gamma' \in \Gamma} (\gamma\gamma' \cdot h)(\gamma \cdot h)\, d\gamma\, d\gamma' (\gamma\gamma')^{-1} \\ &= -\sum_{\gamma \in \Gamma} d^{\hat{M}}(\gamma^{-1} \cdot h)\gamma^{-1}\, d\gamma - \sum_{\gamma, \gamma' \in \Gamma} \big((\gamma\gamma')^{-1} \cdot h\big)\big((\gamma')^{-1} \cdot h\big)(\gamma\gamma')^{-1}\, d\gamma\, d\gamma'.\end{aligned} \tag{2.39}$$

Note that if Γ is infinite then Θ does not lie in $\Omega^2(\hat{M}, \mathbb{C}\Gamma)$, as the sums involved are infinite. Nevertheless, it sends $C_c^\infty(\hat{M})$ to $\Omega^2(\hat{M}, \mathbb{C}\Gamma) \otimes_{C_c^\infty(\hat{M}) \rtimes \Gamma} C_c^\infty(\hat{M})$. Put

$$\mathrm{ch}(\nabla^{\mathrm{can}}) = e^{-\frac{\Theta}{2\pi i}} \in \mathrm{End}_{\Omega^*(\hat{M}, \mathbb{C}\Gamma)}\big(\Omega^*(\hat{M}, \mathbb{C}\Gamma) \otimes_{C_c^\infty(\hat{M}) \rtimes \Gamma} C_c^\infty(\hat{M})\big). \tag{2.40}$$

Then the abelianization of $\mathrm{ch}(\nabla^{\mathrm{can}})$ is closed. This can be seen by writing $\Theta = dA - A^2$, where

$$A = \sum_{\gamma \in \Gamma} d\gamma\, h\gamma^{-1} = -\sum_{\gamma \in \Gamma} (\gamma^{-1} \cdot h)\gamma^{-1}\, d\gamma. \tag{2.41}$$

Then $d\Theta = -[\Theta, A]$ and $d\,\mathrm{ch}(\nabla^{\mathrm{can}}) = -[\mathrm{ch}(\nabla^{\mathrm{can}}), A]$. Also, the cohomology class of $\mathrm{ch}(\nabla^{\mathrm{can}})$ is independent of the choice of h.

In the construction of $\mathrm{ch}(\nabla^{\mathrm{can}})$, we can allow h to be a Lipschitz function on $\hat{M}$ (see [20], Lemma 4, where $\mathrm{ch}(\nabla^{\mathrm{can}})$ is called $\tilde{\omega}_h$). Let $E\Gamma$ be the bar simplicial complex for Γ (with degenerate simplices collapsed ([5], Chapter 1.5, Exercise 3b)). We formally replace $\hat{M}$ by $E\Gamma$. There is a complex $\Omega^*(E\Gamma)$ of $\mathbb{C}$-valued polynomial forms on $E\Gamma$ defined as in [26], p. 297. Let $j \in C(E\Gamma)$ be the barycentric coordinate corresponding to the vertex $e \in E\Gamma$ ([20], (94)). Then $\sum_{\gamma \in \Gamma} \gamma \cdot j = 1$ ([20], Lemma 5). (The support of j may not be compact, but this will not be a problem.) Define ∇^{univ} as in (2.37), replacing h by j.

Let $\mathrm{ch}(\nabla^{\mathrm{univ}}) = e^{-\frac{\Theta}{2\pi i}}$ denote the explicit form constructed using j. Then $\mathrm{ch}(\nabla^{\mathrm{univ}})$ lies in

$$\prod_{k=0}^{\infty} \bigoplus_{l=0}^{\infty} \Omega^{k-l}(E\Gamma) \hat{\otimes} \Omega^{k+l}(\mathbb{C}\Gamma). \tag{2.42}$$

Looking at the formula for $\mathrm{ch}(\nabla^{\mathrm{univ}})$, we see that in fact it lies in

$$(2.43) \qquad \prod_{k=0}^{\infty}\left[\left(Z^k(E\Gamma)\,\hat{\otimes}\,\Omega^k(\mathbb{C}\Gamma)\right)\oplus\bigoplus_{l=1}^{\infty}\left(\Omega^{k-l}(E\Gamma)\,\hat{\otimes}\,\Omega^{k+l}(\mathbb{C}\Gamma)\right)\right].$$

Let $\eta\in\bigoplus_{a+b=n}C^a(\Gamma)\otimes C_b(B)$ be a graded n-trace on $\Omega^*(B,\mathbb{C}\Gamma)$. We can pair $\mathrm{ch}(\nabla^{\mathrm{univ}})$ and η with respect to $\mathbb{C}\Gamma$, to obtain

$$(2.44) \qquad \langle \mathrm{ch}(\nabla^{\mathrm{univ}}),\eta\rangle\in\bigoplus_{k=0}^{\infty}\left[\left(Z^k(E\Gamma)\otimes C_{n-k}(B)\right)\oplus\bigoplus_{l=1}^{\infty}\left(\Omega^{k-l}(E\Gamma)\otimes C_{n-k-l}(B)\right)\right].$$

End of important digression.

Lemma 6. *The element constructed in* (2.44) *is* Γ*-invariant.*

Proof. With respect to (2.42), let us write $\mathrm{ch}(\nabla^{\mathrm{univ}})$ in the form $\mathrm{ch}(\nabla^{\mathrm{univ}})=\sum_i\omega_i\,\hat{\otimes}\,\omega_i'$, with $\omega_i\in\Omega^*(E\Gamma)$ and $\omega_i'\in\Omega^*(\mathbb{C}\Gamma)$. As Θ commutes with Γ, it follows that for all $\gamma\in\Gamma$,

$$(2.45) \qquad \sum_i\omega_i\,\hat{\otimes}\,\omega_i'=\sum_i(\gamma\cdot\omega_i)\,\hat{\otimes}\,\gamma\omega_i'\gamma^{-1}.$$

Given $\phi\in\Omega_c^*(B)$, we can define the pairing $\langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}),\eta\rangle,\phi\rangle\in\Omega^*(E\Gamma)$. Then for any $\gamma\in\Gamma$,

$$\begin{aligned}(2.46) \qquad \langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}),\eta\rangle,\gamma\cdot\phi\rangle &=\sum_i\omega_i\,\eta(\gamma\cdot\phi\,\hat{\otimes}\,\omega_i')\\
&=\sum_i\gamma\cdot\omega_i\,\eta(\gamma\cdot\phi\,\hat{\otimes}\,\gamma\omega_i'\gamma^{-1})\\
&=\sum_i\gamma\cdot\omega_i\,\eta\big(\gamma(\phi\,\hat{\otimes}\,\omega_i')\gamma^{-1}\big)\\
&=\sum_i\gamma\cdot\omega_i\,\eta\big([\gamma,(\phi\,\hat{\otimes}\,\omega_i')\gamma^{-1}]+\phi\,\hat{\otimes}\,\omega_i'\big)\\
&=\sum_i\gamma\cdot\omega_i\,\eta(\phi\,\hat{\otimes}\,\omega_i')\\
&=\gamma\cdot\langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}),\eta\rangle,\phi\rangle.\end{aligned}$$

This proves the lemma. □

Equivalently, define a complex $\mathscr{C}_*(\Gamma)$ by

$$(2.47) \qquad \mathscr{C}_k(\Gamma)=Z^k(E\Gamma)\oplus\bigoplus_{l=1}^{\infty}\Omega^{k-2l}(E\Gamma),$$

with the natural chain map of degree -1. Then

$$(2.48) \qquad \langle \mathrm{ch}(\nabla^{\mathrm{univ}}),\eta\rangle\in\bigoplus_{a+b=n}\left(\mathscr{C}_a(\Gamma)\otimes C_b(B)\right)^{\Gamma}.$$

To take into account the normalization of $\Omega^*(\mathbb{C}\Gamma)$, let $E\Gamma_{(0)}$ denote the vertices of $E\Gamma$ and put

$$(2.49)\qquad \overline{\mathscr{C}}_k(\Gamma) = \mathrm{Ker}\left(\left(Z^k(E\Gamma) \oplus \bigoplus_{l=1}^{\infty} \Omega^{k-2l}(E\Gamma)\right) \to \left(Z^k(E\Gamma_{(0)}) \oplus \bigoplus_{l=1}^{\infty} \Omega^{k-2l}(E\Gamma_{(0)})\right)\right).$$

Consider the complex $\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)$, where the factor $\mathbb{C}$ is in degree zero. Then pairing with $\mathrm{ch}(\nabla^{\mathrm{univ}})$ gives a linear map from the graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ to the total space of the double complex $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^{\Gamma}$.

Lemma 7. *Pairing with* $\mathrm{ch}(\nabla^{\mathrm{univ}})$ *gives a morphism from the complex of graded traces on* $\Omega^*(B, \mathbb{C}\Gamma)$ *to the total complex of the double complex* $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^{\Gamma}$.

Proof. Given $\phi \in \Omega_c^*(B)$, we have an equality in $\Omega^*(E\Gamma)$:

$$(2.50)\qquad d\langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}), \eta\rangle, \phi\rangle - \langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}), \eta\rangle, d\phi\rangle - \langle\langle \mathrm{ch}(\nabla^{\mathrm{univ}}), d^t\eta\rangle, \phi\rangle = \langle\langle d\,\mathrm{ch}(\nabla^{\mathrm{univ}}), \eta\rangle, \phi\rangle = \langle\langle [\mathrm{ch}(\nabla^{\mathrm{univ}}), A], \eta\rangle, \phi\rangle.$$

Let us write $\mathrm{ch}(\nabla^{\mathrm{univ}}) = \sum_i \omega_i \mathbin{\hat{\otimes}} \omega_i'$ and $A = \sum_j a_j \mathbin{\hat{\otimes}} a_j'$, with $\omega_i, a_j \in \Omega^*(E\Gamma)$ and $\omega_i', a_j' \in \Omega^*(\mathbb{C}\Gamma)$. Note that $\mathrm{ch}(\nabla^{\mathrm{univ}})$ and A are concentrated at the identity conjugacy class of Γ, so we can assume the same about ω_i' and a_j'. Then

$$(2.51)\qquad [\mathrm{ch}(\nabla^{\mathrm{univ}}), A] = \sum_{i,j} [\omega_i \mathbin{\hat{\otimes}} \omega_i', a_j \mathbin{\hat{\otimes}} a_j'] = \pm \sum_{i,j} \omega_i a_j \mathbin{\hat{\otimes}} [\omega_i', a_j'].$$

Hence

$$(2.52)\qquad \langle\langle [\mathrm{ch}(\nabla^{\mathrm{univ}}), A], \eta\rangle, \phi\rangle = \pm \sum_{i,j} \omega_i a_j \langle [\omega_i', \phi \mathbin{\hat{\otimes}} a_j'], \eta\rangle,$$

which vanishes as η is a graded trace. The lemma follows. □

We note that by the construction of $\overline{\mathscr{C}}_*$, the E^1-term $E^1_{k,l}$ of the double complex $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^{\Gamma}$, i.e. the k-th homology of $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_l(B)\right)^{\Gamma}$ with respect to the differential of $\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)$, is isomorphic to (2.34).

Lemma 8. *Pairing with* $\mathrm{ch}(\nabla^{\mathrm{univ}})$ *induces an isomorphism from the* E^1*-term of the double complex* $\mathscr{T}_{*,*}$ *to the* E^1*-term of the double complex* $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^{\Gamma}$.

Proof. For simplicity of notation, we only address the case when $k > 0$. Consider first the component $\overline{\mathrm{H}}^k\left(\Gamma; C_l(B)\right)$ of the $E^1_{k,l}$-term of the double complex $\mathscr{T}_{*,*}$. It follows from [20], Proposition 13 that pairing with $\mathrm{ch}(\nabla^{\mathrm{univ}})$ induces an isomorphism from this component to the corresponding component of the $E^1_{k,l}$-term of the double complex $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^{\Gamma}$.

Next, we remark that for both double complexes, there is a (reduced) S-operator $\overline{S} : E^1_{k,l} \to E^1_{k+2,l}$ which sends $\overline{\mathrm{H}}^{k-2i}\left(\Gamma; C_l(B)\right)$ to itself. In the case of the double complex $\mathscr{T}_{*,*}$, this S-operator is essentially the dual of the one that acts on the right-hand-side of

(2.3). (A formula for the reduced S-operator, as opposed to the S-operator, is given in [21], (55).) In the case of the double complex $\left(\overline{\mathscr{C}}_*(\Gamma) \otimes C_*(B)\right)^\Gamma$, the S-operator is induced from the natural map from $\left(\overline{\mathscr{C}}_k(\Gamma) \otimes C_l(B)\right)^\Gamma$ to $\left(\overline{\mathscr{C}}_{k+2}(\Gamma) \otimes C_l(B)\right)^\Gamma$.

On the other hand, after adjusting the coefficients of its terms, $\mathrm{ch}(\nabla^{\mathrm{univ}})$ is $\overline{S}$-invariant ([21], Proposition 28). Then using the known isomorphism on the $\overline{\mathrm{H}}^k\left(\Gamma; C_l(B)\right)$ component of $E^1_{k,l}$, along with the $\overline{S}$-operator, it follows that pairing with $\mathrm{ch}(\nabla^{\mathrm{univ}})$ induces an isomorphism on all of $E^1_{k,l}$. $\square$

Theorem 1. *The homology of $GT_{*,\langle e\rangle}$, the complex of graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ which are concentrated at the identity conjugacy class, is isomorphic to the homology of the total complex of the double complex $\left(\left(\mathbb{C} \oplus \overline{\mathscr{C}}_*(\Gamma)\right) \otimes C_*(B)\right)^\Gamma$.*

Proof. By Proposition 1, the homology of $GT_{*,\langle e\rangle}$ is isomorphic to the homology of the total complex of the double complex $\mathscr{T}_{*,*}$. We have shown that pairing with $\mathrm{ch}(\nabla^{\mathrm{univ}})$ gives a morphism of double complexes, which is an isomorphism between the E^1-terms. The differentials on the E^1-terms are both induced by ∂. It follows that there is an isomorphism between the E^∞-terms. $\square$

Using the isomorphism between the homology of $GT_{*,\langle e\rangle}$ and the homology of the double complex $\mathscr{T}_{*,*}$, we can periodize with respect to $\overline{S}$ to define the periodic homology $\mathrm{H}^{\mathrm{per}}_*(GT_{*,\langle e\rangle})$, with $* \in \mathbb{Z}/2\mathbb{Z}$. Let $\mathrm{H}^*_\tau\left((E\Gamma \times B)/\Gamma\right)$ denote the cohomology of $(E\Gamma \times B)/\Gamma$, twisted by the orientation bundle of B ([8], Section II.7). Let $\overline{\mathrm{H}}^*_\tau\left((E\Gamma \times B)/\Gamma\right)$ denote the cohomology relative to $(E\Gamma_{(0)} \times B)/\Gamma = B$.

Corollary 1. *There is an isomorphism between*

$$\mathrm{H}^{\mathrm{per}}_*(GT_{*,\langle e\rangle}) \quad \textit{and} \quad \overline{\mathrm{H}}^{*+\dim(B)+2\mathbb{Z}}_\tau\left((E\Gamma \times B)/\Gamma\right).$$

Proof. As homology commutes with direct limits, Theorem 1 implies that $\mathrm{H}^{\mathrm{per}}_*(GT_{*,\langle e\rangle})$ is isomorphic to the homology of the $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\mathrm{Ker}\left(\left(\Omega^*(E\Gamma) \otimes C_*(B)\right)^\Gamma \to \left(\Omega^*(E\Gamma_{(0)}) \otimes C_*(B)\right)^\Gamma\right). \tag{2.53}$$

Dualizing with respect to B, this is isomorphic to the $\mathbb{Z}/2\mathbb{Z}$-graded complex

$$\mathrm{Ker}\left(\left(\Omega^*(E\Gamma) \otimes \Omega^{*+\dim(B)}_\tau(B)\right)^\Gamma \to \left(\Omega^*(E\Gamma_{(0)}) \otimes \Omega^{*+\dim(B)}_\tau(B)\right)^\Gamma\right), \tag{2.54}$$

where $\Omega^*_\tau(B)$ consists of the differential forms on B with distributional coefficients and with value in the orientation bundle $o(TB)$. The homology of this complex is $\overline{\mathrm{H}}^{*+\dim(B)+2\mathbb{Z}}_\tau\left((E\Gamma \times B)/\Gamma\right)$. $\square$

Remark. If $\Gamma = \{e\}$ then the homology of the graded traces on $\Omega^*_c(B)$ is the *homology* of the currents on B. If B is a point then the homology of the graded traces on $\Omega^*(\mathbb{C}\Gamma)$ is essentially the group *cohomology* of Γ. In order to put these together into one object, we have used Poincaré duality to convert the homology of B into the twisted cohomology of B. In this way we write the periodic homology of graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$ in terms of the twisted cohomology of $(E\Gamma \times B)/\Gamma$. However, this uniform description only exists after periodizing, because of the grading reversal in the Poincaré duality. For the unperiodized homology of graded traces, we must use the setup of Theorem 1.

The closed graded traces coming from $(\mathbb{C} \otimes C_*(B))^\Gamma$ are also relevant; they correspond exactly to the homology of the Γ-invariant currents on B. In general, forgetting about reduced cohomology, we have constructed a map which sends a closed graded n-trace η on $\Omega^*(E\Gamma) \otimes \Omega^*(\mathbb{C}\Gamma)$ to an element $\Phi_\eta \in \mathrm{H}_\tau^{n+\dim(B)+2\mathbb{Z}}((E\Gamma \times B)/\Gamma)$.

3. Fiberwise operators and traces

In this section we first consider smoothing operators on $\hat{M}$ which act fiberwise, preserve compact support and commute with Γ. We define a $C_c^\infty(B)$-valued trace $\mathrm{Tr}_{\langle e \rangle}$ on such operators. We then make various extensions of $\mathrm{Tr}_{\langle e \rangle}$. First, we extend it to an $\Omega^*(B, \mathbb{C}\Gamma)_{\mathrm{ab}}$-valued trace on form-valued operators. Next, we extend it to a supertrace on operators on $\mathbb{Z}_2$-graded vector bundles. Finally, we extend it to an $\Omega^*(B, \mathscr{B}^\omega)_{\mathrm{ab}}$-valued trace on smoothing operators whose Schwartz kernels have sufficiently rapid decay.

Let $\hat{M}$ be a smooth manifold on which Γ acts freely, properly discontinuously and cocompactly. Put $M = \hat{M}/\Gamma$. Let B be a smooth manifold on which Γ acts, not necessarily freely or properly. Suppose that there is a Γ-invariant submersion $\pi : \hat{M} \to B$, a fiber of which we denote by Z. Then M is foliated by the images of Z under the map $\hat{M} \to M$. That is, given $b \in B$, put $Z_b = \pi^{-1}(b)$. Then the corresponding leaf of the foliation $\mathscr{F}$ is $Z_b/\Gamma_b \subset \hat{M}/\Gamma$.

Let TZ denote the vertical tangent bundle of $\hat{M} \to B$, a vector bundle on $\hat{M}$. Let g^{TZ} be a Γ-invariant Euclidean inner product on TZ. Give Z_b the corresponding Riemannian metric and induced metric space structure d. As Γ acts cocompactly on $\hat{M}$, preserving the submersion structure, it follows that $\{Z_b\}_{b \in B}$ has bounded geometry. That is, there is a uniform upper bound on the absolute values of the sectional curvatures, and a uniform lower bound on the injectivity radii. Let $d\mathrm{vol}_Z$ denote the Riemannian volume forms on the fibers $\{Z_b\}_{b \in B}$.

An element K of $\mathrm{End}_{C_c^\infty(B) \rtimes \Gamma}(C_c^\infty(\hat{M}))$ has a Schwartz kernel $K(z, w)$, with respect to its fiberwise action, so that we can write

$$(KF)(z) = \int_{Z_{\pi(z)}} K(z, w) F(w)\, d\mathrm{vol}_{Z_{\pi(z)}}(w) \tag{3.1}$$

for $F \in C_c^\infty(\hat{M})$.

Definition 1. $\mathrm{End}^\infty_{C_c^\infty(B) \rtimes \Gamma}(C_c^\infty(\hat{M}))$ is the subalgebra of $\mathrm{End}_{C_c^\infty(B) \rtimes \Gamma}(C_c^\infty(\hat{M}))$ consisting of elements K with a smooth integral kernel in $C^\infty(\hat{M} \times \hat{M})$.

Note that for each $b \in B$ and each $w \in Z_b$, the function $K_w(z) = K(z, w)$ has compact support in z. To simplify notation, if $K \in \mathrm{End}^\infty_{C_c^\infty(B) \rtimes \Gamma}(C_c^\infty(\hat{M}))$ then we will write the action of K on $C_c^\infty(\hat{M})$ by

$$(KF)(z) = \int_Z K(z, w) F(w)\, d\mathrm{vol}_Z(w). \tag{3.2}$$

That is, $\int_Z$ denotes fiberwise integration. The convolution product on $\mathrm{End}^\infty_{C^\infty_c(B)\rtimes\Gamma}\big(C^\infty_c(\hat{M})\big)$ is given by

$$(3.3)\qquad (KK')(z,w) = \int_Z K(z,u)K(u,w)\,d\mathrm{vol}_Z(u).$$

In this way $\mathrm{End}^\infty_{C^\infty_c(B)\rtimes\Gamma}\big(C^\infty_c(\hat{M})\big)$ is an algebra over $\mathbb{C}$, possibly without unit.

Let $\phi \in C^\infty_c(\hat{M})$ satisfy $\sum_{\gamma\in\Gamma} \gamma\cdot\phi = 1$. Given $K \in \mathrm{End}^\infty_{C^\infty_c(B)\rtimes\Gamma}\big(C^\infty_c(\hat{M})\big)$ and $b \in B$, put

$$(3.4)\qquad \mathrm{Tr}(K)(b) = \sum_{\gamma\in\Gamma_b}\left(\int_{Z_b} \phi(w)K(w\gamma^{-1},w)\,d\mathrm{vol}_{Z_b}(w)\right)\gamma.$$

From the support condition on K, $\mathrm{Tr}(K)(b) \in \mathbb{C}\Gamma_b$.

To express the range of Tr in a better way, let $(B^\gamma)^{\mathbb{C}}$ be the complex-valued functions on B^γ. There is an inclusion $(B^\gamma)^{\mathbb{C}} \subset B^{\mathbb{C}}$ coming from extension by zero. Then $\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma$ is an algebra, as a subalgebra of $B^{\mathbb{C}} \rtimes \Gamma$. Put

$$(3.5)\qquad \left(\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma\right)_{\mathrm{ab}} = \frac{\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma}{\left[\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma,\ \bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma\right]}.$$

Consider Tr from (3.4).

Proposition 3. *Tr* : $\mathrm{End}^\infty_{C^\infty_c(B)\rtimes\Gamma}\big(C^\infty_c(\hat{M})\big) \to \left(\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma\right)_{\mathrm{ab}}$ *is a trace.*

Proof. Let $\{O_\alpha\}$ be the orbits of Γ in B and let $b_\alpha \in O_\alpha$ be representative elements. Put $(\mathbb{C}\Gamma_{b_\alpha})_{\mathrm{ab}} = \dfrac{\mathbb{C}\Gamma_{b_\alpha}}{[\mathbb{C}\Gamma_{b_\alpha},\mathbb{C}\Gamma_{b_\alpha}]}$. Then there is an isomorphism

$$(3.6)\qquad I : \left(\bigoplus_{\gamma\in\Gamma}(B^\gamma)^{\mathbb{C}}\gamma\right)_{\mathrm{ab}} \to \prod_\alpha (\mathbb{C}\Gamma_{b_\alpha})_{\mathrm{ab}}.$$

Namely, given $\gamma\in\Gamma$ and $f \in (B^\gamma)^{\mathbb{C}}$,

$$(3.7)\qquad I([f\gamma]) = \prod_\alpha \sum_{b\in b_\alpha\Gamma} f(b)[\gamma'_b\gamma(\gamma'_b)^{-1}],$$

where $\gamma'_b \in \Gamma$ is such that $b = b_\alpha\gamma'_b$. Under this isomorphism, we obtain

$$(3.8)\qquad I\big(\mathrm{Tr}(K)\big) = \prod_\alpha \sum_{b\in b_\alpha\Gamma}\sum_{\gamma\in\Gamma_b}\left(\int_{Z_b}\phi(w)K(w\gamma^{-1},w)\,d\mathrm{vol}_{Z_b}(w)\right)[\gamma'_b\gamma(\gamma'_b)^{-1}].$$

Thus it is enough to show that for each α, if we put

$$(3.9)\quad I_\alpha\big(\mathrm{Tr}(K)\big) = \sum_{b \in b_\alpha \Gamma} \sum_{\gamma \in \Gamma_b} \left(\int_{Z_b} \phi(w) K(w\gamma^{-1}, w)\, d\mathrm{vol}_{Z_b}(w) \right) [\gamma'_b \gamma (\gamma'_b)^{-1}]$$

then $I_\alpha\big(\mathrm{Tr}(KK')\big) = I_\alpha\big(\mathrm{Tr}(K'K)\big)$.

Let $\{\gamma_\beta\}_{\beta=1}^\infty$ be a sequence of elements of Γ such that $\Gamma_{b_\alpha} \backslash \Gamma = \{\Gamma_{b_\alpha} \gamma_\beta\}_{\beta=1}^\infty$. Then writing $b = b_\alpha \gamma_\beta$ and $\gamma = \gamma_\beta^{-1} \gamma_\alpha \gamma_\beta$,

$$\begin{aligned}(3.10)\qquad I_\alpha\big(\mathrm{Tr}(K)\big) &= \sum_{\beta=1}^\infty \sum_{\gamma_\alpha \in \Gamma_{b_\alpha}} \left(\int_{Z_{b_\alpha \gamma_\beta}} \phi(w) K(w\gamma_\beta^{-1} \gamma_\alpha^{-1} \gamma_\beta, w)\, d\mathrm{vol}_{Z_{b_\alpha \gamma_\beta}}(w) \right) [\gamma_\alpha] \\ &= \sum_{\beta=1}^\infty \sum_{\gamma_\alpha \in \Gamma_{b_\alpha}} \left(\int_{Z_{b_\alpha \gamma_\beta}} \phi(w) K(w\gamma_\beta^{-1} \gamma_\alpha^{-1}, w\gamma_\beta^{-1})\, d\mathrm{vol}_{Z_{b_\alpha \gamma_\beta}}(w) \right) [\gamma_\alpha] \\ &= \sum_{\beta=1}^\infty \sum_{\gamma_\alpha \in \Gamma_{b_\alpha}} \left(\int_{Z_{b_\alpha}} \phi(w\gamma_\beta) K(w\gamma_\alpha^{-1}, w)\, d\mathrm{vol}_{Z_{b_\alpha}}(w) \right) [\gamma_\alpha].\end{aligned}$$

Define $\phi_\alpha \in C_c^\infty(Z_{b_\alpha})$ by $\phi_\alpha(w) = \sum_{\beta=1}^\infty \phi(w\gamma_\beta)$. Then $\sum_{\gamma_\alpha \in \Gamma_{b_\alpha}} \gamma_\alpha \cdot \phi_\alpha = 1$ and

$$(3.11)\qquad I_\alpha\big(\mathrm{Tr}(K)\big) = \sum_{\gamma_\alpha \in \Gamma_{b_\alpha}} \left(\int_{Z_{b_\alpha}} \phi_\alpha(w) K(w\gamma_\alpha^{-1}, w)\, d\mathrm{vol}_{Z_{b_\alpha}}(w) \right) [\gamma_\alpha].$$

It now follows from [20], Prop. 7 that $I_\alpha \circ \mathrm{Tr}$ is a trace. (The formula in [20], Prop. 7 is slightly different because [20] considers function spaces to be right Γ-modules instead of left Γ-modules.) This proves the proposition. $\square$

One can show that Tr is independent of the choice of ϕ.

We can decompose $\mathrm{Tr}(K)$ according to the conjugacy classes of Γ. In particular, the component corresponding to the conjugacy class of $e \in \Gamma$ is

$$(3.12)\qquad \mathrm{Tr}_{\langle e \rangle}(K)(b) = \left(\int_{Z_b} \phi(w) K(w, w)\, d\mathrm{vol}_{Z_b}(w) \right).$$

We see that $\mathrm{Tr}_{\langle e \rangle}$ is a trace on $\mathrm{End}^\infty_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M})\big)$ which takes values in the co-invariants $\big(C_c^\infty(B)\big)_\Gamma$.

We will need some slight extensions of Tr. First, consider the $\mathbb{Z}$-graded algebra

$$(3.13)\qquad \mathrm{Hom}^\infty_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B) \rtimes \Gamma} C_c^\infty(\hat{M})\big)$$

consisting of elements of $\mathrm{Hom}_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B) \rtimes \Gamma} C_c^\infty(\hat{M})\big)$ with a smooth integral kernel. An element K of

$$\mathrm{Hom}^\infty_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}), \big(\Omega^k(B) \otimes \Omega^l(\mathbb{C}\Gamma)\big) \otimes_{C_c^\infty(B) \rtimes \Gamma} C_c^\infty(\hat{M})\big)$$

can be represented as a finite sum

$$K = \sum_{g_1,\dots,g_l \in \Gamma} dg_1 \dots dg_l K_{g_1,\dots,g_l}, \tag{3.14}$$

where $K_{g_1,\dots,g_l}$ has a smooth integral kernel $K_{g_1,\dots,g_l}(z,w) \in \Lambda^k(T^*_{\pi(z)}B)$. It acts on $C_c^\infty(\hat{M})$ by

$$(KF)(z) = \sum_{g_1,\dots,g_l} dg_1 \dots dg_l \left(\int_Z K_{g_1,\dots,g_l}(z,w) F(w)\, d\mathrm{vol}_Z(w) \right). \tag{3.15}$$

Then we define Tr to act on $\mathrm{Hom}^\infty_{C_c^\infty(B)\rtimes\Gamma}\big(C_c^\infty(\hat{M}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\hat{M})\big)$ by the formula

$$\begin{aligned} &\mathrm{Tr}(K)(b) \\ &\quad = \sum_{g_0, g_1,\dots,g_l \in \Gamma:\ g_0\dots g_l \in \Gamma_b} (dg_1 \dots dg_l) g_0 \left(\int_{Z_b} \phi(w) K_{g_1,\dots,g_l}(w g_0^{-1}, w)\, d\mathrm{vol}_{Z_b}(w) \right). \end{aligned} \tag{3.16}$$

(Compare [20], (36).)

Example. Suppose that $\hat{M} = \Gamma$ and that $B = \mathrm{pt}$. The action of Γ on $C_c^\infty(\hat{M})$ is given by $g \cdot \delta_h = \delta_{hg^{-1}}$. There is an isomorphism of left $\mathbb{C}\Gamma$-modules $\mathbb{C}\Gamma \to C_c^\infty(\hat{M})$ which sends h to $\delta_{h^{-1}}$. Consider $h_0\, dh_1 \in \Omega^1(\mathbb{C}\Gamma)$ and the corresponding element $K \in \mathrm{Hom}_{\mathbb{C}\Gamma}\big(\mathbb{C}\Gamma, \Omega^*(\mathbb{C}\Gamma) \otimes_{\mathbb{C}\Gamma} \mathbb{C}\Gamma\big)$ given by

$$K(h) = h(h_0\, dh_1) = d(hh_0h_1)e - d(hh_0)h_1. \tag{3.17}$$

Then under the above isomorphism, $K \in \mathrm{Hom}_{\mathbb{C}\Gamma}\big(C_c^\infty(\hat{M}), \Omega^*(\mathbb{C}\Gamma) \otimes_{\mathbb{C}\Gamma} C_c^\infty(\hat{M})\big)$ acts by

$$K(\delta_{h^{-1}}) = d(hh_0h_1)\delta_e - d(hh_0)\delta_{h_1^{-1}}. \tag{3.18}$$

Thus

$$K_{g_1}(z,w) = \delta_{w^{-1}h_0h_1, g_1}\delta_{z,e} - \delta_{w^{-1}h_0, g_1}\delta_{z, h_1^{-1}} \tag{3.19}$$

and

$$\sum_{g_0, g_1} dg_1\, g_0 K_{g_1}(w g_0^{-1}, w) = d(w^{-1}h_0h_1)w - d(w^{-1}h_0)h_1 w = w^{-1}(h_0\, dh_1)w. \tag{3.20}$$

If $\phi = \delta_x$ then we get

$$\mathrm{Tr}(K)(\mathrm{pt.}) = x^{-1}(h_0\, dh_1)x, \tag{3.21}$$

which is equivalent to $h_0\, dh_1$ in $\Omega^*(\mathbb{C}\Gamma)_{\mathrm{ab}}$, as it should be.

End of example. As before, we can decompose $\mathrm{Tr}(K)$ according to the conjugacy classes of Γ. In this paper we will only be concerned with the component of $\mathrm{Tr}(K)$ associated to the conjugacy class of $e \in \Gamma$, namely

$$\mathrm{Tr}_{\langle e\rangle}(K)(b) = \sum_{g_0, g_1, \ldots, g_l \in \Gamma:\ g_0 \ldots g_l = e} (dg_1 \ldots dg_l) g_0 \times \left(\int_{Z_b} \phi(w) K_{g_1, \ldots, g_l}(w g_0^{-1}, w)\, d\mathrm{vol}_{Z_b}(w) \right). \tag{3.22}$$

Then one sees from (3.22) that $\mathrm{Tr}_{\langle e\rangle}$ is a trace on

$$\mathrm{Hom}^{\infty}_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\hat{M})\big) \tag{3.23}$$

which takes values in $\Omega^*(B, \mathbb{C}\Gamma)_{\mathrm{ab}}$.

Next, let $\hat{E}$ be a Γ-equivariant $\mathbb{Z}_2$-graded Hermitian vector bundle on $\hat{M}$. Define

$$\mathrm{End}^{\infty}_{C_c^\infty(B)\rtimes\Gamma}\big(C_c^\infty(\hat{M}; \hat{E})\big) \tag{3.24}$$

as before, except with $K(z, w) \in \mathrm{Hom}(\hat{E}_w, \hat{E}_z)$. Also define

$$\mathrm{Hom}^{\infty}_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}; \hat{E}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\hat{M}; \hat{E})\big) \tag{3.25}$$

as before, except with $K_{g_1, \ldots, g_l}(z, w) \in \Lambda^k(T^*_{\pi(z)} B) \otimes \mathrm{Hom}(\hat{E}_w, \hat{E}_z)$. Then we obtain a supertrace

$$\mathrm{Tr}_{s, \langle e\rangle} : \mathrm{Hom}^{\infty}_{C_c^\infty(B) \rtimes \Gamma}\big(C_c^\infty(\hat{M}; \hat{E}), \Omega^*(B, \mathbb{C}\Gamma) \otimes_{C_c^\infty(B)\rtimes\Gamma} C_c^\infty(\hat{M}; \hat{E})\big) \to \Omega^*(B, \mathbb{C}\Gamma)_{\mathrm{ab}} \tag{3.26}$$

by

$$\mathrm{Tr}_{s, \langle e\rangle}(K)(b) = \sum_{g_0, g_1, \ldots, g_l \in \Gamma:\ g_0 \ldots g_l = e} (dg_1 \ldots dg_l) g_0 \times \left(\int_{Z_b} \phi(w)\, \mathrm{tr}_s\big(K_{g_1, \ldots, g_l}(w g_0^{-1}, w)\big)\, d\mathrm{vol}_{Z_b}(w) \right). \tag{3.27}$$

Finally, choose a finite generating set for Γ and consider the corresponding right-invariant word metric $\|\cdot\|$. Let $\mathscr{B}^\omega$ be the formal sums $\sum_{\gamma\in\Gamma} c_\gamma \gamma$ such that $|c_\gamma|$ decreases faster than any exponential in $\|\gamma\|$ (see [20], Lemma 2). Then $\mathscr{B}^\omega$ is a locally convex Fréchet algebra ([20], Prop. 4). The notation "ω" comes from the fact that if $\Gamma = \mathbb{Z}$ then $\mathscr{B}^\omega$ can be identified with the holomorphic functions on $\mathbb{C} - 0$.

Put

$$C^\infty(B, \mathscr{B}^\omega) = \mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} \big(C_c^\infty(B) \rtimes \Gamma\big), \tag{3.28}$$

i.e. the quotient of the locally convex topological vector space $\mathscr{B}^\omega \otimes \big(C_c^\infty(B) \rtimes \Gamma\big)$ by $\mathrm{span}\{(a, f) - (\gamma a, \gamma \cdot f)\}$, where $a \in \mathscr{B}^\omega$, $f \in C_c^\infty(B) \rtimes \Gamma$ and $\gamma \in \Gamma$. Then $C^\infty(B, \mathscr{B}^\omega)$ is a locally convex topological algebra which has $C_c^\infty(B) \rtimes \Gamma$ as a dense subalgebra. We can write an element of $C^\infty(B, \mathscr{B}^\omega)$ as an infinite sum $\sum_{\gamma\in\Gamma} f_\gamma \gamma$, where the functions on B $\{\gamma^{-1} \cdot f_\gamma\}_{\gamma\in\Gamma}$ all have support in some compact set $K \subset B$ and have the decay property that for all $q \in \mathbb{Z}^+$,

$$\sup_{b \in K, \gamma \in \Gamma} e^{q\|\gamma\|} |f_\gamma(b\gamma^{-1})| < \infty, \tag{3.29}$$

along with the analogous statement for the derivatives of f.

Similarly, we define a locally convex GDA $\Omega^*(B, \mathscr{B}^\omega)$ by saying that an element of type (k,l) is an infinite sum $\sum_{\gamma_0, \dots, \gamma_l \in \Gamma} \omega_{\gamma_0, \dots, \gamma_l} \gamma_0 \, d\gamma_1 \dots d\gamma_l$, where the k-forms on B $\{(\gamma_0 \dots \gamma_l)^{-1} \cdot \omega_{\gamma_0, \dots, \gamma_l}\}_{\gamma_0, \dots, \gamma_l \in \Gamma}$ all have support in some compact set $K \subset B$ and have the decay property that for all $q \in \mathbb{Z}^+$,

$$\sup_{b \in K, \gamma_0, \dots, \gamma_l \in \Gamma} e^{q(\|\gamma_0\| + \dots + \|\gamma_l\|)} \left| \omega_{\gamma_0, \dots, \gamma_l}\big(b(\gamma_0 \dots \gamma_l)^{-1}\big) \right| < \infty, \tag{3.30}$$

along with the analogous statement for the derivatives. Then $\Omega^*(B, \mathscr{B}^\omega)$ has $\Omega^*(B, \mathbb{C}\Gamma)$ as a dense subalgebra.

Put

$$C^\infty_{\mathscr{B}^\omega}(\hat{M}) = \mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} C^\infty_c(\hat{M}). \tag{3.31}$$

Then $C^\infty_{\mathscr{B}^\omega}(\hat{M})$ is a left $C^\infty(B, \mathscr{B}^\omega)$-module. As in [20], Prop. 5 and using the cocompactness of the Γ-action on $\hat{M}$, the elements of $C^\infty_{\mathscr{B}^\omega}(\hat{M})$ can be characterized as the elements $F \in C^\infty(\hat{M})$ such that for any $b \in B$, $m_0 \in Z_b$ and $q \in \mathbb{Z}^+$,

$$\sup_{z \in Z_b} e^{q\, d(z, m_0)} |F(z)| < \infty, \tag{3.32}$$

along with the analogous property for the covariant derivatives of f. Let $\mathrm{End}^\infty_{C^\infty(B, \mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M})\big)$ be the subalgebra of $\mathrm{End}_{C^\infty(B, \mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M})\big)$ consisting of elements K with a smooth integral kernel $K(z,w)$. Then as in [20], Prop. 6, the elements of $\mathrm{End}^\infty_{C^\infty(B, \mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M})\big)$ can be characterized as the Γ-invariant elements $K(z,w) \in C^\infty(\hat{M} \times \hat{M})$ such that for any $b \in B$ and $q \in \mathbb{Z}^+$,

$$\sup_{z, w \in Z_b} e^{q\, d(z,w)} |K(z,w)| < \infty, \tag{3.33}$$

along with the analogous property for the covariant derivatives of K. The convolution product in $\mathrm{End}^\infty_{C^\infty(B, \mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M})\big)$ is given by the same expression as (3.3), and makes sense because of the bounded geometry of $\{Z_b\}_{b \in B}$. With the natural definition of

$$\mathrm{Hom}^\infty_{C^\infty(B, \mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M}), \Omega^*(B, \mathscr{B}^\omega) \otimes_{C^\infty(B, \mathscr{B}^\omega)} C^\infty_{\mathscr{B}^\omega}(\hat{M})\big), \tag{3.34}$$

an element K can be written as a formal Γ-invariant sum (3.14). In particular, the individual terms have the decay property that for any $b \in B$ and $q \in \mathbb{Z}^+$,

$$\sup_{w \in Z_b, z \in Z_{bg_l \dots g_1}} e^{q\, d(z, wg_l \dots g_1)} K_{g_1, \dots, g_l}(z,w) < \infty. \tag{3.35}$$

The formula (3.12) extends to a trace

$$\mathrm{Tr}_{\langle e\rangle} : \mathrm{End}^{\infty}_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M})\big) \to \frac{C^\infty(B,\mathscr{B}^\omega)}{[C^\infty(B,\mathscr{B}^\omega), C^\infty(B,\mathscr{B}^\omega)]}$$

which is concentrated at the identity conjugacy class. The formula (3.22) extends to a trace

$$(3.36)\qquad \mathrm{Tr}_{\langle e\rangle} : \mathrm{Hom}^{\infty}_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M}), \Omega^*(B,\mathscr{B}^\omega) \otimes_{C^\infty(B,\mathscr{B}^\omega)} C^\infty_{\mathscr{B}^\omega}(\hat{M})\big) \to \Omega^*(B,\mathscr{B}^\omega)_{\mathrm{ab}}.$$

If $\hat{E}$ is a $\mathbb{Z}_2$-graded Γ-invariant Hermitian vector bundle on $\hat{M}$, with an invariant Hermitian connection, then we can define $\mathrm{End}^{\infty}_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big)$ and a supertrace

$$(3.37)\qquad \mathrm{Tr}_{s,\langle e\rangle} : \mathrm{Hom}^{\infty}_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}), \Omega^*(B,\mathscr{B}^\omega) \otimes_{C^\infty(B,\mathscr{B}^\omega)} C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big) \to \Omega^*(B,\mathscr{B}^\omega)_{\mathrm{ab}}.$$

4. Superconnections and small-s asymptotics

In this section we define the superconnection A_s and compute the small-s limit of the supertrace of $e^{-A_s^2}$, thereby obtaining the right-hand-side of the index theorem.

Let $\pi : \hat{M} \to B$ be a Γ-invariant submersion as in the previous section. We choose a Γ-invariant vertical Riemannian metric g^{TZ} on TZ and a Γ-invariant horizontal distribution $T^H\hat{M}$ on $\hat{M}$.

Suppose that Z is even-dimensional. Let $\hat{E}$ be a Γ-invariant Clifford bundle on $\hat{M}$ which is equipped with a Γ-invariant connection. For simplicity of notation, we asssume that $\hat{E} = S^Z \hat{\otimes} \hat{V}$, where S^Z is a vertical spinor bundle and $\hat{V}$ is an auxiliary vector bundle on $\hat{M}$. More precisely, suppose that the vertical tangent bundle TZ has a spin structure. Let S^Z be the vertical spinor bundle, a Γ-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on $\hat{M}$. Let $\hat{V}$ be another Γ-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on $\hat{M}$ which is equipped with a Γ-invariant Hermitian connection $\nabla^{\hat{V}}$. Then we put $\hat{E} = S^Z \hat{\otimes} \hat{V}$. The case of general Γ-invariant Clifford bundles $\hat{E}$ can be treated in a way completely analogous to what follows.

Let Q denote the vertical Dirac-type operator acting on $C^\infty_c(\hat{M};\hat{E})$. From finite-propagation-speed estimates as in [20], Pf. of Prop. 8, along with the bounded geometry of $\{Z_b\}_{b\in B}$, for any $s > 0$ we have

$$(4.1)\qquad e^{-s^2Q^2} \in \mathrm{End}^{\infty}_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big).$$

Let

$$(4.2)\qquad A_s^{\mathrm{Bismut}} : C^\infty_c(\hat{M};\hat{E}) \to \Omega^*_c(B) \otimes_{C^\infty_c(B)} C^\infty_c(\hat{M};\hat{E})$$

denote the Bismut superconnection ([2], Proposition 10.15). In the cited reference it is constructed for fiber bundles with compact fibers. However, being a differential operator, it makes sense for any submersion. It is of the form

$$(4.3)\qquad A_s^{\mathrm{Bismut}} = sQ + \nabla^u - \frac{1}{4s}c(T),$$

where ∇^u is a certain Hermitian connection and $c(T)$ is Clifford multiplication by the curvature 2-form T of the horizontal distribution $T^H\hat{M}$. We also denote by

$$A_s^{\mathrm{Bismut}} : C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}) \to \big(\mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} \Omega_c^*(B)\big) \otimes_{(\mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} C_c^\infty(B))} C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}) \tag{4.4}$$

its extension to $C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})$. One can use finite-propagation-speed estimates, along with the bounded geometry of $\{Z_b\}_{b\in B}$ and the Duhamel expansion as in [2], Theorem 9.48, to show that we obtain a well-defined element

$$e^{-(A_s^{\mathrm{Bismut}})^2} \in \mathrm{Hom}^\infty_{(\mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} C_c^\infty(B))}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}), \big(\mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} \Omega_c^*(B)\big) \otimes_{(\mathscr{B}^\omega \otimes_{\mathbb{C}\Gamma} C_c^\infty(B))} C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big). \tag{4.5}$$

We now couple A_s^{Bismut} to the connection ∇^{can} of Section 2 in order to obtain a superconnection

$$A_s : C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}) \to \Omega^*(B,\mathscr{B}^\omega) \otimes_{C^\infty(B,\mathscr{B}^\omega)} C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}). \tag{4.6}$$

Explicitly,

$$A_s = sQ + \nabla^u - \frac{1}{4s}c(T) + \sum_{\gamma\in\Gamma} d\gamma \otimes h\gamma^{-1}. \tag{4.7}$$

Let $\mathscr{R}$ be the rescaling operator on $\Omega^{\mathrm{even}}(B,\mathscr{B}^\omega)_{\mathrm{ab}}$ which multiplies an element of $\Omega^{2k}(B,\mathscr{B}^\omega)_{\mathrm{ab}}$ by $(2\pi i)^{-k}$. Doing a Duhamel expansion around $e^{-(A_s^{\mathrm{Bismut}})^2}$ and using the fact that h has compact support, we can define

$$e^{-A_s^2} \in \mathrm{Hom}^\infty_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}), \Omega^*(B,\mathscr{B}^\omega) \otimes_{C^\infty(B,\mathscr{B}^\omega)} C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big) \tag{4.8}$$

and hence also define $\mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2}) \in \Omega^*(B,\mathscr{B}^\omega)_{\mathrm{ab}}$. From the superconnection formalism [2], Chapter 1.4, $\mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2})$ is closed and its cohomology class is independent of $s>0$; see [14], Theorem 3.1 for a detailed proof in the analogous case of $\mathscr{R}\,\mathrm{Tr}_s(e^{-(A_s^{\mathrm{Bismut}})^2})$.

Theorem 2.

$$\lim_{s\to 0} \mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2}) = \int_Z \phi\hat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\hat{V}})\,\mathrm{ch}(\nabla^{\mathrm{can}}) \in \Omega^*(B,\mathscr{B}^\omega)_{\mathrm{ab}}. \tag{4.9}$$

Proof. We use a variation of the proof of [2], Theorem 10.23. As in [2], Theorem 10.23, we must first establish a Lichnerowicz-type formula for A_s^2. Let $\{e_i\}_{i=1}^{\dim(Z)}$ be a local orthonormal basis for TZ and let $\{c^i\}_{i=1}^{\dim(Z)}$ be Clifford algebra generators, with $(c^i)^2 = -1$. Let $\{\tau^\alpha\}_{\alpha=1}^{\dim(B)}$ be a local basis of T^*B and let E^α denote exterior multiplication by τ^α. Bismut proved a Lichnerowicz-type formula for $(A_s^{\mathrm{Bismut}})^2$ ([2], Theorem 10.17), namely

$$(A_s^{\mathrm{Bismut}})^2 = s^2\mathscr{D}^*\mathscr{D} + s^2\frac{1}{4}r^Z + \frac{1}{4}\sum_{i,j} F_{ij}(\hat{V})[c^i,c^j] + \sum_{i,\alpha} F_{\alpha i}(\hat{V})E^\alpha c^i + \frac{1}{4}\sum_{\alpha,\beta} F_{\alpha\beta}(\hat{V})[E^\alpha,E^\beta], \tag{4.10}$$

where $\mathscr{D}$ is a certain vertical differentiation operator and $r^Z \in C^\infty(\hat{M})$ is the scalar curvature function of the fibers. Then from the formula (4.7) for A_s, one finds

$$
\begin{aligned}
(4.11)\qquad A_s^2 = {} & s^2\mathscr{D}^*\mathscr{D} + \frac{1}{4}s^2 r^Z + \frac{1}{4}\sum_{i,j} F_{ij}(\hat{V})[c^i, c^j] + \sum_{\alpha,i} F_{\alpha i}(\hat{V})E^\alpha c^i \\
& + \frac{1}{4}\sum_{\alpha,\beta} F_{\alpha\beta}(\hat{V})[E^\alpha, E^\beta] - s\sum_{\gamma\in\Gamma} d\gamma\big(c(d^{\mathrm{vert}}h) + E(d^{\mathrm{hor}}h)\big)\gamma^{-1} \\
& - \sum_{\gamma,\gamma'}(\gamma\gamma'\cdot h)(\gamma\cdot h)\, d\gamma\, d\gamma'(\gamma\gamma')^{-1}.
\end{aligned}
$$

We now perform a Getzler rescaling, as in [2], p. 342. Explicitly, we send $\partial_{x^j} \to s^{-1}\partial_{x^j}$, $c^j \to E^j - I^j \to s^{-1}E^j - sI^j$ and $\tau^\alpha \to \tau^\alpha$. Then following [2], Proposition 10.28, one finds that in the rescaling limit A_s^2 becomes

$$
(4.12)\qquad -\sum_{j=1}^{\dim(Z)}\left(\partial_{x^j} - \frac{1}{8}\sum_{j=1}^{\dim(Z)}\sum_{a,b=1}^{\dim(\hat{M})} R^Z_{abij}x^j E^a E^b\right)^2 + (\nabla^{\hat{V}})^2 + (\nabla^{\mathrm{can}})^2.
$$

The rest of the proof now proceeds as in the proof of [2], Theorem 10.21; compare [20], Pf. of Prop. 12. □

Let us note that the right-hand-side of (4.9) pairs with closed graded traces on $\Omega^*(B, \mathbb{C}\Gamma)$, and not just closed graded traces on $\Omega^*(B, \mathscr{B}^\omega)$. In the construction of $\mathrm{ch}(\nabla^{\mathrm{can}})$, we can allow h to be a Lipschitz function on $\hat{M}$ (see [20], Lemma 4 where $\mathrm{ch}(\nabla^{\mathrm{can}})$ is called $\tilde{\omega}_h$). Let $c : \hat{M} \to E\Gamma$ be a Γ-equivariant classifying map for the fiber bundle $\hat{M} \to M$. It is defined up to Γ-homotopy. As M is compact, we may assume that c is Lipschitz with respect to a piecewise Euclidean Γ-invariant metric on the simplicial complex $E\Gamma$. Let η be a closed graded n-trace on $\Omega^*(B, \mathbb{C}\Gamma)$. Then we can describe $\left\langle \int_Z \phi(z)\hat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\hat{V}})\,\mathrm{ch}(\nabla^{\mathrm{can}}), \eta\right\rangle$ as follows.

First, let us dualize (2.44) with respect to B to write

$$
(4.13)\qquad \langle \mathrm{ch}(\nabla^{\mathrm{univ}}), \eta\rangle \in \left(\bigoplus_{a+b=n}\bigoplus_{l=0}^{\infty}\Omega^{a-2l}(E\Gamma)\,\hat{\otimes}\,\Omega_\tau^{\dim(B)-b}(B)\right)^\Gamma,
$$

where $\Omega_\tau^*(B)$ denotes the differential forms on B with distributional coefficients and with value in the flat orientation line bundle $o(TB)$. Passing to a $\mathbb{Z}_2$-grading, we obtain

$$
(4.14)\qquad \langle \mathrm{ch}(\nabla^{\mathrm{univ}}), \eta\rangle \in \big(\Omega_\tau^{n+\dim(B)+2\mathbb{Z}}(E\Gamma \times B)\big)^\Gamma.
$$

By construction, the form that we have obtained is closed, so we have an element $\Phi_\eta \in \mathrm{H}_\tau^{n+\dim(B)+2\mathbb{Z}}\big((E\Gamma\times B)/\Gamma\big)$.

The map $(c, \pi) : \hat{M} \to E\Gamma \times B$ is Γ-equivariant and so descends to a classifying map $\nu : M \to (E\Gamma \times B)/\Gamma$. Let $T\mathscr{F}$ denote the leafwise tangent bundle on M with respect to the foliation $\mathscr{F}$, a vector bundle on M. Put $V = \hat{V}/\Gamma$. Then we claim that

$$
(4.15)\qquad \left\langle \int_Z \phi(z)\hat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\hat{V}})\,\mathrm{ch}(\nabla^{\mathrm{can}}), \eta\right\rangle = \int_M \hat{A}(T\mathscr{F})\,\mathrm{ch}(V)\nu^*\Phi_\eta.
$$

To see this, take the pairing of $\mathrm{ch}(\nabla^{\mathrm{can}})$ and η with respect to $\mathbb{C}\Gamma$, to get a Γ-invariant element of $\Omega^*(\hat{M}) \hat{\otimes} C_*(B)$. Dualizing with respect to B, we obtain an element of $\Omega^*(\hat{M}) \hat{\otimes} \Omega^*_\tau(B)$. Applying the product $(\omega_1, \omega_2) \to \omega_1 \wedge \pi^* \omega_2$ to this, we finally obtain a closed Γ-invariant element of $\Omega^*_\tau(\hat{M})$, the latter being the differential forms on $\hat{M}$ with distributional coefficients and with value in $\pi^*\big(o(TB)\big)$. Hence we have an element of $\mathrm{H}^*_\tau(M)$ (where the τ now refers to the flat orientation line bundle $o(N\mathscr{F})$), which we denote by $\langle \mathrm{ch}(\nabla^{\mathrm{can}}), \eta \rangle$. Let $*\big(\hat{A}(T\mathscr{F})\,\mathrm{ch}(V)\big) \in \mathrm{H}^\tau_*(M)$ be the Poincaré dual of $\hat{A}(T\mathscr{F})\,\mathrm{ch}(V) \in \mathrm{H}^*(M;\mathbb{R})$. Then the left-hand-side of (4.15) is the pairing between $\langle \mathrm{ch}(\nabla^{\mathrm{can}}), \eta \rangle \in \mathrm{H}^*_\tau(M)$ and $*\big(\hat{A}(T\mathscr{F})\,\mathrm{ch}(V)\big) \in \mathrm{H}^\tau_*(M)$. Now we may also compute $\langle \mathrm{ch}(\nabla^{\mathrm{can}}), \eta \rangle$ by using the Lipschitz function c^*j instead of h in constructing ∇^{can}. (We may have to smooth ω_1 before taking the product.) Then by naturality, $\langle \mathrm{ch}(\nabla^{\mathrm{can}}), \eta \rangle = \nu^* \Phi_\eta$, which proves the claim.

5. The index and the superconnection Chern character

In this section we prove Theorem 3, relating the Chern character of A_s to the Chern character of the index. We define the index by means of the index projection and show that its Chern character can be computed by means of a connection ∇. We then show that the Chern character of the index can also be written as the supertrace of $e^{-(\nabla')^2}$ for a certain $\mathbb{Z}_2$-graded connection ∇'.

We relate the Chern character of the index to the superconnection Chern character by means of a homotopy from ∇' to A_s. This is done in three cases. In the first case, that of finitely-generated projective modules, the naive homotopy argument works. In the second case, that of the families index theorem, we show that smoothing factors in the homotopy allow the naive argument to be carried through. In the third case, that of Theorem 3, we again justify the naive homotopy argument. We then give some geometric consequences of Theorem 3.

Let $\mathfrak{A}$ be an algebra over $\mathbb{C}$ and let Ω^* be a GDA equipped with a homomorphism $\rho : \mathfrak{A} \to \Omega^0$. Let $\mathscr{E}$ be a left $\mathfrak{A}$-module and let $\nabla : \mathscr{E} \to \Omega^1 \otimes_{\mathfrak{A}} \mathscr{E}$ be a connection on $\mathscr{E}$.

Let $\tilde{\Omega}^*$ be a subalgebra of the graded algebra $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}, \Omega^* \otimes_{\mathfrak{A}} \mathscr{E})$. Put $\tilde{\mathfrak{A}} = \tilde{\Omega}^0$. We assume that $\tilde{\Omega}^*$ is closed under ∇ and that the curvature $\Theta = \nabla^2 \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}, \Omega^2 \otimes_{\mathfrak{A}} \mathscr{E})$ of the connection lies in $\tilde{\Omega}^2$. Then ∇ extends to a covariant differentiation $\tilde{\nabla} : \tilde{\Omega}^* \to \tilde{\Omega}^{*+1}$ on $\tilde{\Omega}^*$ which satisfies $\tilde{\nabla}^2(\tilde{\omega}) = \Theta\tilde{\omega} - \tilde{\omega}\Theta$. Let $\tilde{\eta} : \tilde{\Omega}^* \to \mathbb{C}$ be an even graded trace which satisfies $\tilde{\eta}(\tilde{\nabla}\tilde{\omega}) = 0$ for all $\tilde{\omega} \in \tilde{\Omega}^*$.

As in [8], Chapter III.3, Lemma 9, let X be a new formal odd variable of degree 1 and put

$$\tilde{\tilde{\Omega}}^* = \tilde{\Omega} \oplus X\tilde{\Omega}^* \oplus \tilde{\Omega}^* X \oplus X\tilde{\Omega}^* X \tag{5.1}$$

with the new multiplication rules $(\tilde{\omega}_1 X)\tilde{\omega}_2 = \tilde{\omega}_1(X\tilde{\omega}_2) = 0$ and $(\tilde{\omega}_1 X)(X\tilde{\omega}_2) = \tilde{\omega}_1 \Theta \tilde{\omega}_2$. Define a graded trace $\tilde{\tilde{\eta}}$ on $\tilde{\tilde{\Omega}}^*$ by

$$\tilde{\tilde{\eta}}(\tilde{\omega}_1 + X\tilde{\omega}_2 + \tilde{\omega}_3 X + X\tilde{\omega}_4 X) = \tilde{\eta}(\tilde{\omega}_1) + (-1)^{|\tilde{\omega}_4|}\tilde{\eta}(\tilde{\omega}_4). \tag{5.2}$$

Define a differential d on $\tilde{\tilde{\Omega}}^*$ which is generated by the relations

$$d\tilde{\tilde{\omega}} = \tilde{\nabla}\tilde{\omega} + X\tilde{\omega} + (-1)^{\tilde{\omega}}\tilde{\omega}X \tag{5.3}$$

and $dX = 0$. One can check that $d^2 = 0$ and $\tilde{\tilde{\eta}}(d\tilde{\tilde{\omega}}) = 0$ for $\tilde{\tilde{\omega}} \in \tilde{\tilde{\Omega}}^*$. That is, $(\tilde{\tilde{\Omega}}^*, d, \tilde{\tilde{\eta}})$ defines a cycle over $\tilde{\mathfrak{A}}$ in the sense of [8], Chapter III.1.α, Definition 1.

Suppose that $\tilde{\mathfrak{A}}$ is unital. The cycle structure induces a map from $\mathrm{K}_0(\tilde{\mathfrak{A}})$ to $\mathbb{C}$ ([8], Chapter III.3, Proposition 2). To fix normalizations, let $p \in \tilde{\mathfrak{A}}$ be a projection with corresponding class $[p] \in \mathrm{K}_0(\tilde{\mathfrak{A}})$. Then the pairing of the Chern character of $[p]$ with $\tilde{\eta}$ is defined to be

$$\langle \mathrm{ch}([p]), \tilde{\eta}\rangle = (2\pi i)^{-\deg(\tilde{\eta})/2}\tilde{\tilde{\eta}}(pe^{-p\,dp\,dp}p). \tag{5.4}$$

One can check that $p\,dp\,dp\,p = p(\tilde{\nabla}p)(\tilde{\nabla}p)p + p\Theta p$, which in turn equals the curvature of the connection $p \circ \nabla \circ p$. Thus

$$\langle \mathrm{ch}([p]), \tilde{\eta}\rangle = (2\pi i)^{-\deg(\tilde{\eta})/2}\tilde{\eta}(pe^{-(p\circ\nabla\circ p)^2}p). \tag{5.5}$$

This is consistent with well-known formulae if $\mathscr{E}$ is a finitely-generated projective $\mathfrak{A}$-module, but we have not assumed that $\mathscr{E}$ is finitely-generated projective. The equation extends to $p \in M_n(\tilde{\mathfrak{A}})$ in an obvious way.

We will need an extension of this formula to the nonunital case. We suppose again that we have the algebra $\tilde{\Omega}^*$ and the connection $\tilde{\nabla}$ on it. In general, $\tilde{\nabla}^2$ may not be given in terms of an element of $\tilde{\Omega}^2$. Instead, as in [25], Section 2, we make the weaker assumption that $\tilde{\nabla}^2$ comes from a multiplier (l, r) of $\tilde{\Omega}^*$. This means that l and r are linear maps from $\tilde{\Omega}^*$ to itself such that for all $\tilde{\omega}_1, \tilde{\omega}_2 \in \tilde{\Omega}^*$, we have $l(\tilde{\omega}_1\tilde{\omega}_2) = l(\tilde{\omega}_1)\tilde{\omega}_2$, $r(\tilde{\omega}_1\tilde{\omega}_2) = \omega_1 r(\tilde{\omega}_2)$ and $\tilde{\omega}_1 l(\tilde{\omega}_2) = r(\tilde{\omega}_1)\tilde{\omega}_2$. Then we assume that $\tilde{\nabla}^2(\tilde{\omega}) = l(\tilde{\omega}) - r(\tilde{\omega})$ for some (l, r) of degree 2. (If $\tilde{\mathfrak{A}}$ is unital then we recover Θ by $\Theta = l(1) = r(1)$, and (l, r) are given in terms of Θ by $l(\tilde{\omega}) = \Theta\tilde{\omega}$, $r(\tilde{\omega}) = \tilde{\omega}\Theta$.) With this understanding, $p\,dp\,dp\,p = p(\tilde{\nabla}p)(\tilde{\nabla}p)p + pl(p)$ and equation (5.5) still makes sense for $p \in \tilde{\mathfrak{A}}$.

Next, recall that if $\tilde{\mathfrak{A}}$ is nonunital and $\tilde{\mathfrak{A}}^+$ is the algebra obtained by adding a unit to $\tilde{\mathfrak{A}}$, with canonical homomorphism $\pi : \tilde{\mathfrak{A}}^+ \to \mathbb{C}$, then $\mathrm{K}_0(\tilde{\mathfrak{A}}) = \mathrm{Ker}(\pi_* : \mathrm{K}_0(\tilde{\mathfrak{A}}^+) \to \mathrm{K}_0(\mathbb{C}))$. Thus an element of $\mathrm{K}_0(\tilde{\mathfrak{A}})$ can be represented as $p - p_0$ with the projections $p, p_0 \in M_n(\tilde{\mathfrak{A}}^+)$ satisfying $\pi_*(p) = \pi_*(p_0) \in M_n(\mathbb{C})$. Then the equation

$$\langle \mathrm{ch}([p - p_0]), \tilde{\eta}\rangle = (2\pi i)^{-\deg(\tilde{\eta})/2}\tilde{\eta}(pe^{-(p\circ\nabla\circ p)^2}p - p_0e^{-(p_0\circ\nabla\circ p_0)^2}p_0) \tag{5.6}$$

gives a well-defined map on $\mathrm{K}_0(\tilde{\mathfrak{A}})$.

5.1. Finitely-generated projective $\mathfrak{A}$-modules. Now suppose that $\mathscr{E} = \mathscr{E}^+ \oplus \mathscr{E}^-$ is $\mathbb{Z}_2$-graded, with $\mathscr{E}^\pm$ finitely-generated projective $\mathfrak{A}$-modules. We assume that ∇ preserves the grading. Put $\tilde{\Omega}^* = \mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}, \Omega^* \otimes_{\mathfrak{A}} \mathscr{E})$. We assume that $\tilde{\mathfrak{A}} = \mathrm{End}_{\mathfrak{A}}(\mathscr{E})$ has a holomorphic functional calculus. For example, it suffices that $\mathfrak{A}$ be a complete locally convex topological algebra. Put $e = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

Given $D \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}^+, \mathscr{E}^-)$ and $D^* \in \mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}^-, \mathscr{E}^+)$, we assume that the spectra of DD^* and D^*D are contained in the nonnegative reals. We construct an index projection following [9] and [23]. Let $u \in C^\infty(\mathbb{R})$ be an even function such that $w(x) = 1 - x^2u(x)$ is a Schwartz function and the Fourier transforms of u and w have compact support [23], Lemma 2.1. Define $\bar{u} \in C^\infty\big([0,\infty)\big)$ by $\bar{u}(x) = u(x^2)$. Put $\mathscr{P} = \bar{u}(D^*D)D^*$, which we will think of as a parametrix for D, and put $S_+ = I - \mathscr{P}D$, $S_- = I - D\mathscr{P}$. Consider the operator

$$l = \begin{pmatrix} S_+ & -(I+S_+)\mathscr{P} \\ D & S_- \end{pmatrix}, \tag{5.7}$$

with inverse

$$l^{-1} = \begin{pmatrix} S_+ & \mathscr{P}(I+S_-) \\ -D & S_- \end{pmatrix}. \tag{5.8}$$

The index projection is defined by

$$p = l\frac{e+v}{2}l^{-1} = \begin{pmatrix} S_+^2 & S_+(I+S_+)\mathscr{P} \\ S_-D & I - S_-^2 \end{pmatrix}. \tag{5.9}$$

Put

$$p_0 = \frac{e-v}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \tag{5.10}$$

We note that the supertrace $\mathrm{Tr}_s : \mathrm{End}_{\mathfrak{A}}(\mathscr{E}) \to \mathfrak{A}/[\mathfrak{A},\mathfrak{A}]$ is given by

$$\begin{aligned} \mathrm{Tr}_s(M) &= \mathrm{Tr}\left(\frac{e+v}{2}M\frac{e+v}{2}\right) - \mathrm{Tr}\left(\frac{e-v}{2}M\frac{e-v}{2}\right) \\ &= \mathrm{Tr}(plMl^{-1}p) - \mathrm{Tr}(p_0Mp_0). \end{aligned} \tag{5.11}$$

Let us define a new connection on $\mathscr{E}$ by

$$\nabla' = \left(\frac{e+v}{2}l^{-1} \circ \nabla \circ l\frac{e+v}{2}\right) + \left(\frac{e-v}{2}\nabla\frac{e-v}{2}\right). \tag{5.12}$$

Then by construction,

$$l \circ \nabla' \circ l^{-1} = (p \circ \nabla \circ p) + \big((1-p)l \circ \nabla \circ l^{-1}(1-p)\big). \tag{5.13}$$

In particular,

$$\mathrm{Tr}(pe^{-(p\circ\nabla\circ p)^2}p) = \mathrm{Tr}(ple^{-(\nabla')^2}l^{-1}p). \tag{5.14}$$

Also, from (5.12), we have

$$\mathrm{Tr}(p_0e^{-\nabla^2}p_0) = \mathrm{Tr}(p_0e^{-(\nabla')^2}p_0). \tag{5.15}$$

Using (5.11), we see that we can write $\mathrm{ch}([p-p_0])$ as the Chern character form of a connection on $\mathscr{E}$, namely

$$\mathrm{ch}([p-p_0]) = \mathscr{R}\,\mathrm{Tr}_s(e^{-(\nabla')^2}). \tag{5.16}$$

For future use, we note that $(\nabla')^+ = S_+\nabla^+S_+ + \mathscr{P}(I+S_-)\nabla^-D$ and $(\nabla')^- = \nabla^-$.

So far in this section we have been working with algebraic tensor products. If $\mathfrak{A}$ is a complete locally convex topological algebra then it is straightforward to extend the statements to the topological setting.

Lemma 9. *Suppose that A is a superconnection on $\mathscr{E}$. Then for any closed graded trace $\tilde{\eta}$ on* $\mathrm{Hom}_{\mathfrak{A}}(\mathscr{E}, \Omega^* \otimes_{\mathfrak{A}} \mathscr{E})$,

$$\langle \mathrm{ch}([p - p_0]), \tilde{\eta} \rangle = \tilde{\eta}\big(\mathrm{ch}(A)\big). \tag{5.17}$$

Proof. In general, if $\{A(t)\}_{t \in [0,1]}$ is a smooth 1-parameter family of superconnections on $\mathscr{E}$ then

$$\mathrm{ch}\big(A(1)\big) - \mathrm{ch}\big(A(0)\big) = \mathscr{R} d \int_{[0,1]} \mathrm{Tr}_s\left(\frac{dA}{dt} e^{-A(t)^2} \right) dt. \tag{5.18}$$

Thus it suffices to construct a smooth path in the space of superconnections between ∇' and A, for example the linear homotopy $A(t) = tA + (1-t)\nabla'$. □

5.2. Fiber bundles. Let $\pi : M \to B$ be a fiber bundle with closed even-dimensional fiber Z. Endow the fiber bundle with a vertical Riemannian metric g^{TZ} and a horizontal distribution $T^H M$. Let E be a Hermitian vector bundle on M which is a fiberwise Clifford bundle, with compatible connection ∇^E. Let $\mathscr{E}$ be the smooth sections of the $\mathbb{Z}_2$-graded vector bundle $\pi_!(E)$ on B, whose fiber over b is $C^\infty(Z_b; E|_{Z_b})$.

Take $\mathfrak{A} = C^\infty(B)$, $\Omega^* = \Omega^*(B)$ and let $\nabla^\pm$ be the natural Hermitian connection on $\mathscr{E}^\pm$ ([2], Proposition 10.10). Let $\tilde{\Omega}^*$ be the subalgebra of $\mathrm{Hom}_{C^\infty(B)}\big(\mathscr{E}, \Omega^*(B) \otimes_{C^\infty(B)} \mathscr{E}\big)$ consisting of elements with a smooth fiberwise integral kernel $K(z, w)$. Put $\tilde{\mathfrak{A}} = \tilde{\Omega}^0$, a nonunital algebra if $\dim(Z) > 0$. Given a closed current $\eta \in Z_{\mathrm{even}}(B; \mathbb{R})$, let $\tilde{\eta}$ be the graded trace on $\tilde{\Omega}^*$ given by $\tilde{\eta}(K) = \int_\eta \left(\int_Z K(z,z)\, d\mathrm{vol}_Z \right)$.

Let $D : \mathscr{E}^+ \to \mathscr{E}^-$ be the vertical Dirac operator. Define the index projection $p \in \tilde{\mathfrak{A}}$ as in (5.9). Define e and v as before. Then the index of D is defined to be $\mathrm{Ind}(D) = [p - p_0] \in \mathrm{K}_0(\tilde{\mathfrak{A}})$. The Chern character of $\mathrm{Ind}(D)$ pairs with η by (5.6).

Let us note that although $pe^{-(p \circ \nabla \circ p)^2} p$ and $p_0 e^{-(p_0 \circ \nabla \circ p_0)^2} p_0$ may not individually lie in $\tilde{\Omega}^*$, their difference does. For example, the component in $\tilde{\Omega}^0$ is

$$\begin{pmatrix} S_+^2 & S_+(I + S_+)\mathscr{P} \\ S_- D & I - S_-^2 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} S_+^2 & S_+(I + S_+)\mathscr{P} \\ S_- D & -S_-^2 \end{pmatrix}. \tag{5.19}$$

This is related to the fact that the K-theory of a nonunital algebra is defined in terms of the K-theory of the algebra obtained by adding a unit.

For $s > 0$, let A_s^{Bismut} denote the Bismut superconnection on $\pi_! E$.

Proposition 4. $\big\langle \mathrm{ch}\big(\mathrm{Ind}(D)\big), \tilde{\eta} \big\rangle = \tilde{\eta}\big(\mathrm{ch}(A_s^{\mathrm{Bismut}})\big).$

Proof. We can homotop from D to sD in the definition of the index projection without changing the K-theory class and then everywhere change sD to D, to easily reduce to the case $s = 1$. Define ∇' as in (5.12). As in the proof of Lemma 9, we wish to homotop from ∇' to A_1^{Bismut} and then apply (5.18). The only issue is to write things in a way so that the formal expressions are well-defined.

First, for $t \in [0,1]$ put

$$A(t) = \begin{pmatrix} (\nabla')^+ & tD^* \\ tD & (\nabla')^- \end{pmatrix}. \tag{5.20}$$

Write

$$e^{-A(t)^2} = \begin{pmatrix} e^{-A(t)^2}{}_{11} & e^{-A(t)^2}{}_{12} \\ e^{-A(t)^2}{}_{21} & e^{-A(t)^2}{}_{22} \end{pmatrix}. \tag{5.21}$$

Put

$$\mathrm{ch}\big(A(t)\big) = \mathscr{R}\Big(\mathrm{Tr}(S_+ e^{-A(t)^2}{}_{11} S_+) + \mathrm{Tr}\big(D e^{-A(t)^2}{}_{11} \mathscr{P}(I + S_-) - e^{-A(t)^2}{}_{22}\big)\Big). \tag{5.22}$$

Formally, the right-hand-side of (5.22) equals $\mathscr{R}\big(\mathrm{Tr}(e^{-A(t)^2}{}_{11}) - \mathrm{Tr}(e^{-A(t)^2}{}_{22})\big)$. To see that the traces in the right-hand-side of (5.22) make sense, let us compute $A(t)^2$. In an ungraded notation, we have

$$A(t)^2 = \begin{pmatrix} \big((\nabla')^+\big)^2 + t^2 D^* D & t[(\nabla')^-, D^*] + t\big((\nabla')^+ - (\nabla')^-\big)D^* \\ t[(\nabla')^+, D] - t\big((\nabla')^+ - (\nabla')^-\big)D & \big((\nabla')^-\big)^2 + t^2 DD^* \end{pmatrix}. \tag{5.23}$$

The term in the lower left-hand corner of (5.23) is

$$\begin{aligned} & t[(\nabla')^+, D] - t\big((\nabla')^+ - (\nabla')^-\big)D \\ &= t[(\nabla')^-, D] - tD\big((\nabla')^+ - (\nabla')^-\big) \\ &= t[\nabla^-, D] - tD\big(S_+ \nabla^+ S_+ + \mathscr{P}(I + S_-)\nabla^- D - \nabla^-\big) \\ &= t(S_-^2 \nabla^- D - DS_+ \nabla^+ S_+). \end{aligned} \tag{5.24}$$

Then modulo uniformly smoothing operators,

$$A(t)^2 \equiv \begin{pmatrix} \big((\nabla')^+\big)^2 + t^2 D^* D & t[(\nabla')^-, D^*] + t\big((\nabla')^+ - (\nabla')^-\big)D^* \\ 0 & D\big[\big((\nabla')^+\big)^2 + t^2 D^* D\big]\mathscr{P} \end{pmatrix} \tag{5.25}$$

and

$$e^{-A(t)^2} \equiv \begin{pmatrix} e^{-(((\nabla')^+)^2 + t^2 D^* D)} & \mathscr{L} \\ 0 & De^{-(((\nabla')^+)^2 + t^2 D^* D)}\mathscr{P} \end{pmatrix}, \tag{5.26}$$

where

$$(5.27)\qquad \mathscr{X} = -\int_0^1 e^{-u(((\nabla')^+)^2+t^2D^*D)}\big(t[(\nabla')^-, D^*] + t\big((\nabla')^+ - (\nabla')^-\big)D^*\big)$$
$$e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du.$$

It follows that the right-hand-side of (5.22) is well-defined.

We claim that $\langle \mathrm{ch}\big(\mathrm{Ind}(D)\big), \tilde{\eta}\rangle = \tilde{\eta}\big(\mathrm{ch}\big(A(0)\big)\big)$. To see this, we have

$$(5.28)\qquad pe^{-(p\circ\nabla\circ p)^2}p = l\begin{pmatrix} e^{-((\nabla')^+)^2} & 0 \\ 0 & 0 \end{pmatrix} l^{-1}$$
$$= \begin{pmatrix} S_+e^{-((\nabla')^+)^2}S_+ & S_+e^{-((\nabla')^+)^2}\mathscr{P}(I+S_-) \\ De^{-((\nabla')^+)^2}S_+ & De^{-((\nabla')^+)^2}\mathscr{P}(I+S_-) \end{pmatrix}.$$

Then

$$(5.29)\qquad \mathrm{Tr}(pe^{-(p\circ\nabla\circ p)^2}p - p_0e^{-(p_0\circ\nabla\circ p_0)^2}p_0) = \mathrm{Tr}(S_+e^{-((\nabla')^+)^2}S_+)$$
$$+ \mathrm{Tr}\big(De^{-((\nabla')^+)^2}\mathscr{P}(I+S_-) - e^{-(\nabla^-)^2}\big),$$

from which the claim follows.

Let us note that the terms being traced in (5.22) are in fact uniformly smoothing with respect to t, due to factors of the form $S_\pm$.

We now wish to write the analog of equation (5.18). Although $\mathrm{ch}\big(A(t)\big)$ is well-defined, it is not clear in the present setting that the integrand in (5.18) is integrable for small t. Let us first do a formal calculation. With respect to (5.20), (5.26) and (5.27), we have

$$(5.30)\qquad \frac{dA}{dt} = \begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$$

and

$$(5.31)\qquad \mathrm{Tr}_s\left(\frac{dA}{dt}\begin{pmatrix} e^{-(((\nabla')^+)^2+t^2D^*D)} & \mathscr{X} \\ 0 & De^{-(((\nabla')^+)^2+t^2D^*D)}\mathscr{P} \end{pmatrix}\right) = -\mathrm{Tr}(D\mathscr{X})$$
$$= t\,\mathrm{Tr}\bigg(D\int_0^1 e^{-u(((\nabla')^+)^2+t^2D^*D)}\big([(\nabla')^-, D^*] + \big((\nabla')^+ - (\nabla')^-\big)D^*\big)$$
$$e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du\bigg).$$

Modulo smoothing operators,

$$(5.32)\qquad D\int_0^1 e^{-u(((\nabla')^+)^2+t^2D^*D)}\big([(\nabla')^-,D^*]+\big((\nabla')^+-(\nabla')^-\big)D^*\big)e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du$$

$$\equiv \int_0^1 e^{-u((\nabla^-)^2+t^2DD^*)}D\big([(\nabla')^-,D^*]+\big((\nabla')^+-(\nabla')^-\big)D^*\big)$$

$$\times\, e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du$$

$$\equiv \int_0^1 e^{-u((\nabla^-)^2+t^2DD^*)}[\nabla^-,DD^*]e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du.$$

Then

$$(5.33)\qquad \mathrm{Tr}\left(\int_0^1 e^{-u((\nabla^-)^2+t^2DD^*)}[\nabla^-,DD^*]e^{-(1-u)((\nabla^-)^2+t^2DD^*)}\,du\right)$$

$$= \mathrm{Tr}\big([\nabla^-,DD^*]e^{-((\nabla^-)^2+t^2DD^*)}\big) = -t^{-2}d\,\mathrm{Tr}\big(e^{-((\nabla^-)^2+t^2DD^*)}\big).$$

The upshot is that we can write

$$(5.34)\qquad \mathrm{ch}\big(A(1)\big)-\mathrm{ch}\big(A(0)\big)$$

$$= \mathscr{R}d\int_{[0,1]}\left(\mathrm{Tr}_s\left(\frac{dA}{dt}e^{-A(t)^2}\right)+t^{-1}d\,\mathrm{Tr}\big(e^{-((\nabla^-)^2+t^2DD^*)}\big)\right)dt,$$

where the integrand in the right-hand-side of (5.34), after the terms are appropriately grouped, is the trace of a smoothing operator that is continuous in t. Thus $\tilde{\eta}\big(\mathrm{ch}\big(A(0)\big)\big)=\tilde{\eta}\big(\mathrm{ch}\big(A(1)\big)\big)$.

Next, we perform a linear homotopy from $A(1)$ to A_1^{Bismut}. As the 0-th order part of the superconnection always equals $\begin{pmatrix} 0 & D^* \\ D & 0 \end{pmatrix}$ during this homotopy, it is easy to justify the formal superconnection argument, using (5.18), that $\tilde{\eta}\big(\mathrm{ch}\big(A(1)\big)\big)=\tilde{\eta}\big(\mathrm{ch}(A_1^{\mathrm{Bismut}})\big)$. This proves the proposition. □

5.3. Fiber bundles over cross-product groupoids. Using the notation of Section 4, put $\mathfrak{A}=C^\infty(B,\mathscr{B}^\omega)$, $\Omega^*=\Omega^\infty(B,\mathscr{B}^\omega)$, $\mathscr{E}=C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})$, $\tilde{\mathfrak{A}}=\mathrm{End}^\infty_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big)$ and

$$\tilde{\Omega}^*=\mathrm{Hom}^\infty_{C^\infty(B,\mathscr{B}^\omega)}\big(C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E}),\Omega^*(B,\mathscr{B}^\omega)\otimes_{C^\infty(B,\mathscr{B}^\omega)}C^\infty_{\mathscr{B}^\omega}(\hat{M};\hat{E})\big).$$

Define the index projection as in (5.9). Let η be a closed graded trace on $\Omega^*(B,\mathscr{B}^\omega)$. Then we can go through the same steps as in the proof of Proposition 4 to conclude

Theorem 3.

$$(5.35)\qquad \big\langle \mathrm{ch}\big(\mathrm{Ind}(D)\big),\tilde{\eta}\big\rangle = \langle \mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2}),\eta\rangle.$$

We note that we use finite-propagation-speed estimates in order to know that we can carry out the arguments in $\tilde{\Omega}^*$. That is, we use the fact that if $f \in \Sigma^{(\infty)}(\mathbb{R})$ is a function whose Fourier transform $\hat{f}(k)$ has exponential decay in $|k|$ then the Schwartz kernel $f(sQ)(z,w)$ has exponential decay in $d(z,w)$. In order to obtain uniform decay estimates in the analog of the last step of Proposition 4, as in [2], Theorem 9.48 we use the fact that in the n-th order term of the Duhamel expansion of $e^{-A_s^2}$, there is always a factor of the form $e^{-rs^2Q^2}$ with $r \geqq \dfrac{1}{n+1}$.

Putting together Theorems 2 and 3, we obtain

Theorem 4.

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \tilde{\eta} \rangle = \int_M \hat{A}(T\mathscr{F})\,\mathrm{ch}(V) v^* \Phi_\eta. \tag{5.36}$$

If the fiber Z is instead odd-dimensional then one can prove Theorem 4 by a standard trick involving taking the product with a circle.

Corollary 2. *Let $\mathscr{A}$ be a subalgebra of the reduced cross-product C^*-algebra $C_0(B) \rtimes_r \Gamma$ which is stable under the holomorphic functional calculus in $C_0(B) \rtimes_r \Gamma$ and which contains $C^\infty(B, \mathscr{B}^\omega)$. Let η be a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$ which extends to give a cyclic cocycle on $\mathscr{A}$. Suppose that TZ is spin and that g^{TZ} has fiberwise positive scalar curvature. Then $\int_M \hat{A}(T\mathscr{F}) v^* \Phi_\eta = 0$.*

Proof. Let D be the pure Dirac operator. As η is a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$, it gives rise to a cyclic cocycle on $C_c^\infty(B) \rtimes \Gamma$ through its character ([8], Section III.1.α). By assumption, this has an extension η' to $\mathscr{A}$. Now we have $\mathrm{Ind}(D) \in \mathrm{K}_*(C_0(B) \rtimes_r \Gamma) \cong \mathrm{K}_*(\mathscr{A})$. Then (5.36) becomes

$$\langle \mathrm{Ind}(D), \eta' \rangle = \int_M \hat{A}(T\mathscr{F}) v^* \Phi_\eta. \tag{5.37}$$

However, by the Lichnerowicz argument, $\mathrm{Ind}(D) = 0$. The corollary follows. □

Suppose that $B = S^1$, with Γ acting by orientation-preserving diffeomorphisms. There is a left action of Γ on $\Omega^1(B)$. Let $v \in \Omega^1(B)$ be a volume form. Define a closed graded trace on $\Omega^*(B, \mathbb{C}\Gamma)$ by

$$\eta(fg_0\, dg_1\, dg_2) = \int_B f \left(\ln \frac{v}{g_0 g_1 \cdot v}\, d \ln \frac{g_0 g_1 \cdot v}{g_0 \cdot v} - \ln \frac{g_0 g_1 \cdot v}{g_0 \cdot v}\, d \ln \frac{v}{g_0 g_1 \cdot v} \right). \tag{5.38}$$

Then $v^* \Phi_\eta$ is proportionate to the Godbillon-Vey class $GV \in \mathrm{H}^3(M; \mathbb{R})$ ([8], Chapter III.6.β, Theorem 17). Furthermore, the hypotheses of Corollary 2 are satisfied ([8], Chapter III.7.β).

Corollary 3. *Suppose that $B = S^1$, Γ acts on B by orientation-preserving diffeomorphisms, TZ is spin and g^{TZ} has fiberwise positive scalar curvature. Then $\int_M \hat{A}(T\mathscr{F}) GV = 0$.*

In general, if $\dim(B) = q$ and the action of Γ on B is orientation-preserving then one can write down a closed graded trace η on $\Omega^*(B, \mathbb{C}\Gamma)$ so that $\nu^*\Phi_\eta$ is proportionate to the Godbillon-Vey class $GV \in \mathrm{H}^{2q+1}(M; \mathbb{R})$, and the above results extend.

6. Étale groupoids

In this section we generalize the results of the previous sections from cross-product groupoids to general smooth Hausdorff étale groupoids. In Subsection 6.1 we explain in detail how, in the case of cross-product groupoids, the expressions of this section specialize to the expressions of the previous sections.

We follow the groupoid conventions of [8], Sections II.5 and III.2.δ. Let G be a smooth Hausdorff groupoid, with units $G^{(0)}$. We suppose that G is étale, i.e. that the range map $r : G \to G^{(0)}$ and the source map $s : G \to G^{(0)}$ are local diffeomorphisms. To construct the product of $\gamma_0, \gamma_1 \in G$, we must have $s(\gamma_0) = r(\gamma_1)$. Then $r(\gamma_0\gamma_1) = r(\gamma_0)$ and $s(\gamma_0\gamma_1) = s(\gamma_1)$. Given $x \in G^{(0)}$, put $G^x = r^{-1}(x)$, $G_x = s^{-1}(x)$ and $G^x_x = G^x \cap G_x$.

Given $f_0, f_1 \in C^\infty_c(G)$, the convolution product is

$$(f_0 f_1)(\gamma) = \sum_{\gamma_0\gamma_1=\gamma} f_0(\gamma_0) f_1(\gamma_1). \tag{6.1}$$

The sum in (6.1) is finite.

We write $G^{(n)}$ for the n-chains of composable elements of G, i.e.

$$G^{(n)} = \{(\gamma_1, \ldots, \gamma_n) \in G^n : s(\gamma_1) = r(\gamma_2), \ldots, s(\gamma_{n-1}) = r(\gamma_n)\}. \tag{6.2}$$

As G is étale, $G^{(n)}$ is a manifold of the same dimension as G. As in [8], Section III.2.δ, we define a double complex by letting $\Omega^{m,n}_c(G)$ be the quotient of $\Omega^m_c(G^{(n+1)})$ by the forms which are supported on $\{(\gamma_0, \ldots, \gamma_n) : \gamma_j \text{ is a unit for some } j > 0\}$. The product of $\omega_1 \in \Omega^{*,n_1}_c(G)$ and $\omega_2 \in \Omega^{*,n_2}_c(G)$ is given by

$$\begin{aligned}(\omega_1\omega_2)(\gamma_0, \ldots, \gamma_{n_1+n_2}) = &\sum_{\gamma\gamma'=\gamma_{n_1}} \omega_1(\gamma_0, \ldots, \gamma_{n_1-1}, \gamma) \wedge \omega_2(\gamma', \gamma_{n_1+1}, \ldots, \gamma_{n_1+n_2}) \\ &- \Bigg((-1)^{n_1-1} \sum_{\gamma\gamma'=\gamma_0} \omega_1(\gamma, \gamma', \gamma_1, \ldots, \gamma_{n_1-1}) \\ &\quad + (-1)^{n_1-2} \sum_{\gamma\gamma'=\gamma_1} \omega_1(\gamma_0, \gamma, \gamma', \gamma_2, \ldots, \gamma_{n_1-1}) \\ &\quad + \cdots + \sum_{\gamma\gamma'=\gamma_{n_1-1}} \omega_1(\gamma_0, \ldots, \gamma_{n_1-2}, \gamma, \gamma') \Bigg) \\ &\wedge \omega_2(\gamma_{n_1}, \ldots, \gamma_{n_1+n_2}).\end{aligned} \tag{6.3}$$

In forming the wedge product in (6.3), the maps r and s are used to identify cotangent spaces. The first differential d_1 on $\Omega_c^{*,*}(G)$ is the de Rham differential. To define the second differential d_2, let $\chi_{G^{(0)}} \in C^\infty(G)$ be the characteristic function for the units. Then

$$(d_2\omega)(\gamma_0, \dots, \gamma_{n+1}) = \chi_{G^{(0)}}(\gamma_0)\omega(\gamma_1, \dots, \gamma_n). \tag{6.4}$$

We let $\Omega_c^*(G)$ denote the GDA formed by the total complex of $\Omega_c^{*,*}(G)$.

Let P be a smooth G-manifold ([8], Section II.10.α, Definition 1). That is, first of all, there is a submersion $\pi : P \to G^{(0)}$. Given $x \in G^{(0)}$, we write $Z_x = \pi^{-1}(x)$. Putting

$$P \times_{G^{(0)}} G = \{(p, \gamma) \in P \times G : p \in Z_{r(\gamma)}\}, \tag{6.5}$$

we must also have a map $P \times_{G^{(0)}} G \to P$, denoted $(p, \gamma) \to p\gamma$, such that $p\gamma \in Z_{s(\gamma)}$ and $(p\gamma_1)\gamma_2 = p(\gamma_1\gamma_2)$ for all $(\gamma_1, \gamma_2) \in G^{(2)}$. It follows that for each $\gamma \in G$, the map $p \to p\gamma$ gives a diffeomorphism from $Z_{r(\gamma)}$ to $Z_{s(\gamma)}$. The groupoid $\mathscr{G} = P \rtimes G$ has underlying space $P \times_{G^{(0)}} G$, units $\mathscr{G}^{(0)} = P$ and maps $r(p, \gamma) = p$ and $s(p, \gamma) = p\gamma$.

We assume that P is a proper G-manifold, i.e. that the map $P \times_{G^{(0)}} G \to P \times P$ given by $(p, \gamma) \to (p, p\gamma)$ is proper. Then $\mathscr{G} = P \rtimes G$ is a proper groupoid, i.e. the map $\mathscr{G} \to \mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$ given by $\gamma \to \big(r(\gamma), s(\gamma)\big)$ is proper ([8], Section II.10.α, Definition 2). We also assume that G acts cocompactly on P, i.e. that the quotient of P by the equivalence relation ($p \sim p'$ if $p = p'\gamma$ for some $\gamma \in G$) is compact. Equivalently, $\mathscr{G} = P \rtimes G$ is a cocompact groupoid, i.e. the quotient of $\mathscr{G}^{(0)}$ by the equivalence relation ($x \sim x'$ if $(x, x') = \big(r(\gamma), s(\gamma)\big)$ for some $\gamma \in \mathscr{G}$) is compact. Finally, we assume that G acts freely on P, i.e. that the preimage of the diagonal in $P \times P$ under the map $P \times_{G^{(0)}} G \to P \times P$ equals $P \times_{G^{(0)}} G^{(0)}$. Equivalently, $\mathscr{G} = P \rtimes G$ is a free groupoid, i.e. the preimage of the diagonal in $\mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$ under the map $(r, s) : \mathscr{G} \to \mathscr{G}^{(0)} \times \mathscr{G}^{(0)}$ equals $\mathscr{G}^{(0)}$.

Now let $\mathscr{G}$ be any proper cocompact étale groupoid. The product

$$C_c^\infty(\mathscr{G}) \times C_c^\infty(\mathscr{G}^{(0)}) \to C_c^\infty(\mathscr{G}^{(0)})$$

is given explicitly by

$$(fF)(x) = \sum_{\gamma \in \mathscr{G}^x} f(\gamma)F\big(s(\gamma)\big). \tag{6.6}$$

We wish to define a connection

$$\nabla^{\mathrm{can}} : C_c^\infty(\mathscr{G}^{(0)}) \to \Omega_c^1(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)}). \tag{6.7}$$

To do so, we use isomorphisms

$$\Omega_c^{1,0}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)}) \equiv \Omega_c^1(\mathscr{G}^{(0)}) \text{ and } \Omega_c^{0,1}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)}) \equiv C_c^\infty(\mathscr{G})/C_c^\infty(\mathscr{G}^{(0)}).$$

The latter isomorphism is realized by saying that the image of $\omega \otimes F$ in $C_c^\infty(\mathscr{G})/C_c^\infty(\mathscr{G}^{(0)})$ is given by

$$(\omega F)(\gamma_0) = \sum_{\gamma\gamma' = \gamma_0} \omega(\gamma, \gamma')F\big(s(\gamma_0)\big) - \sum_{\gamma \in \mathscr{G}^{s(\gamma_0)}} \omega(\gamma_0, \gamma)F\big(s(\gamma)\big) \tag{6.8}$$

for $\gamma_0 \notin \mathscr{G}^{(0)}$. Then with this isomorphism, the multiplication

$$(6.9) \quad C_c^\infty(\mathscr{G}) \times \bigl(\Omega_c^{0,1}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)})\bigr) \to \Omega_c^{0,1}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)}),$$

i.e. the multiplication

$$(6.10) \qquad C_c^\infty(\mathscr{G}) \times \frac{C_c^\infty(\mathscr{G})}{C_c^\infty(\mathscr{G}^{(0)})} \to \frac{C_c^\infty(\mathscr{G})}{C_c^\infty(\mathscr{G}^{(0)})}$$

is given by

$$(6.11) \qquad (f\mathscr{F})(\gamma_0) = \sum_{\gamma\gamma'=\gamma_0} f(\gamma)\mathscr{F}(\gamma') - f(\gamma_0) \sum_{\gamma \in \mathscr{G}^{s(\gamma_0)}} \mathscr{F}(\gamma)$$

for $\gamma_0 \notin \mathscr{G}^{(0)}$.

More generally, there is an isomorphism between $\Omega_c^{m,n}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)})$ and the quotient of $\Omega_c^m(\mathscr{G}^n)$ by the forms which are supported on

$$\{(\gamma_0, \dots, \gamma_{n-1}) : \gamma_j \text{ is a unit for some } j \geqq 0\},$$

under which the image of $\omega \otimes F$ is given by

$$\begin{aligned}(6.12) \quad (\omega F)(\gamma_0, \dots, \gamma_{n-1}) = &\sum_{\gamma\gamma'=\gamma_0} \omega(\gamma, \gamma', \gamma_1, \dots, \gamma_{n-1}) F\bigl(s(\gamma_{n-1})\bigr) \\ &- \sum_{\gamma\gamma'=\gamma_1} \omega(\gamma_0, \gamma, \gamma', \gamma_2, \dots, \gamma_{n-1}) F\bigl(s(\gamma_{n-1})\bigr) \\ &+ \dots + (-1)^{n-1} \sum_{\gamma\gamma'=\gamma_{n-1}} \omega(\gamma_0, \dots, \gamma_{n-2}, \gamma, \gamma') F\bigl(s(\gamma_{n-1})\bigr) \\ &+ (-1)^n \sum_{\gamma \in \mathscr{G}^{s(\gamma_{n-1})}} \omega(\gamma_0, \dots, \gamma_{n-1}, \gamma) F\bigl(s(\gamma)\bigr)\end{aligned}$$

for $\gamma_0, \dots, \gamma_{n-1} \notin \mathscr{G}^{(0)}$.

Now let $h \in C_c^\infty(\mathscr{G}^{(0)})$ satisfy

$$(6.13) \qquad \sum_{\gamma \in \mathscr{G}^x} h\bigl(s(\gamma)\bigr) = 1$$

for all $x \in \mathscr{G}^{(0)}$. Then there is a connection

$$(6.14) \qquad \nabla^{\mathrm{can}} = \nabla^{1,0} \oplus \nabla^{0,1}$$

on $C_c^\infty(\mathscr{G}^{(0)})$ where $\nabla^{1,0}(F) \in \Omega_c^1(\mathscr{G}^{(0)})$ is the de Rham differential of $F \in C_c^\infty(\mathscr{G}^{(0)})$ and $\nabla^{0,1}(F) \in \Omega_c^{0,1}(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)}) \equiv C_c^\infty(\mathscr{G})/C_c^\infty(\mathscr{G}^{(0)})$ is given by

$$(6.15) \qquad \bigl(\nabla^{0,1}(F)\bigr)(\gamma_0) = F\bigl(r(\gamma_0)\bigr) h\bigl(s(\gamma_0)\bigr)$$

for $\gamma_0 \notin \mathscr{G}^{(0)}$.

One sees that

$$(\nabla^{\mathrm{can}})^2 \in \mathrm{Hom}_{C_c^\infty(\mathscr{G})}\big(C_c^\infty(\mathscr{G}^{(0)}), \Omega_c^2(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)})\big)$$

acts on $C_c^\infty(\mathscr{G}^{(0)})$ as left multiplication by a 2-form Θ which commutes with $C_c^\infty(\mathscr{G})$. Explicitly, $\Theta = \Theta^{1,1} + \Theta^{0,2}$ where

$$\Theta^{1,1}(\gamma_0, \gamma_1) = -\chi_{\mathscr{G}^{(0)}}(\gamma_0\gamma_1) d^{\text{de Rham}} h\big(s(\gamma_0)\big) \tag{6.16}$$

and

$$\Theta^{0,2}(\gamma_0, \gamma_1, \gamma_2) = -\chi_{\mathscr{G}^{(0)}}(\gamma_0\gamma_1\gamma_2) h\big(s(\gamma_0)\big) h\big(s(\gamma_1)\big) \tag{6.17}$$

for $\gamma_1, \gamma_2 \notin \mathscr{G}^{(0)}$. Put

$$\mathrm{ch}(\nabla^{\mathrm{can}}) = e^{-\frac{\Theta}{2\pi i}} \in \mathrm{End}_{\Omega_c^*(\mathscr{G})}\big(\Omega_c^*(\mathscr{G}) \otimes_{C_c^\infty(\mathscr{G})} C_c^\infty(\mathscr{G}^{(0)})\big). \tag{6.18}$$

Then the abelianization of $\mathrm{ch}(\nabla^{\mathrm{can}})$ is closed and its cohomology class is independent of the choice of h.

Now suppose that G acts properly and cocompactly on P. Give P a G-invariant fiberwise Riemannian metric. An element K of $\mathrm{End}_{C_c^\infty(G)}\big(C_c^\infty(P)\big)$ has a Schwartz kernel $K(p|p')$ with respect to its fiberwise action, so that we can write

$$(KF)(p) = \int_{Z_{\pi(p)}} K(p|p') F(p')\, d\mathrm{vol}_{Z_{\pi(p)}}. \tag{6.19}$$

Let $\mathrm{End}^\infty_{C_c^\infty(G)}\big(C_c^\infty(P)\big)$ denote the subalgebra of $\mathrm{End}_{C_c^\infty(G)}\big(C_c^\infty(P)\big)$ consisting of elements with a smooth integral kernel.

Let $\phi \in C_c^\infty(P)$ satisfy

$$\sum_{\gamma \in G^{\pi(p)}} \phi(p\gamma) = 1 \tag{6.20}$$

for all $p \in P$. Define a trace $\mathrm{Tr}_{\langle e\rangle}$ on $\mathrm{End}^\infty_{C_c^\infty(G)}\big(C_c^\infty(P)\big)$ by

$$\mathrm{Tr}_{\langle e\rangle}(K)(\gamma_0) = \int_{Z_{\gamma_0}} \phi(p) K(p|p)\, d\mathrm{vol}_{Z_{\gamma_0}}, \tag{6.21}$$

for $\gamma_0 \in G^{(0)}$. Then $\mathrm{Tr}_{\langle e\rangle}$ takes value in $\dfrac{C_c^\infty(G)}{[C_c^\infty(G), C_c^\infty(G)]}$ and is concentrated at the units.

Put

$$G^n \times_s P = \{(\gamma_0, \ldots, \gamma_{n-1}, p) \in G^n \times P : p \in Z_{s(\gamma_{n-1})}\}. \tag{6.22}$$

There is an isomorphism, as in (6.12), between $\Omega_c^{m,n}(G) \otimes_{C_c^\infty(G)} C_c^\infty(P)$ and the quotient of $\Omega_c^m(G^n \times_s P)$ by the forms which are supported on

$$\{(\gamma_0, \ldots, \gamma_{n-1}, p) : \gamma_j \text{ is a unit for some } j \geqq 0\}. \tag{6.23}$$

Consider the $\mathbb{Z}$-graded algebra

$$\mathrm{Hom}^{\infty}_{C_c^{\infty}(G)}\big(C_c^{\infty}(P), \Omega_c^*(G) \otimes_{C_c^{\infty}(G)} C_c^{\infty}(P)\big) \tag{6.24}$$

consisting of elements K of $\mathrm{Hom}_{C_c^{\infty}(G)}\big(C_c^{\infty}(P), \Omega_c^*(G) \otimes_{C_c^{\infty}(G)} C_c^{\infty}(P)\big)$ with a smooth integral kernel. Using the above isomorphism, the kernel of an element K can be written in the form

$$K(\gamma_0, \ldots, \gamma_{n-1}, p|p') \in \Lambda^m(T^*_{r(\gamma_0)} G^{(0)}) \tag{6.25}$$

where $(\gamma_0, \ldots, \gamma_{n-1}, p) \in G^n \times_s P$, $\gamma_0, \ldots, \gamma_{n-1} \notin G^{(0)}$, and $p' \in Z_{r(\gamma_0)}$. The action of K on $C_c^{\infty}(P)$ is given by

$$(KF)(\gamma_0, \ldots, \gamma_{n-1}, p) = \int_{Z_{r(\gamma_0)}} K(\gamma_0, \ldots, \gamma_{n-1}, p|p') F(p')\, d\mathrm{vol}_{Z_{r(\gamma_0)}} \tag{6.26}$$

for $\gamma_0, \ldots, \gamma_{n-1} \notin G^{(0)}$. Then there is a trace

$$\mathrm{Tr}_{\langle e \rangle} : \mathrm{Hom}^{\infty}_{C_c^{\infty}(G)}\big(C_c^{\infty}(P), \Omega_c^*(G) \otimes_{C_c^{\infty}(G)} C_c^{\infty}(P)\big) \to \Omega_c^*(G)_{\mathrm{ab}} \tag{6.27}$$

given by

$$\begin{aligned}
&\mathrm{Tr}_{\langle e \rangle}(K)(\gamma_0, \ldots, \gamma_n) \\
&\quad = \chi_{G^{(0)}}(\gamma_0 \cdots \gamma_n) \int_{Z_{r(\gamma_0)}} \phi(p) \\
&\qquad \Big[\chi_{G^{(0)}}(\gamma_0) \sum_{\gamma\gamma' = \gamma_n} K\big(\gamma_1, \ldots, \gamma_{n-1}, \gamma, p(\gamma')^{-1} \,|\, p\big) \\
&\qquad - \chi_{G^{(0)}}(\gamma_0) \sum_{\gamma\gamma' = \gamma_{n-1}} K\big(\gamma_1, \ldots, \gamma_{n-2}, \gamma, \gamma', p(\gamma_n)^{-1} \,|\, p\big) \\
&\qquad + \chi_{G^{(0)}}(\gamma_0) \sum_{\gamma\gamma' = \gamma_{n-2}} K\big(\gamma_1, \ldots, \gamma_{n-3}, \gamma, \gamma', \gamma_{n-1}, p(\gamma_n)^{-1} \,|\, p\big) \\
&\qquad + \cdots + (-1)^{n-1} \chi_{G^{(0)}}(\gamma_0) \sum_{\gamma\gamma' = \gamma_1} K\big(\gamma, \gamma', \gamma_2, \ldots, \gamma_{n-1}, p(\gamma_n)^{-1} \,|\, p\big) \\
&\qquad + (-1)^n K\big(\gamma_0, \ldots, \gamma_{n-1}, p(\gamma_n)^{-1} \,|\, p\big)\Big]\, d\mathrm{vol}_{Z_{r(\gamma_0)}}.
\end{aligned} \tag{6.28}$$

Let L be a topological space which is the total space of a fiber bundle $\sigma : L \to G^{(0)}$. We suppose that each fiber $L_x = \sigma^{-1}(x)$ is a complete length space with metric d_x. We also assume that G acts isometrically, properly and cocompactly on L. Let $i : G \to L$ be a G-equivariant map, not necessarily continuous. That is, for each $x \in G^{(0)}$, i sends G_x to L_x and for each $\gamma \in G$, the composite map $G_{r(\gamma)} \xrightarrow{\cdot\gamma} G_{s(\gamma)} \xrightarrow{i} L_{s(\gamma)}$ equals the composite map $G_{r(\gamma)} \xrightarrow{i} L_{r(\gamma)} \xrightarrow{\cdot\gamma} L_{s(\gamma)}$. We assume that i is proper in the sense that the preimage of a compact set has compact closure. Note that i gives a possibly-discontinuous section of σ. We assume in addition that for any compact subset K of $G^{(0)}$, $i(K)$ has compact closure. Define a "length function" on G by

$$l(\gamma) = d_{s(\gamma)}\big(i\big(s(\gamma)\big), i(\gamma)\big), \tag{6.29}$$

where we think of γ and $s(\gamma) \in G^{(0)}$ as living in $G_{s(\gamma)}$. Then $l(\gamma_0\gamma_1) \leqq l(\gamma_0) + l(\gamma_1)$. Furthermore, for each $x \in G^{(0)}$, the restriction of l to G_x is proper.

Let $C^\infty_\omega(G)$ be the set of $f \in C^\infty(G)$ such that

1. s^*f has support in some compact subset K of $G^{(0)}$, and,
2. for all $q \in \mathbb{Z}^+$,

$$\sup_{x \in K} \sup_{\gamma \in G_x} e^{ql(\gamma)}|f(\gamma)| < \infty, \tag{6.30}$$

along with the analogous property for derivatives. Then $C^\infty_\omega(G)$ is an algebra with the same formal multiplication as in (6.1), and is independent of the choices of L and i; compare [20], Proposition 3. We define $\Omega^*_\omega(G)$ similarly. That is, first define s_n on $G^{(n+1)}$ by $s_n(\gamma_0, \dots, \gamma_n) = s(\gamma_n)$. Let $\tilde{\Omega}^{m,n}_\omega(G)$ be the elements ω of $\Omega^m(G^{n+1})$ such that

1. $s_n^*\omega$ has support in some compact subset K of $G^{(0)}$, and,
2. for all $q \in \mathbb{Z}^+$,

$$\sup_{x \in K} \sup_{(\gamma_0, \dots, \gamma_n) \in s_n^{-1}(x)} e^{q(l(\gamma_0) + \cdots + l(\gamma_n))}|\omega(\gamma_0, \dots, \gamma_n)| < \infty, \tag{6.31}$$

along with the analogous property for derivatives. Let $\Omega^*_\omega(G)$ be the quotient of $\tilde{\Omega}^{m.n}_\omega(G)$ by the forms which are supported on $\{(\gamma_0, \dots, \gamma_n) : \gamma_j \text{ is a unit for some } j > 0\}$. Then $\Omega^*_\omega(G)$ is a GDA, with the same formal multiplication as in (6.3).

Suppose now that G acts properly and cocompactly on P as before. Put

$$C^\infty_\omega(P) = C^\infty_\omega(G) \otimes_{C^\infty_c(G)} C^\infty_c(P). \tag{6.32}$$

Using the cocompactness of the G-action on P, the elements of $C^\infty_\omega(P)$ can be characterized as elements $F \in C^\infty(P)$ such that for any $x \in G^{(0)}$, $p \in Z_x$ and $q \in \mathbb{Z}^+$, we have

$$\sup_{z \in Z_x} e^{qd(z,x)}|F(z)| < \infty, \tag{6.33}$$

along with the analogous property for the covariant derivatives of F. Let $\mathrm{End}^\infty_{C^\infty_\omega(G)}\big(C^\infty_\omega(P)\big)$ be the subalgebra of $\mathrm{End}_{C^\infty_\omega(G)}\big(C^\infty_\omega(P)\big)$ consisting of elements K with a smooth integral kernel $K(z,w)$. Then the elements of $\mathrm{End}^\infty_{C^\infty_\omega(G)}\big(C^\infty_\omega(P)\big)$ can be characterized as the G-invariant elements $K(z|w) \in C^\infty(P \times_{G^{(0)}} P)$ such that for any $x \in G^{(0)}$ and $q \in \mathbb{Z}^+$,

$$\sup_{z,w \in Z_x} e^{qd(z,w)}|K(z|w)| < \infty, \tag{6.34}$$

along with the analogous property for the covariant derivatives of K. With the natural definition of $\mathrm{Hom}^{\infty}_{C^{\infty}_{\omega}(G)}\big(C^{\infty}_{\omega}(P), \Omega^{*}_{\omega}(G) \otimes_{C^{\infty}_{\omega}(G)} C^{\infty}_{\omega}(P)\big)$, an element K has a kernel as in (6.25). The formula (6.21) extends to a trace $\mathrm{Tr}_{\langle e\rangle} : \mathrm{End}^{\infty}_{C^{\infty}_{\omega}(G)} \to \frac{C^{\infty}_{\omega}(G)}{[C^{\infty}_{\omega}(G), C^{\infty}_{\omega}(G)]}$. The formula (6.28) extends to a trace

$$\mathrm{Tr}_{\langle e\rangle} : \mathrm{Hom}^{\infty}_{C^{\infty}_{\omega}(G)}\big(C^{\infty}_{\omega}(P), \Omega^{*}_{\omega}(G) \otimes_{C^{\infty}_{\omega}(G)} C^{\infty}_{\omega}(P)\big) \to \Omega^{*}_{\omega}(G)_{\mathrm{ab}}. \tag{6.35}$$

If E is a $\mathbb{Z}_2$-graded G-invariant Hermitian vector bundle on P, with an invariant Hermitian connection, then we can define $C^{\infty}_{\omega}(P;E)$ and a supertrace

$$\mathrm{Tr}_{s,\langle e\rangle} : \mathrm{Hom}^{\infty}_{C^{\infty}_{\omega}(G)}\big(C^{\infty}_{\omega}(P;E), \Omega^{*}_{\omega}(G) \otimes_{C^{\infty}_{\omega}(G)} C^{\infty}_{\omega}(P;E)\big) \to \Omega^{*}_{\omega}(G)_{\mathrm{ab}}. \tag{6.36}$$

We now choose a G-invariant vertical Riemannian metric g^{TZ} on the submersion $\pi : P \to G^{(0)}$ and a G-invariant horizontal distribution $T^{H}P$. Suppose that Z is even-dimensional. Let $\hat{E}$ be a Γ-invariant Clifford bundle on P which is equipped with a G-invariant connection. For simplicity of notation, we asssume that $\hat{E} = S^{Z} \hat{\otimes} \hat{V}$, where S^{Z} is a vertical spinor bundle and $\hat{V}$ is an auxiliary vector bundle on P. More precisely, suppose that the vertical tangent bundle TZ has a spin structure. Let S^{Z} be the vertical spinor bundle, a G-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on P. Let $\hat{V}$ be another G-invariant $\mathbb{Z}_2$-graded Hermitian vector bundle on P which is equipped with a G-invariant Hermitian connection. Then we put $\hat{E} = S^{Z} \hat{\otimes} \hat{V}$. The case of general G-invariant Clifford bundles $\hat{E}$ can be treated in a way completely analogous to what follows.

Let Q denote the vertical Dirac-type operator acting on $C^{\infty}_{c}(P;\hat{E})$. From finite-propagation-speed estimates as in [20], Pf. of Prop. 8, along with the bounded geometry of $\{Z_x\}_{x \in G^{(0)}}$, for any $s > 0$ we have

$$e^{-s^2 Q^2} \in \mathrm{End}^{\infty}_{C^{\infty}_{\omega}(G)}\big(C^{\infty}_{\omega}(P;\hat{E})\big). \tag{6.37}$$

Let

$$A^{\mathrm{Bismut}}_{s} : C^{\infty}_{c}(P;\hat{E}) \to \Omega^{*}_{c}(G^{(0)}) \otimes_{C^{\infty}_{c}(G^{(0)})} C^{\infty}_{c}(P;\hat{E}) \tag{6.38}$$

denote the Bismut superconnection on the submersion $\pi : P \to G^{(0)}$ ([2], Proposition 10.15). It is of the form

$$A^{\mathrm{Bismut}}_{s} = sQ + \nabla^{u} - \frac{1}{4s} c(T), \tag{6.39}$$

where ∇^{u} is a certain Hermitian connection and $c(T)$ is Clifford multiplication by the curvature 2-form T of the horizontal distribution $T^{H}P$. We also denote by

$$A^{\mathrm{Bismut}}_{s} : C^{\infty}_{\omega}(P;\hat{E}) \to \Omega^{*}_{\omega}(G) \otimes_{C^{\infty}_{\omega}(G)} C^{\infty}_{\omega}(P;\hat{E}) \tag{6.40}$$

its extension to $C^{\infty}_{\omega}(P;\hat{E})$. One can use finite-propagation-speed estimates, along with the bounded geometry of $\{Z_x\}_{x \in G^{(0)}}$ and the Duhamel expansion as in [2], Theorem 9.48, to show that we obtain a well-defined element $e^{-(A^{\mathrm{Bismut}}_{s})^{2}}$.

We now couple A_s^{Bismut} to the connection ∇^{can} in order to obtain a superconnection

$$(6.41) \qquad A_s : C_\omega^\infty(P; \hat{E}) \to \Omega_\omega^*(G) \otimes_{C_\omega^\infty(G)} C_\omega^\infty(P; \hat{E}).$$

Let $\mathscr{R}$ be the rescaling operator on $\Omega_\omega^{\mathrm{even}}(G)_{\mathrm{ab}}$ which multiplies an element of $\Omega_\omega^{2k}(G)_{\mathrm{ab}}$ by $(2\pi i)^{-k}$. Doing a Duhamel expansion around $e^{-(A_s^{\mathrm{Bismut}})^2}$ and using the fact that h has compact support, we can define

$$(6.42) \qquad e^{-A_s^2} \in \mathrm{Hom}^\infty_{C_\omega^\infty(G)}\big(C_\omega^\infty(P; \hat{E}), \Omega_\omega^*(G) \otimes_{C_\omega^\infty(G)} C_\omega^\infty(P; \hat{E})\big)$$

and hence also define $\mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2}) \in \Omega_\omega^*(G)_{\mathrm{ab}}$. From the superconnection formalism ([2], Chapter 1.4), $\mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2})$ is closed and its cohomology class is independent of $s > 0$; see [14], Theorem 3.1 for a detailed proof in the analogous case of $\mathscr{R}\,\mathrm{Tr}_s(e^{-(A_s^{\mathrm{Bismut}})^2})$.

The proof of the next theorem is analogous to that of Theorem 2.

Theorem 5.

$$(6.43) \qquad \lim_{s\to 0} \mathscr{R}\,\mathrm{Tr}_{s,\langle e\rangle}(e^{-A_s^2}) = \int_Z \phi(z)\hat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\hat{V}})\,\mathrm{ch}(\nabla^{\mathrm{can}}) \in \Omega_\omega^*(G)_{\mathrm{ab}}.$$

Let us note that the right-hand-side of (6.43) pairs with closed graded traces on $\Omega_c^*(G)$, and not just closed graded traces on $\Omega_\omega^*(G)$. Let η be a closed graded trace on $\Omega_c^*(G)$.

Let EG denote the bar construction of a universal space on which G acts freely. That is, EG is the geometric realization of a simplicial manifold given by $E_nG = G^{(n+1)}$, with face maps

$$(6.44) \qquad d_i(\gamma_0, \dots, \gamma_n) = \begin{cases} (\gamma_1, \dots, \gamma_n) & \text{if } i = 0, \\ (\gamma_0, \dots, \gamma_{i-1}\gamma_i, \dots, \gamma_n) & \text{if } 1 \leqq i \leqq n \end{cases}$$

and degeneracy maps

$$(6.45) \qquad s_i(\gamma_0, \dots, \gamma_n) = (\gamma_0, \dots, \gamma_i, 1, \gamma_{i+1}, \dots \gamma_n), \quad 0 \leqq i \leqq n.$$

Here 1 denotes a unit. The action of G on EG is induced from the action on E_nG given by $(\gamma_0, \dots, \gamma_n)\gamma = (\gamma_0, \dots, \gamma_n\gamma)$. Let BG be the quotient space. Let $\pi : EG \to G^{(0)}$ be the map induced from the maps $E_nG \to G^{(0)}$ given by $(\gamma_0, \dots, \gamma_n) \to s(\gamma_n)$. Let $J \in C(EG)$ be the "barycentric coordinate" corresponding to the units $G^{(0)} \subset E_0G$. That is, for each $x \in G^{(0)}$, $\pi^{-1}(x)$ is a simplicial complex and $J|_{\pi^{-1}(x)}$ is the function on $\pi^{-1}(x)$ defined as in [20], (94), with respect to the vertex x instead of the vertex e. Then for all $p \in EG$, $\sum_{\gamma \in G^{\pi(p)}} J(p\gamma) = 1$. Let ∇^{univ} be the connection constructed as in (6.14), using J in place of h. Then pairing $\mathrm{ch}(\nabla^{\mathrm{univ}})$ with η, we construct an element $\Phi_\eta \in \mathrm{H}_\tau^*(BG)$.

Put $M = P/G$, a compact manifold. It inherits a foliation $\mathscr{F}$ from the submersion $\pi : P \to G^{(0)}$. Let $v : M \to BG$ be the classifying map for the G-action on $\hat{M}$. Put $V = \hat{V}/G$, a vector bundle on M. By naturality,

$$(6.46) \qquad \left\langle \int_Z \phi(z)\hat{A}(\nabla^{TZ})\,\mathrm{ch}(\nabla^{\hat{V}})\,\mathrm{ch}(\nabla^{\mathrm{can}}), \eta \right\rangle = \int_M \hat{A}(T\mathscr{F})\,\mathrm{ch}(V) v^*\Phi_\eta.$$

As in the proof of Theorem 4, we obtain

Theorem 6. *Let η be a closed graded trace on $\Omega^*_\omega(G)$. Then*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \tilde{\eta} \rangle = \int_M \hat{A}(T\mathscr{F})\, \mathrm{ch}(V) v^* \Phi_\eta. \tag{6.47}$$

Remark. Theorem 6 also follows from [8], Section III.7.γ, Theorem 12.

Corollary 4. *Let M^n be a compact manifold with a codimension-q foliation $\mathscr{F}$. Let V be a vector bundle on M and let D be a leafwise Dirac-type operator coupled to V. Let $\mathrm{H}^*(\mathrm{Tr}\,\mathscr{F})$ denote the Haefliger cohomology of $(M, \mathscr{F})$ ([13]). Recall that there is a linear map $\int_{\mathscr{F}} : \mathrm{H}^*(M) \to \mathrm{H}^{*-n+q}(\mathrm{Tr}\,\mathscr{F})$. Let η be a holonomy-invariant closed transverse current. Then*

$$\langle \mathrm{ch}(\mathrm{Ind}(D)), \tilde{\eta} \rangle = \left\langle \int_{\mathscr{F}} \hat{A}(T\mathscr{F})\, \mathrm{ch}(V), \eta \right\rangle. \tag{6.48}$$

Proof. Let $\mathscr{H}$ be the holonomy groupoid of $\mathscr{F}$, with source and range maps $s_{\mathscr{H}}, r_{\mathscr{H}} : \mathscr{H} \to M$ [8], Section II.8.α. Let T be a complete transversal for $\mathscr{F}$. That is, T is a q-dimensional submanifold of M, not necessarily connected, which is transverse to $\mathscr{F}$ and has the property that every leaf of $(M, \mathscr{F})$ intersects T. Put $G = r_{\mathscr{H}}^{-1}(T) \cap s_{\mathscr{H}}^{-1}(T)$, the reduced holonomy groupoid. That is, an element of G is an equivalence class of smooth leafwise paths in M from T to T, where two paths are equivalent if they have the same endpoints and the same holonomy. The units are $G^{(0)} = T$.

Put $P = s_{\mathscr{H}}^{-1}(T)$. Define $\pi : P \to G^{(0)}$ to be the restriction of $s_{\mathscr{H}}$ to P. Then for $x \in T$, $\pi^{-1}(x)$ is the holonomy cover of the leaf through x, which we give the induced Riemannian metric. One can see that G acts freely, properly and cocompactly on P.

Put $L = P$, $\sigma = \pi$ and let $i : G \to L$ be the inclusion from $r_{\mathscr{H}}^{-1}(T) \cap s_{\mathscr{H}}^{-1}(T)$ to $s_{\mathscr{H}}^{-1}(T)$. It is easy to check that (L, σ, i) satisfies the requirements to define $\Omega^*_\omega(G)$. Then η defines a closed graded trace on $\Omega^*_\omega(G)$. The right-hand-side of (6.47) becomes the right-hand-side of (6.48). $\square$

Remark. In order to prove Corollary 4, we do not have to assume that the holonomy groupoid is Hausdorff. This is because the pairing with the transverse current η amounts to an integration over $G^{(0)} = T$. Because of this we are effectively dealing with forms of the type $\Omega^{*,0}_\omega(G)$, and so the Hausdorffness of G does not play a role.

Remark. To see the relationship between Corollary 4 and Connes' index theorem for a foliation with a holonomy-invariant transverse measure μ ([8], Section I.5.γ, Theorem 7), let $RS \in \mathrm{H}_{n-q}(M; \mathbb{R})$ denote the Ruelle-Sullivan current associated to μ ([8], Section I.5.β). Then $\left\langle \int_{\mathscr{F}} \hat{A}(T\mathscr{F})\, \mathrm{ch}(V), \mu \right\rangle = \langle \hat{A}(T\mathscr{F})\, \mathrm{ch}(V), RS \rangle$.

Remark. In some cases of foliations, a heat equation proof of Corollary 4, using the Bismut superconnection, was given in [15].

Corollary 5. *Let $\mathscr{A}$ be a subalgebra of the reduced groupoid C^*-algebra $C^*_r(G)$ which is stable under the holomorphic functional calculus in $C^*_r(G)$ and which contains*

$C^\infty_\omega(G)$. Let η be a closed graded trace on $\Omega^*_c(G)$ which extends to give a cyclic cocycle on $\mathscr{A}$. Suppose that TZ is spin and that g^{TZ} has fiberwise positive scalar curvature. Then $\int_M \hat{A}(T\mathscr{F})\nu^*\Phi_\eta = 0$.

Suppose that $\dim(G^{(0)}) = 1$, with G acting on $G^{(0)}$ so as to preserve orientation. Let $v \in \Omega^1(G^{(0)})$ be a volume form. With a hopefully-clear notation, define a closed graded trace on $\Omega^*_c(G)$, concentrated on $\Omega^{0,2}_c(G)$, by

$$\eta(\omega) = \int_{\gamma_0\gamma_1\gamma_2 \in G^{(0)}} \omega(\gamma_0,\gamma_1,\gamma_2)\left(\ln\frac{v}{\gamma_0\gamma_1\cdot v}\, d\ln\frac{\gamma_0\gamma_1\cdot v}{\gamma_0\cdot v} - \ln\frac{\gamma_0\gamma_1\cdot v}{\gamma_0\cdot v}\, d\ln\frac{v}{\gamma_0\gamma_1\cdot v}\right). \tag{6.49}$$

Then $\nu^*\Phi_\eta$ is proportionate to the Godbillon-Vey class $GV \in \mathrm{H}^3(M;\mathbb{R})$ ([8], Chapter III.6.γ, Theorem 17). Furthermore, the hypotheses of Corollary 5 are satisfied ([8], Chapter III.7.β).

Corollary 6. *Suppose that* $\dim(G^{(0)}) = 1$, *G acts on $G^{(0)}$ so as to preserve orientation, TZ is spin and g^{TZ} has fiberwise positive scalar curvature. Then $\int_M \hat{A}(T\mathscr{F})GV = 0$.*

In general, if $\dim(G^{(0)}) = q$ and the action of G on $G^{(0)}$ is orientation-preserving then one can write down a closed graded trace η on $\Omega^*_c(G)$ so that $\nu^*\Phi_\eta$ is proportionate to the Godbillon-Vey class $GV \in \mathrm{H}^{2q+1}(M;\mathbb{R})$, and the above results extend.

Corollary 7. *Let M be a compact manifold with a codimension-q foliation $\mathscr{F}$. We assume that the foliation is tranversally orientable and that $T\mathscr{F}$ is spin. We also assume that the holonomy groupoid of the foliation is Hausdorff. Let $g^{T\mathscr{F}}$ be a leafwise metric on $(M,\mathscr{F})$. If $g^{T\mathscr{F}}$ has positive scalar curvature on the leaves then $\int_M \hat{A}(T\mathscr{F})GV = 0$.*

Proof. Let T be a complete transversal for $\mathscr{F}$. Let G be the reduced holonomy groupoid. Then the corollary is an application of Theorem 6. □

Remark. Corollary 7 also follows from [8], Section III.7.β, Corollary 10.

6.1. Translation. In this subsection we show how the results of Section 6 specialize to those of Section 3, in the case when the groupoid comes from the action of Γ on B. We use the notation of Section 3.

We put $G = B \times \Gamma$, with $G^{(0)} \cong B$, $r(b,\gamma) = b$ and $s(b,\gamma) = b\gamma$. A form

$$\sum_{g_0,\ldots,g_n} \eta_{g_0,\ldots,g_n} g_0\, dg_1 \ldots dg_n \in \Omega^*(B, \mathbb{C}\Gamma) \tag{6.50}$$

gets translated to the form $\omega \in \Omega^*(G)$ given by

$$\omega\big((b_0,g_0),\ldots,(b_n,g_n)\big) = \eta_{g_0,\ldots,g_n}(b_0). \tag{6.51}$$

Then the product (6.3) is equivalent to the calculation

$$(6.52)\quad \left(\left(\sum_{g_0,\dots,g_n} \eta_{g_0,\dots,g_n} g_0\, dg_1 \dots dg_n\right) \cdot \left(\sum_{g'_0,\dots,g'_{n'}} \eta'_{g'_0,\dots,g'_{n'}} g'_0\, dg'_1 \dots dg'_{n'}\right)\right)(b)$$
$$= \sum_{g_0,\dots,g_n,g'_0,\dots,g'_{n'}} \eta_{g_0,\dots,g_n}(b)\eta'_{g'_0,\dots,g'_n}(bg'_0 \dots g'_{n'}) g_0\, dg_1 \dots dg_n\, g'_0\, dg'_1 \dots dg'_{n'}$$
$$= \sum_{g_0,\dots,g_n,g'_0,\dots,g'_{n'}} \eta_{g_0,\dots,g_n}(b)\eta'_{g'_0,\dots,g'_n}(bg'_0 \dots g'_{n'})$$
$$\cdot [g_0\, dg_1 \dots d(g_n g'_0)\, dg'_1 \dots dg'_{n'}$$
$$+ \dots + (-1)^n g_0 g_1\, dg_2 \dots dg_n\, dg'_0 \dots dg'_{n'}].$$

The differential d, given by

$$(6.53)\quad d\left(\sum_{g_0,\dots,g_n} \eta_{g_0,\dots,g_n} g_0\, dg_1 \dots dg_n\right) = \sum_{g_0,\dots,g_n} (d^{\text{de Rham}} \eta_{g_0,\dots,g_n}) g_0\, dg_1 \dots dg_n$$
$$+ (-1)^{|\eta|} \sum_{g_0,\dots,g_n} \eta_{g_0,\dots,g_n} 1\, dg_0\, dg_1 \dots dg_n$$

becomes the sum of d_1 and the differential d_2 of (6.4).

Take $P = \hat{M}$. Then $\mathscr{G} = \hat{M} \times \Gamma$. The product (6.6) becomes

$$(6.54)\quad \left(\left(\sum_g f_g g\right) F\right)(p) = \sum_g f_g(p) F(pg).$$

We illustrate the right-hand-side of (6.54) by the diagram $p \overset{g}{\leftarrow} pg$. Equation (6.8) is the translation of

$$(6.55)\quad \left(\sum_{g_0,g_1} f_{g_0,g_1} g_0\, dg_1\right) F = \left(\sum_{g_0,g_1} g_0\, dg_1 (g_0 g_1)^{-1} \cdot f_{g_0,g_1}\right) F$$
$$= \left(\sum_{g_0,g_1} [d(g_0 g_1) - dg_0\, g_1](g_0 g_1)^{-1} \cdot f_{g_0,g_1}\right) \cdot F$$
$$= \sum_{g_0,g_1} d(g_0 g_1)\big((g_0 g_1)^{-1} \cdot f_{g_0,g_1}\big) F$$
$$- \sum_{g_0,g_1} dg_0 (g_0^{-1} \cdot f_{g_0,g_1})(g_1 \cdot F)$$
$$= \sum_{g_0 g_1 = g} dg(g^{-1} \cdot f_{g_0,g_1}) F - \sum_{g,g'} dg(g^{-1} \cdot f_{g,g'})(g' \cdot F),$$

or

$$(6.56)\quad \Big(\Big(\sum_{g_0,g_1} f_{g_0,g_1} g_0\, dg_1\Big)F\Big)(p) = \sum_{g_0 g_1 = g} dg f_{g_0,g_1}(pg^{-1})F(p) - \sum_{g,g'} dg f_{g,g'}(pg^{-1})F(pg').$$

We illustrate the right-hand-side of (6.56) by the diagrams $pg^{-1} \overset{g_0}{\leftarrow} pg_1^{-1} \overset{g_1}{\leftarrow} p$ and $pg^{-1} \overset{g}{\leftarrow} p \overset{g'}{\leftarrow} pg'$. Equation (6.11) is the translation of

$$\begin{aligned}(6.57)\quad \Big(\sum_g f_g g\Big)\Big(\sum_{g'} dg' \mathscr{F}_{g'}\Big) &= \sum_{g,g'} f_g g\, dg' \mathscr{F}_{g'} \\ &= \sum_{g,g'} g\, dg' \big((gg')^{-1} \cdot f_g\big)\mathscr{F}_{g'} \\ &= \sum_{g,g'} [d(gg') - dg\, g'] \big((gg')^{-1} \cdot f_g\big)\mathscr{F}_{g'} \\ &= \sum_{g,g'} d(gg')\big((gg')^{-1} \cdot f_g\big)\mathscr{F}_{g'} - \sum_{g,g'} dg(g^{-1} \cdot f_g)(g' \cdot \mathscr{F}_{g'})\end{aligned}$$

or

$$\begin{aligned}(6.58)\quad \Big(\Big(\sum_g f_g g\Big)\Big(\sum_{g'} dg' \mathscr{F}_{g'}\Big)\Big)(p) &= \sum_{gg' = g_0} dg_0\, f_g(pg_0^{-1})\mathscr{F}_{g'}(p) \\ &\quad - \sum_g dg \cdot f_g(pg^{-1}) \sum_{g'} \mathscr{F}_{g'}(pg').\end{aligned}$$

We illustrate the right-hand-side of (6.58) by the diagrams $pg_0^{-1} \overset{g}{\leftarrow} p(g')^{-1} \overset{g'}{\leftarrow} p$ and $pg^{-1} \overset{g}{\leftarrow} p \overset{g'}{\leftarrow} pg'$. Equation (6.12) is the translation of

$$\begin{aligned}(6.59)\quad (\eta_{g_0,\dots,g_n} g_0\, dg_1 \dots dg_n)F &= [d(g_0 g_1) \dots dg_n - dg_0\, d(g_1 g_2) \dots dg_n \\ &\quad + \dots + (-1)^{n-1} dg_0 \dots d(g_{n-1} g_n) + (-1)^n dg_0 \dots dg_{n-1} g_n] \\ &\quad \big((g_0 \dots g_n)^{-1} \cdot \eta_{g_0,\dots,g_n}\big)F \\ &= [d(g_0 g_1) \dots dg_n - dg_0\, d(g_1 g_2) \dots dg_n \\ &\quad + \dots + (-1)^{n-1} dg_0 \dots d(g_{n-1} g_n)]\big((g_0 \dots g_n)^{-1} \cdot \eta_{g_0,\dots,g_n}\big)F \\ &\quad + (-1)^n dg_0 \dots dg_{n-1}\big((g_0 \dots g_{n-1})^{-1} \cdot \eta_{g_0,\dots,g_n}\big)(g_n \cdot F).\end{aligned}$$

Equation (6.15) is the translation of

$$(6.60)\qquad (\nabla^{(0,1)} F)(p) = \Big(\sum_g dg\, h(g^{-1} \cdot F)\Big)(p) = \sum_g dg\, h(p)F(pg^{-1}).$$

We illustrate the right-hand-side of (6.60) by the diagram $pg^{-1} \overset{g}{\leftarrow} p$. Equations (6.16) and (6.17) are the translations of (2.39). Equation (6.21) is the translation of (3.12). Equation (6.28) is the translation of

$$\begin{aligned}(6.61)\quad & \mathrm{Tr}_{\langle e\rangle}(K)(b)\\ &= \sum_{g_0,g_1,\ldots,g_l\in\Gamma:\ g_0\cdots g_l=e}(dg_1\ldots dg_l)g_0\left(\int_{Z_b}\phi(w)K_{g_1,\ldots,g_l}(wg_0^{-1},w)\,d\mathrm{vol}_{Z_b}(w)\right)\\ &= \sum_{g_0,g_1,\ldots,g_l\in\Gamma:\ g_0\cdots g_l\in\Gamma_b}[dg_1\ldots d(g_lg_0)-dg_1\ldots d(g_{l-1}g_l)\,dg_0\\ &\quad+\cdots+(-1)^{l-1}d(g_1g_2)\ldots dg_l\,dg_0+(-1)^l g_1\,dg_2\ldots dg_0]\\ &\quad\left(\int_{Z_b}\phi(w)K_{g_1,\ldots,g_l}(wg_0^{-1},w)\,d\mathrm{vol}_{Z_b}(w)\right).\end{aligned}$$

Choose a finite generating set for Γ. Let $\mathscr{C}$ be the corresponding Cayley graph, on which Γ acts on the right by isometries. If B is compact, put $L = B \times \mathscr{C}$. Let $i : G \to L$ be the natural inclusion $B \times \Gamma \to B \times \mathscr{C}$. (In this case, the requirements on L and i are satisfied because Γ is finitely-generated.) Then $C^\infty_\omega(G) = C^\infty(B, \mathscr{B}^\omega)$, $\Omega^*_\omega(G) = \Omega^*(B, \mathscr{B}^\omega)$ and $C^\infty_\omega(P) = C^\infty_{\mathscr{B}^\omega}(\hat{M})$.

We have $EG = E\Gamma \times B$. If $p_0 : E\Gamma \times B \to E\Gamma$ is the projection map then $J = p_0^*(j)$.

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