Homework 2, Math 277, due Monday, March 2.

The *strong minimum principle* is an addendum to the weak minimum principle. The setup is that a smooth function $u(x,t)$ satisfies

$$\frac{\partial u}{\partial t} \geq \triangle_{g(t)} u, \tag{0.1}$$

for x in a connected manifold M and $t \in (0,T)$. (Here $(M, g(t))$ need not be complete.) The strong minimum principle says that if there is some point (x_0, t_0) where u attains its minimum in $M \times (0,T)$, i.e. $u(x_0, t_0) = \inf_{(x,t) \in M \times (0,T)} u(x,t)$, then $u(x,t)$ is constant in x and t.

Recall that the weak minimum principle said that if M is compact then $\min_{x \in M} u(x,t)$ is nondecreasing in t. The strong minimum principle says what happens if $\min_{x \in M} u(x,t)$ is not increasing.

1.a. Using an evolution equation that we've already seen, apply the strong minimum principle to show that a steady Ricci soliton on a compact manifold is Ricci-flat.
b. Where your argument does your argument fail for the cigar soliton?

2.a. Show that in three dimensions, the Ricci tensor at p along with the metric at p determines the entire curvature tensor at p. (Hint : choose an appropriate orthonormal basis at a point p.)
b. Conclude that in three dimensions, an Einstein metric has constant sectional curvature.
c. Show that this conclusion fails in four dimensions.

3. Given a Riemannian manifold with a smooth positive measure $dm = e^{-f}$ dvol, Perelman defines the modified scalar curvature by

$$R^m = R + 2\triangle f - |\nabla f|^2.$$

Show that if M is compact then the $\mathcal{F}$-function is the total scalar curvature associated to R^m, i.e. $\mathcal{F} = \int_M R^m \, dm$.