ON THE SPECTRUM OF A FINITE-VOLUME NEGATIVELY-CURVED MANIFOLD

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Abstract. We show that a noncompact manifold with bounded sectional curvature, whose ends are sufficiently Gromov-Hausdorff close to rays, has a finite dimensional space of square-integrable harmonic forms. In the special case of a finite-volume manifold with pinched negative sectional curvature, we show that the essential spectrum of the p-form Laplacian is the union of the essential spectra of a collection of ordinary differential operators associated to the ends. We give examples of such manifolds with curvature pinched arbitrarily close to -1 and with an infinite number of gaps in the spectrum of the function Laplacian.

1. Introduction. In this paper we consider Riemannian manifolds of finite volume and pinched negative sectional curvature. We give results about the kernel of the differential form Laplacian and about its essential spectrum.

Our first result is the finite dimensionality of the space of square-integrable harmonic forms for a more general class of Riemannian manifolds, which can be roughly characterized as those with bounded sectional curvature and with ends that are sufficiently Gromov-Hausdorff close to rays. Let M be a complete connected n-dimensional Riemannian manifold with a basepoint m. Let $B_r(m)$ denote the distance ball around m and let $S_r(m) = \partial B_r(m)$ be the distance sphere around m. Put

$$\mathcal{D}_r(m) = \sup_{\Sigma_r} \operatorname{diam}(\Sigma_r), \tag{1.1}$$

where Σ_r ranges over the connected components of $S_r(m)$ which intersect a ray through m.

For $p \in \mathbb{Z} \cap [0, n]$, let $\triangle_p^M$ be the p-form Laplacian on M. A harmonic p-form on M is an element of $\operatorname{Ker}(\triangle_p^M)$. Let $\mathcal{H}_{(2)}^p(M)$ denote the vector space of square-integrable harmonic p-forms on M.

Theorem 1. *There is a number $\delta = \delta(n) > 0$ with the property that if for some $b > 0$ the sectional curvatures of M are all bounded in absolute value by b^2, and*

$$\limsup_{r\to\infty} \mathcal{D}_r(m) \leq \delta b^{-1}, \tag{1.2}$$

then for all $p \in \mathbb{Z} \cap [0, n]$ the dimension of $\mathcal{H}_{(2)}^p(M)$ is finite.

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COROLLARY 1. *Let M be a complete connected n-dimensional Riemannian manifold of finite volume whose sectional curvatures satisfy $-b^2 \leq K \leq -a^2$, with $0 < a \leq b$. Then for all $p \in \mathbb{Z} \cap [0, n]$, the dimension of $\mathcal{H}^p_{(2)}(M)$ is finite.*

Corollary 1 was previously known to be true when $p \notin \{\frac{n-1}{2}, \frac{n}{2}, \frac{n+1}{2}\}$ and $\frac{b}{a} < \frac{n-1}{2\min(p, n-p)}$, and when $p = \frac{n}{2}$ and $\frac{b}{a}$ satisfies a certain inequality for which we refer to [4].

The other results in this paper concern manifolds M as in Corollary 1. Recall that the essential spectrum of $\triangle^M_p$ consists of all numbers in the spectrum of $\triangle^M_p$ other than those which are both isolated in the spectrum and have a finite-dimensional eigenspace.

Label the ends of M by $I \in \{1, \ldots, B\}$. An end of M has a neighborhood U_I whose closure is homeomorphic to $[0, \infty) \times N_I$, with N_I an infranilmanifold and the parameter $s \in [0, \infty)$ being a Busemann function for the end. As will be explained, if U_I lies far enough out the end then the differential forms on each fiber $\{s\} \times N_I$ decompose into a finite-dimensional space $E_I(s)$, consisting of "bounded energy" forms, and its orthogonal complement $E_I(s)^\perp$, consisting of "high energy" forms. The vector spaces $\{E_I(s)\}_{s \in [0,\infty)}$ fit together to form a vector bundle E_I on $[0, \infty)$. Let P_0 be orthogonal projection from $\bigoplus_{I=1}^B \Omega^*(\overline{U_I})$ to $\bigoplus_{I=1}^B \Omega^*([0, \infty); E_I)$. Put $\mathcal{A} = P_0 d^M P_0$. Consider the operator $\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A}$ corresponding to the quadratic form

$$Q(\omega) = \int_0^\infty \left[|\mathcal{A}\omega|^2 + |\mathcal{A}^*\omega|^2 \right] ds, \tag{1.3}$$

where $\omega \in \bigoplus_{I=1}^B \Omega^*([0, \infty); E_I)$ satisfies the boundary condition that its pullback to $\{0\}$ vanishes. Then $\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A}$ is a second-order ordinary differential operator. Let $(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A})_p$ denote its restriction to elements of total degree p.

THEOREM 2. *Suppose that M is as in Corollary 1. Then for all $p \in \mathbb{Z} \cap [0, n]$, the essential spectrum of $\triangle^M_p$ equals the essential spectrum of $(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A})_p$.*

Theorem 2 was previously known in the case when M is a finite-volume rank-1 locally symmetric space [11].

As an example of Theorem 2, we consider the case of the Laplacian on functions. It is well-known that if M is a noncompact finite-volume hyperbolic manifold then the spectrum of its function Laplacian is the union of $\left[\frac{(n-1)^2}{4}, \infty\right)$ with a finite subset of $\left[0, \frac{(n-1)^2}{4}\right)$. In particular, there is a finite number of gaps in the spectrum. We show that for manifolds with sectional curvature pinched close to -1, the situation can be very different.

THEOREM 3. *For any $\epsilon > 0$, there is a complete connected noncompact finite-volume Riemannian manifold whose sectional curvatures lie in $[-1-\epsilon, -1+\epsilon]$ and whose function Laplacian has an infinite number of gaps in its spectrum. The gaps tend toward infinity.*

Theorems 1 and 2 continue to hold if one allows M to be altered within a compact region. The proofs go through without change.

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2. Proof of Theorem 1. The vector space $\mathcal{H}^p_{(2)}(M)$ is isomorphic to the p-dimensional (reduced) L^2-cohomology of M. For background on L^2-cohomology, we refer to [8], [9, Section 2] and references therein.

Suppose that the sectional curvatures of M are all bounded in absolute value by b^2. From [14, Theorem 1], if the number δ is sufficiently small and M satisfies (1.2) then M has finite topological type, i.e., is homeomorphic to the interior of a compact manifold-with-boundary $\overline{M}$. (In fact, for this conclusion it is enough to just have the lower bound on the sectional curvatures [13].) In particular, if $\{N_I\}_{I=1}^B$ are the connected components of $\partial\overline{M}$ then there is a compact set $K \subset M$ such that the closures $\{\overline{U_I}\}_{I=1}^B$ of the connected components of $M - K$ are homeomorphic to $\{[0,\infty) \times N_I\}_{I=1}^B$.

From [9, Proposition 5], the dimension of $\mathcal{H}^p_{(2)}(M)$ is finite if and only if the dimension of $\mathcal{H}^p_{(2)}(\overline{U_I})$ is finite for each I. Here $\mathcal{H}^p_{(2)}(\overline{U_I})$ can be defined either as the p-dimensional (reduced) L^2-cohomology of $\overline{U_I}$ or as the space of square-integrable harmonic p-forms on $\overline{U_I}$ satisfying absolute boundary conditions on $\partial\overline{U_I}$.

From [14, Theorem 2], N_I is diffeomorphic to an infranilmanifold. The proof of [14, Theorem 2] uses the collapsing results of Cheeger, Fukaya and Gromov, as given for example in [1]. In particular, it uses Fukaya's fibration theorem, along with the fact that U_I is Gromov-Hausdorff close to a ray which passes through it. Strictly speaking, as in the proof of [14, Theorem 2], one may have to shrink U_I a bit in order to apply the fibration theorem.

In fact, [1] describes a model metric which is uniformly C^0-close to that of $\overline{U_I}$. However, reduced L^2-cohomology is bi-Lipschitz invariant (see, for example, [9, Proposition 1]). Hence it suffices to compute the (reduced) L^2-cohomology of $\overline{U_I}$ with the model metric. We now describe the model metric.

The infranilmanifold N_I is F_I-covered by a nilmanifold $\Gamma_I \backslash \mathfrak{N}_I$ where $\mathfrak{N}_I$ is a simply-connected connected nilpotent Lie group, Γ_I is a lattice in $\mathfrak{N}_I$ and F_I is a finite group of automorphisms of $\mathfrak{N}_I$ which preserve Γ_I. From [1, Proposition 4.9], the model metric on $\overline{U_I}$ is that of a certain Riemannian submersion from $\overline{U_I}$ to $[0,\infty)$ which is invariant under a local action of $\mathfrak{N}_I$. In particular, the flow of the horizontal vector field for the Riemannian submersion $\overline{U_I} \to [0,\infty)$ preserves the affine structures on the fibers. By integrating the vector field, the model metric can be written in the form

$$g = ds^2 + h(s), \tag{2.1}$$

where for each $s \in [0,\infty)$, $h(s)$ is a smooth metric on N_I which comes from an F_I-invariant left-invariant metric on $\mathfrak{N}_I$. Furthermore, if $S(s)$ denotes the second fundamental form of $\{s\} \times N_I$ then we can assume that $\{S(s)\}_{s\in[0,\infty)}$ are uniformly bounded with respect to $\{h(s)\}_{s\in[0,\infty)}$. In what follows we will allow ourselves to reduce the end by making finite shifts of the interval $[0,\infty)$, without change of notation.

There is a canonical flat connection ∇^{aff} on TN_I coming from the flat connection on $T\mathfrak{N}_I$ for which left-invariant vector fields are parallel. Let $\mathcal{E}_I$ be the finite-dimensional vector space of differential forms on N_I which are parallel with respect to ∇^{aff}. Let P: $\Omega^*(N_I) \to \Omega^*(N_I)$ be an orthogonal projection onto $\mathcal{E}_I$, using $h(s)$. From [10, Proposition 1], P is actually independent of s and arises from an averaging procedure over the group $\mathfrak{N}_I$. Let $\widehat{d}$ denote the exterior derivative on $\Omega^*(N_I)$, let $\widehat{d}^*$ denote its adjoint with respect to $h(s)$ and put

$$\widehat{\triangle} = \widehat{d}\widehat{d}^* + \widehat{d}^*\widehat{d}. \tag{2.2}$$

The operators $\widehat{d}$, $\widehat{d}^*$ and $\widehat{\triangle}$ are diagonal with respect to the decomposition

$$\Omega^*(N_I) = \mathcal{E}_I \oplus \mathcal{E}_I^{\perp}. \tag{2.3}$$

We extend $\widehat{d}$, $\widehat{d}^*$ and $\widehat{\triangle}$ to act on $\Omega^*(N_I) \bigoplus (ds \wedge \Omega^*(N_I))$, separately in each factor.

Let $\{x^i\}$ be local coordinates on N_I. Let E^i denote exterior multiplication by dx^i and let I_i denote interior multiplication by ∂_{x^i}.

LEMMA 1. *One has*

$$\partial_s \widehat{d}^* = \left[\widehat{d}^*, V\right], \tag{2.4}$$

where

$$V = 2\sum_{i,j} S^j_i E^i I_j - \sum_i S^i_i. \tag{2.5}$$

Proof. With our conventions, $\partial_s h = -2S$. Given $\omega, \eta \in \Omega^*(N_I)$, let $\langle \omega, \eta\rangle \in C^\infty(N_I)$ be the inner product constructed using $h(s)$. One can check that $\partial_s \langle \omega, \eta\rangle = \langle X\omega, \eta\rangle$, where $X = 2\sum_{i,j} S^j_i E^i I_j$. In addition, the derivative of the volume form is given by $\partial_s d\mathrm{vol} = Y d\mathrm{vol}$, where $Y = -\sum_i S^i_i$. Differentiating the equation

$$\int_{N_I} \langle \widehat{d}^* \omega, \eta\rangle d\mathrm{vol} = \int_{N_I} \langle \omega, \widehat{d}\eta\rangle d\mathrm{vol} \tag{2.6}$$

with respect to s gives

$$\int_{N_I} \langle X\widehat{d}^* \omega, \eta\rangle d\mathrm{vol} + \int_{N_I} \langle \partial_s \widehat{d}^* \omega, \eta\rangle d\mathrm{vol} + \int_{N_I} \langle Y\widehat{d}^* \omega, \eta\rangle d\mathrm{vol} \tag{2.7}$$

$$= \int_{N_I} \langle X\omega, \widehat{d}\eta\rangle d\mathrm{vol} + \int_{N_I} \langle Y\omega, \widehat{d}\eta\rangle d\mathrm{vol}.$$

As ω and η are arbitrary, it follows that

$$X\widehat{d}^* + \partial_s \widehat{d}^* + Y\widehat{d}^* = \widehat{d}^* X + \widehat{d}^* Y, \tag{2.8}$$

or

$$\partial_s \widehat{d}^* = [\widehat{d}^*, X + Y]. \tag{2.9}$$

The lemma follows. □

Here V is also diagonal with respect to the decomposition (2.3).

It follows from Malcev's theorem that the harmonic forms on $(N_I, h(s))$ are parallel with respect to ∇^{aff}. In particular, $\widehat{\triangle}$ is invertible on $\mathcal{E}_I^\perp$. (Here $\mathcal{E}_I^\perp$ is also independent of s.) Let G denote the corresponding Green's operator, which is the inverse of $\widehat{\triangle}$ on $\mathcal{E}_I^\perp$ and which vanishes on $\mathcal{E}_I$.

LEMMA 2. *One has*

$$\partial_s(\widehat{d}^* G) = -\left[\widehat{d}, G\widehat{d}^* V \widehat{d}^* G\right]. \tag{2.10}$$

Proof. Differentiating the equations

$$\widehat{\triangle} G = G\widehat{\triangle} = 1 - P \tag{2.11}$$

and

$$PG = GP = 0 \tag{2.12}$$

with respect to s gives

$$\partial_s G = -G(\partial_s \widehat{\triangle})G. \tag{2.13}$$

From (2.2),

$$\partial_s \widehat{\triangle} = \widehat{d}(\partial_s \widehat{d}^*) + (\partial_s \widehat{d}^*)\widehat{d}. \tag{2.14}$$

Then

$$\begin{aligned} \partial_s(\widehat{d}^* G) &= [\widehat{d}^*, V]G - \widehat{d}^* G \left(\widehat{d}[\widehat{d}^*, V] + [\widehat{d}^*, V]\widehat{d}\right) G \\ &= \widehat{d}^* VG - V\widehat{d}^* G \\ &\quad - \widehat{d}^* G\widehat{d}\widehat{d}^* VG + \widehat{d}^* G\widehat{d} V\widehat{d}^* G - \widehat{d}^* G\widehat{d}^* V\widehat{d} G + \widehat{d}^* GV\widehat{d}^*\widehat{d} G \\ &= \widehat{d}^* \left(I - G\widehat{d}\widehat{d}^*\right) VG - \left(I - \widehat{d}^* G\widehat{d}\right) V\widehat{d}^* G + \widehat{d}^* GV\widehat{d}^*\widehat{d} G \end{aligned} \tag{2.15}$$

$$
\begin{aligned}
&= \widehat{d}^* \left(I - G\widehat{\triangle}\right) VG - \left(I - G\widehat{\triangle} + \widehat{d}G\widehat{d}^*\right) V\widehat{d}^*G + \widehat{d}^*GV\widehat{d}^*\widehat{d}G \\
&= \widehat{d}^*PVG - PV\widehat{d}^*G - \widehat{d}G\widehat{d}^*V\widehat{d}^*G + \widehat{d}^*GV\widehat{d}^*\widehat{d}G \\
&= \widehat{d}^*VPG - VP\widehat{d}^*G - \widehat{d}G\widehat{d}^*V\widehat{d}^*G + G\widehat{d}^*V\widehat{d}^*G\widehat{d} \\
&= -[\widehat{d}, G\widehat{d}^*V\widehat{d}^*G].
\end{aligned}
$$

This proves the lemma. □

Let $e(ds)$ denote exterior multiplication by ds. Define $\mathcal{K}\colon\ \Omega^*(\overline{U_I}) \to \Omega^*(\overline{U_I})$ by

$$\mathcal{K} = \widehat{d}^*G - e(ds)G\widehat{d}^*V\widehat{d}^*G. \tag{2.16}$$

Lemma 3. *Acting on $\Omega^*(\overline{U_I})$, one has*

$$d\mathcal{K} + \mathcal{K}d = 1 - P. \tag{2.17}$$

(In this last equation, P acts fiberwise.)

Proof. Using the fact that

$$d = \widehat{d} + e(ds)\partial_s, \tag{2.18}$$

we have

$$
\begin{aligned}
d\mathcal{K} + \mathcal{K}d &= \left(\widehat{d} + e(ds)\partial_s\right)\left(\widehat{d}^*G - e(ds)G\widehat{d}^*V\widehat{d}^*G\right) \\
&\quad + \left(\widehat{d}^*G - e(ds)G\widehat{d}^*V\widehat{d}^*G\right)\left(\widehat{d} + e(ds)\partial_s\right) \\
&= \widehat{d}\widehat{d}^*G + \widehat{d}^*G\widehat{d} + e(ds)\left([\partial_s, \widehat{d}^*G] + [\widehat{d}, G\widehat{d}^*V\widehat{d}^*G]\right) \\
&= I - P.
\end{aligned} \tag{2.19}
$$

This proves the lemma. □

Consider the trivial vector bundle $\mathcal{W}_I = [0,\infty) \times \mathcal{E}_I$ over $[0,\infty)$. Let $\widehat{d}^{inv}$ be the restriction of $\widehat{d}$ to $\mathcal{E}_I \subset \Omega^*(N_I)$ and consider the flat superconnection A_I on $\mathcal{W}_I$ whose action on $\Omega^*([0,\infty);\mathcal{W}_I)$ is given by

$$A_I = \widehat{d}^{inv} + e(ds)\partial_s. \tag{2.20}$$

That is, A_I is simply the restriction of d from $\Omega^*(\overline{U_I})$ to $\Omega^*([0,\infty);\mathcal{W}_I)$. Then (2.17) gives a homotopy equivalence between the cochain complexes $\left(\Omega^*(\overline{U_I}), d\right)$ and $(\Omega^*([0,\infty);\mathcal{W}_I), A_I)$.

From the Gauss-Codazzi equation and the results of [1], we can assume that there is a uniform upper bound on the absolute values of the sectional curvatures of the fibers $(N_I, h(s))$, of the form const.b^2. Then from [10, Proposition 2], it follows that if δ is small enough then there is a uniform positive lower bound on the eigenvalues of $\widehat{\triangle}|_{\mathcal{E}^\perp}$. Hence $\mathcal{K}$ is a bounded operator. Then it follows as in [9, Lemma 1] that the (reduced) L^2-cohomology of $\overline{U_I}$ is isomorphic to the (reduced) L^2-cohomology of $(\Omega^*([0,\infty); \mathcal{W}_I), A_I)$, where $\Omega^*([0,\infty); \mathcal{W}_I)$ acquires an L^2-inner product from $\Omega^*(\overline{U_I})$. From Hodge theory, the (reduced) L^2-cohomology of $(\Omega^*([0,\infty); \mathcal{W}_I), A_I)$ is isomorphic to the vector space of square-integrable solutions to the equation

$$(A_I A_I^* + A_I^* A_I)\psi = 0 \tag{2.21}$$

on $[0,\infty)$, with absolute boundary conditions at $\{0\}$. However, as $A_I A_I^* + A_I^* A_I$ is a second-order ordinary differential operator, the solution space of (2.21) is finite-dimensional. This proves the theorem.

3. Geometry of finite-volume negatively-curved manifolds. We review some results from [6] and [7]. Let (M, g) be a complete connected Riemannian manifold of finite volume whose sectional curvatures satisfy $-b^2 \leq K \leq -a^2$, with $0 < a \leq b$. Then M is diffeomorphic to the interior of a smooth compact connected manifold-with-boundary $\overline{M}$. The boundary components of $\overline{M}$ are diffeomorphic to infranilmanifolds. If N is such a boundary component then there is a corresponding end E of M. Let s be a Busemann function for a ray exiting E. Then after changing s by a constant if necessary, there are a neighborhood U of E and a C^1-diffeomorphism $\mathcal{F}$: $(0,\infty) \times N \to U$ so that

$$\mathcal{F}^* \left(g|_U\right) = ds^2 + h(s), \tag{3.1}$$

where for $s \in (0,\infty)$, $h(s)$ is a Riemannian metric on N. We will think of s as a coordinate function on U. The slices $N(s) = \{s\} \times N$ are projections of horospheres in the universal cover $\widetilde{M}$. *A priori*, the Busemann function is only C^2-smooth on M and the Riemannian metric $h(s)$ is only C^1-smooth on N. Given $n \in N$, the curve $s \to (s, n)$ is a unit-speed geodesic which intersects the slices orthogonally. All of the rays in M which exit E arise in this way.

As s is C^2-smooth, the second fundamental form $S(s)$ of $N(s)$ exists and is continuous on $N(s)$. From Jacobi field estimates, it satisfies

$$ah(s) \leq S(s) \leq bh(s) \tag{3.2}$$

and the metric $h(s)$ satisfies

$$e^{-2bs}h(0) \leq h(s) \leq e^{-2as}h(0). \tag{3.3}$$

4. Infranilmanifolds. Let N be a boundary component of $\overline{M}$. It has a regular covering by a nilmanifold $\Gamma\backslash\mathfrak{N}$, with covering group F. Here $\mathfrak{N}$ is a simply-connected connected nilpotent Lie group, Γ is a lattice in $\mathfrak{N}$ and F is a finite group of automorphisms of $\mathfrak{N}$ which preserve Γ. Let $\mathfrak{n}$ be the Lie algebra of $\mathfrak{N}$. Let $\Lambda^*(\mathfrak{n}^*)^F$ denote the F-invariant subspace of $\Lambda^*(\mathfrak{n}^*)$. Let $\triangle_{N(s)}$ denote the differential form Laplacian on $N(s)$ (which can be defined using quadratic forms [12, Vol. I, Theorem VIII.15] even if $h(s)$ is only C^1-smooth). Given $\lambda \in [0, \infty)$, let $P_{N(s)}(\lambda)$ denote the spectral projection onto the direct sum of the eigenspaces of $\triangle_{N(s)}$ with eigenvalue less than or equal to λ.

PROPOSITION 1. *There are constants $c_1, c_2 > 0$ such that for all sufficiently large s, the images of $P_{N(s)}(c_1^2 b^2)$ and $P_{N(s)}(c_2^2 a^2 e^{2as})$ are isomorphic to $\Lambda^*(\mathfrak{n}^*)^F$.*

Proof. Suppose first that the parametrization $\mathcal{F}\colon (0, \infty) \times N \to U$ is smooth. From the Gauss-Codazzi equation, the intrinsic sectional curvatures $R^{N(s)}$ of $N(s)$ are bounded in absolute value by a universal constant times b^2. From [1], there is an $\epsilon > 0$ such that for all $s \in [1, \infty)$, there is a metric $h_0(s)$ on $N(s)$, coming from an F-invariant left-invariant inner product on $\mathfrak{N}$, with

$$e^{-\epsilon} h_0(s) \leq h(s) \leq e^{\epsilon} h_0(s). \tag{4.1}$$

By [2], there is an integer $J > 0$ such that the jth eigenvalue $\lambda_{p,j}$ of the p-form Laplacian satisfies

$$e^{-J\epsilon} \lambda_{p,j}(h_0(s)) \leq \lambda_{p,j}(h(s)) \leq e^{J\epsilon} \lambda_{p,j}(h_0(s)). \tag{4.2}$$

Thus without loss of generality, we may assume that $h(s)$ comes from a left-invariant inner product on $\mathfrak{N}$.

The vector space of F-invariant left-invariant differential forms on $\mathfrak{N}$ is isomorphic to $\Lambda^*(\mathfrak{n}^*)^F$. These differential forms push down to comprise a vector space $\mathcal{V}$ of differential forms on $N(s)$. The Laplacian $\triangle_{N(s)}$ has an orthogonal direct sum decomposition

$$\triangle_{N(s)} = \triangle_{\mathcal{V}} \oplus \triangle_{\mathcal{V}^\perp}. \tag{4.3}$$

From [10, Proposition 2], there is a constant $c_2 > 0$ such that for sufficiently large s, the eigenvalues of $\triangle_{\mathcal{V}^\perp}$ are greater than $c_2^2 a^2 e^{2as}$.

It remains to show that there is a constant $c_1 > 0$ such that the eigenvalues of $\triangle_{\mathcal{V}}$ are less than or equal to $c_1^2 b^2$, uniformly in s. We follow the notation of [10, Section 3]. Let $\{e_i\}$ be the orthonormal basis of $\mathfrak{n}$ described in [10, Section 3], with dual basis $\{\tau^i\}$. Let E^i denote exterior multiplication by τ^i and let I_i denote interior multiplication by e_i. The exterior derivative d, acting on $\Omega^*(N(s))$, can be written as $d = \sum_i E^i \nabla^{N(s)}_{e_i}$, and its adjoint can be written as $d^* = -\sum_i I^i \nabla^{N(s)}_{e_i}$. Now $\nabla^{N(s)}_{e_i}$ acts on $\mathcal{V}$ as $\sum_{j,k} \omega^j_{ki} E^j I^k$, where $\{\omega^j_{ki}\}$ are the components of the Levi-Civita connection 1-form ω of the left-invariant metric. Hence $\triangle_{\mathcal{V}}$ is quadratic in ω. From [10, Lemma 3], there is a constant, which only depends on $\dim(N)$, such that

$$\|\omega\|_\infty^2 \leq \text{const.} \|R^{N(s)}\|_\infty. \tag{4.4}$$

The proposition follows, under the assumption that the parametrization $\mathcal{F}\colon (0,\infty)\times N \to U$ is smooth.

In the general case, thinking of $N(s)$ as the graph of a C^2-function on N, for any $\epsilon > 0$ we can find a smooth hypersurface N' of M which is ϵ-close to $N(s)$ in the C^2-topology. Then the proposition holds for N'. Using the continuity of the eigenvalues with respect to the metric, in the C^0-topology, as in (4.2), the proposition follows. In fact, we can take c_1 and c_2 to only depend on $\dim(N)$, although we will not need this. □

Let $\{N_I\}_{I=1}^B$ be the boundary components of $\overline{M}$, with corresponding ends E_I and neighborhoods of the ends U_I. By reducing U_I if necessary, we may assume that Proposition 1 holds for all $s \geq 0$, with $c_1 b < c_2 a$. As in [3, Proposition 2.1], the essential spectrum of $\triangle_p^M$ is invariant under compactly-supported changes of the metric. Thus without loss of generality, we may assume that the metric on $\overline{U_I}$ is a product near $\{0\} \times N_I$, with Proposition 1 still holding for $s \geq 0$. Let Ω_I^* denote the smooth compactly-supported forms on $[0,\infty) \times N_I$ which satisfy relative boundary conditions at $\{0\} \times N_I$. Let H' be the L^2-completion of $\bigoplus_{I=1}^B \Omega_I^*$. The Laplacian $\triangle' = dd^* + d^*d$, defined initially on $\bigoplus_{I=1}^B \Omega_I^*$, is a densely-defined self-adjoint operator on H' and corresponds to relative boundary conditions.

For later use, we write d and d^* more explicitly. Fix I. Let $\{x^i\}_{i=1}^{n-1}$ be local coordinates on N_I and write the metric on U_I as $ds^2 + \sum_{i,j} h_{ij} dx^i dx^j$. We think of $s = x^0$ as another coordinate. Let S_{ij} be the second fundamental form of $\{s\} \times N_I$. We let Greek letters run over $\{0,\ldots,n-1\}$ and we let Roman letters run over $\{1,\ldots,n-1\}$. The nonzero Christoffel symbols are

$$\begin{aligned} \Gamma_{ijk} &= \Gamma_{ijk}(h), \\ \Gamma_{0ij} &= S_{ij}, \\ \Gamma_{ij0} &= -S_{ij}, \\ \Gamma_{i0j} &= -S_{ij}. \end{aligned} \tag{4.5}$$

Let E^α denote exterior multiplication by dx^α and let I_α denote interior multiplication by ∂_{x^α}. Covariant differentiation on forms is given in local coordinates by

$$\nabla_{\partial_{x^\alpha}} = \partial_{x^\alpha} - \sum_{\beta,\gamma} \Gamma^\gamma_{\beta\alpha} E^\beta I_\gamma. \tag{4.6}$$

Let $\widehat{\nabla}$ denote the covariant derivative on $N_I(s)$. Then

$$\begin{aligned} \nabla_{\partial_{x^i}} &= \widehat{\nabla}_{\partial_{x^i}} - \sum_j S_{ij} E^j I_0 + \sum_j S^j_i E^0 I_j, \\ \nabla_{\partial_s} &= \partial_s + \sum_{i,j} S^i_j E^j I_i. \end{aligned} \tag{4.7}$$

Let $\widehat{d} = \sum_i E^i \widehat{\nabla}_{\partial_{x^i}}$ be the exterior derivative on $N_I(s)$, extended to act on $\Omega^*(N_I(s)) \bigoplus (ds \wedge \Omega^*(N_I(s)))$, and let $\widehat{d}^* = -\sum_i I^i \widehat{\nabla}_{\partial_{x^i}}$ be its adjoint. Then

$$d = \widehat{d} + E^0 \partial_s \tag{4.8}$$

and

$$\begin{aligned} d^* &= -\sum_\alpha I^\alpha \nabla_{\partial_{x^\alpha}} \\ &= \widehat{d}^* - I^0 \left(\partial_s + \sum_{i,j} S^i_j (E^j I_i - I_i E^j) \right). \end{aligned} \tag{4.9}$$

5. Boundedness of the off-diagonal operators. Given I, consider N_I to be an infranilmanifold which is F_I-covered by a nilmanifold $\Gamma_I \backslash \mathfrak{N}_I$ and let $\mathfrak{n}_I$ be the Lie algebra of $\mathfrak{N}_I$. Let $E_I = [0, \infty) \times \Lambda^*(\mathfrak{n}_I^*)^{F_I}$ be the trivial vector bundle on $[0, \infty)$ with fiber $\Lambda^*(\mathfrak{n}_I^*)^{F_I}$.

Let $\Omega^*([0,\infty); E_I)$ be the smooth compactly-supported forms on $[0,\infty)$, with value in E_I. Using Proposition 1, there is an embedding of $\Omega^*([0,\infty); E_I)$ into $\Omega^*(\overline{U_I})$. Let $\Omega^*_{rel}([0,\infty); E_I)$ be the subspace of $\Omega^*([0,\infty); E_I)$ consisting of elements which satisfy relative boundary conditions at $\{0\}$. Let H_0 be the completion of $\bigoplus_{I=1}^B \Omega^*_{rel}([0,\infty); E_I)$ in H' and let H_1 be its orthogonal complement. Roughly speaking, the elements of H_0 correspond to fiberwise low-energy forms and the elements of H_1 correspond to fiberwise high-energy forms.

Let $P_0 \colon H' \to H_0$ and $P_1 \colon H' \to H_1$ be the orthogonal projections. With respect to the orthogonal decomposition $H' = H_0 \bigoplus H_1$, write

$$d = \begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix}. \tag{5.1}$$

Then

$$d^* = \begin{pmatrix} \mathcal{A}^* & \mathcal{C}^* \\ \mathcal{B}^* & \mathcal{D}^* \end{pmatrix} \tag{5.2}$$

and

$$\triangle' = \begin{pmatrix} \mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + \mathcal{B}\mathcal{B}^* + \mathcal{C}^*\mathcal{C} & \mathcal{A}\mathcal{C}^* + \mathcal{B}\mathcal{D}^* + \mathcal{A}^*\mathcal{B} + \mathcal{C}^*\mathcal{D} \\ \mathcal{C}\mathcal{A}^* + \mathcal{D}\mathcal{B}^* + \mathcal{B}^*\mathcal{A} + \mathcal{D}^*\mathcal{C} & \mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D} + \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^* \end{pmatrix}. \tag{5.3}$$

PROPOSITION 2. *The operators $\mathcal{B}$: $H_1 \to H_0$ and $\mathcal{C}$: $H_0 \to H_1$ are bounded.*

Proof. We have

$$\mathcal{B} = P_0 d P_1 = \left(P_1 d^* P_0\right)^* \tag{5.4}$$

and

$$\mathcal{C} = P_1 d P_0. \tag{5.5}$$

From (4.8) and (4.9), in order to show that $\mathcal{B}$ and $\mathcal{C}$ are bounded it is enough to show that

$$P_1 \partial_s P_0 = P_1(\partial_s P_0). \tag{5.6}$$

is bounded. Let γ be the circle of radius $c_1 b$ around the origin in $\mathbb{C}$, oriented counterclockwise. From Proposition 1,

$$P_0(s) = \oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1} \frac{d\lambda}{2\pi i}. \tag{5.7}$$

Here $P_0(s)$ is a projection on $\bigoplus_{I=1}^{B} (\Omega^*(N_I(s)) \bigoplus (ds \wedge \Omega^*(N_I(s))))$. We note that the Hilbert space structure on $\bigoplus_{I=1}^{B} (\Omega^*(N_I(s)) \bigoplus (ds \wedge \Omega^*(N_I(s))))$ depends on s, but the underlying topological vector space structure on $\bigoplus_{I=1}^{B} (\Omega^*(N_I) \bigoplus (ds \wedge \Omega^*(N_I)))$ does not. Hence it makes sense to differentiate (5.7) with respect to s, giving

$$\begin{aligned} \partial_s P_0 &= \oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1} \partial_s(\widehat{d} + \widehat{d}^*)(\lambda - \widehat{d} - \widehat{d}^*)^{-1} \frac{d\lambda}{2\pi i} \\ &= \oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1} \partial_s \widehat{d}^* (\lambda - \widehat{d} - \widehat{d}^*)^{-1} \frac{d\lambda}{2\pi i} \\ &= \oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1} [\widehat{d}^*, V](\lambda - \widehat{d} - \widehat{d}^*)^{-1} \frac{d\lambda}{2\pi i}, \end{aligned} \tag{5.8}$$

where V is as in (2.5).

As $P_1(s) = 1 - P_0(s)$, it follows from differentiating $P_0^2(s) = P_0(s)$ that

$$P_1(\partial_s P_0) = (\partial_s P_0) P_0. \tag{5.9}$$

If $\eta_0 \in \mathrm{Im}(P_0)$ is an eigenform for $\widehat{d} + \widehat{d}^*$ with eigenvalue λ_0 and $\eta_1 \in \mathrm{Im}(P_1)$ is an eigenform for $\widehat{d} + \widehat{d}^*$ with eigenvalue λ_1 then

$$\left\langle \eta_1, \oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1} [\widehat{d}^*, V](\lambda - \widehat{d} - \widehat{d}^*)^{-1} \frac{d\lambda}{2\pi i} \eta_0 \right\rangle \tag{5.10}$$

$$= \oint_\gamma \langle \eta_1, (\lambda - \lambda_1)^{-1}[\widehat{d}^*, V](\lambda - \lambda_0)^{-1}\eta_0\rangle \frac{d\lambda}{2\pi i}$$
$$= -\frac{1}{\lambda_1 - \lambda_0}\langle \eta_1, [\widehat{d}^*, V]\eta_0\rangle.$$

It follows that

$$\oint_\gamma (\lambda - \widehat{d} - \widehat{d}^*)^{-1}[\widehat{d}^*, V](\lambda - \widehat{d} - \widehat{d}^*)^{-1}\frac{d\lambda}{2\pi i}\eta_0 = -\left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s))} - \lambda_0\right)^{-1} P_1(s)[\widehat{d}^*, V]\eta_0. \tag{5.11}$$

Using (5.8) and (5.11), in order to prove the proposition it suffices to show that as η_0 runs over unit-length eigenforms of $(\widehat{d} + \widehat{d}^*)\Big|_{Im(P_0(s))}$, one has a bound on the norm of

$$\left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} P_1(s)[\widehat{d}^*, V]\eta_0$$

which is uniform in s. Writing

$$\begin{aligned}\left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} P_1(s)[\widehat{d}^*, V]\eta_0 &= \left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} P_1(s)\widehat{d}^*(V\eta_0) \\ &- \left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} P_1(s)V\widehat{d}^*\eta_0,\end{aligned} \tag{5.12}$$

we know from (2.5) and Proposition 1 that we have bounds on $\left|V\eta_0\right|$ and $\left|V\widehat{d}^*\eta_0\right|$ given by const.b and const.b^2, respectively. Hence it suffices to show that the operators

$$\left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} \widehat{d}^* P_1(s) \tag{5.13}$$

and

$$\left((\widehat{d} + \widehat{d}^*)\Big|_{Im(P_1(s)))} - \lambda_0\right)^{-1} P_1(s) \tag{5.14}$$

have uniform bounds on their operator norms.

Put $\widehat{\triangle} = (\widehat{d} + \widehat{d}^*)^2$. As (5.13) vanishes on $\mathrm{Im}(\widehat{d}^*)$, using the Hodge decomposition it is enough to consider its action on $\mathrm{Im}(\widehat{d})$. Given $\eta \in \mathrm{Im}(P_1(s))$, we have

$$\left((\widehat{d}+\widehat{d}^*)-\lambda_0\right)^{-1}\widehat{d}^*\widehat{d}\eta = \frac{\widehat{d}+\widehat{d}^*+\lambda_0}{\widehat{\triangle}-\lambda_0^2}\widehat{d}^*\widehat{d}\eta = \frac{\widehat{\triangle}}{\widehat{\triangle}-\lambda_0^2}\widehat{d}\eta + \frac{\lambda_0}{\widehat{\triangle}-\lambda_0^2}\widehat{d}^*\widehat{d}\eta. \tag{5.15}$$

By Proposition 1, the operator norm of $\frac{\widehat{\triangle}}{\widehat{\triangle}-\lambda_0^2}$, acting on $\mathrm{Im}(P_1(s))$, is at most $\frac{c_2^2a^2e^{2as}}{c_2^2a^2e^{2as}-c_1^2b^2}$. If $\mathcal{T} = \frac{\lambda_0}{\widehat{\triangle}-\lambda_0^2}\widehat{d}^*$ then $\mathcal{T}^*\mathcal{T} = \frac{\lambda_0^2\widehat{d}\widehat{d}^*}{(\widehat{\triangle}-\lambda_0^2)^2}$ which, acting on $\mathrm{Im}(\widehat{d})$, equals $\frac{\lambda_0^2\widehat{\triangle}}{(\widehat{\triangle}-\lambda_0^2)^2}$. Hence the norm of $\mathcal{T}$, acting on $\mathrm{Im}(\widehat{d})\bigcap\mathrm{Im}(P_1(s))$, is at most $\frac{c_1bc_2ae^{as}}{c_2^2a^2e^{2as}-c_1^2b^2}$.

By Proposition 1, the operator norm of (5.14) is at most $\frac{1}{c_2ae^{as}-c_1b}$. The proposition follows. □

6. High energy forms.

PROPOSITION 3. *The operator* $\mathcal{D}\mathcal{D}^*+\mathcal{D}^*\mathcal{D}+\mathcal{B}^*\mathcal{B}+\mathcal{C}\mathcal{C}^*$ *has vanishing essential spectrum.*

Proof. Without loss of generality, we consider the neighborhood U_I of a single end. By standard arguments [3], it suffices to show that as $c\to\infty$, the infimum of

$$|\mathcal{D}J|^2+|\mathcal{D}^*J|^2+|\mathcal{B}J|^2+|\mathcal{C}^*J|^2, \tag{6.1}$$

where J runs over smooth unit-length elements of H_1 with compact support in $[c,\infty)\times N_I$, goes to infinity. In this proof, all norms will be L^2-norms on $\overline{U_I}=[0,\infty)\times N_I$.

Taking $c>0$, we can ignore boundary terms in the following equations. We have

$$|\mathcal{D}J|^2+|\mathcal{D}^*J|^2 = |dJ-\mathcal{B}J|^2+|d^*J-\mathcal{C}^*J|^2. \tag{6.2}$$

From Proposition 2, $\mathcal{B}$ and $\mathcal{C}^*$ are bounded. In terms of the two-component vector $(dJ-\mathcal{B}J, d^*J-\mathcal{C}^*J)$, we can write

$$\begin{aligned} |dJ-\mathcal{B}J|^2+|d^*J-\mathcal{C}^*J|^2 &= |(dJ-\mathcal{B}J, d^*J-\mathcal{C}^*J)|^2 && (6.3)\\ &= |(dJ,d^*J)-(\mathcal{B}J,\mathcal{C}^*J)|^2 \\ &\geq \left(|(dJ,d^*J)|-|(\mathcal{B}J,\mathcal{C}^*J)|\right)^2 \\ &\geq \left(\max\left(\sqrt{|dJ|^2+|d^*J|^2}-\mathrm{const.}|J|,0\right)\right)^2, && (6.4)\end{aligned}$$

where "const." in this proof will denote something that is independent of c. Hence it suffices to consider $|dJ|^2 + |d^*J|^2$.

From Bochner's equation,

$$|dJ|^2 + |d^*J|^2 = |\nabla J|^2 + \int_{U_I} \sum_{p,q,r,s=0}^{n-1} R^M_{pqrs} \langle E^p I^q J, E^r I^s J \rangle \tag{6.5}$$
$$\geq |\nabla J|^2 - \text{const.} |J|^2.$$

Letting ∇^{vert} denote covariant differentiation in vertical directions, we have

$$|\nabla J|^2 = |\nabla^{vert} J|^2 + |\nabla_{\partial_s} J|^2. \tag{6.6}$$

Thus

$$|dJ|^2 + |d^*J|^2 \geq |\nabla^{vert} J|^2 - \text{const.} |J|^2 \tag{6.7}$$
$$\geq \left(|\widehat{\nabla} J| - |(\nabla^{vert} - \widehat{\nabla}) J| \right)^2 - \text{const.} |J|^2.$$

Using (4.7), we obtain

$$|dJ|^2 + |d^*J|^2 \geq \max \left(|\widehat{\nabla} J| - \text{const.} |J|, 0 \right)^2 - \text{const.} |J|^2. \tag{6.8}$$

Applying Bochner's equation fiberwise gives

$$|\widehat{d} J|^2 + |\widehat{d}^* J|^2 = |\widehat{\nabla} J|^2 + \int_{U_I} \sum_{p,q,r,s=1}^{n-1} R^Z_{pqrs} \langle E^p I^q J, E^r I^s J \rangle. \tag{6.9}$$

From the Gauss-Codazzi equation,

$$\int_{U_I} \sum_{p,q,r,s=1}^{n-1} R^Z_{pqrs} \langle E^p I^q J, E^r I^s J \rangle \tag{6.10}$$
$$= \int_{U_I} \sum_{p,q,r,s=1}^{n-1} (R^M_{pqrs} + S_{pr} S_{qs} - S_{ps} S_{qr}) \langle E^p I^q J, E^r I^s J \rangle.$$

Hence

$$|\widehat{\nabla} J|^2 \geq |\widehat{d} J|^2 + |\widehat{d}^* J|^2 - \text{const.} |J|^2. \tag{6.11}$$

(We note that the right-hand side of (6.10) makes sense even if the Busemann function s is only C^2-smooth. Hence (6.11) is valid in this generality.) From

Proposition 1, we have

$$|\widehat{d}J|^2 + |\widehat{d}^*J|^2 \geq c_2^2 a^2 e^{2ac}|J|^2. \tag{6.12}$$

Taking $c \to \infty$, the proposition follows. □

7. Proof of Theorem 2. We will use the general identity that

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}^{-1} = \begin{pmatrix} (\alpha - \beta\delta^{-1}\gamma)^{-1} & -(\alpha - \beta\delta^{-1}\gamma)^{-1}\beta\delta^{-1} \\ -\delta^{-1}\gamma(\alpha - \beta\delta^{-1}\gamma)^{-1} & \delta^{-1} + \delta^{-1}\gamma(\alpha - \beta\delta^{-1}\gamma)^{-1}\beta\delta^{-1} \end{pmatrix}, \tag{7.1}$$

provided that δ and $\alpha - \beta\delta^{-1}\gamma$ are invertible.

By a standard argument as in [3, Proposition 2.1], the essential spectra of $\triangle_p^M$ and $\triangle_p'$ are the same. For simplicity, we will omit the subscript p. Using Proposition 3, it is enough to show that $\triangle'$ and

$$\mathcal{L} = \begin{pmatrix} \mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} & 0 \\ 0 & \mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D} + \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^* \end{pmatrix} \tag{7.2}$$

have the same essential spectra. To show this, from [12, Vol. IV, Theorem XIII.14] it suffices to show that $\left(\triangle' + kI\right)^{-1} - (\mathcal{L} + kI)^{-1}$ is compact for some $k > 0$.

We put

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \triangle' + kI. \tag{7.3}$$

Explicitly,

$$\begin{aligned} \alpha &= \mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + \mathcal{B}\mathcal{B}^* + \mathcal{C}^*\mathcal{C} + kI, \\ \beta &= \mathcal{A}\mathcal{C}^* + \mathcal{B}\mathcal{D}^* + \mathcal{A}^*\mathcal{B} + \mathcal{C}^*\mathcal{D} \\ \gamma &= \mathcal{C}\mathcal{A}^* + \mathcal{D}\mathcal{B}^* + \mathcal{B}^*\mathcal{A} + \mathcal{D}^*\mathcal{C} \\ \delta &= \mathcal{D}\mathcal{D}^* + \mathcal{D}^*\mathcal{D} + \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^* + kI. \end{aligned} \tag{7.4}$$

As $k > 0$, the operators α and δ are invertible, with $\|\alpha^{-1}\| \leq k^{-1}$ and $\|\delta^{-1}\| \leq k^{-1}$. By Proposition 3, δ^{-1} is compact. By an elementary argument, $\|\mathcal{D}\delta^{-1/2}\| \leq 1$, $\|\mathcal{D}^*\delta^{-1/2}\| \leq 1$, $\|\delta^{-1/2}\mathcal{D}\| \leq 1$ and $\|\delta^{-1/2}\mathcal{D}^*\| \leq 1$. Then $\mathcal{D}\delta^{-1}$, $\mathcal{D}^*\delta^{-1}$, $\delta^{-1}\mathcal{D}$ and $\delta^{-1}\mathcal{D}^*$ are compact with norm at most $k^{-1/2}$.

We claim that $\alpha - \beta\delta^{-1}\gamma$ is invertible if k is large enough. To see this, we write

$$\alpha - \beta\delta^{-1}\gamma = \alpha^{1/2}\left(I - \alpha^{-1/2}\beta\delta^{-1}\gamma\alpha^{-1/2}\right)\alpha^{1/2}. \tag{7.5}$$

Then it suffices to show that $\|\alpha^{-1/2}\beta\delta^{-1}\gamma\alpha^{-1/2}\| < 1$ if k is large enough. Writing things out, we have

$$\alpha^{-1/2}\beta\delta^{-1}\gamma\alpha^{-1/2} = \alpha^{-1/2}\left(\mathcal{A}\mathcal{C}^* + \mathcal{B}\mathcal{D}^* + \mathcal{A}^*\mathcal{B} + \mathcal{C}^*\mathcal{D}\right)\delta^{-1} \times \left(\mathcal{C}\mathcal{A}^* + \mathcal{D}\mathcal{B}^* + \mathcal{B}^*\mathcal{A} + \mathcal{D}^*\mathcal{C}\right)\alpha^{-1/2}. \tag{7.6}$$

Now the operators $\alpha^{-1/2}\mathcal{A}$, $\alpha^{-1/2}\mathcal{A}^*$, $\mathcal{A}^*\alpha^{-1/2}$, $\mathcal{A}\alpha^{-1/2}$, $\mathcal{D}^*\delta^{-1}\mathcal{D}$, $\mathcal{D}\delta^{-1}\mathcal{D}^*$, $\mathcal{D}\delta^{-1}\mathcal{D}$ and $\mathcal{D}^*\delta^{-1}\mathcal{D}^*$ all have norm at most one. From Proposition 2, $\mathcal{B}$ and $\mathcal{C}$ are bounded. Writing out (7.6) into its sixteen terms, we see that the structure is such that by taking k large, we can make the norm of any individual term as small as desired. Hence by taking k large enough, we can make $\alpha - \beta\delta^{-1}\gamma$ invertible.

Writing

$$\left(\alpha - \beta\delta^{-1}\gamma\right)^{-1}\mathcal{A} = \alpha^{-1/2}\left(I - \alpha^{-1/2}\beta\delta^{-1}\gamma\alpha^{-1/2}\right)^{-1}\alpha^{-1/2}\mathcal{A}, \tag{7.7}$$

we see that $\left(\alpha - \beta\delta^{-1}\gamma\right)^{-1}\mathcal{A}$ is bounded. Similarly, $(\alpha - \beta\delta^{-1}\gamma)^{-1}\mathcal{A}^*$, $\mathcal{A}(\alpha - \beta\delta^{-1}\gamma)^{-1}$ and $\mathcal{A}^*\left(\alpha - \beta\delta^{-1}\gamma\right)^{-1}$ are bounded. It now follows from (7.1) that all components of $\left(\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\right)^{-1}$ except for the upper left component are compact. We note that the same statement is true about $(\mathcal{L} + kI)^{-1}$. It remains to show that

$$\left(\alpha - \beta\delta^{-1}\gamma\right)^{-1} - \left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1} \tag{7.8}$$

is compact.

Let us write

$$\alpha - \beta\delta^{-1}\gamma = \mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI - \left(\beta\delta^{-1}\gamma - \mathcal{B}\mathcal{B}^* - \mathcal{C}^*\mathcal{C}\right). \tag{7.9}$$

Then formally,

$$\left(\alpha - \beta\delta^{-1}\gamma\right)^{-1} = \left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2} \times (I - X)^{-1}\left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2}, \tag{7.10}$$

where

$$X = \left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2} \times \left(\beta\delta^{-1}\gamma - \mathcal{B}\mathcal{B}^* - \mathcal{C}^*\mathcal{C}\right)\left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2}. \tag{7.11}$$

It follows that

$$(7.12)\qquad \left(\alpha - \beta\delta^{-1}\gamma\right)^{-1} - \left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1} = \left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2}\left(\sum_{i=1}^{\infty} X^i\right)\left(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A} + kI\right)^{-1/2},$$

provided that the sum converges. We will show that X is compact and that the sum norm-converges if k is large enough, which will prove the theorem.

We have

$$(7.13)\quad \beta\delta^{-1}\gamma = \left(\mathcal{A}\mathcal{C}^* + \mathcal{B}\mathcal{D}^* + \mathcal{A}^*\mathcal{B} + \mathcal{C}^*\mathcal{D}\right)\delta^{-1}\left(\mathcal{C}\mathcal{A}^* + \mathcal{D}\mathcal{B}^* + \mathcal{B}^*\mathcal{A} + \mathcal{D}^*\mathcal{C}\right).$$

Consider first the terms of (7.13) that are explicitly quadratic in $\mathcal{D}$, namely

$$(7.14)\qquad \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{D}\mathcal{B}^* + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{D}\mathcal{B}^*.$$

As $d^2 = 0$, we have $\mathcal{A}\mathcal{B} = -\mathcal{B}\mathcal{D}$, $\mathcal{C}\mathcal{A} = -\mathcal{D}\mathcal{C}$ and $\mathcal{D}^2 = -\mathcal{C}\mathcal{B}$. Then (7.14) equals

$$(7.15)\qquad \begin{aligned} &\mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{D}\mathcal{B}^* + \mathcal{B}\mathcal{D}\delta^{-1}\mathcal{D}^*\mathcal{B}^* + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{C}^*\mathcal{D}^*\delta^{-1}\mathcal{D}\mathcal{C} \\ &+ \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{D}\mathcal{B}^* - \mathcal{A}\mathcal{B}\delta^{-1}\mathcal{B}^*\mathcal{A}^* - \mathcal{A}^*\mathcal{C}^*\delta^{-1}\mathcal{C}\mathcal{A}. \end{aligned}$$

Thus

$$(7.16)\quad \begin{aligned} \beta\delta^{-1}\gamma - \mathcal{B}\mathcal{B}^* - \mathcal{C}^*\mathcal{C} = {} & \mathcal{B}\left(\mathcal{D}^*\delta^{-1}\mathcal{D} + \mathcal{D}\delta^{-1}\mathcal{D}^* - I\right)\mathcal{B}^* \\ & + \mathcal{C}^*\left(\mathcal{D}^*\delta^{-1}\mathcal{D} + \mathcal{D}\delta^{-1}\mathcal{D}^* - I\right)\mathcal{C} \\ & + \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{D}\mathcal{B}^* + O(\mathcal{D}), \end{aligned}$$

where $O(\mathcal{D})$ denotes the terms that are linear in $\mathcal{D}$.

We have

$$(7.17)\qquad \begin{aligned} &\mathcal{D}^*\delta^{-1}\mathcal{D} + \mathcal{D}\delta^{-1}\mathcal{D}^* - I \\ &\quad = \left(\mathcal{D}^*\mathcal{D} + \mathcal{D}\mathcal{D}^* - \delta\right)\delta^{-1} + D^*\delta^{-1}(\mathcal{D}\delta - \delta\mathcal{D})\delta^{-1} \\ &\quad\quad + D\delta^{-1}(\mathcal{D}^*\delta - \delta\mathcal{D}^*)\delta^{-1}, \end{aligned}$$

$$(7.18)\qquad \begin{aligned} \mathcal{D}\delta^{-1}\mathcal{D} &= \mathcal{D}^2\delta^{-1} + \mathcal{D}\delta^{-1}(\mathcal{D}\delta - \delta\mathcal{D})\delta^{-1} \\ &= -\mathcal{C}\mathcal{B}\delta^{-1} + \mathcal{D}\delta^{-1}(\mathcal{D}\delta - \delta\mathcal{D})\delta^{-1} \end{aligned}$$

and

$$(7.19)\qquad \begin{aligned} \mathcal{D}^*\delta^{-1}\mathcal{D}^* &= (\mathcal{D}^*)^2\delta^{-1} + \mathcal{D}^*\delta^{-1}(\mathcal{D}^*\delta - \delta\mathcal{D}^*)\delta^{-1} \\ &= -\mathcal{B}^*\mathcal{C}^*\delta^{-1} + \mathcal{D}^*\delta^{-1}(\mathcal{D}^*\delta - \delta\mathcal{D}^*)\delta^{-1}. \end{aligned}$$

Furthermore,

$$\mathcal{D}^*\mathcal{D} + \mathcal{D}\mathcal{D}^* - \delta = -\mathcal{B}^*\mathcal{B} - \mathcal{C}\mathcal{C}^* - kI, \tag{7.20}$$

$$\begin{aligned} \mathcal{D}\delta - \delta\mathcal{D} &= [\mathcal{D}^2, \mathcal{D}^*] + [\mathcal{D}, \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^*] \\ &= -[\mathcal{C}\mathcal{B}, \mathcal{D}^*] + [\mathcal{D}, \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^*] \end{aligned} \tag{7.21}$$

and

$$\begin{aligned} \mathcal{D}^*\delta - \delta\mathcal{D}^* &= [(\mathcal{D}^*)^2, \mathcal{D}] + [\mathcal{D}^*, \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^*] \\ &= -[\mathcal{B}^*\mathcal{C}^*, \mathcal{D}] + [\mathcal{D}^*, \mathcal{B}^*\mathcal{B} + \mathcal{C}\mathcal{C}^*]. \end{aligned} \tag{7.22}$$

Substituting (7.20) – (7.22) into (7.17) – (7.19), we see that $\mathcal{D}^*\delta^{-1}\mathcal{D} + \mathcal{D}\delta^{-1}\mathcal{D}^* - I$, $\mathcal{D}\delta^{-1}\mathcal{D}$ and $\mathcal{D}^*\delta^{-1}\mathcal{D}^*$ are compact. Substituting (7.16) into (7.11), we see that the contributions to X of the terms listed in (7.16) are all compact.

Next, from (7.13), the terms in $\beta\delta^{-1}\gamma - \mathcal{B}\mathcal{B}^* - \mathcal{C}^*\mathcal{C}$ that are explicitly proportionate to $\mathcal{D}$ are

$$\begin{aligned} &\mathcal{A}\mathcal{C}^*\delta^{-1}\mathcal{D}\mathcal{B}^* + \mathcal{A}\mathcal{C}^*\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{C}\mathcal{A}^* + \mathcal{B}\mathcal{D}^*\delta^{-1}\mathcal{B}^*\mathcal{A} \\ &\quad + \mathcal{A}^*\mathcal{B}\delta^{-1}\mathcal{D}\mathcal{B}^* + \mathcal{A}^*\mathcal{B}\delta^{-1}\mathcal{D}^*\mathcal{C} + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{C}\mathcal{A}^* + \mathcal{C}^*\mathcal{D}\delta^{-1}\mathcal{B}^*\mathcal{A}. \end{aligned} \tag{7.23}$$

One sees that their contributions to (7.11) are all compact. Finally, the remaining terms in $\beta\delta^{-1}\gamma - \mathcal{B}\mathcal{B}^* - \mathcal{C}^*\mathcal{C}$ that are constant in $\mathcal{D}$ are

$$\left(\mathcal{A}\mathcal{C}^* + \mathcal{A}^*\mathcal{B}\right)\delta^{-1}\left(\mathcal{C}\mathcal{A}^* + \mathcal{B}^*\mathcal{A}\right) - \mathcal{A}\mathcal{B}\delta^{-1}\mathcal{B}^*\mathcal{A}^* - \mathcal{A}^*\mathcal{C}^*\delta^{-1}\mathcal{C}\mathcal{A}. \tag{7.24}$$

Their contributions to (7.11) are all compact.

One can show as before that the norm of X can be made arbitrarily small by making k large enough.

This proves Theorem 2 under our assumption that the metric on M is a product near each $\{0\} \times N_I$. However, as in [3, Proposition 2.1], the essential spectrum of $\triangle_p^M$ is invariant under a compactly-supported change of the metric. Furthermore, the essential spectrum of a self-adjoint ordinary differential operator on $[0, \infty)$ is independent of the choice of (self-adjoint) boundary condition at $\{0\}$ [5, Volume II, Chapter XIII.7, Corollary 3] and is also unchanged by a compactly-supported perturbation of the operator. Thus Theorem 2 also holds for the original metric on M.

8. Proof of Theorem 3.

We now specialize to the case of functions. In this case, E_I is a trivial real line bundle on $[0, \infty)$. Consider the quadratic form (1.3) in the case $B = 1$, with $f \in C^\infty([0, \infty))$ and $f(0) = 0$.

Let $v(s)$ denote the volume of $(N, h(s))$. Then

$$\frac{dv}{ds} = -\int_N \sum_i S^i_i d\mathrm{vol}(s). \tag{8.1}$$

If $\mathcal{F}\colon (0,\infty) \times N \to U$ is smooth then the Gauss-Codazzi equation gives

$$-\partial_s \sum_i S^i_i + \sum_{ij} S^{ij} S_{ij} = -\mathrm{Ric}(\partial_s, \partial_s), \tag{8.2}$$

which in turn implies that

$$\frac{d^2 v}{ds^2} = \int_N \left[-\mathrm{Ric}(\partial_s, \partial_s) - \sum_{ij} S^{ij} S_{ij} + \left(\sum_i S^i_i \right)^2 \right] d\mathrm{vol}(s). \tag{8.3}$$

This last equation makes sense even if $\mathcal{F}$ is not smooth, showing that v is C^2-smooth in s.

Lemma 4. $(\mathcal{A}\mathcal{A}^* + \mathcal{A}^*\mathcal{A})_0$ *is unitarily equivalent to the operator*

$$-\frac{d^2}{ds^2} + \frac{1}{2}\frac{d^2 \ln v}{ds^2} + \frac{1}{4}\left(\frac{d \ln v}{ds}\right)^2, \tag{8.4}$$

which is densely-defined and self-adjoint on $L^2([0,\infty))$*, with Dirichlet boundary conditions.*

Proof. Putting $k(s) = v(s)^{1/2} f$, we have

$$\langle f, f \rangle = \langle k, k \rangle_{L^2} \tag{8.5}$$

and

$$\begin{aligned} Q(f) &= \int_0^\infty \left(\frac{d}{ds}(v^{-1/2}k) \right)^2 v(s) ds \\ &= \int_0^\infty \left(v^{-1/2}\frac{dk}{ds} - \frac{1}{2} v^{-3/2} \frac{dv}{ds} k \right)^2 v(s) ds \\ &= \int_0^\infty \left(\frac{dk}{ds} - \frac{1}{2} v^{-1} \frac{dv}{ds} k \right)^2 ds \\ &= \int_0^\infty \left[\left(\frac{dk}{ds} \right)^2 - v^{-1} \frac{dv}{ds} k \frac{dk}{ds} + \frac{1}{4} \left(v^{-1} \frac{dv}{ds} \right)^2 k^2 \right] ds \\ &= \int_0^\infty \left[\left(\frac{dk}{ds} \right)^2 - \frac{1}{2} \frac{d \ln v}{ds} \frac{dk^2}{ds} + \frac{1}{4} \left(\frac{d \ln v}{ds} \right)^2 k^2 \right] ds \end{aligned} \tag{8.6}$$

$$= \int_0^\infty \left[\left(\frac{dk}{ds} \right)^2 + \left(\frac{1}{2} \frac{d^2 \ln v}{ds^2} + \frac{1}{4} \left(\frac{d \ln v}{ds} \right)^2 \right) k^2 \right] ds.$$

The lemma follows. □

Now let P be an even periodic element of $C^\infty(\mathbb{R})$ which is not real-analytic. Put

$$V_P = \frac{1}{2} \frac{dP}{ds} + \frac{1}{4} P^2. \tag{8.7}$$

Let $\mathcal{O}'_P$ be the operator $-\frac{d^2}{ds^2} + V_P$ acting on $L^2([0, \infty))$, with Dirichlet boundary conditions at 0.

LEMMA 5. *$\mathcal{O}'_P$ has an infinite number of gaps in its essential spectrum.*

Proof. As $\frac{dP}{ds}$ is odd and P^2 is even, if V_P were real-analytic then $\frac{dP}{ds}$ would be real-analytic, which would imply that P is real-analytic. Thus V_P is not real-analytic. From [12, Vol. IV, Thm. XIII.91(d)], the operator $\mathcal{O}_P = -\frac{d^2}{ds^2} + V_P$ on $L^2(\mathbb{R})$ has an absolutely continuous spectrum which consists of an infinite number of disjoint closed intervals in $[0, \infty)$, tending toward infinity. Let $\mathcal{O}''_P$ be the operator $-\frac{d^2}{ds^2} + V_P$ acting on $L^2((-\infty, 0])$, again with Dirichlet boundary conditions at 0. Then the essential spectrum of $\mathcal{O}_P$ is the union of the essential spectra of $\mathcal{O}'_P$ and $\mathcal{O}''_P$. As the essential spectra of both $\mathcal{O}'_P$ and $\mathcal{O}''_P$ tend toward infinity, the lemma follows. □

Proof of Theorem 3. Start with a complete finite-volume hyperbolic metric on a punctured 2-torus. On the cusp, the metric is $ds^2 + e^{-2s} d\theta^2$ for $s \in [s_0, \infty)$, with $s_0 > 0$.

Let p be an even periodic element of $C^\infty(\mathbb{R})$ which is not real-analytic. Let $\phi \in C_0^\infty([0, \infty))$ be a nonincreasing function which is identically one on $[0, 1]$ and identically zero on $[2, \infty)$. For $\delta > 0$ and $s \geq s_0$, put

$$v_\delta(s) = e^{-s - \delta \int_0^{s - s_0} p(u)(1 - \phi(\delta u)) du}. \tag{8.8}$$

Keep the metric on the complement of the cusp unaltered and change the metric on the cusp to $ds^2 + v_\delta(s)^2 d\theta^2$. From Theorem 2 and Lemma 4, the essential spectrum of the Laplacian of the new metric is the same as the essential spectrum of the operator $\mathcal{O}'_{-1-\delta p}$. Then from Lemma 5, the Laplacian of the new metric has an infinite number of gaps in its essential spectrum. Hence it has an infinite number of gaps in its spectrum. One can check that as $\delta \to 0$, the sectional curvatures of the new metric become pinched arbitrarily close to -1.

Remark. It seems likely that by taking p to be almost-periodic, one can find similar examples in which the essential spectrum is a Cantor set.

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