Quantum Signal Processing and Nonlinear Fourier Analysis

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Background: Quantum Singular Value Transformation (QSVT) [9] is one of the most important developments in quantum algorithms in the past decade. It has a wide range of applications in quantum computation, such as Hamiltonian simulation [15, 11], linear system of equations [11, 14, 16], eigenvalue problems [13, 6], Gibbs states preparation [11], Petz recovery channel [10], benchmarking quantum systems [3, 8], to name a few. At the heart of QSVT is an innovative polynomial representation called Quantum Signal Processing (QSP) [15], which can encode a target polynomial of definite parity as the $(1,1)$-entry of the product of a sequence of parameterized SU(2) matrices. Given a target polynomial f of degree d, the corresponding parameters, denoted by Ψ, are called phase factors. Despite the significant progress in this field, algorithms for finding phase factors are either numerically unstable (i.e., requires high precision arithmetic operations) [11, 12], can only be proved to converge within some limited parameter regimes [7, 18, 4], or do not have provable performance guarantees [2, 19, 17, 5].

There is further evidence suggesting that our current understanding of QSP is incomplete. Specifically, for a given polynomial f, the choice of the phase factors Ψ is highly non-unique. This non-uniqueness poses a challenge when $f : [-1,1] \to \mathbb{R}$ is a smooth function rather than a polynomial, as there is no clear approach for defining an infinite sequence of phase factors to represent f. Although the work on infinite quantum signal processing (iQSP) by some of the authors [5] provides a partial positive solution in certain parameter regimes, it does not fully address this issue.

In this work, we argue that nonlinear Fourier analysis provides a natural framework for understanding both QSP and iQSP. This allows us to solve the two aforementioned problems simultaneously:

(1) Among the combinatorially many possible phase factors, we identify a **unique** choice that satisfies a nonlinear generalization of the Plancherel equality from classical Fourier analysis. This generalization remains well-defined for smooth functions beyond polynomials.

(2) We develop a Riemann–Hilbert–Weiss (RHW) algorithm to evaluate phase factors. To the best of our knowledge, this is the **first** provably numerically stable algorithm for almost all functions that admit a QSP representation. In addition, the RHW algorithm exhibits a surprising property: each component of the phase factor can be computed independently of the others, a feature not found in any other algorithm for determining phase factors to date.

Problem Setup: Standard QSP only represents polynomials of definite parity[2]. Without loss of generality, we restrict discussion to even functions f, and the case of odd functions can be treated similarly. Let $\mathbf{P}$ denote the space of infinite sequences $\Psi = (\psi_k)_{k\in\mathbb{N}}$ with $\psi_k \in [-\pi/2, \pi/2]$ and X, Z be Pauli matrices. For $x \in [0,1]$, define $W(x) = e^{\mathrm{i}\arccos(x)X} = \begin{pmatrix} x & \mathrm{i}\sqrt{1-x^2} \\ \mathrm{i}\sqrt{1-x^2} & x \end{pmatrix}$. Given any $\Psi \in \mathbf{P}$ and $x \in [0,1]$, we can recursively define a sequence of unitary matrices

$$\begin{aligned} U_0(x,\Psi) &= e^{\mathrm{i}\psi_0 Z} \\ U_d(x,\Psi) &= e^{\mathrm{i}\psi_d Z} W(x) U_{d-1}(x,\Psi) W(x) e^{\mathrm{i}\psi_d Z}. \end{aligned} \tag{1}$$

The central question of this work is as follows:

Let $f : [0,1] \to [-1,1]$ be an even function in a suitable function class. Let $u_d(x,\Psi)$ denote the upper left entry of $U_d(x,\Psi)$.

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[2]Though this is partially relaxed by the recent development of generalized QSP [17].

(1) Is there a $\Psi \in \ell^2(\mathbb{N})$ such that $\mathrm{Im}[u_d(x,\Psi)]$ converges to $f(x)$? Is this Ψ unique in some sense?
(2) When f is a polynomial of degree d, is there a provably numerically stable algorithm to compute Ψ, i.e., an algorithm which uses $\mathrm{polylog}(d/\varepsilon)$ bits of precision and has a cost of $\mathrm{poly}(d\log(1/\varepsilon))$?

Connection between QSP and the nonlinear Fourier transform: Similar to QSP, nonlinear Fourier analysis operates on a product of countably many unitary matrices. Given a compactly supported sequence $F = (F_n)_{n\in\mathbb{Z}}$ of complex numbers, for each $n \in \mathbb{Z}$, define a pair of Laurent polynomials $(a_n(z), b_n(z))$ via the recurrence relation

$$\begin{pmatrix} a_n(z) & b_n(z) \\ -b_n^*(z) & a_n^*(z) \end{pmatrix} = \begin{pmatrix} a_{n-1}(z) & b_{n-1}(z) \\ -b_{n-1}^*(z) & a_{n-1}^*(z) \end{pmatrix} \frac{1}{\sqrt{1+|F_n|^2}} \begin{pmatrix} 1 & F_n z^n \\ -\overline{F_n} z^{-n} & 1 \end{pmatrix} \tag{2}$$

with the initial condition

$$\begin{pmatrix} a_{-\infty}(z) & b_{-\infty}(z) \\ -b_{-\infty}^*(z) & a_{-\infty}^*(z) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{3}$$

Here, $a^*(z) := \overline{a(\overline{z^{-1}})}$ for any function a and $\overline{w}$ refers to the complex conjugate of w for any w. The nonlinear Fourier transform (NLFT) of the sequence F is defined as the pair of Laurent series $(a(z), b(z)) := (a_\infty(z), b_\infty(z))$.

The NLFT and quantum signal processing are closely related. Specifically, we can convert the original problem of finding for given f some phase factors with $\mathrm{Im}[u_d(x,\Psi)] = f(x)$ into that of determining the infinite sequence $(F_n)_{n\in\mathbb{Z}}$ for some data (a, b) with $\mathrm{i}f(x) = b(z)$ (see Lemma 3 in the manuscript).

In classical Fourier analysis, when $f(x)$ is a periodic function on $[-\pi, \pi]$, and $\hat{f}_n$ represents its Fourier series coefficients, the renowned Plancherel equality is an identity connecting the L^2 norm of the function in the Fourier and real space:$2\pi \sum_{n=-\infty}^{\infty} |\hat{f}_n|^2 = \int_{-\pi}^{\pi} |f(x)|^2\, dx$.

In nonlinear Fourier analysis, the Plancherel equality is generalized into an inequality as below.

Lemma 1 (Nonlinear Plancherel inequality, Lemma 15 in the manuscript)**.** *If (a, b) is the NLFT of some $F \in \ell^2(\mathbb{Z})$, then*

$$\sum_n \log(1+|F_n|^2) \geq -\int_{\mathbb{T}} \log(1-|b(z)|^2)\,. \tag{4}$$

Furthermore, the equality holds if and only if a^* is an *outer function* (see Section 2 of the manuscript for backgrounds in Hardy functions and the precise definition of outer functions).

It is this condition, when the equality is reached, that provides the mathematical foundation allowing us to **uniquely** specify the phase factors.

Main Results: In this work, we provide positive answers to both questions above, which constitute a **complete solution** of the iQSP problem. Let $\mathbf{S}$ denote the set of all Szegő functions, which are functions satisfying

$$\|f\|_{\mathbf{S}} := \left(\frac{2}{\pi} \int_0^1 |f(x)|^2 \frac{dx}{\sqrt{1-x^2}} \right)^{\frac{1}{2}} < \infty. \tag{5}$$

This is a natural class of functions admitting a NLFT representation. For each $\eta \in (0, 1)$, we define

$$\mathbf{S}_\eta = \{ f \in \mathbf{S} \mid \|f\|_\infty \leq 1-\eta \}\,. \tag{6}$$

Theorem 2 (Short version of Theorem 1 in the manuscript)**.** *For each $f \in \mathbf{S}$, there exists a unique sequence $\Psi \in \mathbf{P}$ such that both the L^2 convergence criterion in*

$$\lim_{d\to\infty} \|\mathrm{Im}[u_d(x,\Psi)] - f(x)\|_{\mathbf{S}} = 0, \tag{7}$$

and the nonlinear Plancherel equality in Eq. (4) hold.

A fundamental result in Fourier analysis is that the mapping from the function to its Fourier / Chebyshev coefficients is a linear functional, and hence each Fourier / Chebyshev coefficient can be evaluated independently from the others using a single inner product. Can the phase factors ψ_k be evaluated independently as well? The proof of Theorem 2 provides several useful tools for characterizing the phase sequence Ψ. From Lemma 6 in the manuscript, we can compute an individual phase factor via the formula

$$\psi_k = \arctan \frac{(B_k z^{-k})(0)}{iA_k(\infty)}\,, \tag{8}$$

where A_k is a Laurent series on $\mathbb{C}$, and $B_k z^{-k}$ is a Taylor series on $\mathbb{C}$, both depending on f, and the pair (A_k, B_k) is the unique solution to a linear system . By solving this linear system, we obtain an algorithm, dubbed the Riemann-Hilbert-Weiss algorithm, that is able to compute each individual phase factor ψ_k independently. This is in sharp contrast to *all* algorithms in the literature, where phase factors need to be computed in an interdependent fashion.

Theorem 3 (Short version of Theorem 2 in the manuscript)**.** *Let $0 < \varepsilon < 1$, $0 < \eta < \frac{1}{2}$ and let k and d be integers satisfying $d \geq 1$ and $0 \leq k \leq d$. The Riemann-Hilbert-Weiss algorithm computes the k-th phase factor ψ_k for any even input polynomial $f \in \mathbf{S}_\eta$ with degree $2d$ to precision ε with a computational cost of $\mathcal{O}\left(d^3 + \frac{d\log(d/(\eta\varepsilon))}{\eta}\right)$ and using $\mathcal{O}(\log(d/(\eta\varepsilon)))$ bits. To determine all $\mathcal{O}(d)$ phase factors, the algorithm results in a cumulative cost of $\mathcal{O}\left(d^4 + \frac{d\log(d/(\eta\varepsilon))}{\eta}\right)$, and the bit requirement remains $\mathcal{O}(\log(d/(\eta\varepsilon)))$.*

It is worth noting that the main purpose of Theorem 3 is to provide a numerically stable, polynomial scaling that works for arbitrarily small η.

Technical overview: Based on the interpretation of QSP in terms of NLFT, our Riemann–Hilbert–Weiss (RHW) consists of two main steps.

(1) **The Weiss algorithm to compute the "maximal" complementary function a^*.** In QSP and iQSP, most methods of computing phase factors for the polynomial f require the construction of a complementary polynomial, which after the change of variables, corresponds to the function a in the NLFT. Constructing a complementary polynomial was often costly. If f is Szegő, then as in [1], we take a^* to be an outer function on $\mathbb{D}$, which corresponds to the unique *maximal* complementary polynomial for f. Given b, we compute a^* using the Fast Fourier transform, an algorithm which we call the Weiss algorithm.

(2) **Solving the Riemann-Hilbert problem to obtain a phase factor.** We must show that (a, b) is a NLFT, and compute its kth nonlinear Fourier coefficient F_k. Following Ref. [1], we show there exists a unique solution $(A_k, B_k) \in H^2(\mathbb{D}^*) \times z^k H^2(\mathbb{D})$ to

$$(\mathrm{Id} + M_k)\begin{pmatrix} A_k \\ B_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \text{ where } \quad M_k = \begin{pmatrix} 0 & P_{\mathbb{D}^*}\frac{b^*}{a^*} \\ -z^k P_{\mathbb{D}} z^{-k}\frac{b}{a} & 0 \end{pmatrix}, \tag{9}$$

and then we can solve for the phase factor via the formula

$$F_k = \frac{(B_k z^{-k})(0)}{A_k^*(0)}\,.$$

Ref. [1] showed that when $\|f\|_\infty = \|b\|_\infty < \frac{1}{\sqrt{2}}$, then M_k has small operator norm and one can invert $\mathrm{Id} + M_k$ by the von Neumann series, yielding (A_k, B_k) exists and is unique. Using some spectral theory, we go past the $\frac{1}{\sqrt{2}}$ restriction and show that so long f is Szegő then $\mathrm{Id} + M_k$ is invertible, meaning the linear system (9) has a unique solution (A_k, B_k). Furthermore, our proof shows that if $|f|$ is bounded away from 1, then we have stability of the phase factors with respect to f.

In practice, we invert $\mathrm{Id} + M_k$ using the fact that on the off-diagonal, $\mathrm{Id} + M_k$ looks like a Toeplitz matrix for the function $\frac{b}{a}$. The computational cost of $\mathcal{O}(d^3)$ in Theorem 3 arises from solving the linear system Eq. (9), which is of size $\mathcal{O}(d)$ when f is a polynomial of degree $2d$.

References

[1] M. Alexis, G. Mnatsakanyan, and C. Thiele. Quantum signal processing and nonlinear fourier analysis. *arXiv preprint arXiv:2310.12683*, 2023.

[2] R. Chao, D. Ding, A. Gilyén, C. Huang, and M. Szegedy. Finding Angles for Quantum Signal Processing with Machine Precision. *arXiv preprint arXiv:2003.02831*, 2020.

[3] Y. Dong and L. Lin. Random circuit block-encoded matrix and a proposal of quantum linpack benchmark. *Phys. Rev. A*, 103(6):062412, 2021.

[4] Y. Dong, L. Lin, H. Ni, and J. Wang. Infinite quantum signal processing. *arXiv:2209.10162*, 2022.

[5] Y. Dong, L. Lin, H. Ni, and J. Wang. Robust iterative method for symmetric quantum signal processing in all parameter regimes. *arXiv:2307.12468*, 2023.

[6] Y. Dong, L. Lin, and Y. Tong. Ground state preparation and energy estimation on early fault-tolerant quantum computers via quantum eigenvalue transformation of unitary matrices. *arXiv preprint arXiv:2204.05955*, 2022.

[7] Y. Dong, X. Meng, K. B. Whaley, and L. Lin. Efficient phase factor evaluation in quantum signal processing. *Phys. Rev. A*, 103:042419, 2021.

[8] Y. Dong, K. B. Whaley, and L. Lin. A quantum hamiltonian simulation benchmark. *arXiv preprint arXiv:2108.03747*, 2021.

[9] A. Gilyén, S. Arunachalam, and N. Wiebe. Optimizing quantum optimization algorithms via faster quantum gradient computation. In *Proceedings of the Thirtieth Annual ACM-SIAM Symposium on Discrete Algorithms*, pages 1425–1444, 2019.

[10] A. Gilyén, S. Lloyd, I. Marvian, Y. Quek, and M. M. Wilde. Quantum algorithm for petz recovery channels and pretty good measurements. *Phys. Rev. Lett.*, 128(22):220502, 2022.

[11] A. Gilyén, Y. Su, G. H. Low, and N. Wiebe. Quantum singular value transformation and beyond: exponential improvements for quantum matrix arithmetics. In *Proceedings of the 51st Annual ACM SIGACT Symposium on Theory of Computing*, pages 193–204, 2019.

[12] J. Haah. Product decomposition of periodic functions in quantum signal processing. *Quantum*, 3:190, 2019.

[13] L. Lin and Y. Tong. Near-optimal ground state preparation. *Quantum*, 4:372, 2020.

[14] L. Lin and Y. Tong. Optimal quantum eigenstate filtering with application to solving quantum linear systems. *Quantum*, 4:361, 2020.

[15] G. H. Low and I. L. Chuang. Optimal hamiltonian simulation by quantum signal processing. *Phys. Rev. Lett.*, 118:010501, 2017.

[16] J. M. Martyn, Z. M. Rossi, A. K. Tan, and I. L. Chuang. A grand unification of quantum algorithms. *arXiv:2105.02859*, 2021.

[17] D. Motlagh and N. Wiebe. Generalized quantum signal processing. *arXiv preprint arXiv:2308.01501*, 2023.

[18] J. Wang, Y. Dong, and L. Lin. On the energy landscape of symmetric quantum signal processing. *Quantum*, 6:850, 2022.

[19] L. Ying. Stable factorization for phase factors of quantum signal processing. *arXiv preprint arXiv:2202.02671*, 2022.

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