Quantum numerical linear algebra

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Outline

Solve nature with nature

There is perhaps a widespread belief that a talk on quantum computation should start with a picture of Feynman..

Figure. A superposition of Feynmans

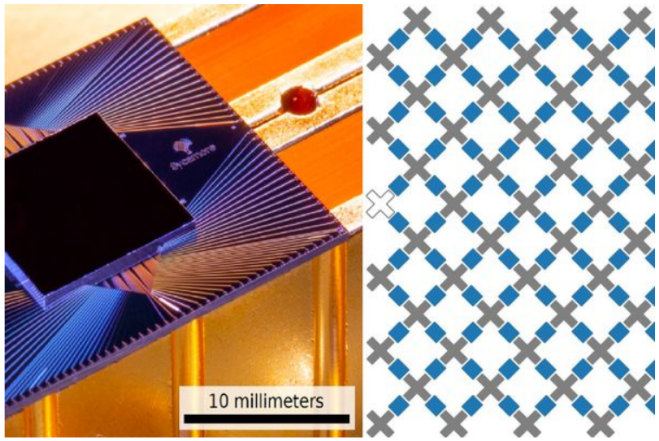
... if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy.

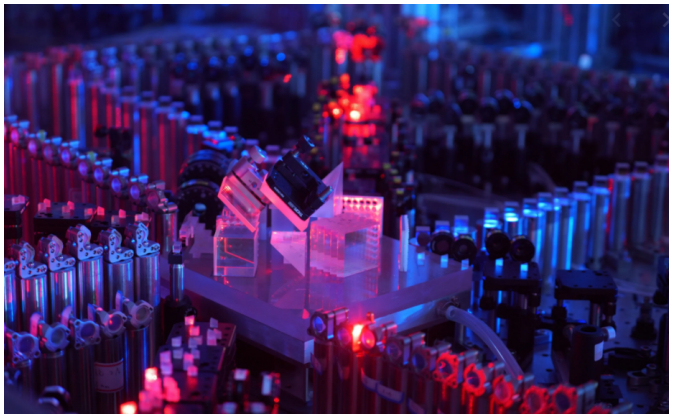
– Richard P. Feynman (1981) 1st Conference on Physics and Computation, MIT

Quantum computation meets public attention

Google, Nature 2019
Random circuit sampling
Theory: [Boixo et al, 2018]

USTC, Science 2020
Boson sampling.
Theory: [Aaronson–Arkhipov, 2011]





- After about four decades, quantum supremacy has been reached: the point *where quantum computers can do things that classical computers cannot, regardless of whether those tasks are useful.*

- *Is controlling large-scale quantum systems merely really, really hard, or is it ridiculously hard?* – John Preskill (2012)

- Quantum computer does anything useful? called quantum advantage.

Axes swung at quantum supremacy experiments

Simulating the Sycamore quantum supremacy circuits

arXiv:2103.03074v1

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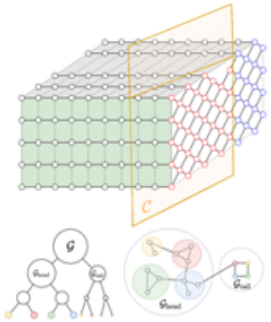
We propose a general tensor network method for simulating quantum circuits. The method is massively more efficient in computing a large number of correlated bitstring amplitudes and probabilities than existing methods. As an application, we study the sampling problem of Google's Sycamore circuits, which are believed to be beyond the reach of classical supercomputers and have been used to demonstrate quantum supremacy. Using our method, employing a small computational cluster containing 60 graphical processing units (GPUs), we have generated one million correlated bitstrings with some entries fixed, from the Sycamore circuit with 53 qubits and 20 cycles, with *linear cross-entropy benchmark* (XEB) fidelity equals 0.739, which is much higher than those in Google's quantum supremacy experiments.



arXiv:2109.11525v1

Efficient approximation of experimental Gaussian boson sampling

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(Dated: September 24, 2021)

Two recent landmark experiments have performed Gaussian boson sampling (GBS) with a non-programmable linear interferometer and threshold detectors on up to 144 output modes (see Refs. 1 and 2). Here we give classical sampling algorithms with better total variation distance than these experiments and a computational cost quadratic in the number of modes. Our method samples from a distribution that approximates the single-mode and two-mode ideal marginals of the given Gaussian boson sampler, which are calculated efficiently. One implementation sets the parameters of a Boltzmann machine from the calculated marginals using a mean field solution. This is a 2nd order approximation, with the uniform and thermal approximations corresponding to the 0th and 1st order, respectively. The kth order approximation reproduces marginals and correlations only up to order k with a cost exponential in k and high precision, while the experiment exhibits higher order correlations with lower precision. This methodology, like other polynomial approximations introduced previously, does not apply to random circuit sampling because the kth order approximation would simply result in the uniform distribution, in contrast to GBS.

A fast growing industry

Numerical linear algebra

- Linear systems of equation $Ax = b$
- Least squares problem $\min_x \|Ax - b\|_2$
- Eigenvalue decomposition $Av_i = \lambda_i v_i$
- Singular value decomposition $Av_i = u_i \sigma_i$
- Preconditioner $M^{-1}Ax = M^{-1}b$
- Matrix exponentiation e^{iHt} (Hamiltonian simulation)
- Other matrix functions: $\sqrt{A}, \log A, \ldots$
- Machine learning, e.g. kernel ridge regression $\alpha = (K + \tilde{I})^{-1} y$
- ...

Quantum numerical linear algebra

- Solving numerical linear algebra problems on a quantum computer.
- Many interesting, exciting progresses in the past few years.
- Reasonable way towards "quantum advantage".
- Quantum linear system problem (QLSP)

$$A\,|x\rangle \propto |b\rangle$$

- $A \in \mathbb{C}^{N\times N}$: cost can be $\mathcal{O}(\mathrm{polylog}(N))$.

What is a quantum computer (mathematically)

- $|\psi\rangle \in \mathbb{C}^N \cong (\mathbb{C}^2)^{\otimes n}, \quad N = 2^n$. n : number of qubits.
- Normalization condition $\langle\psi|\psi\rangle = 1$.
- $U \in \mathbb{C}^{N\times N}$ is unitary. $U|\psi\rangle$ is efficient to apply ($\mathrm{poly}(n)$).
- Quantum computer: $U_K \cdots U_1 |\psi\rangle$, and then classical output by measuring one or a few qubits M times.
- Quantum cost: $MK\mathrm{poly}(n)$. Potential exponential speedup.

Quantum circuit: “graphical” tensor linear algebra

- State vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Pauli matrices

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$|0\rangle \;—\boxed{X}—\; |1\rangle \qquad |1\rangle \;—\boxed{Z}—\; -|1\rangle$$

- Hadamard gate $H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$, $H|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) := |+\rangle$

$$|0\rangle \;—\boxed{H}—\; |+\rangle$$

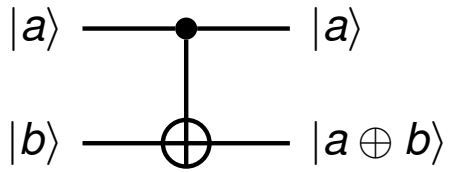
Quantum circuit: “graphical” tensor linear algebra

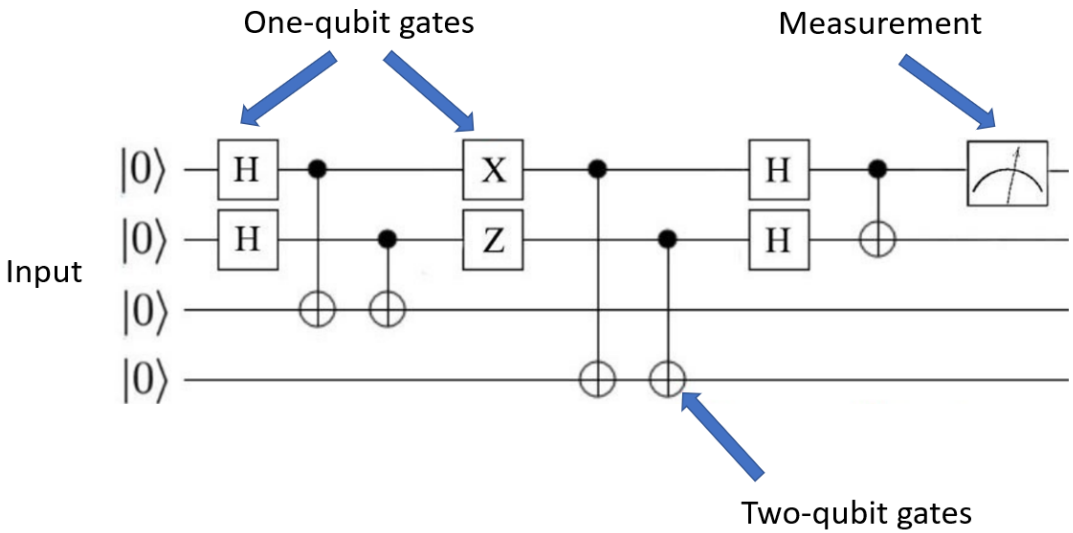
- Tensor product

$$|00\rangle = \begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \quad |01\rangle = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \quad |10\rangle = \begin{pmatrix}0\\0\\1\\0\end{pmatrix}, \quad |11\rangle = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

- CNOT gate

$$\text{CNOT} = \begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}$$

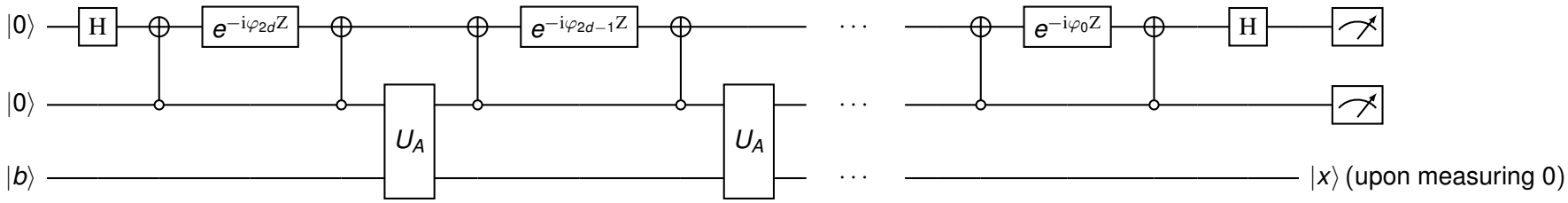




A toy linear system problem

$$A = \frac{1}{4}X + \frac{3}{4}I = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}, \quad |b\rangle = |0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

- Goal: obtain $|x\rangle = A^{-1}|b\rangle / \left\| A^{-1}|b\rangle \right\|_2 = \begin{pmatrix} 0.949 \\ -0.316 \end{pmatrix}$.
- (One possible) quantum circuit



- Does not look like any classical direct or iterative algorithm.
- $d = 80$, error of approximation $\begin{pmatrix} -7.020 \times 10^{-11} \\ -2.106 \times 10^{-10} \end{pmatrix}$

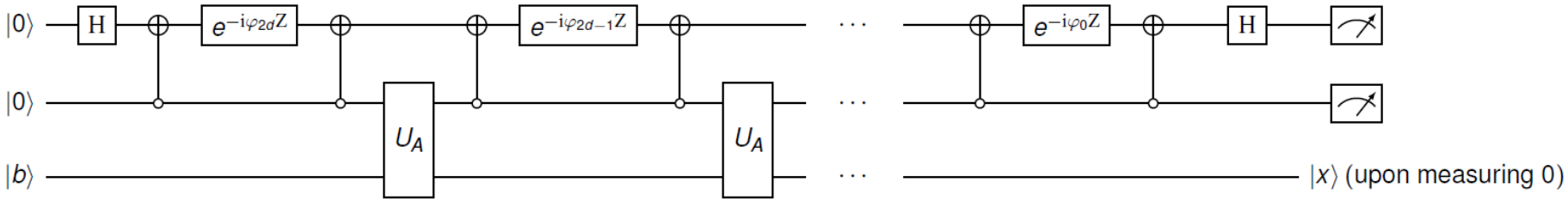
How to query A on a quantum computer?

- X, I are unitaries. A is a linear combination of unitaries (LCU), and is itself non-unitary. $\kappa(A) = \|A\|_2 \left\|A^{-1}\right\|_2 = 2$.
- Idea: extend 1-qubit non-unitary matrix to a 2-qubit unitary matrix

$$U_A = \begin{pmatrix} A & \cdot \\ \cdot & \cdot \end{pmatrix}$$

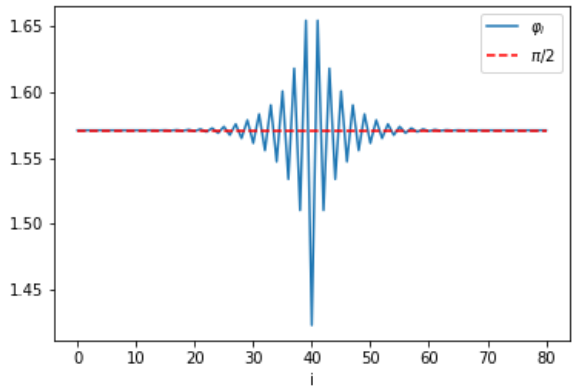
- Block-encoding (Low-Chuang, 2016; called “standard form” initially)

Procedure to construct $U_{A^{-1}}\,|b\rangle$



(Simplified circuit using that U_A is Hermitian, $A \succ 0$;$\{\varphi_i\}_{i=0}^{2d}$ are called phase factors).

The same circuit works for arbitrarily large matrix ($|b\rangle$ is n-qubit).



$d = 80$. Error for approximating A^{-1}/α

$$\begin{pmatrix} -2.046 \times 10^{-11} & 2.532 \times 10^{-11} \\ 2.532 \times 10^{-11} & -2.046 \times 10^{-11} \end{pmatrix}$$

What is under the hood?

- Polynomial approximation $A^{-1}/\alpha \approx \sum_{k=0}^{d} c_k T_k(A) := P(A)$ on $[\kappa^{-1}, 1]$ (after possible rescaling)
- Key step: parameterized polynomial representation in SU(2) $\Rightarrow$ Quantum singular value transformation circuit (QSVT) (Gilyén-Su-Low-Wiebe, 2019).

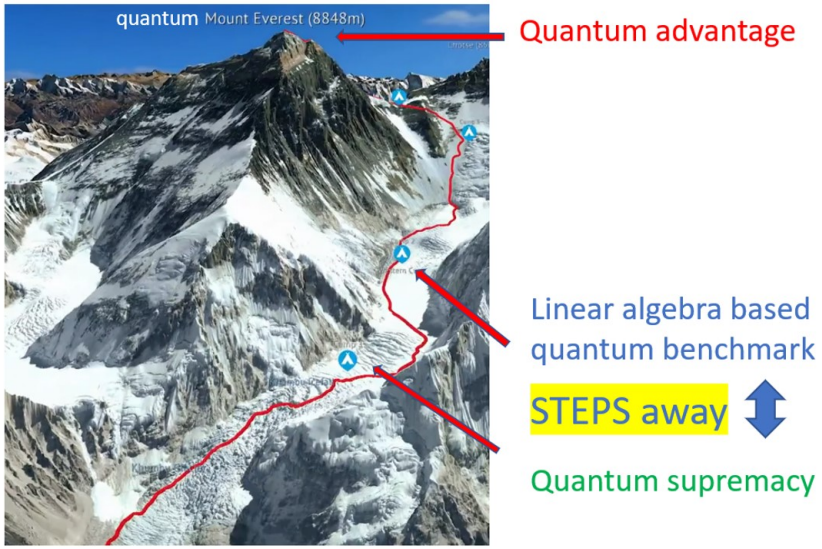
$$\mathrm{Re}\,\langle 0| e^{\mathrm{i}\phi_0 Z} e^{\mathrm{i}\arccos(x) X} e^{\mathrm{i}\phi_1 Z} e^{\mathrm{i}\arccos(x) X} \ldots e^{\mathrm{i}\phi_{d-1} Z} e^{\mathrm{i}\arccos(x) X} e^{\mathrm{i}\phi_d Z} |0\rangle = P(x)$$

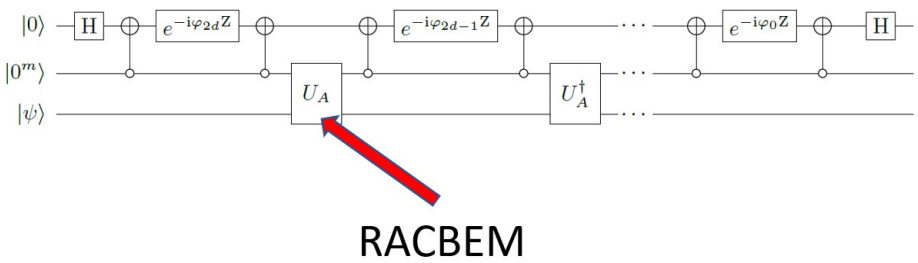
- Use the same circuit (but different parameters) for various (generalized) matrix functions.
- This circuit is much simpler than almost all other alternatives
- One of the most interesting developments in quantum algorithms in the past decade.

Outline

Linear algebra based quantum benchmarks

Climbing the Quantum Mount Everest





RACBEM

- Given block-encoding U_A, QSVT provides (so far) the simplest implementation of linear algebra tasks; but U_A is hard to construct for real applications
- Maintain the structure of the circuit, and replace U_A by RAndom Circuit Block-Encoded Matrix (RACBEM)
- Whole machine, application based, scalable benchmark

Quantum benchmarks

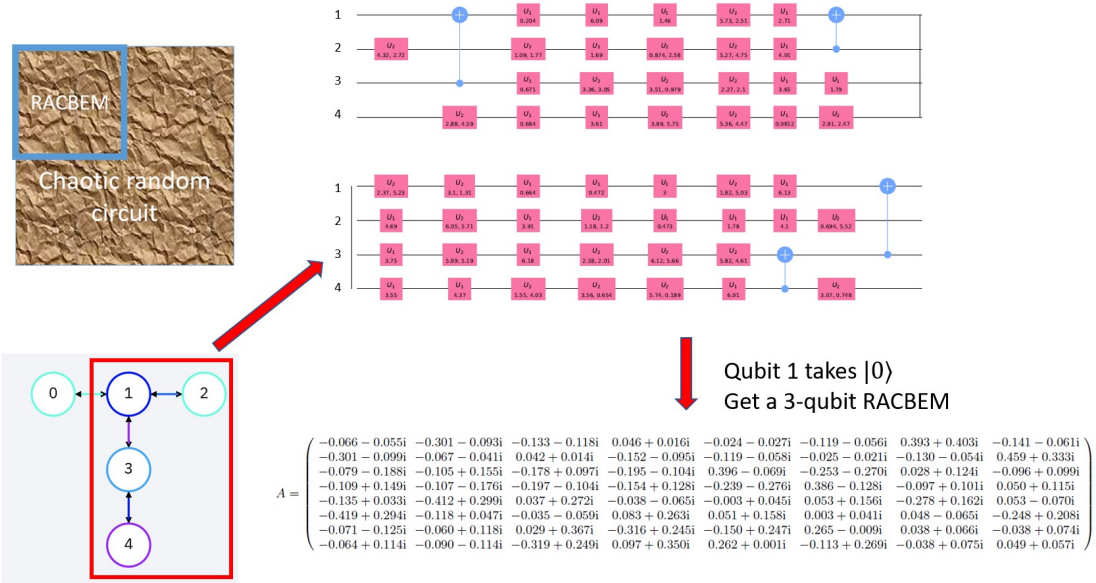
Randomized benchmarking [Magesan et al PRL 2011]
Gate based tomography [Blume-Kohout et al, Nat. Comm. 2017]

Quantum volume [Bishop et al, 2017; Cross et al, PRA 2019]
Cross-entropy [Boixo et al Nat Phys 2018; Arute et al Nature 2019]
Mirror circuit benchmark [Proctor et al, 2008.11294]
Schrodinger's microscope etc [Cornelissen et al, 2104.10698]

LINPACK benchmark [Dong, L. 2006.04010, PRA 2021]
Hamiltonian simulation benchmark [Dong, Whaley, L. 2108.03747]

LINPACK: Solve $Ax = b$ with some pseudo-random matrix A.
(Benchmarking classical supercomputers: TOP500 since 1993)

RAndom Circuit Block-Encoded Matrix (RACBEM)



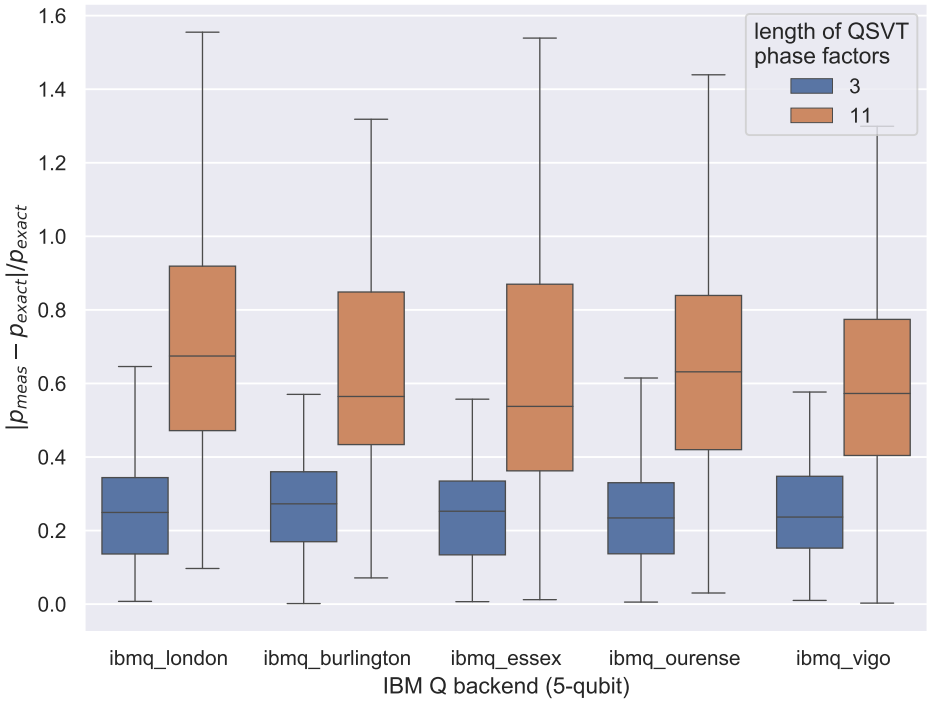
$$A = \begin{pmatrix} -0.066-0.055i & -0.301-0.093i & -0.133-0.118i & 0.046+0.016i & -0.024-0.027i & -0.119-0.056i & 0.393+0.403i & -0.141-0.061i \\ -0.301-0.099i & -0.067-0.041i & 0.042+0.014i & -0.152-0.095i & -0.119-0.058i & -0.025-0.021i & -0.130-0.054i & 0.459+0.333i \\ -0.079-0.188i & -0.105+0.155i & -0.178+0.097i & -0.195-0.104i & 0.396-0.069i & -0.253-0.270i & 0.028+0.124i & -0.096+0.099i \\ -0.109+0.149i & -0.107-0.176i & -0.197-0.104i & -0.154+0.128i & -0.239-0.276i & 0.386-0.128i & -0.097+0.101i & 0.050+0.115i \\ -0.135+0.033i & -0.412+0.299i & 0.037+0.272i & -0.038-0.065i & -0.003+0.045i & 0.053+0.156i & -0.278+0.162i & 0.053-0.070i \\ -0.419+0.294i & -0.118+0.047i & -0.035-0.059i & 0.083+0.263i & 0.051+0.158i & 0.003+0.041i & 0.048-0.065i & -0.248+0.208i \\ -0.071-0.125i & -0.060+0.118i & 0.029+0.367i & -0.316+0.245i & -0.150+0.247i & 0.265-0.009i & 0.038+0.066i & -0.038+0.074i \\ -0.064+0.114i & -0.090-0.114i & -0.319+0.249i & 0.097+0.350i & 0.262+0.001i & -0.113+0.269i & -0.038+0.075i & 0.049+0.057i \end{pmatrix}$$

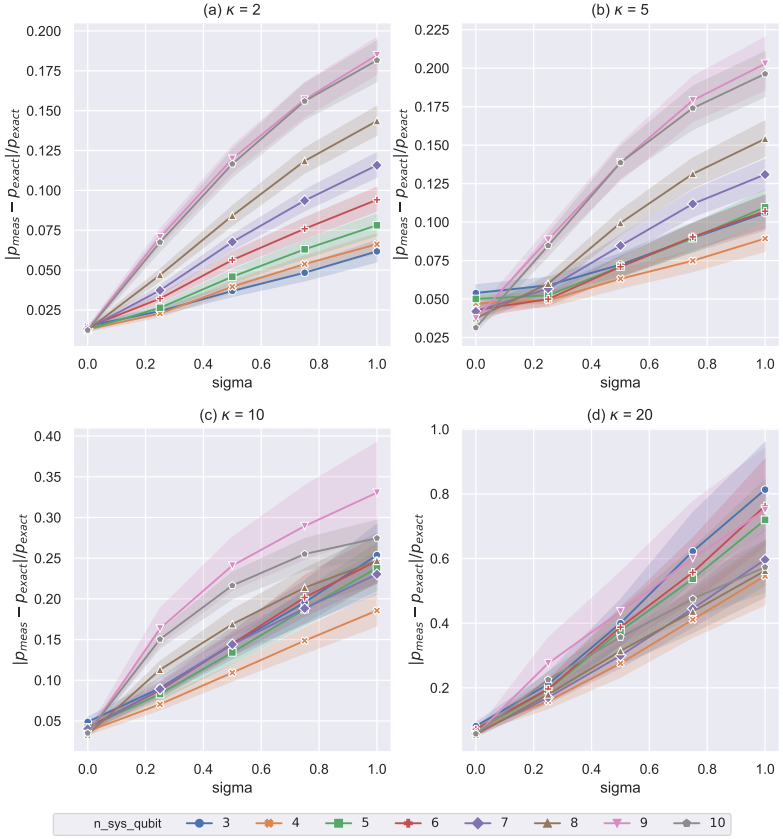
- A very flexible way to construct a non-unitary matrix with respect to any coupling map of the quantum architecture.
- Take upper-left diagonal block: measure one-qubit. $A = (\langle 0| \otimes I_n)\, U_A\, (|0\rangle \otimes I_n)$
- Explicit formulation of the Hermitian matrix by tracing out ancilla $\mathfrak{H} = [-2\sin(2\varphi_0)\sin\varphi_1]\, A^\dagger A + \cos(2\varphi_0 - \varphi_1)$.
- Fully adjustable condition number.

(Dong, L., Phys. Rev. A, 2021; 2006.04010); https://github.com/qsppack/racbem

Solving linear system on IBM Q and QVM

Compute $\left\|\mathfrak{H}^{-1}\left|0^n\right\rangle\right\|_2^2$. (sigma: noise level on QVM)
Well conditioned linear system

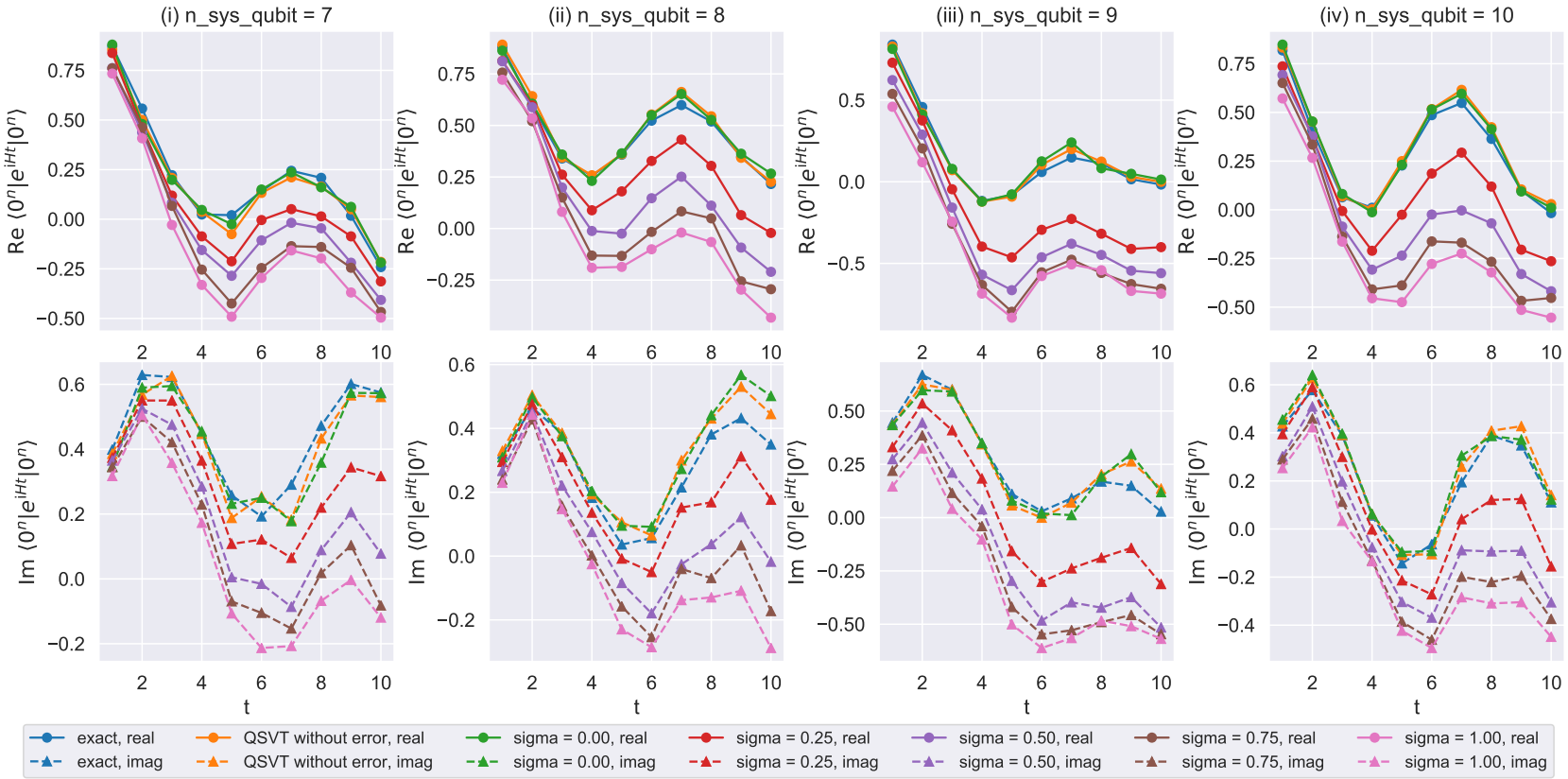




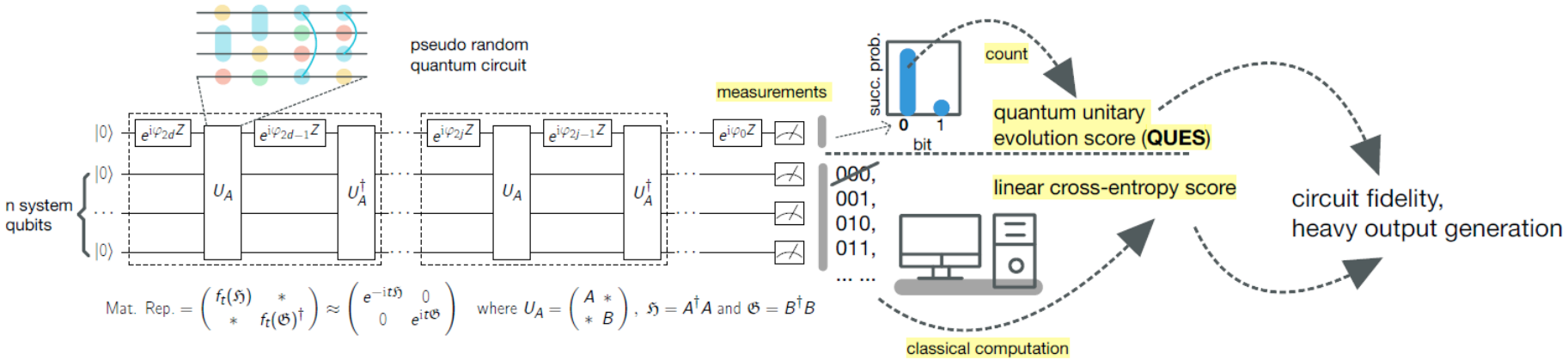
Time series (no Trotter)

$$s(t) = \langle \psi | e^{i\mathfrak{H}t} | \psi \rangle$$

QVM:



Quantum Hamiltonian simulation benchmark



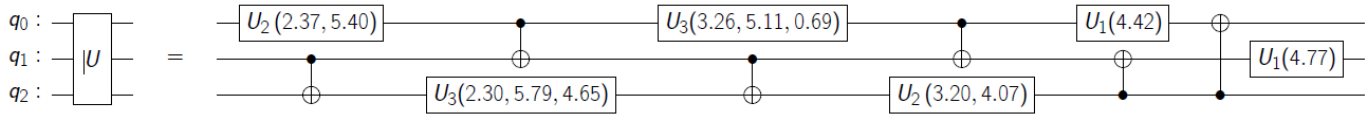
- Even simpler circuit (one ancilla in total) for Hamiltonian simulation $e^{it\mathfrak{H}}$; Remove the QSVT control block; Can use any 2-qubit gate
- Perhaps the simplest circuit for HS given block-encoding
- Quantum unitary evolution score (QUES)

$$\mathrm{QUES}(n, d) := \mathbb{E}\left(P_{\exp}(U_A)\right),$$

- Supremacy regime: similar to [Aaronson-Gunn 2019] (reduction from heavy output generation to a quantum threshold assumption)
- In progress on Google's Sycamore machine

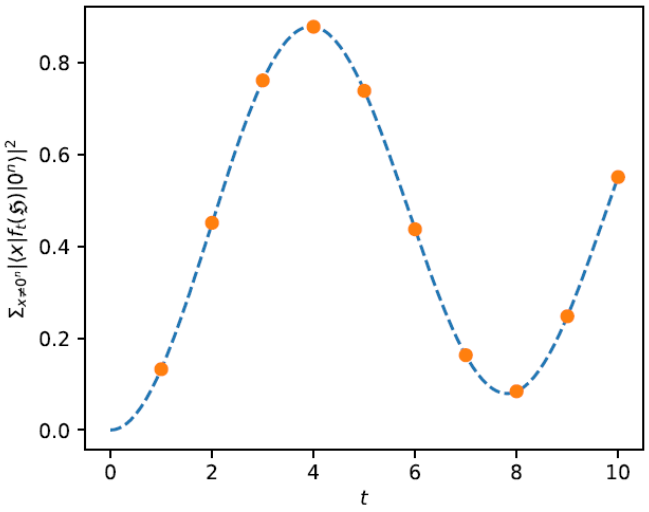
(Dong, Whaley, L. 2108.03747)

Illustration of Hamiltonian simulation benchmark

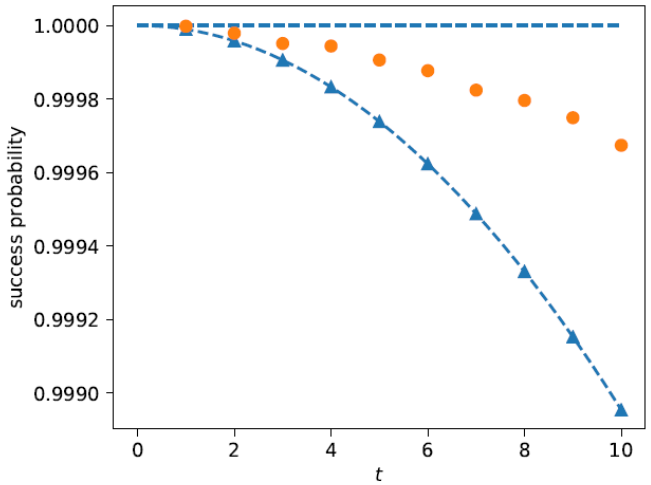


$$U = \begin{pmatrix} A = \begin{bmatrix} -0.04-0.01i & -0.01-0.02i & 0.13+0.25i & 0.63+0.14i \\ -0.17-0.23i & -0.15+0.63i & -0.01+0.04i & -0.01-0.01i \\ -0.06-0.52i & 0.13+0.45i & -0.01-0.03i & 0.00+0.03i \\ -0.02-0.02i & -0.03-0.02i & -0.40-0.34i & -0.31-0.36i \end{bmatrix} & \begin{matrix} 0.03-0.02i & 0.02+0.01i & 0.28+0.06i & 0.51-0.40i \\ 0.28+0.01i & -0.39-0.51i & 0.03+0.03i & -0.02+0.00i \\ -0.44-0.28i & 0.43+0.18i & 0.03+0.01i & -0.03-0.02i \\ -0.03+0.00i & -0.03+0.01i & 0.52-0.10i & 0.47-0.01i \end{matrix} \\ \begin{matrix} -0.03+0.00i & 0.03-0.01i & 0.46-0.07i & -0.53-0.00i \\ -0.27+0.39i & -0.22+0.48i & 0.02-0.03i & 0.02-0.02i \\ -0.29+0.58i & -0.28+0.06i & -0.02+0.01i & -0.01+0.04i \\ -0.01+0.01i & 0.04+0.01i & -0.64-0.07i & 0.20-0.20i \end{matrix} & \begin{matrix} -0.02+0.03i & 0.01-0.03i & -0.24+0.41i & 0.34-0.40i \\ 0.13+0.45i & 0.23+0.47i & 0.01+0.03i & 0.01+0.03i \\ -0.26-0.59i & 0.13-0.25i & -0.01+0.02i & 0.02+0.03i \\ -0.00-0.02i & -0.03+0.03i & -0.46+0.46i & -0.03-0.28i \end{matrix} \end{pmatrix}$$

(a)

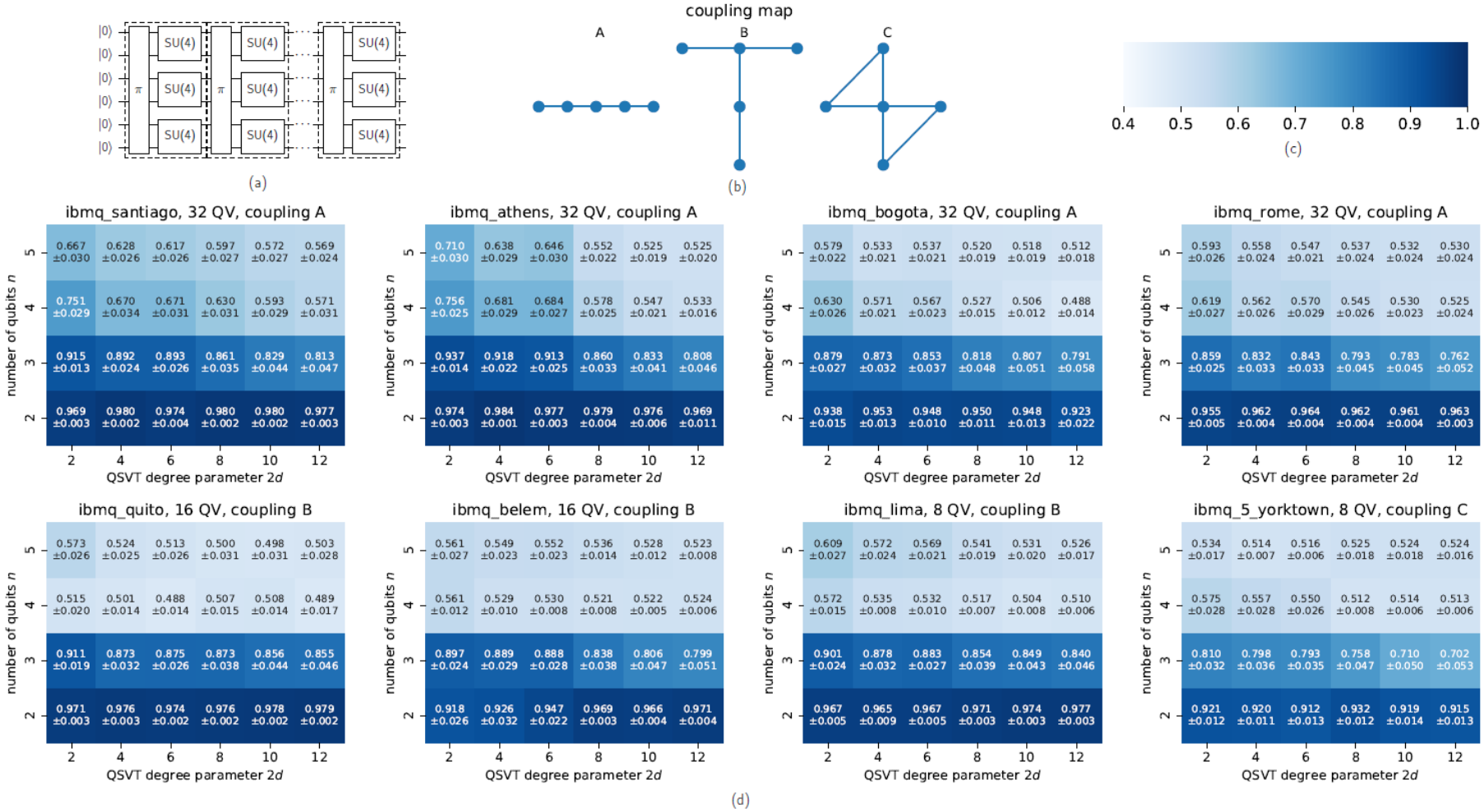


(b)

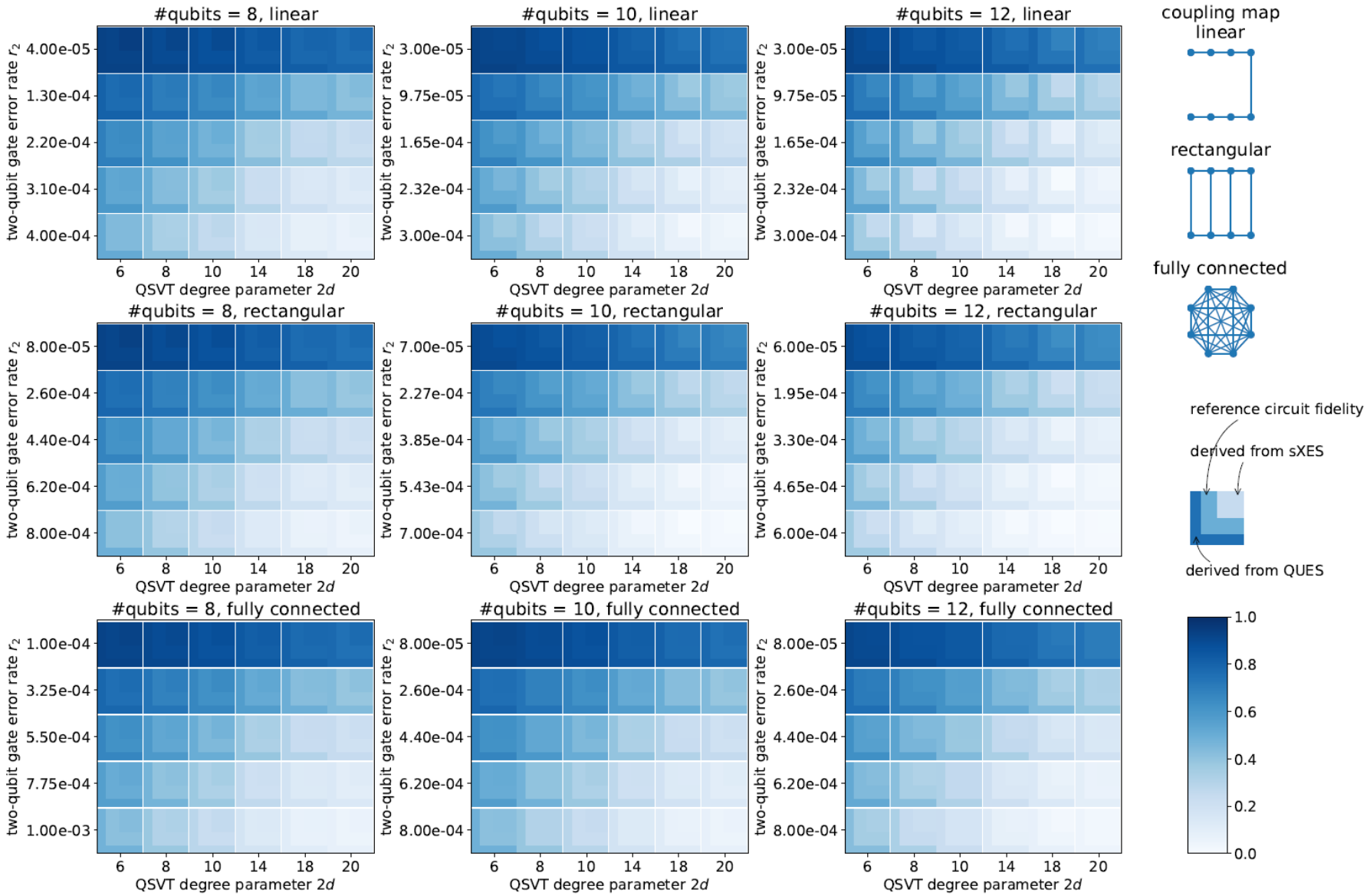


(c)

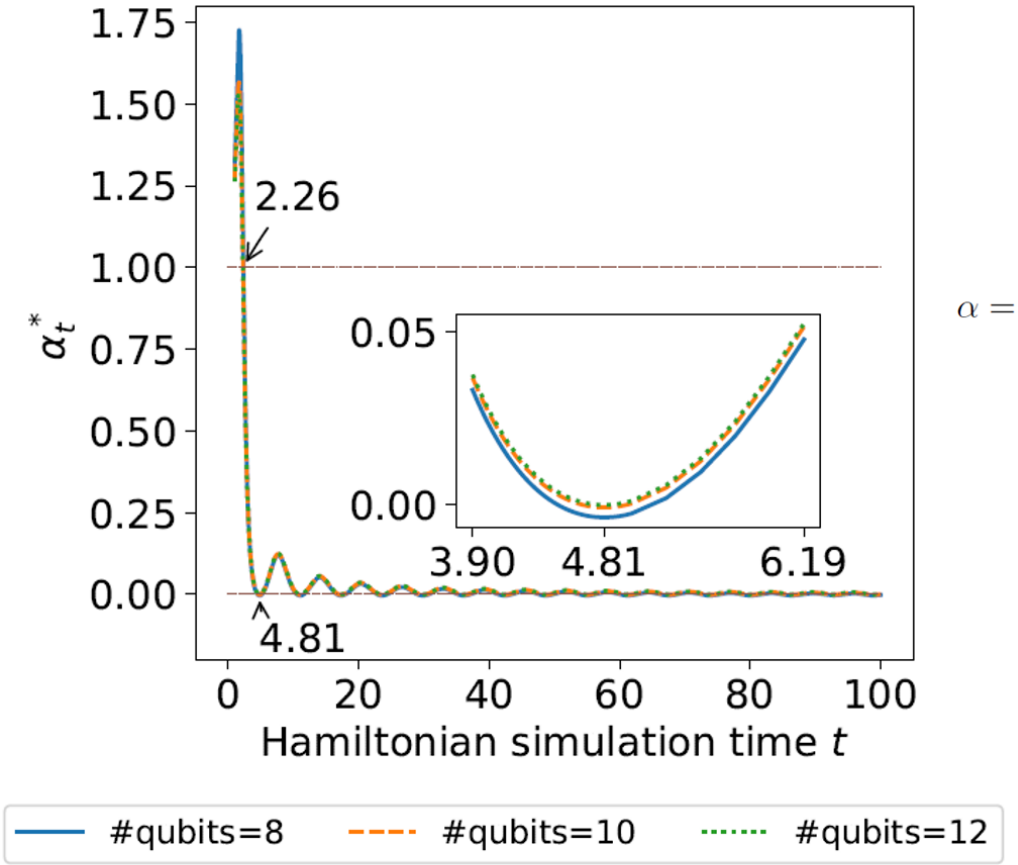
Quantum unitary evolution score (QUES) on IBMQ



Circuit fidelity from Hamiltonian simulation benchmark



Characterizing heavy output generation (HOG) regime



$$\alpha = \frac{\mathbb{E}\left(\sum_{x\neq 0} p_{\exp}(U,x)p(U,x)\right) - \frac{1}{2N}\mathbb{E}\left(\sum_{x\neq 0} p(U,x)\right)}{\mathbb{E}\left(\sum_{x\neq 0} p(U,x)^2\right) - \frac{1}{2N}\mathbb{E}\left(\sum_{x\neq 0} p(U,x)\right)}.$$

$\alpha \geq \alpha^*(t)$: HOG regime

(Dong, Whaley, L. 2108.03747)

Outline

QLSP: quantum and classical (iterative) solvers

- Positive definite matrix. Error in A-norm. $N = 2^n$
- Steepest descent: $\mathcal{O}(N\kappa \log(1/\epsilon))$; Conjugate gradient: $\mathcal{O}(N\sqrt{\kappa} \log(1/\epsilon))$
- Quantum algorithms can scale better in N but worse in κ.
- Lower bound: Quantum solver cannot generally achieve $\mathcal{O}(\kappa^{1-\delta})$ complexity for any $\delta > 0$ (Harrow-Hassadim-Lloyd, 2009)
- Goal of near-optimal quantum linear solver: $\widetilde{\mathcal{O}}(\kappa \mathrm{polylog}(1/\epsilon))$ complexity.

Compare the complexities of QLSP solvers

Significant progress in the past few years: Near-optimal complexity matching lower bounds. All with the promise of $\mathrm{poly}(n)$ complexity for matrix of size 2^n.

Algorithm	Query complexity
Quantum phase estimation (HHL) (Harrow-Hassidim-Lloyd, 2009)	$\widetilde{\mathcal{O}}(\kappa^2/\epsilon)$
Linear combination of unitaries (LCU) (Childs-Kothari-Somma, 2017)	$\widetilde{\mathcal{O}}(\kappa^2 \mathrm{polylog}(1/\epsilon))$
Quantum singular value transformation (QSVT) (Gilyén-Su-Low-Wiebe, 2019)	$\widetilde{\mathcal{O}}(\kappa^2 \log(1/\epsilon))$
Randomization method (RM) (Subasi-Somma-Orsucci, 2019)	$\widetilde{\mathcal{O}}(\kappa/\epsilon)$
Time-optimal adiabatic quantum computing (AQC(exp)) (An-L., 2019, 1909.05500)	$\widetilde{\mathcal{O}}(\kappa \,\mathrm{poly}\log(1/\epsilon))$
Eigenstate filtering[1] (L.-Tong, 1910.14596, Quantum 2020)	$\widetilde{\mathcal{O}}(\kappa \log(1/\epsilon))$

[1] When combined with a procedure inspired by quantum Zeno effects (a variant of adiabatic computation), achieve near-optimal complexity without any complex subroutines)

Preconditioned quantum linear system solver

- QLSP with $\|A\| \gg \|B\|$

$$(A + B)\,|x\rangle \sim |b\rangle$$

- Preconditioner: A^{-1}

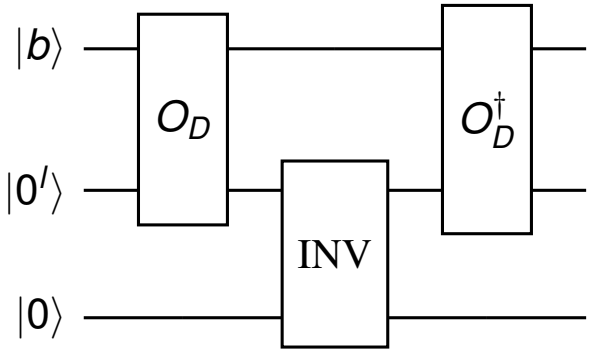
$$(I + A^{-1}B)\,|x\rangle \sim A^{-1}\,|b\rangle$$

- Condition number:
  $\kappa(I + A^{-1}B) \leq \left(1 + \|(A+B)^{-1}\|\|B\|\right)\left(1 + \|A^{-1}\|\|B\|\right)$

- Circuit depth: independent of $\|A\|$

Key: Fast inversion of diagonal matrices

- $D = \mathrm{diag}(D_{ii})$: $\left\|D^{-1}\right\| = \min |D_{ii}| = \Omega(1)$, $\|D\| = \max |D_{ii}| \gg 1$
- Assume $O_D \left|i\right\rangle \left|0^l\right\rangle = \left|i\right\rangle \left|D_{ii}\right\rangle, \quad i \in [N]$
- Circuit U_D' for the block-encoding of D^{-1} (classical arithmetic)



- Circuit depth is independent of $\|D\|$

(Tong-An-Wiebe-L., Phys. Rev. A, 2021; 2008.13295)

Example: elliptic partial differential equation

- Consider a 1D Poisson's equation:

$$-\Delta u(r) + u(r) = b(r), \quad r \in \Omega = [0, 1]. \tag{1}$$

- Discretize under planewave (Fourier) basis $\exp(2\pi \mathrm{i} kr)$:

$$\begin{pmatrix} 1 & & & \\ & 1 + (2\pi)^2 & & \\ & & \ddots & \\ & & & 1 + (2\pi N)^2 \end{pmatrix} \begin{pmatrix} \widehat{u}_0 \\ \widehat{u}_1 \\ \vdots \\ \widehat{u}_N \end{pmatrix} = \begin{pmatrix} \widehat{b}_0 \\ \widehat{b}_1 \\ \vdots \\ \widehat{b}_N \end{pmatrix}$$

- Circuit depth of unpreconditioned method depends on $\kappa(D) = \mathcal{O}(N^2)$

- Circuit depth of fast inversion: $\mathcal{O}(1)$.

Fast matrix function evaluation: Gibbs state preparation

- Prepare $\rho_\beta = \frac{1}{Z_\beta} e^{-\beta H}$, $Z_\beta = \mathrm{Tr}(e^{-\beta H})$.
- Purified Gibbs state $|\Psi\rangle = \frac{1}{\sqrt{Z_\beta}} \sum_{x \in [N]} |x\rangle \left(e^{-\beta H/2} |x\rangle\right)$: trace out first register $\Rightarrow$ Obtain ρ_β
- Two new approaches (convert to linear system problems):
  - Cauchy's contour integral formula:

$$e^{-\beta H} = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma e^{-\beta z} (z - H)^{-1} \, \mathrm{d}z$$

  - Inverse transform:

$$e^{-\beta H} = e^{-\beta (H^{-1})^{-1}}$$

Fast algorithm for preparing $\propto e^{-H}|b\rangle$

	Algorithm	Query complexities
w.o. preconditioner	Phase estimation (Poulin-Wocjan, 2009)	$\widetilde{\mathcal{O}}(\frac{\alpha_H}{\xi\epsilon})$
	LCU (van Apeldoorn et al, 2020)	$\widetilde{\mathcal{O}}(\frac{\alpha_H}{\xi}\log(\frac{1}{\epsilon}))$
w. preconditioner	This work (contour integral)	$\widetilde{\mathcal{O}}(\frac{\alpha_B}{\xi\widetilde{\sigma}'^2_{\min}}\log(\frac{1}{\epsilon}))$
	This work (inverse transformation)	$\widetilde{\mathcal{O}}\left(\frac{\alpha_B}{\xi\widetilde{\sigma}^2_{\min}}\left[\log\left(\frac{1}{\epsilon}\right)\right]^5\right)$

$H = A + B, \|A\| \gg \|B\|, \alpha_H \sim \|H\|, \alpha_B \sim \|B\|$

$\xi = \|e^{-H}|b\rangle\|, \widetilde{\sigma}'_{\min} = \Omega(1/\alpha_B), \widetilde{\sigma}_{\min} = \Omega(1/(1 + \|(A+B)^{-1}\|\|B\|))$

(Tong-An-Wiebe-L., Phys. Rev. A, 2021; 2008.13295)

Outline

Conclusion

- Large-scale fully error-corrected quantum computer remains really, really hard in the near future. Think about both near-term and long-term quantum algorithms.
- Many interesting, exciting progresses in the past few years on quantum linear algebra. Many more are coming.
- Linear system, preconditioning, eigenvalue problem (ground state), benchmark
- (Random) sparse matrix collection?
- Quantum Hamiltonian simulation benchmark on Sycamore
- Problem with more structural information. Unbounded operators.

Advertisements

IPAM workshop on “Quantum numerical linear algebra”

Quantum Numerical Linear Algebra

JANUARY 24 - 27, 2022

OVERVIEW | LODGING | APPLICATION & REGISTRATION

Overview

With the rapid development of quantum computers, a number of quantum algorithms have been developed and tested on both superconducting qubits based machines and trapped-ion hardware. The recent development of quantum algorithms has significantly pushed forward the frontier of using quantum computers for performing a wide range of numerical linear algebra tasks, such as solving linear systems, eigenvalue decomposition, singular value decomposition, matrix function evaluation etc. While many quantum algorithms aim at future fault-tolerant quantum architecture, some of such numerical linear algebra algorithms have already demonstrated promise for being implemented on near term quantum devices. This workshop brings together leading experts in quantum numerical linear algebra, to discuss the recent development of quantum algorithms to perform linear algebra tasks for solving challenging problems in science and engineering and for various industrial and technological applications.

This workshop will include a poster session; a request for posters will be sent to registered participants in advance of the workshop.

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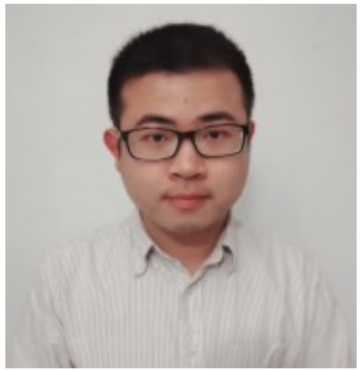

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Thank you for your attention!

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U.S. DEPARTMENT OF ENERGY | Office of Science
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Google AI Quantum

Block-encoding

Definition

Given an n-qubit matrix A, if we can find $\alpha, \epsilon \in \mathbb{R}_+$*, and an* $(m+n)$*-qubit unitary matrix* U_A *so that that*

$$\|A - \alpha\left(\langle 0^m| \otimes I_n\right) U_A \left(|0^m\rangle \otimes I_n\right)\| \leq \epsilon,$$

then U_A *is called an* (α, m, ϵ)*-block-encoding of A.*

- A "gray box" for the read-in problem.
- Many examples of block-encoding: density operators, POVM operators, *d*-sparse matrices, addition and multiplication of block-encoded matrices (Gilyén-Su-Low-Wiebe, 2019)

HOG and QUATH

Definition (HOG, or Heavy Output Generation)

Given a circuit $\mathcal{U}_{f,U}$ and only nonzero bit-string whose ancilla components equal 0 is accepted, generate k nonzero distinct samples $\{0x_i : 0^n \neq x_i \in \{0,1\}^n, i = 1, \cdots, k\}$ so that $\frac{1}{k}\sum_{i=1}^{k} p(U, x_i) \geq b2^{-n}$ for some $b \geq 1$.

Definition (QUATH, or Quantum Threshold Assumption)

There is no polynomial-time classical algorithm that takes as input a quantum circuit $U \in \mathrm{U}(2^{n+1})$ and produces an estimate p of $p_1 := p(U, 1)$ so that

$$\mathbb{E}\left((p_1 - p)^2\right) = \mathbb{E}\left((p_1 - 2^{-n})^2\right) - \Omega(2^{-3n}).$$

HOG and QUATH

Analogous to [Aaronson-Gunn, 2019]

Theorem

Assuming QUATH, no polynomial-time classical algorithm can solve HOG with probability $s > \frac{1}{2} + \frac{1}{2b}$*, and*

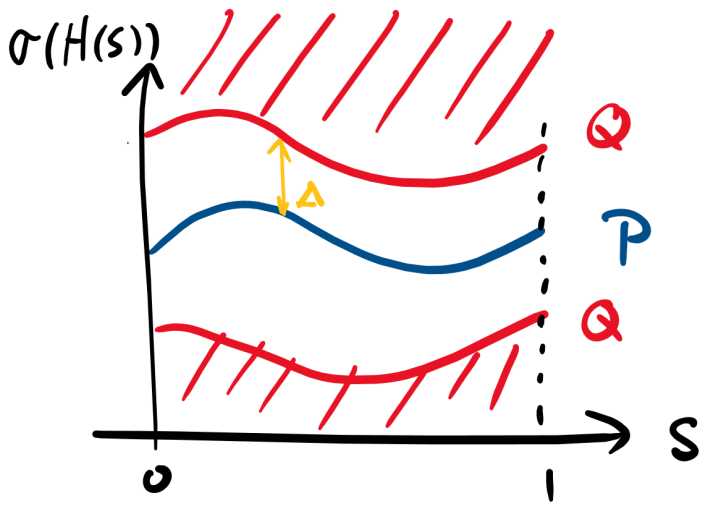
$$k \geq \frac{1}{((2s-1)b-1)(b-1)}.$$

Adiabatic computation

- (Born-Fock, 1928)

  A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.

- Albash, Avron, Babcock, Cirac, Cerf, Elgart, Hagedorn, Jansen, Kato, Lidar, Nenciu, Roland, Ruskai, Seiler, Wiebe...



$$H(s) = (1-s)H_0 + sH_1,$$

$$\frac{1}{T}\mathrm{i}\partial_s \left|\psi_T(s)\right\rangle = H(s)\left|\psi_T(s)\right\rangle, \quad \left|\psi_T(0)\right\rangle = \left|\psi_0\right\rangle$$

$\left|\psi_T(1)\right\rangle \approx \psi(1)$ (up to a phase factor), T sufficiently large?

Reformulating QLSP into an eigenvalue problem

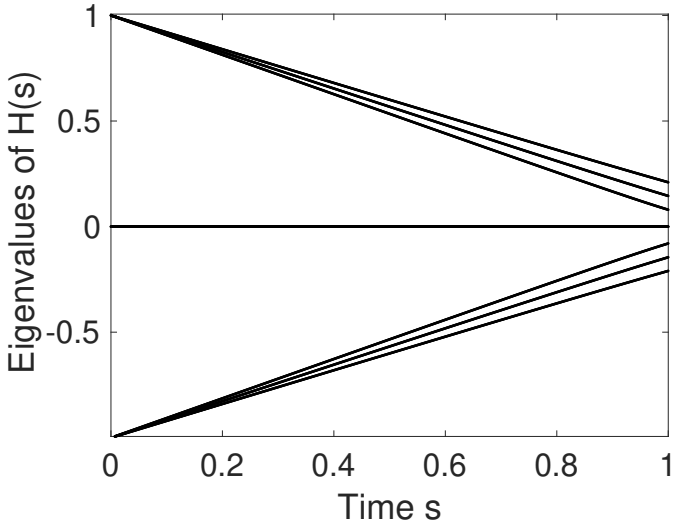
- Weave together linear system, eigenvalue problem, differential equation (Subasi-Somma-Orsucci, 2019)
- $Q_b = I_N - |b\rangle\langle b|$. If $A|x\rangle = |b\rangle \quad \Rightarrow \quad Q_b A|x\rangle = Q_b|b\rangle = 0$
- Then

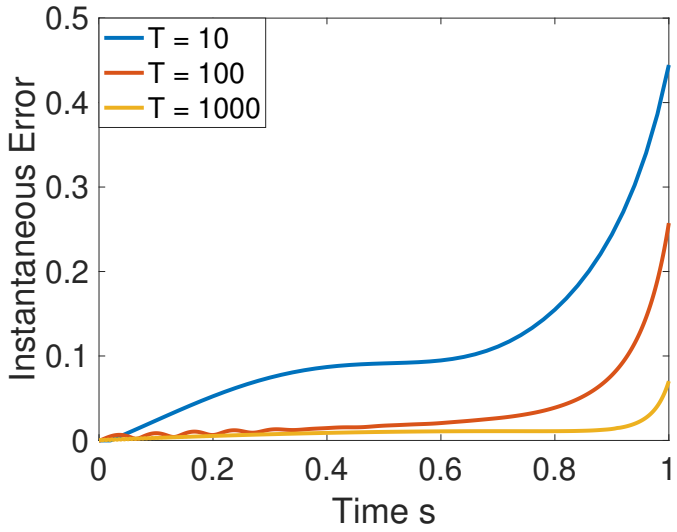
$$H_1 = \begin{pmatrix} 0 & AQ_b \\ Q_b A & 0 \end{pmatrix}, \quad |\widetilde{x}\rangle = |0\rangle|x\rangle = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\mathrm{Null}(H_1) = \mathrm{span}\{|\widetilde{x}\rangle, |\bar{b}\rangle\}, \quad |\bar{b}\rangle = |1\rangle|b\rangle = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

- QLSP $\quad \Rightarrow \quad$ Find an eigenvector of H_1 with eigenvalue 0.

Slow convergence with respect to T





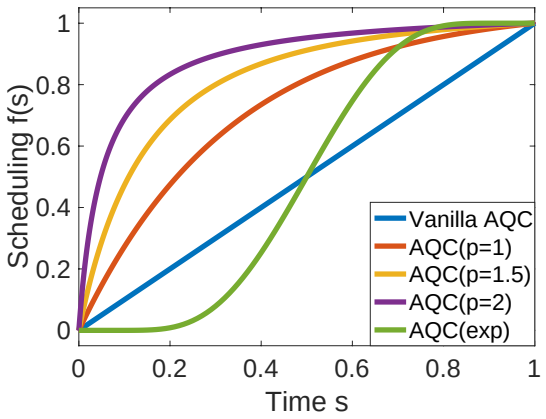
Algorithm	Time complexity (T)
Direct (Jansen-Ruskai-Seiler, 2007)	$\mathcal{O}(\kappa^3/\epsilon)$
Randomization method (RM) (Subasi-Somma-Orsucci, 2019)	$\mathcal{O}(\kappa \log(\kappa)/\epsilon)$
Time-optimal adiabatic quantum computing (AQC(exp)) (An-L., 2019)	$\mathcal{O}(\kappa \operatorname{poly} \log(\kappa/\epsilon))$

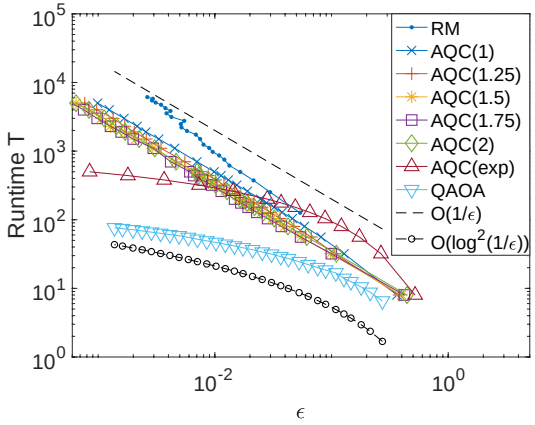
AQC(exp): exponential improvement w.r.t. ϵ

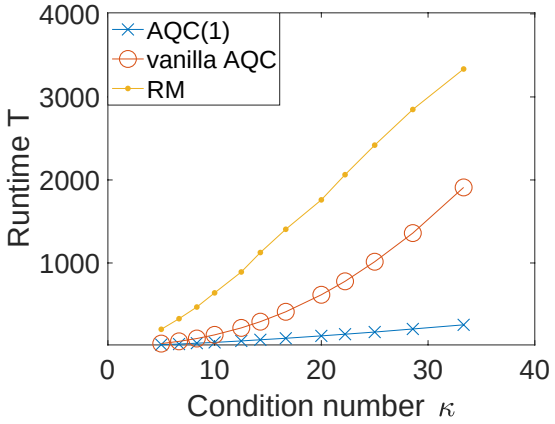
- Adiabatic evolution with $H(f(s)) = (1 - f(s))H_0 + f(s)H_1$

$$f(s) = c_e^{-1} \int_0^s \exp\left(-\frac{1}{s'(1-s')}\right) \mathrm{d}s'$$

- allow $H(f(s))$ to slow down when the gap is close to 0, to cancel with the vanishing gap.

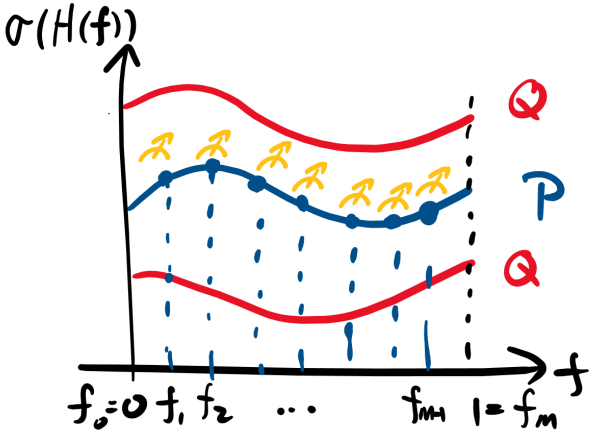




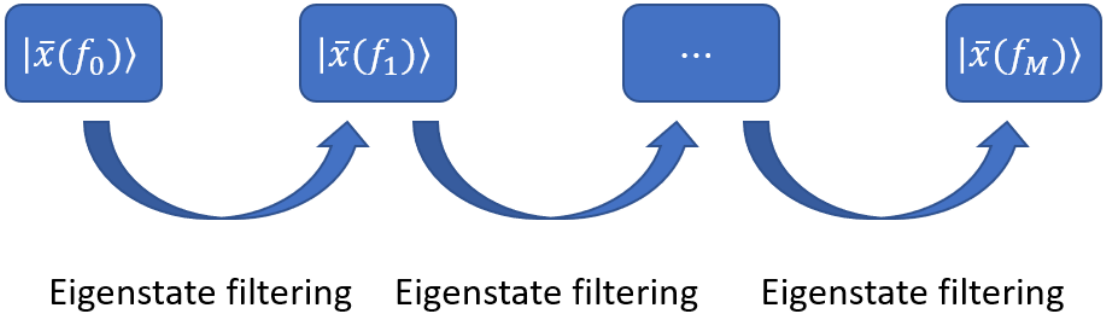


(An-L., 2019, 1909.05500)

Solving QLSP via eigenstate filtering and quantum Zeno effect



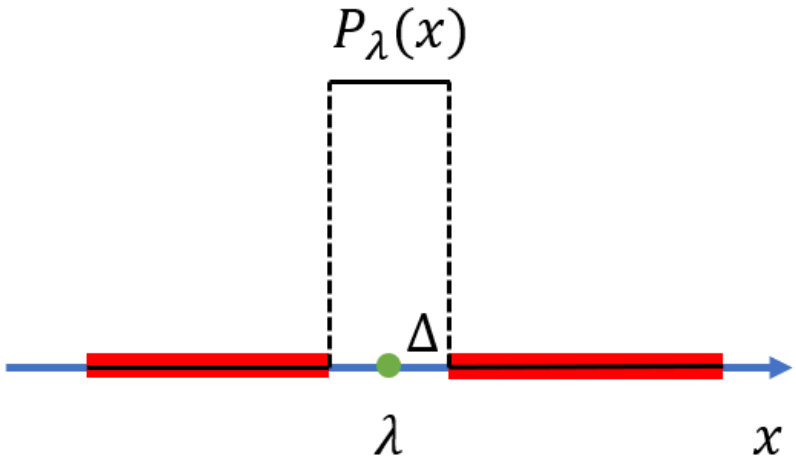
- QZE: variant of adiabatic computation. Frequent measurement $\Rightarrow$ Projection
- Measure the state $|\bar{x}(f_{j-1})\rangle$ in the eigenbasis of $H(f_j)$.
- Fidelity approaches 1 as step size decreases.
- Replace measurement with eigenstate filtering (projection).

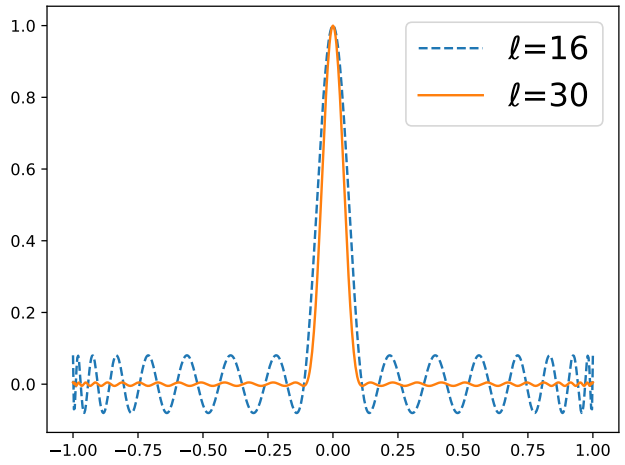


(L.-Tong, 1910.14596, Quantum 2020)

Eigenstate filtering problem

- H is Hermitian. λ is an eigenvalue of H, separated from the rest of the spectrum by a gap Δ.
- P_λ: projection operator into the λ-eigenspace of H. How to find a polynomial P to approximate P_λ?
- Requirement: $P(\lambda) = 1$ and $|P(\lambda')|$ is small for $\lambda' \in \sigma(H) \backslash \{\lambda\}$.





Application of eigenstate filtering: Solving QLSP via quantum Zeno effect (QZE)

Theorem (L.-Tong, 1910.14596)

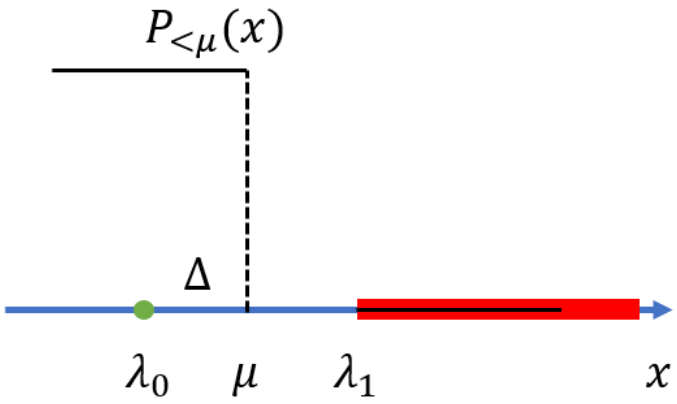
A is a d-sparse Hermitian matrix with condition number κ, $\|A\|_2 \leq 1$. Then $|x\rangle \propto A^{-1}|b\rangle$ can be obtained with fidelity $1-\epsilon$ using

1. $\mathcal{O}\left(d\kappa\left(\log(\kappa)\log\log(\kappa)+\log(\frac{1}{\epsilon})\right)\right)$ *queries to $A, |b\rangle$,*
2. $\mathcal{O}\left(nd\kappa\left(\log(\kappa)\log\log(\kappa)+\log(\frac{1}{\epsilon})\right)\right)$ *other primitive gates,*
3. $\mathcal{O}(n)$ *qubits.*

- Fully-gate based implementation.
- No need for time-dependent Hamiltonian simulation.
- Successive projection along the carefully scheduled adiabatic path.
- Near-optimal complexity!

Ground state energy

- $H\,|\psi_0\rangle = \lambda_0\,|\psi_0\rangle$. w.o. assumption the problem is QMA-complete.
- Initial state $|\phi_0\rangle$ prepared by unitary U_I, w. assumptions
  (P1) Lower bound for the overlap: $|\,\langle\phi_0|\psi_0\rangle\,| \geq \gamma$,
  (P2) Bounds for the ground energy and spectral gap:
  $\lambda_0 \leq \mu - \Delta/2 < \mu + \Delta/2 \leq \lambda_1$.



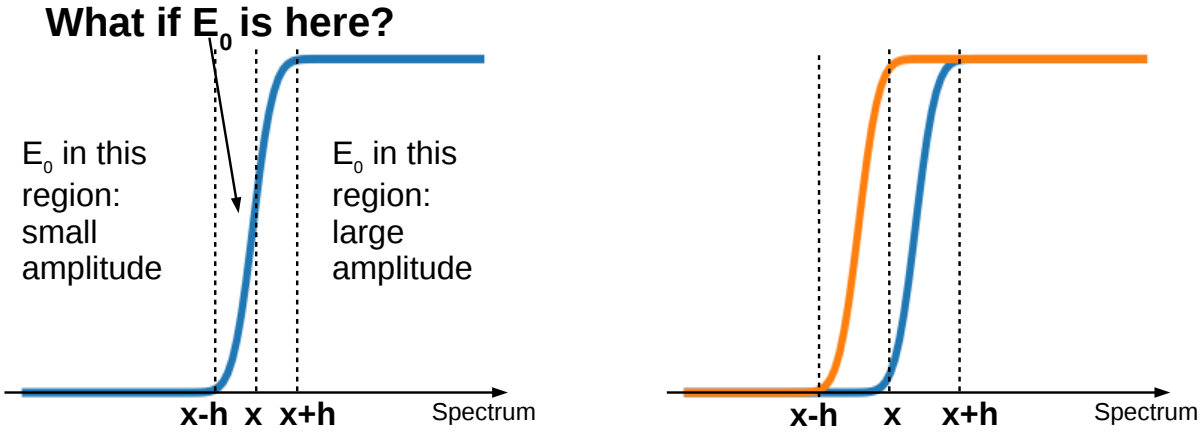
Near-optimal algorithm for finding ground state energy

		Preparation (bound known)	Ground energy	Preparation (bound unknown)
U_H	Our work	$\mathcal{O}\left(\frac{\alpha}{\gamma\Delta}\log(\frac{1}{\epsilon})\right)$	$\widetilde{\mathcal{O}}\left(\frac{\alpha}{\gamma h}\log(\frac{1}{\vartheta})\right)$	$\widetilde{\mathcal{O}}\left(\frac{\alpha}{\gamma\Delta}\log(\frac{1}{\vartheta\epsilon})\right)$
	GTC19	$\widetilde{\mathcal{O}}\left(\frac{\alpha}{\gamma\Delta}\right)$	$\widetilde{\mathcal{O}}\left(\frac{\alpha^{3/2}}{\gamma h^{3/2}}\right)$	$\widetilde{\mathcal{O}}\left(\frac{\alpha^{3/2}}{\gamma\Delta^{3/2}}\right)$
U_I	Our work	$\mathcal{O}\left(\frac{1}{\gamma}\right)$	$\widetilde{\mathcal{O}}\left(\frac{1}{\gamma}\log(\frac{\alpha}{h})\log(\frac{1}{\vartheta})\right)$	$\widetilde{\mathcal{O}}\left(\frac{1}{\gamma}\log(\frac{\alpha}{\Delta})\log(\frac{1}{\vartheta})\right)$
	GTC19	$\widetilde{\mathcal{O}}\left(\frac{1}{\gamma}\right)$	$\widetilde{\mathcal{O}}\left(\frac{1}{\gamma}\sqrt{\frac{\alpha}{h}}\right)$	$\widetilde{\mathcal{O}}\left(\frac{1}{\gamma}\sqrt{\frac{\alpha}{\Delta}}\right)$
Extra qubits	Our work	$\mathcal{O}(1)$	$\mathcal{O}(\log(\frac{1}{\gamma}))$	$\mathcal{O}(\log(\frac{1}{\gamma}))$
	GTC19	$\mathcal{O}(\log(\frac{1}{\Delta}\log(\frac{1}{\epsilon})))$	$\mathcal{O}(\log(\frac{1}{h}))$	$\mathcal{O}(\log(\frac{1}{\Delta}\log(\frac{1}{\epsilon})))$

Well-known result: phase estimation; Previous best results: (Ge-Tura-Cirac, 2019)
h: precision of the ground energy estimate; $1-\vartheta$: success probability
Lower bound for the overlap: $|\langle\phi_0|\psi_0\rangle| \geq \gamma$,
Bounds for the ground energy and spectral gap: $\lambda_0 \leq \mu - \Delta/2 < \mu + \Delta/2 \leq \lambda_1$.

(L.-Tong, 2002.12508, Quantum 2020)

Binary search for ground energy



- Construct filtering polynomial with cost $\mathcal{O}(\frac{1}{h}\log(\frac{1}{\epsilon}))$ by approximating erf (Low-Chuang, 2017)
- Apply two shifted polynomials. Return with high confidence: $E_0 \geq x - h$ or $E_0 \leq x + h$.
- Perform binary search for E_0.