Quantum numerical linear algebra

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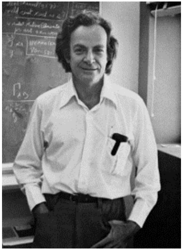
Solve nature with nature: Where are we today after four decades?

Nature evolves quantum mechanics in real time.

Use nature to solve quantum mechanical problems.



"... if you want to make a simulation of nature, you'd better make it quantum mechanical, and by golly it's a wonderful problem, because it doesn't look so easy."'

– Richard P. Feynman (1981) 1st Conference on Physics and Computation, MIT

What does nature do

- Evolution of the Schrödinger equation

$$\mathrm{i}\frac{\partial}{\partial t}\left|\psi(t)\right\rangle = H\left|\psi(t)\right\rangle$$

- Solution

$$\left|\psi(t)\right\rangle = e^{-\mathrm{i}Ht}\left|\psi(0)\right\rangle := U(t)\left|\psi(0)\right\rangle .$$

- $U(t)$: unitary matrix.

Shor's algorithm for prime factorization

- $n = p \cdot q$ (p, q are prime numbers)
- Classical algorithm with best asymptotic complexity (super-polynomial): General number field sieve
  $\mathcal{O}\left(\exp\left[c(\log n)^{\frac{1}{3}}(\log\log n)^{\frac{2}{3}}\right]\right)$
- (Shor, 1994) Quantum algorithm achieves polynomial complexity
  $\mathcal{O}\left((\log n)^2(\log\log n)(\log\log\log n)\right)$
- "First wave" of interests on quantum computing

MIT and quantum computation

Chuang, Farhi, Goldstone, Harrow, Lloyd, Shapiro, Shor..

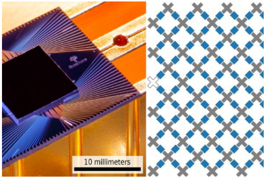
Quantum supremacy

- John Preskill coined the term “Quantum supremacy” in 2012.
- Describe the point *where quantum computers can do things that classical computers cannot, regardless of whether those tasks are useful.*
- It was not clear whether this “moderate” goal can be reached..

  Is controlling large-scale quantum systems merely really, really hard, or is it ridiculously hard? – John Preskill (2012)
- Noisy Intermediate-Scale Quantum (NISQ) technology.

Quantum supremacy

- Quantum supremacy has been reached! (It is merely really, really hard) (Martinis et al, Nature, 2019) Google's 54-qubit machine *Sycamore*

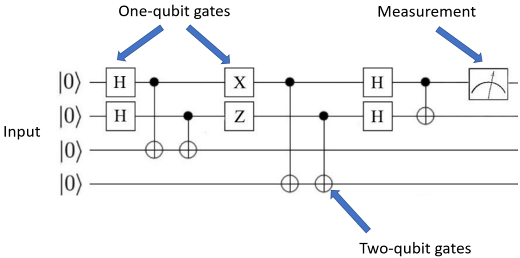


- Finding the probability of bit-strings from a random quantum circuit. Exponentially hard for classical computer w.r.t. n.
- When quantum computer can do something useful, it is called "Quantum advantage".
- We are in the "second wave" of interests on quantum computing.

What is a quantum computer (mathematically)

- $|\psi\rangle \in \mathbb{C}^N \cong (\mathbb{C}^2)^{\otimes n}, \quad N = 2^n$. n : number of qubits.
- $U \in \mathbb{C}^{N\times N}$ is unitary. $U\,|\psi\rangle$ is efficient to apply.
- $U_N \cdots U_1\,|\psi\rangle$, and then measure one or a few qubits (with output $0/1$).
- Quantum computer: cost $\mathrm{poly}(n)$, potential exponential speedup.

Quantum circuit: “graphical” linear algebra



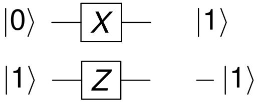
Quantum circuit: “graphical” linear algebra

- State vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Pauli matrices

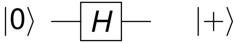
$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad Y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$



Quantum circuit: "graphical" linear algebra

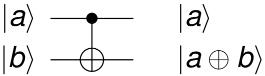
- Hadamard gate $H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

$$H\,|0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) := |+\rangle$$



- CNOT gate

$$\text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$



- See more in the classical textbook (Nielsen-Chuang, 2000), or simply take a course at MIT..

Quantum numerical linear algebra

- Solving linear systems, eigenvalue problems, matrix exponentials, least square problems, singular value decompositions etc on a quantum computer.
- Many interesting, exciting progresses in the past few years.
- Reasonable way towards “quantum advantage”.
- Related to “Quantum machine learning”.
- Solving linear equations

$$Ax = b$$

- Quantum linear system problem (QLSP)

$$A|x\rangle \propto |b\rangle$$

Our recent works on quantum numerical linear algebra

Near-optimal QLSP solver (adiabatic computing)	(An-**L.**, 1909.05500)
Near-optimal QLSP solver (eigenstate filtering)	(**L.**-Tong, 1910.14596)
Near-optimal ground energy solver	(**L.**-Tong, 2002.12508)
Quantum signal process phase factor	(Dong-Meng-Whaley-**L.**, 2002.11649)
Policy gradient method and QAOA	(Yao-Bukov-**L.**, 2002.01068)
Proposal of quantum LINPACK benchmark	(Dong-**L.**, 2006.04010)
Fast inversion and preconditioned linear solver	(Tong-An-Wiebe-**L.**, 2008.13295)

A concrete, toy example

$$A = \frac{1}{4}X + \frac{3}{4}I = \begin{pmatrix} 0.75 & 0.25 \\ 0.25 & 0.75 \end{pmatrix}, \quad |b\rangle = |0\rangle .$$

- X, I are unitaries. A is a linear combination of unitaries (LCU), and is itself non-unitary. $\kappa(A) = 2$.
- Idea: extend 1-qubit non-unitary matrix to a 2-qubit unitary matrix

$$U_A = \begin{pmatrix} A & \cdot \\ \cdot & \cdot \end{pmatrix}$$

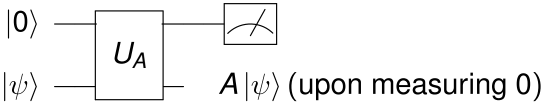
- Block-encoding

A concrete, toy example

- An example of block-encoding. Unitary. Use 1 ancilla qubit.

$$U_A = \begin{pmatrix} \boxed{\begin{matrix} 0.750 & 0.250 \\ 0.250 & 0.750 \end{matrix}} & \begin{matrix} 0.433 & -0.433 \\ -0.433 & 0.433 \end{matrix} \\ \begin{matrix} 0.433 & -0.433 \\ -0.433 & 0.433 \end{matrix} & \begin{matrix} 0.250 & 0.750 \\ 0.750 & 0.250 \end{matrix} \end{pmatrix}$$

- U_A should be viewed as a mapping on $(\mathbb{C}^2)^{\otimes 2}$.
- $(\langle 0| \otimes I) U_A (|0\rangle \otimes I) = A$.



- U_A is our oracle.

A concrete, toy example

- Inverse

$$A^{-1} = \begin{pmatrix} 1.5 & -0.5 \\ -0.5 & 1.5 \end{pmatrix}$$

Note $\|A^{-1}\| = 2 > 1$, no hope to have

$$U_{A^{-1}} = \begin{pmatrix} A^{-1} & \cdot \\ \cdot & \cdot \end{pmatrix}$$

- How about (with $\alpha > 1$)

$$U_{A^{-1}} \approx \begin{pmatrix} A^{-1}/\alpha & \cdot \\ \cdot & \cdot \end{pmatrix}$$

- Construct $U_{A^{-1}}$ using U_A, $U_A^\dagger$, and simple quantum gates (in this case $U_A = U_A^\dagger$).

Such an $U_{A^{-1}}$ exists

$$U_{A^{-1}} = \begin{pmatrix} 0.075 & -0.025 & 0.0 & 0.0 & 0.271j & 0.728j & -0.442j & 0.442j \\ -0.025 & 0.075 & 0.0 & 0.0 & 0.728j & 0.271j & 0.442j & -0.442j \\ 0.0 & 0.0 & 0.075 & -0.025 & -0.442j & 0.442j & -0.271j & -0.728j \\ 0.0 & 0.0 & -0.025 & 0.075 & 0.442j & -0.442j & -0.728j & -0.271j \\ 0.271j & 0.728j & -0.442j & 0.442j & 0.075 & -0.025 & 0.0 & 0.0 \\ 0.728j & 0.271j & 0.442j & -0.442j & -0.025 & 0.075 & 0.0 & 0.0 \\ -0.442j & 0.442j & -0.271j & -0.728j & 0.0 & 0.0 & 0.075 & -0.025 \\ 0.442j & -0.442j & -0.728j & -0.271j & 0.0 & 0.0 & -0.025 & 0.075 \end{pmatrix}$$

- We find

$$A^{-1}/\alpha = \begin{pmatrix} 0.075 & -0.025 \\ -0.025 & 0.075 \end{pmatrix}, \quad \alpha = 20.$$

- Use 2 ancilla qubits.

Procedure to construct $U_{A^{-1}} |b\rangle$

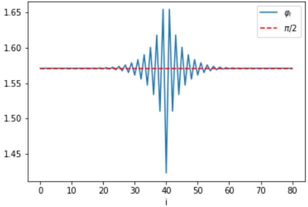
Does not look like any classical direct or iterative algorithm.

- Start from $|0\rangle |0\rangle |b\rangle \equiv (b_1, b_2, 0, 0, 0, 0, 0, 0)^\top$
- Apply $H \otimes I \otimes I$ and obtain $|+\rangle |0\rangle |b\rangle$, where $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$.
- For $i = 0, \ldots, 2d - 1$[1]
  Apply $U_{\varphi_i} \otimes I$, where $U_\varphi = e^{\mathrm{i}\varphi Z} \otimes |0\rangle \langle 0| + e^{-\mathrm{i}\varphi Z} \otimes |1\rangle \langle 1|$.
  Apply $I \otimes U_A$
- Apply $U_{\varphi_{2d}} \otimes I$
- Apply $H \otimes I \otimes I$
- (Optionally, multiply a global phase factor $(-1)^d$)
- Measure the ancilla qubits, i.e. $(\langle 0^2| \otimes I) |\psi\rangle \approx A^{-1}/\alpha |b\rangle$.

[1] This is a simplified procedure using that U_A is Hermitian, $A \succ 0$;$\{\varphi_i\}_{i=0}^{2d}$ are called phase factors.

Accuracy

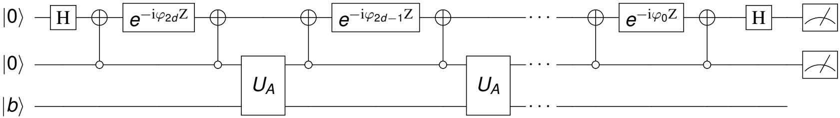
Take $d = 80$, plot the phase factors



- Decay of the phase factor away from the center
- Error for approximating A^{-1}/α

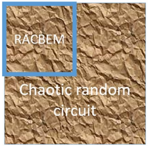
$$\begin{pmatrix} -2.046 \times 10^{-11} & 2.532 \times 10^{-11} \\ 2.532 \times 10^{-11} & -2.046 \times 10^{-11} \end{pmatrix}$$

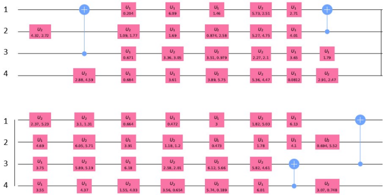
Simplifying the presentation using a quantum circuit

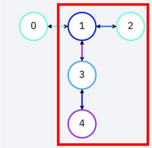


- The same circuit works for arbitrarily large matrix.
- A special case of quantum signal processing (Low-Chuang, 2017) and quantum singular value transformation circuit (QSVT) (Gilyén-Su-Low-Wiebe, 2019).
- One of the most interesting developments in quantum algorithms in the past decade.
- Polynomial eigenvalue transformation and singular value transformation with a definite parity.

RAndom Circuit Block-Encoded Matrix (RACBEM)







Qubit 1 takes |0⟩
Get a 3-qubit RACBEM

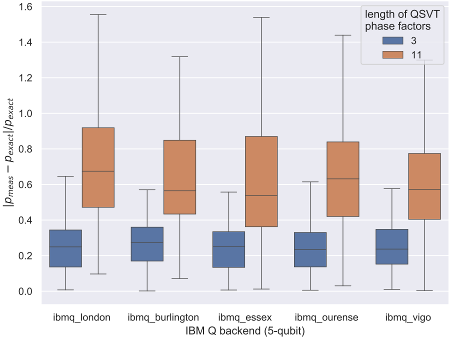
$$A = \begin{pmatrix} -0.066-0.055i & -0.301-0.093i & -0.133-0.118i & 0.046+0.016i & -0.024-0.027i & -0.119-0.056i & 0.393+0.403i & -0.141-0.061i \\ -0.301-0.099i & -0.067-0.041i & 0.042+0.014i & -0.152-0.095i & -0.119-0.058i & -0.025-0.021i & -0.130-0.054i & 0.459+0.333i \\ -0.079-0.188i & -0.105+0.155i & -0.178+0.097i & -0.195-0.104i & 0.396-0.069i & -0.253-0.270i & 0.028+0.124i & -0.096+0.099i \\ -0.109+0.149i & -0.107-0.176i & -0.197-0.104i & -0.154+0.128i & -0.239-0.276i & 0.386-0.128i & -0.097+0.101i & 0.050+0.115i \\ -0.135+0.033i & -0.412+0.299i & 0.037+0.272i & -0.038-0.065i & -0.003+0.045i & 0.053+0.156i & -0.278+0.162i & 0.053-0.070i \\ -0.419+0.294i & -0.118+0.047i & -0.035-0.059i & 0.083+0.263i & 0.051+0.158i & 0.003+0.041i & 0.048-0.065i & -0.248+0.208i \\ -0.071-0.125i & -0.060+0.118i & 0.029+0.367i & -0.316+0.245i & -0.150+0.247i & 0.265-0.009i & 0.038+0.066i & -0.038+0.074i \\ -0.064+0.114i & -0.090-0.114i & -0.319+0.249i & 0.097+0.350i & 0.262+0.001i & -0.113+0.269i & -0.038+0.075i & 0.049+0.057i \end{pmatrix}$$

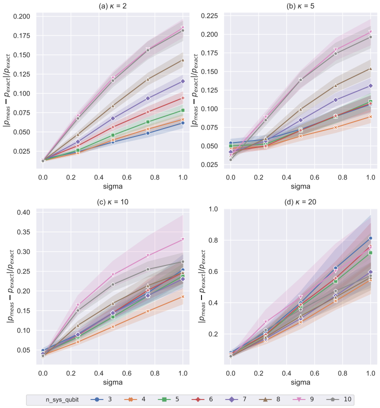
- A very flexible way to construct a non-unitary matrix with respect to any coupling map of the quantum architecture.
- Take upper-left diagonal block: measure one-qubit.
  $A = (\langle 0| \otimes I_n)\, U_A \,(|0\rangle \otimes I_n)$

[1] (Dong, **L.**, arXiv: 2006.04010)

Solving linear system on IBM Q and QVM

Compute $\left\|\mathfrak{H}^{-1}\left|0^n\right\rangle\right\|_2^2$. (sigma: noise level on QVM)





RACBEM

Source Code:
https://github.com/qsppack/racbem

Random circuit block-encoded matrix and
a proposal of quantum LINPACK benchmark

Yulong Dong[1,2] and Lin Lin[3,4*]
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[3] Department of Mathematics, University of California, Berkeley, California 94720 USA and
[4] Computational Research Division, Lawrence Berkeley National Laboratory, Berkeley, CA 94720, USA

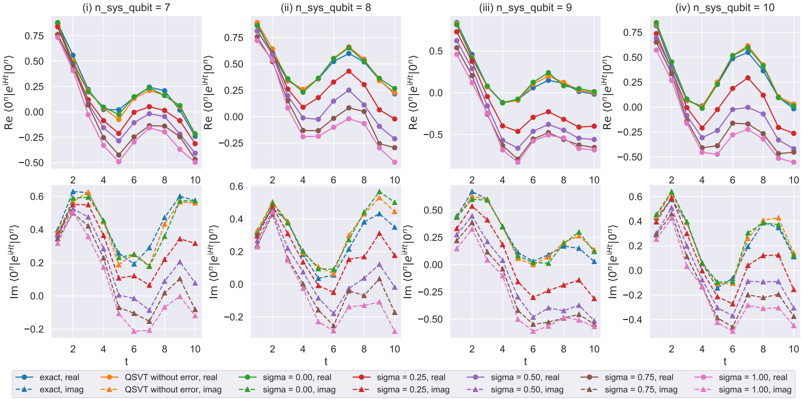
Example:

```
from racbem import *
from qiskit import execute
import numpy as np

n_sys_qubit = 3             # the number of system qubits
n_be_qubit = 1              # the number of block-encoding qubit
n_sig_qubit = 1             # the number of signal qubit
n_tot_qubit = n_sig_qubit+n_be_qubit+n_sys_qubit
n_depth = 15                # the depth of random circuit
prob_one_q_op = 0.5         # the probability of selecting a one-qubit
                            # operation when two_q_op is allowed
n_shots = 2**13             # the number of shots used in measurements

# instances of RACBEM
be = BlockEncoding(n_be_qubit, n_sys_qubit)
qsp = QSPCircuit(n_sig_qubit, n_be_qubit, n_sys_qubit)

# build random circuit with respect to the architecture, including
# 'basis_gates' and 'coupling_map', retrieved from the hardware
be.build_random_circuit(n_depth, basis_gates=basis_gates,
        prob_one_q_op=prob_one_q_op, coupling_map=be_map)

# retrieve the block-encoded matrix
UA = retrieve_unitary_matrix(be.qc)
A = UA[0:2**n_sys_qubit, 0:2**n_sys_qubit]

# build the inverse unitary matrix
be.build_dag()

# build Quantum Signal Processing (QSP) circuit with respect to the
# given set of phase factors 'phi_seq', which determines the functionality
qsp.build_circuit(be.qc, be.qc_dag, phi_seq, realpart=True, measure=True)

# execute the compiled circuit on the quantum backend 'noisy_backend'
job = execute(qsp.qcircuit, backend=noisy_backend, shots=n_shots)

# get the result from the backend
result = job.result()
counts = result.get_counts(qsp.qcircuit)

# success probability of the QSP circuit
prob_meas = np.float(counts['00']) / n_shots
```

Time series (no Trotter)

$$s(t) = \langle \psi | e^{i\mathfrak{H}t} | \psi \rangle$$

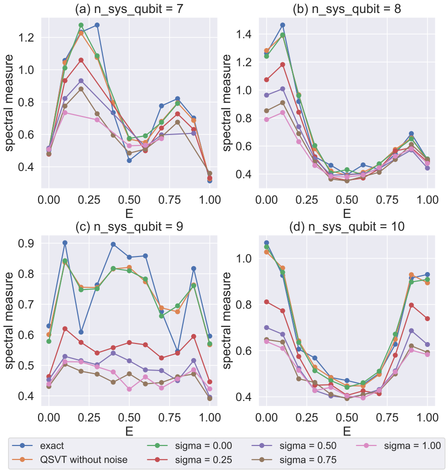
QVM:



Spectral measure

$$s(E) = \langle\psi|\delta(\mathfrak{H} - E)|\psi\rangle \approx \frac{1}{\pi} \operatorname{Im} \langle\psi|(\mathfrak{H} - E - \mathrm{i}\eta)^{-1}|\psi\rangle$$

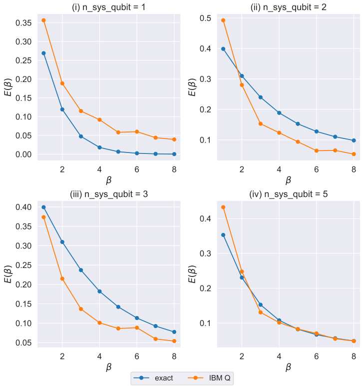
QVM:

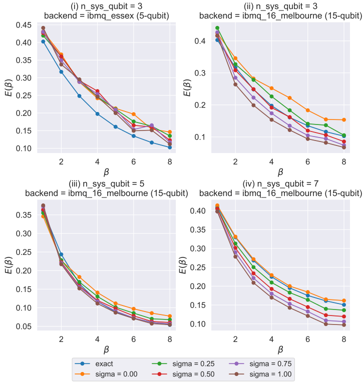


Thermal energy

$$E(\beta) = \frac{\mathrm{Tr}[\mathfrak{H} e^{-\beta \mathfrak{H}}]}{\mathrm{Tr}[e^{-\beta \mathfrak{H}}]}$$

IBM Q (left) and QVM (right)





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[1] Use the minimally entangled typical thermal state (METTS) algorithm (White, 2009) (Motta et al, 2020)

FAQ (usually from a math audience)

1. *Is quantum linear algebra a real thing?* Yes and no (usually works with complex arithmetic..)

2. *How do you get the matrix / vector into the computer?* Read-in problem, e.g. RACBEM, LCU, block-encoding, Trotter

3. *Which quantity do you measure?* Read-out problem. e.g. some success probabilities and/or access to samples

4. *How do you know your answer is correct?* Verification problem. Performance guarantee versus *a posteriori* verification

Quantum linear system problem (QLSP)

- All vectors must be normalized. $A \in \mathbb{C}^{N\times N}, |b\rangle \in \mathbb{C}^N, N = 2^n$.
- $\||b\rangle\|^2 := \langle b|b\rangle = 1$.
- Solution vector
$$|x\rangle = \frac{A^{-1}|b\rangle}{\|A^{-1}|b\rangle\|}.$$
- How to get $A, |b\rangle$ into a quantum computer? read-in problem. Oracular assumption.
- Query complexity: the number of oracles used.

Quantum speedup

- κ: condition number of A. ϵ: target accuracy. Proper assumptions on A (e.g. d-sparse) so that oracles cost $\mathrm{poly}(n)$.
- (Harrow-Hassadim-Lloyd, 2009): $\mathcal{O}(\text{poly}(n)\kappa^2/\epsilon)$. Exponential speedup with respect to n.
- (Childs-Kothari-Somma, 2017): Linear combination of unitary (LCU). $\mathcal{O}(\text{poly}(n)\kappa^2\text{poly}(\log(\kappa/\epsilon)))$
- (Low-Chuang, 2017) (Gilyén-Su-Low-Wiebe, 2019): Quantum signal processing (QSP). $\mathcal{O}(\text{poly}(n)\kappa^2\text{poly}(\log(\kappa/\epsilon)))$

Comparison with classical iterative solvers

- Positive definite matrix. Error in A-norm
- Steepest descent: $\mathcal{O}(N\kappa\log(1/\epsilon))$; Conjugate gradient: $\mathcal{O}(N\sqrt{\kappa}\log(1/\epsilon))$
- Quantum algorithms can scale better in N but worse in κ.
- Lower bound: Quantum solver cannot generally achieve $\mathcal{O}(\kappa^{1-\delta})$ complexity for any $\delta > 0$ (Harrow-Hassadim-Lloyd, 2009)
- Goal of near-optimal quantum linear solver: $\mathcal{O}(\mathrm{poly}(n)\kappa\ \mathrm{polylog}(\kappa/\epsilon))$ complexity.

Compare the complexities of QLSP solvers

Significant progress in the past few years: Near-optimal complexity matching lower bounds.

Algorithm	Query complexity	Remark
HHL,(Harrow-Hassidim-Lloyd, 2009)	$\widetilde{\mathcal{O}}(\kappa^2/\epsilon)$	w. VTAA, complexity becomes $\widetilde{\mathcal{O}}(\kappa/\epsilon^3)$ (Ambainis 2010)
Linear combination of unitaries (LCU),(Childs-Kothari-Somma, 2017)	$\widetilde{\mathcal{O}}(\kappa^2 \mathrm{polylog}(1/\epsilon))$	w. VTAA, complexity becomes $\widetilde{\mathcal{O}}(\kappa \,\mathrm{poly}\log(1/\epsilon))$
Quantum singular value transformation (QSVT) (Gilyén-Su-Low-Wiebe, 2019)	$\widetilde{\mathcal{O}}(\kappa^2 \log(1/\epsilon))$	Queries the RHS only $\widetilde{\mathcal{O}}(\kappa)$ times
Randomization method (RM) (Subasi-Somma-Orsucci, 2019)	$\widetilde{\mathcal{O}}(\kappa/\epsilon)$	Prepares a mixed state; w. repeated phase estimation, complexity becomes $\widetilde{\mathcal{O}}(\kappa \,\mathrm{poly}\log(1/\epsilon))$
Time-optimal adiabatic quantum computing (AQC(exp)) (An-**L.**, 2019, 1909.05500)	$\widetilde{\mathcal{O}}(\kappa \,\mathrm{poly}\log(1/\epsilon))$	No need for any amplitude amplification. Use time-dependent Hamiltonian simulation.
Eigenstate filtering (**L.**-Tong, 2019, 1910.14596)	$\widetilde{\mathcal{O}}(\kappa \log(1/\epsilon))$	No need for any amplitude amplification. Does not rely on any complex subroutines.

Reformulating QLSP into an eigenvalue problem

- Weave together linear system, eigenvalue problem, differential equation (Subasi-Somma-Orsucci, 2019)
- $Q_b = I_N - |b\rangle\langle b|$. If $A|x\rangle = |b\rangle \quad \Rightarrow \quad Q_b A|x\rangle = Q_b|b\rangle = 0$
- Then

$$H_1 = \begin{pmatrix} 0 & AQ_b \\ Q_b A & 0 \end{pmatrix}, \quad |\widetilde{x}\rangle = |0\rangle|x\rangle = \begin{pmatrix} x \\ 0 \end{pmatrix}$$

$$\mathrm{Null}(H_1) = \mathrm{span}\{|\widetilde{x}\rangle, |\bar{b}\rangle\}, \quad |\bar{b}\rangle = |1\rangle|b\rangle = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

- QLSP $\quad \Rightarrow \quad$ Find an eigenvector of H_1 with eigenvalue 0.

Adiabatic computation

- Known eigenstate $H_0 \left|\psi_0\right\rangle = \lambda_0 \left|\psi_0\right\rangle$ for some H_0.
- Interested in some eigenstate $H_1 \left|\psi_1\right\rangle = \lambda_1 \left|\psi_1\right\rangle$
- $H(s) = (1 - s)H_0 + sH_1$,

$$\frac{1}{T}\mathrm{i}\partial_s \left|\psi_T(s)\right\rangle = H(s) \left|\psi_T(s)\right\rangle, \quad \left|\psi_T(0)\right\rangle = \left|\psi_0\right\rangle$$

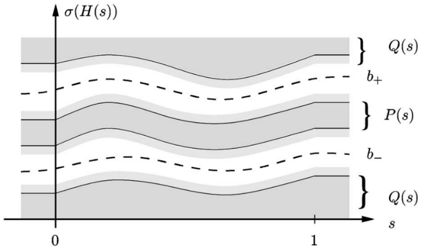
- $\left|\psi_T(1)\right\rangle \approx \psi(1)$ (up to a phase factor), T sufficiently large?
- Gate-based implementation: time-dependent Trotter, for near-optimal complexity (Low-Wiebe, 2019)

Adiabatic computation

- (Born-Fock, 1928)

  A physical system remains in its instantaneous eigenstate if a given perturbation is acting on it slowly enough and if there is a gap between the eigenvalue and the rest of the Hamiltonian's spectrum.

- Albash, Avron, Babcock, Cirac, Cerf, Elgart, Hagedorn, Jansen, Kato, Lidar, Nenciu, Roland, Ruskai, Seiler, Wiebe...



Adiabatic quantum computation (AQC) for QLSP

- Introduce

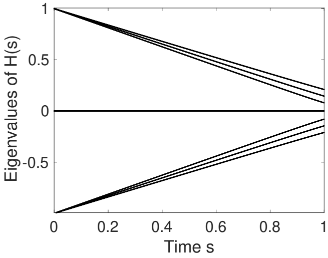
$$H_0 = \begin{pmatrix} 0 & Q_b \\ Q_b & 0 \end{pmatrix}, \quad \mathrm{Null}(H_0) = \mathrm{span}\{|\widetilde{b}\rangle, |\bar{b}\rangle\}$$

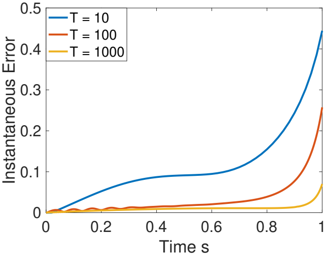
$$|\widetilde{b}\rangle = |0\rangle|b\rangle = \begin{pmatrix} b \\ 0 \end{pmatrix}, \quad |\bar{b}\rangle = |1\rangle|b\rangle = \begin{pmatrix} 0 \\ b \end{pmatrix}$$

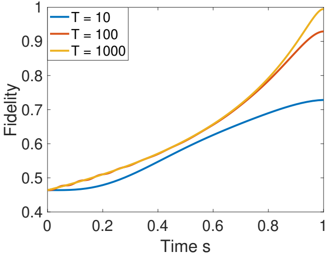
- Adiabatically connecting $|\widetilde{b}\rangle$ (zero eigenvector of H_0) to $|\widetilde{x}\rangle$ (zero eigenvector of H_1) (Subasi-Somma-Orsucci, 2019)

- Only one eigenvector in the null space is of interest: transition to $|\bar{b}\rangle$ is prohibited during dynamics

Eigenvalue gap and fidelity







Fidelity:

$$F(|\varphi\rangle, |\psi\rangle) = |\langle\varphi|\psi\rangle|^2 = \text{Tr}[P_\varphi P_\psi].$$

Adiabatic quantum computation

Theorem (Jansen-Ruskai-Seiler, 2007)

Hamiltonian $H(s)$, $P(s)$ projector to eigenspace of $H(s)$ separated by a gap $\Delta(s)$ from the rest of the spectrum of $H(s)$

$$|1 - \langle \psi_T(s) | P(s) | \psi_T(s) \rangle| \leq \eta^2(s), \quad 0 \leq s \leq 1$$

where

$$\eta(s) = \frac{C}{T} \Big\{ \frac{\|H^{(1)}(0)\|_2}{\Delta^2(0)} + \frac{\|H^{(1)}(s)\|_2}{\Delta^2(s)} + \int_0^s \left(\frac{\|H^{(2)}(s')\|_2}{\Delta^2(s')} + \frac{\|H^{(1)}(s')\|_2^2}{\Delta^3(s')} \right) ds' \Big\}.$$

T: time complexity; $1/T$ convergence.
$\Delta(s) \geq \Delta_*$, $T \sim \mathcal{O}((\Delta_*)^{-3}/\epsilon)$ (worst case)

Implication in QLSP

- Lower bound of gap (Assume $A \succ 0$ for now, can be relaxed)

$$\Delta(s) \geq \Delta_*(s) = 1 - s + s/\kappa \geq \kappa^{-1}$$

- Worst-case time complexity $T \sim \mathcal{O}(\kappa^3/\epsilon)$
- AQC inspired algorithm: randomization method (Subasi-Somma-Orsucci, 2019),

$$T \sim \mathcal{O}(\kappa \log(\kappa)/\epsilon)$$

ϵ : 2-norm error of the density matrix.

- Rescheduled dynamics.

Accelerate AQC for QLSP: Scheduling

- Goal: improve the scaling AQC w.r.t. κ.
- Adiabatic evolution with $H(f(s)) = (1 - f(s))H_0 + f(s)H_1$

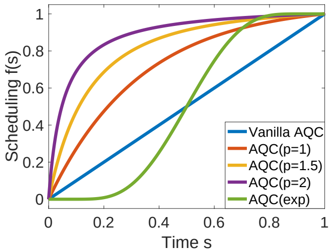
$$\frac{1}{T}\mathrm{i}\partial_s \left|\psi_T(s)\right\rangle = H(f(s)) \left|\psi_T(s)\right\rangle , \quad \left|\psi_T(0)\right\rangle = |\widetilde{b}\rangle$$

- $f(s)$: scheduling function. $0 \leq f(s) \leq 1, f(0) = 0, f(1) = 1$.
- allow $H(f(s))$ to slow down when the gap is close to 0, to cancel with the vanishing gap.
- (Roland-Cerf, 2002) for time-optimal AQC of Grover search.

Choice of scheduling function: AQC(exp)

- AQC(exp): modified schedule (slow at beginning and end)

$$f(s) = c_e^{-1} \int_0^s \exp\left(-\frac{1}{s'(1-s')}\right) \mathrm{d}s'$$



- Intuition: error bound of (Jansen-Ruskai-Seiler, 2007) and integration by parts (Wiebe-Babcock, 2012)

- Rigorous proof of exponential convergence: follow the idea of (Nenciu, 1993), asymptotic expansion of $P(s)$

Near-optimal time complexity

Theorem (An-**L.**, 1909.05500)

$A \succ 0$, condition number κ. Then for large enough $T > 0$, the error of the AQC(exp) scheme is

$$\| P_T(1) - |\widetilde{x}\rangle\langle\widetilde{x}| \|_2 \le C \log(\kappa) \exp\left(-C \left(\frac{\kappa \log^2 \kappa}{T} \right)^{-\frac{1}{4}} \right).$$

Therefore the runtime $T = \mathcal{O}\left(\kappa \log^2(\kappa) \log^4\left(\frac{\log \kappa}{\epsilon}\right)\right)$.

Near-optimal complexity (up to $\mathrm{polylog}$ factors).
Similar results for Hermitian indefinite and non-Hermitian matrices.

Conclusion

- Large-scale fully error-corrected quantum computer remains at least really, really, really hard in the near future. Think about both near-term and long-term for quantum linear algebra.
- Quantum signal processing and quantum singular value transformation: polynomial approximation theory in SU(2).
- Decay properties of phase factors.
- Statistics of random-circuit block-encoded matrix: truncated random unitaries and classical hardness
- Fast thermal state preparation.

NSF Quantum Leap Challenge Institutes, 2021-2025

Full name	Department affiliation	Institutional affiliation
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Thank you for your attention!

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