Math 54: Worksheet

April 18

(1) Find the general solution of
$$y''(t) + y(t) = t\cos(t).$$

(2) True or false: the set of solutions of the ODE in the last problem is a vector space.

(3) Consider the differential equation:
$$y''(t) + y(t) = 0.$$
Derive an equivalent linear system of differential equations in normal form, i.e., in the form $\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{f}(t)$.

(4) Consider the system of differential equations:
$$\begin{cases} y_1''(t) + ty_1'(t) - y_2(t) = e^t \\ y_2'(t) + \cos(t)y_1(t) = 0 \end{cases}$$
Derive an equivalent linear system of differential equations in normal form.

(5) Recall that for $\mathbb{R}^n$-valued functions $\mathbf{x}_1, \ldots, \mathbf{x}_n$, the Wronskian of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$ is the $\mathbb{R}$-valued function $W[\mathbf{x}_1, \ldots, \mathbf{x}_n](t) = \det\left(\mathbf{x}_1(t) \ \cdots \ \mathbf{x}_n(t)\right)$.

Suppose that $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are solutions of $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$ on some open interval I. Then which of the following are possible: (1) the Wronskian is zero on all of I, (2) the Wronskian is never zero on I, (3) the Wronskian takes both zero and nonzero values on I.

In each case, what can we conclude about the linear independence or linear dependence of $\{\mathbf{x}_1, \ldots, \mathbf{x}_n\}$?

(6) Compute the Wronskian $W[\mathbf{x}_1, \mathbf{x}_2]$ determined by
$$\mathbf{x}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix}.$$
$\mathbf{x}_1$ and $\mathbf{x}_2$ happen to be solutions of a system of ODEs $\mathbf{x}'(t) = A\mathbf{x}(t)$. Can you figure out what A must be?

Now consider
$$\mathbf{x}_1(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix}, \quad \mathbf{x}_2(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}.$$
Is it possible that $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of some system $\mathbf{x}'(t) = A\mathbf{x}(t)$? How about $\mathbf{x}'(t) = A(t)\mathbf{x}(t)$? How could this be determined from the Wronskian?

(7) Suppose that $\mathbf{x} : I \to \mathbb{R}^n$ solves $\mathbf{x}'(t) = A\mathbf{x}(t)$, and moreover suppose that we can diagonalize $A = PDP^{-1}$. Define a new function $\mathbf{z} : I \to \mathbb{R}^n$ by $\mathbf{z}(t) = P^{-1}\mathbf{x}(t)$. Verify that $\mathbf{z}$ solves $\mathbf{z}'(t) = D\mathbf{z}(t)$. What is so great about this observation?