Math 54: Worksheet

March 19

This worksheet is meant to provide some practice with concepts related to orthogonality. Since not all of this material will be reviewed via homework before the midterm, take extra care to make sure that you can solve these kinds of problems! Consider this as your crash course in the orthogonality skills that might show up on the midterm. (Unfortunately, I don't know what's on the midterm, so I can't guarantee that there won't be surprises! What I can say is that this is probably my best worksheet so far.) Detailed solutions are available at `https://math.berkeley.edu/~lindsey/math54`

(1) Let

$$W = \text{span}\left[\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right] \subset \mathbb{R}^3.$$

Find an orthogonal basis for W and an orthogonal basis for $W^\perp$. [**Hint**: for the second part, apply the fact that $\text{Col}(A)^\perp = \text{Null}(A^T)$ to an appropriately chosen matrix A.] Then find an orthonormal basis for each.

(2) With W as in question 1, Compute $\text{Proj}_W \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and $\text{Proj}_{W^\perp} \begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$.

(3) With W as in question 1, what is the minimum distance between $\begin{pmatrix} 3 \\ 2 \\ 4 \end{pmatrix}$ and W?

(4) Let $A = \begin{pmatrix} 1 & 2 & 1 \\ 3 & 2 & 3 \\ 2 & 4 & 2 \\ 1 & 1 & 1 \end{pmatrix}$. Find a basis for $\text{Null}(A)^\perp$. [**Hint**: use the fact that $\text{Null}(A)^\perp = \text{Col}(A^T)$.]

(5) [During section, move on from this problem if you're stuck. But you should understand the results and at least look at the solution later.]

Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3 \in \mathbb{R}^3$. Let $A = (\mathbf{a}_1 \ \mathbf{a}_2 \ \mathbf{a}_3)$ and $B = (\mathbf{b}_1 \ \mathbf{b}_2 \ \mathbf{b}_3)$. Explain why

$$A^T B = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 & \mathbf{a}_1 \cdot \mathbf{b}_3 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 & \mathbf{a}_2 \cdot \mathbf{b}_3 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 & \mathbf{a}_3 \cdot \mathbf{b}_3 \end{pmatrix}.$$

In particular,

$$A^T A = \begin{pmatrix} \mathbf{a}_1 \cdot \mathbf{a}_1 & \mathbf{a}_1 \cdot \mathbf{a}_2 & \mathbf{a}_1 \cdot \mathbf{a}_3 \\ \mathbf{a}_2 \cdot \mathbf{a}_1 & \mathbf{a}_2 \cdot \mathbf{a}_2 & \mathbf{a}_2 \cdot \mathbf{a}_3 \\ \mathbf{a}_3 \cdot \mathbf{a}_1 & \mathbf{a}_3 \cdot \mathbf{a}_2 & \mathbf{a}_3 \cdot \mathbf{a}_3 \end{pmatrix}.$$

Use this to explain why a matrix is orthogonal if and only if the columns of the matrix form an orthonormal set.

By the way, notice that $A^T A$ is always **symmetric**! (*Recall*: a matrix M is by definition symmetric if $M = M^T$.) What is an easier way to prove that $A^T A$ is symmetric?

(6) Let $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3 \in \mathbb{R}^3$. Let $A = (\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3)$. Use the result of the last problem to write $\|\mathbf{a}_1 + \mathbf{a}_3\|$ in terms of the entries of $A^T A$. In other words, suppose

$$A^T A = \begin{pmatrix} c_{11} & c_{12} & c_{13} \\ c_{12} & c_{22} & c_{23} \\ c_{13} & c_{23} & c_{33} \end{pmatrix},$$

and come up with a formula for $\|\mathbf{a}_1 + \mathbf{a}_3\|$ in terms of the c_{ij}.

(7) Recall the fact that $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$ for any matrix A. Why does this imply the fact that $\mathrm{Null}(A)^\perp = \mathrm{Col}(A^T)$ for any matrix A?

(8) **Bonus** (for zero points). Suppose that A is a symmetric matrix, and suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors of A with corresponding eigenvalues λ_1 and λ_2. Moreover, suppose that $\lambda_1 \neq \lambda_2$. Show that $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal.