Math 54: Worksheet

February 28

Change of basis

(1) Let $\mathcal{E}$ be the standard basis for $\mathbb{R}^n$. Explain why $(\mathbf{x})_{\mathcal{E}} = \mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^n$.

(2) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for $\mathbb{R}^n$. Write $P_{\mathcal{E} \leftarrow \mathcal{B}}$ in terms of $\mathbf{b}_1, \ldots, \mathbf{b}_n$.

(3) Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ be bases for $\mathbb{R}^n$. Show that $P_{\mathcal{C} \leftarrow \mathcal{B}} P_{\mathcal{B} \leftarrow \mathcal{A}} = P_{\mathcal{C} \leftarrow \mathcal{A}}$.

(4) Use the previous question to show that if $\mathcal{B}$, $\mathcal{C}$ are bases for $\mathbb{R}^n$, then $P_{\mathcal{B} \leftarrow \mathcal{C}} = (P_{\mathcal{C} \leftarrow \mathcal{B}})^{-1}$. (Hint: apply the result of the previous question in the special case $\mathcal{A} = \mathcal{C}$.)

(5) Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ be bases for $\mathbb{R}^n$. Use the results of questions 2, 3, and 4 to show that $P_{\mathcal{C} \leftarrow \mathcal{B}} = (\mathbf{c}_1 \cdots \mathbf{c}_n)^{-1} (\mathbf{b}_1 \cdots \mathbf{b}_n)$.

[For a challenge, connect this result with our usual way of computing change of basis matrices, i.e., row-reducing $(\mathbf{c}_1 \cdots \mathbf{c}_n | \mathbf{b}_1 \cdots \mathbf{b}_n) \longrightarrow (I_n | P_{\mathcal{C} \leftarrow \mathcal{B}})$. Which is more practical?]

With a side of eigenvectors

(1) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$. Compute the characteristic polynomial, eigenvalues, and eigenspaces of A.

(2) In the last question, you should have found two linearly independent eigenvectors. Call them $\mathbf{b}_1$ and $\mathbf{b}_2$ (where $\mathbf{b}_2$ has the smaller eigenvalue), and let $\mathcal{B}$ be the basis $\{\mathbf{b}_1, \mathbf{b}_2\}$ for $\mathbb{R}^2$. Compute $P_{\mathcal{E} \leftarrow \mathcal{B}}$ and $P_{\mathcal{B} \leftarrow \mathcal{E}}$.

(3) Now compute $D := P_{\mathcal{B} \leftarrow \mathcal{E}} A P_{\mathcal{E} \leftarrow \mathcal{B}}$. For short, we can write $P = P_{\mathcal{E} \leftarrow \mathcal{B}}$, so we are computing $D = P^{-1} A P$. Explain the result using the definition of eigenvectors.

(4) Congratulations! You have diagonalized a matrix. Together with matrix inversion (which corresponds to solving linear systems), matrix diagonalization (or more generally, computing eigenvalues and eigenvectors) is the numerical workhorse of the applied sciences. Many people work very hard trying to make these operations faster!