Math 54: Worksheet Solutions

January 31

(1) Solve the sytem

$$\begin{cases} x_1 + 3x_2 - 5x_3 = 4 \\ x_1 + 4x_2 - 8x_3 = 7 \\ -3x_1 - 7x_2 + 9x_3 = -6. \end{cases}$$

Give a geometric interpretation of the solution. Are there other equivalent ways to write down your solution? Can you characterize all possible ways the solution could be written down? (Hint: your answer should have to do with the corresponding *homogeneous* system.)

Solution: Solve the system by writing down the augmented matrix and row-reducing:

$$\begin{bmatrix} 1 & 3 & -5 & 4 \\ 1 & 4 & -8 & 7 \\ -3 & -7 & 9 & -6 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 - R_1} \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ -3 & -7 & 9 & -6 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 + 3R_1} \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 2 & -6 & 6 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - 2R_1} \begin{bmatrix} 1 & 3 & -5 & 4 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_1 \leftarrow R_1 - 3R_2} \begin{bmatrix} 1 & 0 & 4 & -5 \\ 0 & 1 & -3 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus x_3 is a free variable and a general solution can be written in the form

$$\begin{bmatrix} -4x_3 - 5 \\ 3x_3 + 3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} -5 \\ 3 \\ 0 \end{bmatrix}.$$

This means that, geometrically, the solution set is a line. In particular, it is the line passing through the point $(-5, 3, 0)$ in the direction $(-4, 3, 1)$. We could also describe the solution set as the line passing through *any* one particular solution in a direction given by any *nontrivial* solution of the homogeneous system. (More generally, the solution set of any consistent linear system is given by the solution set of the corresponding homogeneous system, translated by any particular solution.)

(2) What is the difference between a matrix of size $m \times n$ and a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$? Given such a matrix, how do we define a corresponding linear transformation? Given such a linear transformation, how do we define a corresponding matrix?

Solution: A matrix of size $m \times n$ is an array of numbers with m rows and n columns. A linear transformation is a function $\mathbb{R}^n \to \mathbb{R}^m$.

To any $m \times n$ matrix A we can associate a linear transformation T_A via the rule $T_A(\mathbf{x}) = A\mathbf{x}$. To any linear transformation $T : \mathbb{R}^n \to \mathbb{R}^m$ we can associate a matrix A_T (the *standard matrix* of the linear transformation) via the rule

$$A_T = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix},$$

where $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are the standard basis vectors of $\mathbb{R}^n$.

(3) Can there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$?

Solution: If there were such a linear transformation, it would have a standard matrix $A_T = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) \end{bmatrix}$. Let us define $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note that since we know the values $T(\mathbf{u})$ and $T(\mathbf{v})$, if we can write $\mathbf{e}_1$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, we can find out what $T(\mathbf{e}_1)$ has to be by linearity. In fact, $\mathbf{e}_1 = -\mathbf{u} + 2\mathbf{v}$, so

$$T(\mathbf{e}_1) = T(-\mathbf{u} + 2\mathbf{v}) = -T(\mathbf{u}) + 2T(\mathbf{v}) = -\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Similarly, we can write $\mathbf{e}_2 = \mathbf{u} - \mathbf{v}$, so

$$T(\mathbf{e}_2) = T(\mathbf{u} - \mathbf{v}) = T(\mathbf{u}) - T(\mathbf{v}) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Therefore we must have

$$A_T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

You can check that this matrix does indeed define a linear transformation T with the specified values, so the answer is yes.

(4) Can there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$?

Solution: Notice that

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

so if there were such a linear transformation, then it would satisfy

$$\begin{aligned} T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \\ &= T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) - T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}. \end{aligned}$$

But it was supposed to satisfy $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, so the answer is no.

(5) Can there exist a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfies $T\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, and $T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

Solution: Omitted. Try to figure this out based on the last two questions.

(6) Suppose that $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are vectors in $\mathbb{R}^3$ and your friend tells you the values $T(\mathbf{u}_1)$, $T(\mathbf{u}_2)$, and $T(\mathbf{u}_3)$ of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$. First of all, is it possible that there does not actually exist a linear transformation with these values? If not, has your friend given you enough information to determine the linear transformation uniquely?

Solution: If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly dependent, it may not be possible to find such a linear transformation. If $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ are linearly independent, then there is a unique linear transformation that satisfies the given data. Try to think about why this is true! (This is a tough one.)

(7) Suppose that A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b} \in \mathbb{R}^m$. Which of the following are possible: $m < n$, $m = n$, $m > n$.

Solution: Recall that $A\mathbf{x} = \mathbf{b}$ is consistent for all $\mathbf{b}$ if and only if the columns of A span $\mathbb{R}^m$. There are n columns, so if $m > n$, this is impossible. Otherwise, it is possible.

(8) Suppose that A is an $m \times n$ matrix and the system $A\mathbf{x} = \mathbf{b}$ is has a unique solution for all $\mathbf{b} \in \mathbb{R}^m$. Which of the following are possible: $m < n$, $m = n$, $m > n$.

Solution: Note that if $A\mathbf{x} = \mathbf{b}$ has a unique solution, then in particular, the system is consistent. So by the last question, $m > n$ is impossible. Also, for there to be a unique solution, we need to have that the columns of A are linearly independent. If $m < n$ then since we are asking for n vectors in $\mathbb{R}^m$ to be linearly independent, this is also impossible. This leaves the case $m = n$. Check that this case is indeed possible! (One need only find a matrix A with linearly independent columns that span $\mathbb{R}^m$.)

(9) **True or false:** if A is an $m \times n$ matrix with $n > m$ and $\mathbf{b} \in \mathbb{R}^m$, then the system $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions.

Solution: False. The system might be inconsistent! (You should find a specific counterexample.)

(10) **True or false:** if A is an $m \times n$ matrix with $n > m$, then the system $A\mathbf{x} = \mathbf{0}$ has infinitely many solutions.

Solution: True. The system is always consistent because $\mathbf{x} = 0$ is a solution. So the question is whether or not A has any free columns. It can have at most m pivot columns (since this is the number of rows). Since $m < n$, there must be at least one free column.

(11) Suppose that A is an $m \times n$ matrix with $n > m$. Can the corresponding linear transformation T_A be onto but not one-to-one? Can it be one-to-one but not onto? Can it be both? Can it be neither?

Solution: It can never be one-to-one, but it can be onto. Try to relate this question to questions 7 and 8. It can also be neither! (To see this, consider the zero matrix of size $m \times n$.) So the answers are yes, no, no, yes.

(12) Same question but for $n < m$.

Solution: It can never be onto, but it can be one-to-one. Try to relate this question to questions 7 and 8. It can also be neither! (To see this, consider the zero matrix of size $m \times n$.) So the answers are no, yes, no, yes.

(13) Same question but for $n = m$.

Solution: Using the reasoning from questions 8 and 9, we can see that it is possible for it to be one-to-one and it is possible for it to be onto. However, is it possible to be one of these and not the other *at the same time*? Actually, no! Note that T_A being onto is the same as the columns of A spanning $\mathbb{R}^n$ (note that $n = m$ here), and T_A being one-to-one is the same as the columns of A being linearly independent. But n vectors in $\mathbb{R}^n$ span $\mathbb{R}^n$ if and only if they are linearly independent, so onto and one-to-one are actually equivalent for linear transformations $\mathbb{R}^n \to \mathbb{R}^n$. Lastly, note that again the zero matrix yields a linear transformation that is neither. So the answers are no, no, yes, yes.