Math 54: Worksheet Solutions

January 22

Mark the boxes that correspond to possible scenarios, i.e., if there exists a linear system with more equations than unknowns and no solutions, then mark the corresponding box. If there does not exist such a linear system, leave the box blank.

	inconsistent	**consistent**	
	no solutions	exactly one solution	infinitely many solutions
# eq. > # unknowns	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
# eq. = # unknowns	$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 1 & 9 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 7 \\ 0 & 0 & 0 \end{bmatrix}$
# eq. < # unknowns	$\begin{bmatrix} 0 & 0 & 7 \end{bmatrix}$		$\begin{bmatrix} 1 & 0 & 7 \end{bmatrix}$

All boxes **except one** (# eq. < # unknowns, exactly one solution) should be marked! In each box we provide the **augmented matrix** of a system that fits the description. (Try to think about what the corresponding linear system is, even if it seems completely trivial.) To see why the blank box is blank, see question 7 below.

Questions

(1) In words, how does one go about determining how many solutions a linear system has (without solving it fully)?

Solution: Put the augmented matrix of the linear system into echelon form. If the last column is a pivot, then the system has no solutions (inconsistent) and we are done. Otherwise, the system is consistent. In this case, if there are no free columns, then the system has exactly one solution, while if there is at least one free column, then the system has infinitely many solutions.

(2) For what values of h is the matrix $\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix}$ the augmented matrix of a consistent linear system?

Solution: Let's put the matrix into echelon form.

$$\begin{bmatrix} 1 & h & -3 \\ -2 & 4 & 6 \end{bmatrix} \xrightarrow{R_2 \leftarrow R_2 + 2R_1} \begin{bmatrix} 1 & h & -3 \\ 0 & 2h+4 & 0 \end{bmatrix}.$$

The matrix is now in echelon form. There are two cases: (1) if $h = -2$, then the matrix is

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Note that the last column is not a pivot column, so the system is consistent. In the second case (2), $h \neq -2$, and the first and second columns are both pivot columns, but the last column still isn't a pivot column! So the system is still consistent. Thus the system is consistent for all h.

(3) Put the following matrices into reduced echelon form, and give the solution set of the corresponding linear system (i.e., the linear system whose augmented matrix is the given matrix).

$$
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ -2 & 3 & 2 & 1 & 5 \end{bmatrix}
\xrightarrow{R_4 \leftarrow R_4 + 2R_1}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 2 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}
$$

$$
\xrightarrow{R_2 \leftarrow \frac{1}{2} R_2}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 3 & 2 & -3 & -1 \end{bmatrix}
$$

$$
\xrightarrow{R_4 \leftarrow R_4 - 3R_2}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 1 \\ 0 & 0 & -1 & -3 & -1 \end{bmatrix}
$$

$$
\xrightarrow{R_3 \leftrightarrow R_4}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 2 & 3 & 1 \end{bmatrix}
$$

$$
\xrightarrow{R_4 \leftarrow R_4 + 2R_3}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -3 & -1 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix} \quad \text{(echelon form)}
$$

$$
\xrightarrow{R_3 \leftarrow R_3 - R_4}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix}
$$

$$
\xrightarrow{R_3 \leftarrow -R_3}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -3 & -1 \end{bmatrix}
$$

$$
\xrightarrow{R_4 \leftarrow -\frac{1}{3} R_4}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}
$$

$$
\xrightarrow{R_2 \leftarrow R_2 - R_3}
\begin{bmatrix} 1 & 0 & 0 & -2 & -3 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix}
$$

$$
\xrightarrow{R_1 \leftarrow R_1 + 2R_4}
\begin{bmatrix} 1 & 0 & 0 & 0 & -\frac{7}{3} \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & \frac{1}{3} \end{bmatrix} \quad \text{(reduced echelon form)}
$$

Thus the corresponding linear system has a unique solution, namely $(x_1, x_2, x_3, x_4) = \left(-\frac{7}{3}, 0, 0, \frac{1}{3}\right)$.

$$\begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ -1 & 7 & -4 & 2 & 7 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 + R_1} \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & -4 & 8 & 12 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -\frac{1}{4} R_3} \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 1 & -2 & -3 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow R_3 - R_2} \begin{bmatrix} 1 & -7 & 0 & 6 & 5 \\ 0 & 0 & 1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(reduced echelon form)}$$

Thus x_2 and x_4 are free variables, and the general solution of the corresponding linear system is given by

$$\left\{ \begin{array}{c} x_1 = 7x_2 - 6x_4 + 5 \\ x_2 \text{ free} \\ x_3 = 2x_4 - 3 \\ x_4 \text{ free} \end{array} \right\}.$$

Now consider the last matrix:

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 9 & 8 & 7 & 6 & 5 \\ 14 & 13 & 12 & 11 & 10 \\ 20 & 18 & 17 & 16 & 15 \end{bmatrix}.$$

We will reduce it a bit more casually. When we subtract the first row from the second, third, and fourth rows, we get

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 5 & 5 & 5 & 5 & 5 \\ 10 & 10 & 10 & 10 & 10 \\ 16 & 15 & 15 & 15 & 15 \end{bmatrix}.$$

Try to convince yourself that we can then row-reduce to

$$\begin{bmatrix} 4 & 3 & 2 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then by subtracting the last row from the first and second rows, and then rearranging, we obtain

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 3 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{R_3 \leftarrow R_3 - 3R_1} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(echelon form)}$$

$$\xrightarrow{R_2 \leftarrow R_2 + R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & -1 & -2 & -3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{R_3 \leftarrow -R_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & -2 \\ 0 & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \quad \text{(reduced echelon form)}$$

Thus x_4 is a free variable, and the general solution of the corresponding linear system is given by

$$\left\{ \begin{array}{c} x_1 = 0 \\ x_2 = x_4 - 2 \\ x_3 = -2x_4 + 3 \\ x_4 \text{ free} \end{array} \right\}.$$

(4) **True or false:** If one row in an echelon form of an augmented matrix is

$$\begin{bmatrix} 0 & 0 & 0 & 2 & 0 \end{bmatrix},$$

then the associated linear system is inconsistent.

Solution: Consider the 1×5 matrix $\begin{bmatrix} 0 & 0 & 0 & 2 & 0 \end{bmatrix}$. This corresponds to a consistent linear system, so the statement is false.

(5) **True or false:** If the last column of the augmented matrix of a linear system is a pivot column, then the system is inconsistent.

Solution: This is true (as discussed in question 1).

(6) If a linear system is consistent, then the solution is unique if and only if ___________. (Fill in the blank with something about pivot columns.)

Solution: all of the columns (except the last one) of the augmented matrix of the linear system are pivot columns

(7) **Always/sometimes/never:** A consistent linear system with more unknowns than equations ___________ has infinitely many solutions.

Solution: *always*. This corresponds to the only blank box in the table question at the beginning of the worksheet. To see why, note that *consistent* is equivalent to the last column of the augmented matrix *not* being a pivot column. A matrix cannot have more pivot columns than rows, so if the system has more unknowns than equations, the *coefficient matrix* has more columns than rows, hence more columns than pivot columns. Thus there is at least one free variable, and the system has infinitely many solutions.