# Math 54: Quiz #9 Solution

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

## Problem 1

Find the solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$, where

$$A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix},$$

that satisfies the initital condition $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$.

**Solution:** First compute the eigenvalues of $A$:

$$\det(A - \lambda I) = \det \begin{pmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & -1 \\ 0 & 1 & -\lambda \end{pmatrix} = (-\lambda)(\lambda^2 + 1),$$

so the eigenvalues are $\lambda_1 = 0$, $\lambda_2 = i$, $\lambda_3 = -i$.

Then compute the eigenvector for $\lambda_1$:

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \to \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

so $\text{Null}(A - \lambda_1 I) = \text{span} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, and we can let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ be our eigenvector.

Then we obtain a solution

$$\mathbf{x}_1(t) = e^{\lambda_1 t}\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

This is actually a constant function, but that's okay.

Now $\lambda_2$ and $\lambda_3$ are a complex-conjugate pair, and we will pick $\lambda_2 = i$ and use it to obtain two more solutions:

$$A - \lambda_2 I = \begin{pmatrix} -i & 0 & 1 \\ 0 & -i & -1 \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -i \\ 0 & 1 & -i \end{pmatrix} \to \begin{pmatrix} 1 & 0 & i \\ 0 & 1 & -i \\ 0 & 0 & 0 \end{pmatrix},$$

so $\text{Null}(A - \lambda_2 I) = \text{span}\begin{pmatrix} -i \\ i \\ 1 \end{pmatrix}$. Then let

$$\mathbf{v} = \begin{pmatrix} -i \\ i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} + i\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} = \mathbf{a} + i\mathbf{b},$$

where $\mathbf{a} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$. Note that we can write the eigenvalue $\lambda_2 = \alpha + i\beta$, where $\alpha = 0$ and $\beta = 1$. Then we use the formula to write down two solutions:

$$\mathbf{x}_2(t) = e^{\alpha t}\left[\cos(\beta t)\mathbf{a} - \sin(\beta t)\mathbf{b}\right] = \begin{pmatrix} \sin(t) \\ -\sin(t) \\ \cos(t) \end{pmatrix}$$

$$\mathbf{x}_3(t) = e^{\alpha t}\left[\sin(\beta t)\mathbf{a} + \cos(\beta t)\mathbf{b}\right] = \begin{pmatrix} -\cos(t) \\ \cos(t) \\ \sin(t) \end{pmatrix}.$$

Then a general solution is given by

$$\begin{aligned} \mathbf{x}(t) &= c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + c_3\mathbf{x}_3(t) \\ &= c_1\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + c_2\begin{pmatrix} \sin(t) \\ -\sin(t) \\ \cos(t) \end{pmatrix} + c_3\begin{pmatrix} -\cos(t) \\ \cos(t) \\ \sin(t) \end{pmatrix} \\ &= \begin{pmatrix} c_1 + c_2\sin(t) - c_3\cos(t) \\ -c_2\sin(t) + c_3\cos(t) \\ c_2\cos(t) + c_3\sin(t) \end{pmatrix} \end{aligned}$$

Now we use the initial condition to determine the values of $c_1, c_2, c_3$. From the above general solution, we have that

$$\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \mathbf{x}(0) = \begin{pmatrix} c_1 + c_2\sin(0) - c_3\cos(0) \\ -c_2\sin(0) + c_3\cos(0) \\ c_2\cos(0) + c_3\sin(0) \end{pmatrix} = \begin{pmatrix} c_1 - c_3 \\ c_3 \\ c_2 \end{pmatrix}.$$

Therefore $c_2 = 1$, $c_3 = 1$, and $c_1 = 2$. Thus our solution is

$$\mathbf{x}(t) = \begin{pmatrix} 2 + \sin(t) - \cos(t) \\ -\sin(t) + \cos(t) \\ \cos(t) + \sin(t) \end{pmatrix}.$$