Math 54: Quiz #8 Solution

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Find the general solution of

$$y''(t) - y'(t) - 6y(t) = 0.$$

Solution: Solve the auxiliary equation:

$$0 = r^2 - r - 6 = (r-3)(r+2),$$

so we obtain two roots $r_1 = 3$, $r_2 = -2$. Then a general solution is given by

$$y(t) = C_1 e^{3t} + C_2 e^{-2t}.$$

Problem 2

Find the general solution of

$$y''(t) - y'(t) - 6y(t) = \cos(t) + e^t.$$

(Use the superposition principle.)

Solution: First compute a particular solution $y_p^{(1)}$ of

$$y''(t) - y'(t) - 6y(t) = \cos(t).$$

We make the guess

$$y_p^{(1)} = A\cos(t) + B\sin(t),$$

and we want

$$\begin{aligned}
\cos(t) &= \left(y_p^{(1)}\right)''(t) - \left(y_p^{(1)}\right)'(t) - 6y_p^{(1)}(t) \\
&= [-A\cos(t) - B\sin(t)] - [-A\sin(t) + B\cos(t)] - 6\,[A\cos(t) + B\sin(t)] \\
&= (-7A - B)\cos(t) + (-7B + A)\sin(t).
\end{aligned}$$

Therefore we want to solve

$$\begin{cases} -7A - B = 0 \\ -7B + A = 0. \end{cases}$$

The second equation gives $A = 7B$; substituting into the first yields $-49B - B = 0$, so $B = -\frac{1}{50}$, and $A = -\frac{7}{50}$, and we have found

$$y_p^{(1)} = -\frac{7}{50}\cos(t) - \frac{1}{50}\sin(t).$$

Next we want to compute a particular solution $y_p^{(2)}$ of

$$y''(t) - y'(t) - 6y(t) = e^t.$$

We make the guess

$$y_p^{(2)} = Ce^t,$$

and we want

$$\begin{aligned}
e^t &= \left(y_p^{(2)}\right)''(t) - \left(y_p^{(2)}\right)'(t) - 6y_p^{(2)}(t) \\
&= Ce^t - Ce^t - 6Ce^t \\
&= -6Ce^t.
\end{aligned}$$

Therefore we want to solve $1 = -6C$, so $C = -\frac{1}{6}$. Then we have found

$$y_p^{(2)} = -\frac{1}{6}e^t.$$

Then, by the superposition principle, $y_p^{(1)} + y_p^{(2)}$ is a particular solution of

$$y''(t) - y'(t) - 6y(t) = \cos(t) + e^t.$$

Moreover, recall that we found the general solution to the homogeneous equation in the last problem. Then a general solution can be written

$$y(t) = C_1e^{3t} + C_2e^{-2t} - \frac{7}{50}\cos(t) - \frac{1}{50}\sin(t) - \frac{1}{6}e^t.$$