Math 54: Quiz #7 Solution

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Compute a singular value decomposition (SVD) of

$$A = \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix},$$

making it clear what it means to find an SVD for A and how you have done so.

Solution: Compute

$$A^T A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}.$$

Then $\mathbf{v}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ are orthonormal eigenvectors of $A^T A$ with eigenvalues 3 and 2, respectively, hence these are right singular vectors for A with singular values $\sigma_1 = \sqrt{3}$ and $\sigma_2 = \sqrt{2}$, respectively, and we form the matrix $V = (\mathbf{v}_1\ \mathbf{v}_2) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. (Note that the ordering is chosen so that $\sigma_1 \geq \sigma_2$.)

Then define $\mathbf{u}_1 = \frac{1}{\sigma_1} A\mathbf{v}_1$, $\mathbf{u}_2 = \frac{1}{\sigma_2} A\mathbf{v}_2$, so

$$\mathbf{u}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

and

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 0 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

These are our first two left singular vectors. We need a third singular vector $\mathbf{u}_3$ that is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$ and of length 1. Note that the orthogonality conditions $\mathbf{u}_1 \cdot \mathbf{u}_3 = 0$ and $\mathbf{u}_2 \cdot \mathbf{u}_3 = 0$ are equivalent to the system

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \mathbf{u}_3 = \mathbf{0}.$$

To solve for $\mathbf{u}_3$, we row-reduce

$$\begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{pmatrix}.$$

So the solution space is

$$\text{span}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

We need to find a vector in this span that is of length 1, so simply normalize $\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$ to obtain

$$\mathbf{u}_3 = \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}.$$

Then our SVD is given by

$$A = U\Sigma V^T = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} \sqrt{3} & 0 \\ 0 & \sqrt{2} \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^T.$$