Math 54: Quiz #6 Solutions

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Let

$$A = \begin{pmatrix} 3 & 0 & 1 \\ 0 & 3 & 0 \\ 1 & 0 & 3 \end{pmatrix}.$$

Diagonalize A (making it clear what it means to diagonalize A and how you have done so). Check that eigenvectors of A with distinct eigenvalues are orthogonal.

Solution: Compute the characteristic polynomial via cofactor expansion along the second column:

$$\begin{aligned}
\det(A - \lambda I) &= \det \begin{pmatrix} 3-\lambda & 0 & 1 \\ 0 & 3-\lambda & 0 \\ 1 & 0 & 3-\lambda \end{pmatrix} \\
&= -0 + (3-\lambda)\det\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} - 0 \\
&= (3-\lambda)\left[(3-\lambda)^2 - 1\right] \\
&= (3-\lambda)\left[\lambda^2 - 6\lambda + 9 - 1\right] \\
&= (3-\lambda)\left[\lambda^2 - 6\lambda + 8\right] \\
&= (3-\lambda)(\lambda-4)(\lambda-2).
\end{aligned}$$

Therefore the eigenvalues are $\lambda_1 = 3$, $\lambda_2 = 4$, $\lambda_3 = 2$. (One could also choose different orderings, corresponding to different valid diagonalizations.)

Then we compute the null spaces of $A - \lambda_1 I$, $A - \lambda_2 I$, and $A - \lambda_3 I$:

$$A - \lambda_1 I = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

A general solution of the homogeneous equation is $x_1 = 0$, x_2 free, $x_3 = 0$. Thus

$$\mathrm{Null}(A - \lambda_1 I) = \mathrm{span}\left(\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}\right).$$

In particular we can choose an eigenvector $\mathbf{v}_1 = (0,1,0)^T$.

Next,

$$A - \lambda_2 I = \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 0 \\ 1 & 0 & -1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A general solution of the homogeneous equation is $x_1 = x_3$, $x_2 = 0$, x_3 free. Thus

$$\mathrm{Null}(A - \lambda_3 I) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}\right).$$

In particular we can choose an eigenvector $\mathbf{v}_2 = (1, 0, 1)^T$.

Finally,

$$A - \lambda_3 I = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

A general solution of the homogeneous equation is $x_1 = -x_3$, $x_2 = 0$, x_3 free. Thus

$$\mathrm{Null}(A - \lambda_3 I) = \mathrm{span}\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\right).$$

In particular we can choose an eigenvector $\mathbf{v}_3 = (1, 0, -1)^T$.

We can compute directly that $\mathbf{v}_1 \cdot \mathbf{v}_2 = \mathbf{v}_1 \cdot \mathbf{v}_3 = \mathbf{v}_2 \cdot \mathbf{v}_3 = 0$, so eigenvectors with distinct eigenvalues are orthogonal. Moreover, we can read off the diagonalization

$$A = PDP^{-1},$$

where

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \quad P = (\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{pmatrix}.$$

Problem 2

Let

$$A = \begin{pmatrix} 3 & 1 \\ 1 & 2 \end{pmatrix}.$$

And consider the basis

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$$

of $\mathbb{R}^2$. (Note that the ordering of the basis matters!) Compute the matrix of A relative to the basis $\mathcal{B}$, i.e., $[A]_{\mathcal{B},\mathcal{B}}$.

Solution: Write $\mathbf{b}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\mathbf{b}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, so $\mathcal{B} = \{\mathbf{b}_1, \mathbf{b}_2\}$. Compute

$$[A]_{\mathcal{B},\mathcal{B}} = [[A\mathbf{b}_1]_{\mathcal{B}}\ [A\mathbf{b}_2]_{\mathcal{B}}] = \left[\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}}\ \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]_{\mathcal{B}}\right].$$

Note that $\begin{pmatrix} 1 \\ 2 \end{pmatrix} = 2\mathbf{b}_1 + 1\mathbf{b}_2$, and $\begin{pmatrix} 3 \\ 1 \end{pmatrix} = 1\mathbf{b}_1 + 3\mathbf{b}_2$, so

$$\left[\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad \left[\begin{pmatrix} 3 \\ 1 \end{pmatrix}\right]_{\mathcal{B}} = \begin{pmatrix} 1 \\ 3 \end{pmatrix},$$

and we can read off the answer as

$$[A]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}.$$