Math 54: Quiz #5 Solutions

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Please give neat and organized answers. Whenever applicable (especially for computational questions), make it clear what strategy you are using.

Problem 1

Find a matrix that is not diagonalizable. (Clearly justify that it is not diagonalizable.)

Solution: Some examples include

$$A_1 := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; A_2 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

One can check that A_1 has 0 as its only eigenvalue (with algebraic multiplicity 2), but the geometric multiplicity of this eigenvalue is only 1.

Meanwhile, the characteristic polynomial of A_2 has no real roots. (Check this!)

Problem 2

Let A be a diagonalizable $n \times n$ matrix, and let Q be an invertible matrix. Show that $Q^{-1}AQ$ is diagonalizable.

Solution: Since A is diagonalizable, this means precisely that we can write $A = P^{-1}DP$, where P is invertible and D is diagonal. Then

$$Q^{-1}AQ = Q^{-1}P^{-1}DPQ.$$

Recall that since P and Q are invertible, PQ is invertible with $(PQ)^{-1} = Q^{-1}P^{-1}$. Therefore

$$Q^{-1}AQ = (PQ)^{-1}D(PQ).$$

Now we have diagonalized the matrix $Q^{-1}AQ$, so we are done.

Bonus: Suppose that A and B are $n \times n$ matrices and that A is invertible. Show that the characteristic polynomial of AB is the same as the characteristic polynomial of BA.

Solution: The characteristic polynomial of AB is

$$\begin{aligned}
\det(AB - \lambda I) &= \det\left(A(B - \lambda A^{-1})\right) \\
&= \det(A)\det(B - \lambda A^{-1}) \\
&= \det(B - \lambda A^{-1})\det(A) \\
&= \det\left((B - \lambda A^{-1})A\right) \\
&= \det\left(BA - \lambda I\right),
\end{aligned}$$

which is the characteristic polynomial of BA.